

PHYSICAL REVIEW A **93**, 052111 (2016)**Class of exact memory-kernel master equations**Salvatore Lorenzo,<sup>1,2</sup> Francesco Ciccarello,<sup>3</sup> and G. Massimo Palma<sup>3</sup><sup>1</sup>*Dipartimento di Fisica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy*<sup>2</sup>*INFN, Gruppo Collegato di Cosenza, Cosenza, Italy*<sup>3</sup>*NEST, Istituto Nanoscienze, Consiglio Nazionale delle Ricerche and Dipartimento di Fisica e Chimica, Università degli Studi di Palermo, Via Archirafi 36, I-90123 Palermo, Italy*

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A well-known situation in which a non-Markovian dynamics of an open quantum system  $S$  arises is when this is coherently coupled to an auxiliary system  $M$  in contact with a Markovian bath. In such cases, while the joint dynamics of  $S$ - $M$  is Markovian and obeys a standard (bipartite) Lindblad-type master equation (ME), this is in general not true for the reduced dynamics of  $S$ . Furthermore, there are several instances (e.g., the dissipative Jaynes-Cummings model) in which a *closed* ME for the  $S$ 's state *cannot* even be worked out. Here, we find a class of bipartite Lindblad-type MEs such that the reduced ME of  $S$  can be derived exactly and in a closed form for any initial product state of  $S$ - $M$ . We provide a detailed microscopic derivation of our result in terms of a mapping between two collision models.

DOI: [10.1103/PhysRevA.93.052111](https://doi.org/10.1103/PhysRevA.93.052111)**I. INTRODUCTION**

The ability to manipulate a single, low-dimensional quantum system  $S$  is key to the emerging field of quantum technologies. However, due to its unavoidable interaction with the surrounding environment, the dynamics of any realistic quantum system  $S$  is *open*, i.e., non-unitary, and this is typically detrimental to the effectiveness of such technologies. A thorough and reliable description of the dynamics of an open quantum system [1] is thus of utmost importance, especially in the case of non-Markovian dynamics, as witnessed by the strong current interest in this topic [2].

In the best case, a full description of the reduced quantum dynamics of  $S$  can be given in terms of a closed, well-behaved, master equation (ME) with the density matrix of  $S$  as the only unknown. Often, however, this is not the case especially for non-Markovian open dynamics. A paradigmatic and relatively simple example is the decay of an atom in a single-mode lossy cavity as described by the well-known dissipative Jaynes-Cummings model. The bipartite atom-mode dynamics is governed by a Kossakowski-Lindblad ME [1], featuring a Hamiltonian term (depending on the atom-mode Hamiltonian) and a Lindbladian dissipator acting on the cavity mode. In this case, tracing out the mode degrees of freedom *does not* lead to a closed ME for the atom [3]. The dissipative Jaynes-Cummings model can be regarded as an instance of a bipartite Lindblad ME in which a quantum system  $S$  is coherently coupled to a second one  $M$ , the “memory,” the latter interacting with a Markovian bath according to an associated Lindbladian superoperator acting on  $M$  only.

In this paper, we present a class of MEs of the type discussed above which, upon trace over the  $M$ 's degrees of freedom and for any initial  $S$ - $M$  product state, yield a closed, exact, ME for  $S$ . This is an integrodifferential memory-kernel ME, defined in terms of the dynamical map of  $S$  that would arise if  $M$  were decoupled from its environment (hence corresponding to a unitary  $S$ - $M$  joint dynamics). The resulting memory-kernel ME is well behaved, meaning that the corresponding dynamics of  $S$  is ensured to be completely positive and trace preserving (CPTP) [1]. Very recently, Chruscinski and Kossakowski [4] studied a parametrization of legitimate memory kernels

entering the general Nakajima-Zwanzig (NZ) ME [1], showing that a number of well-behaved non-Markovian MEs can be arranged in the NZ form. We will show that this is true also for our ME.

Furthermore, this ME is not restricted to the scenario in which the system-environment coupling is mediated via the ancillary degrees of freedom  $M$  but applies to a broader class of non-Markovian dynamics. Indeed, as we will show, the memory-kernel ME for  $S$  discussed in this paper is a generalization of a ME first derived through a collision model of non-Markovian open quantum dynamics [5] (for a different perspective derivation see Ref. [6]). A quantum collision model (CM) [7] is a microscopic framework to describe the open dynamics of a system  $S$  interacting with a reservoir assumed to consist of a large collection of smaller constituents (ancillas). The system is assumed to interact with the environment via a sequence of “collisions” between system and ancillas, each collision being described by the same bipartite quantum map (usually a unitary one). The resulting reduced dynamics of  $S$  is, by construction, a CPTP map. In the limit of weak coupling this leads to a well-behaved ME. CMs have been a useful tool to analyze quantum homogenization and thermalization [8], to derive MEs [9], to study the interaction with small environments and or with random unitaries [10], in quantum thermodynamics [11], and in the study of quantum non-Markovianity [12,13]. Experimentally, a CM can be implemented in all-optical setups [14]. In the second part of this paper, we will illustrate two different environmental memory mechanisms in the context of collision models, both of which lead to reduced open dynamics governed by our ME. We will first show that—when appropriately generalized—the CM without memory  $M$  but with interancillary collisions introduced in Ref. [5] yields, in the continuous-time limit, our memory-kernel ME. In this model, memory effects are to be ascribed to an intraenvironmental incoherent dynamics. Next, we introduce a different CM, with no interancillary collisions, for a bipartite system  $S$ - $M$ , with  $M$  undergoing collisions with the reservoir ancillas. This model mimics situations like the one encountered in the dissipative Jaynes-Cummings model. We will show that both collision models lead to the same discrete open dynamics for  $S$ . Moreover, in the

continuous-time limit, the latter CM leads to the class of bipartite MEs that can be exactly traced over  $M$  to produce our closed memory-kernel ME for  $S$ .

Interestingly, the idea of associating a non-Markovian dynamics with a Lindbladian dynamics on an enlarged space (obtained by adding ancillary degrees of freedom to the system) has been recently investigated [15,16] (see also references therein).

This paper is organized as follows. In Sec. II we formulate without proof our central finding, namely, that a certain class of bipartite Lindbladian MEs—the definition of which is given in detail—under partial trace over  $M$  yields a memory-kernel ME for  $S$ . A direct proof of this result, obtained by carrying out the partial trace of the bipartite ME with the help of the Laplace transform, is given in Sec. III. In Sec. IV, we illustrate a first CM featuring interancillary collisions that leads, in the continuous-time limit, to our reduced memory-kernel ME. In Sec. V, we define a second CM describing the interaction of a bipartite system  $S$ - $M$  with a memoryless reservoir (no interancillary collisions) and show that the resulting discrete dynamics of  $S$  coincides with the one occurring in the CM of Sec. IV. Accordingly, in the continuous-time limit such reduced dynamics is governed by the memory-kernel ME discussed in Secs. II and III. General comments and final conclusions are presented in Sec. VI. Some technical details and proofs are given in the Appendices.

## II. THE CLOSED MEMORY-KERNEL MASTER EQUATION

Let  $S$  be a quantum system of arbitrary dimension the state of which we will denote as “ $\rho$ .” A second quantum system  $M$  (the “memory”) of arbitrary dimension is coupled to  $S$ . Let  $\hat{H}_{SM}$  be the total  $S$ - $M$  Hamiltonian.  $M$  is additionally in contact with a bath such that the evolution of the joint state  $\rho_{SM}(t)$  is governed by the Kossakowski-Lindblad ME (we set  $\hbar = 1$  throughout):

$$\frac{d\rho_{SM}}{dt} = -i[\hat{H}_{SM}, \rho_{SM}] + \Gamma \mathcal{L}_M[\rho_{SM}], \quad (1)$$

where the (dimensionless) Lindblad superoperator  $\mathcal{L}_M$  is given by

$$\mathcal{L}_M[\cdots] = \sum_{\mu\nu} \left( \hat{L}_{\mu\nu} \cdots \hat{L}_{\mu\nu}^\dagger - \frac{1}{2} [\hat{L}_{\mu\nu}^\dagger \hat{L}_{\mu\nu}, \cdots]_+ \right), \quad (2)$$

where  $[\cdots, \cdots]_+$  denotes the anticommutator (the reason for using a double index will become clear soon). In Eq. (2), the jump operators  $\hat{L}_{\mu\nu}$  act on the Hilbert space of  $M$  only, as emphasized by the subscript “ $M$ ” in  $\mathcal{L}_M$ . Physically, Eq. (1) describes the situation in which  $S$  is coherently coupled to  $M$  while the latter is in contact with a Markovian bath. Accordingly, while the joint  $S$ - $M$  dynamics is Markovian, the reduced dynamics of  $S$  is in general non-Markovian. A paradigmatic instance is the well-known dissipative Jaynes-Cummings model [17], where a two-level atom (embodying  $S$ ) is coupled to a lossy cavity mode (embodying  $M$ ).

While—even in cases as (relatively) simple as the aforementioned dissipative Jaynes-Cummins model—a partial trace over the  $M$ ’s degrees of freedom of Eq. (1) does not lead to a closed ME for  $\rho(t) = \text{Tr}\{\rho_{SM}(t)\}$  [3], here, we present a class

of MEs of the form of Eq. (1) where a closed and exact ME for  $\rho(t)$  is found.

Let the initial  $S$ - $M$  state be a product state of the form

$$\rho_{SM}(0) = \rho_0 \otimes \bar{\eta}_M \quad (3)$$

where  $\rho_0$  and  $\bar{\eta}_M$  are arbitrary states of  $S$  and  $M$ , respectively (tensor product symbols will be at times omitted in the remainder of this work). Also, let  $\eta_M$  be an arbitrary state of  $M$  [in general different from  $\bar{\eta}_M$  in Eq. (3)], which we express in a diagonal form in terms of its eigenstates  $\{|v\rangle_M\}$  as

$$\eta_M = \sum_v p_v |v\rangle_M \langle v|, \quad (4)$$

where the probabilities  $\{p_v\}$  are normalized ( $\sum_v p_v = 1$ ).

We will focus on the class of MEs (1) defined by the Lindbladian superoperators  $\mathcal{L}_M$  with associated jump operators

$$\hat{L}_{\mu\nu} = \sqrt{p_\nu} |v\rangle_M \langle \mu|, \quad (5)$$

where  $|\mu\rangle_M$  and  $|v\rangle_M$  are generic eigenstates of  $\eta_M$  while  $p_\nu$  is the eigenvalue corresponding to  $|v\rangle_M$  [cf. Eq. (4)]. Note that the jump operators (5) obey the completeness relation  $\sum_{\mu\nu} \hat{L}_{\mu\nu}^\dagger \hat{L}_{\mu\nu} = \mathbb{1}_M$ , as immediately follows from the completeness of the eigenstates  $|v\rangle_M$  and the normalization of the  $p_\nu$ . Note that  $\mathcal{L}_M$  is defined in terms of the state  $\eta_M$  while  $\hat{H}_{SM}$  [cf. Eq. (1)] is fully arbitrary.

In the next section, we will show that for such a class of bipartite Lindbladian MEs the reduced dynamics of  $S$  obeys exactly the closed memory-kernel ME

$$\begin{aligned} \dot{\rho}(t) = & \Gamma \int_0^t dt' e^{-\Gamma t'} \mathcal{E}(t') [\rho(t-t')] + e^{-\Gamma t} \Phi(t) [\rho_0] \\ & + \Gamma e^{-\Gamma t} [\mathcal{E}(t) - \Phi(t)] [\rho_0], \end{aligned} \quad (6)$$

where  $\Gamma \geq 0$  is a rate while  $\mathcal{E}(t)$  and  $\Phi(t)$  are CPTP quantum maps on  $S$  defined by

$$\mathcal{E}(t)[\rho] = \text{Tr}_M [e^{-i\hat{H}_{SM}t} \rho \otimes \eta_M e^{i\hat{H}_{SM}t}], \quad (7)$$

$$\Phi(t)[\rho] = \text{Tr}_M [e^{-i\hat{H}_{SM}t} \rho \otimes \bar{\eta}_M e^{i\hat{H}_{SM}t}]. \quad (8)$$

ME (6) is a generalization of the memory-kernel ME first introduced in Ref. [5] (see also Ref. [6]). Specifically, the latter is retrieved in the special case  $\mathcal{E}(t) \equiv \Phi(t)$ , namely, for  $\eta_M = \bar{\eta}_M$  [see Eqs. (7) and (8)]. In Appendix A, we will provide a direct proof that the ME (6) entails a CPTP dynamics of  $S$  for any  $\Gamma \geq 0$  and any pair states  $\{\eta_M, \bar{\eta}_M\}$  of  $M$ .

To illustrate the nature of the quantum channel corresponding to the ME (1) and in particular to the Lindbladian  $\mathcal{L}_M$ , consider the case in which  $M$  is a qubit, the Hilbert space of which is spanned by the orthonormal basis  $\{|0\rangle_M, |1\rangle_M\}$ , and  $\eta_M = |0\rangle_M \langle 0|$ . For the sake of simplicity, let us assume here that the  $M$ ’s initial state and the state defining the Lindbladian [see Eqs. (4) and (5)] are the same, i.e.,  $\bar{\eta}_M = \eta_M$ . According to Eq. (5), the Lindbladian  $\mathcal{L}_M$  is then defined by the pair of jump operators

$$\hat{L}_{00} = |0\rangle_M \langle 0| = \frac{\hat{\sigma}_{Mz} + \mathbb{1}_M}{2}, \quad \hat{L}_{10} = |0\rangle_M \langle 1| = \hat{\sigma}_{M-}, \quad (9)$$

where, as usual,  $\hat{\sigma}_{Mz} = |0\rangle_M \langle 0| - |1\rangle_M \langle 1|$  is a Pauli operator,  $\hat{\sigma}_{M-} = \hat{\sigma}_{M+}^\dagger = |0\rangle_M \langle 1|$  are ladder operators, while  $\mathbb{1}_M$  is the

identity. Hence, the Lindbladian  $\mathcal{L}_M$  entering ME (1) takes the explicit form

$$\begin{aligned} \mathcal{L}_M[\rho_{SM}] = & \left( \hat{\sigma}_{M-} \rho_{SM} \hat{\sigma}_{M+} - \frac{1}{2} [\hat{\sigma}_{M+} \hat{\sigma}_{M-}, \rho_{SM}]_+ \right) \\ & + \frac{1}{4} (\hat{\sigma}_{Mz} \rho_{SM} \hat{\sigma}_{Mz} - \rho_{SM}). \end{aligned} \quad (10)$$

The system is thus subject to both dissipation, with jump operator  $\hat{\sigma}_{M-}$ , and dephasing, with jump operator  $\hat{\sigma}_{Mz}$ , both acting on the auxiliary system  $M$ . Remarkably, the corresponding rates of these two decoherence processes must be in the definite 4 : 1 ratio. In the case that  $S$  is also a qubit coupled to  $M$  via an XX-type interaction Hamiltonian (this assigns the form of Hamiltonian  $\hat{H}_{SM}$ ), the resulting ME (6), including its exact solution, has been studied in Ref. [13].

In Sec. III, we provide a direct proof that ME (6) exactly describes the reduced dynamics of  $S$  entailed by Eq. (1) when  $\mathcal{L}_M$  is given by Eq. (5).

### III. DIRECT PROOF

#### A. The master equation in the Laplace space

For the sake of notation compactness, let us define the superoperator  $\mathcal{H}_{SM}[\cdots] = [\hat{H}_{SM}, \cdots]$  so that ME (1) can be written as

$$\dot{\rho}_{SM} = -i\mathcal{H}_{SM}[\rho_{SM}] + \Gamma \mathcal{L}_M[\rho_{SM}] \quad (11)$$

with the initial condition (3). Furthermore, let us note that, thanks to Eqs. (4) and (5), the Lindbladian (2) transforms an arbitrary joint state of  $S$  and  $M$  as

$$\mathcal{L}_M[\rho_{SM}] = \text{Tr}_M\{\rho_{SM}\}\eta_M - \rho_{SM}. \quad (12)$$

Let  $\tilde{\rho}_{SM}(s)$  be the Laplace transform (LT) of  $\rho_{SM}(t)$ , where  $s$  lies on the complex plane. Taking the LT of both sides of Eq. (11) and replacing  $\mathcal{L}_M$  with Eq. (12), we find

$$\begin{aligned} s\tilde{\rho}_{SM}(s) - \rho_{SM}(0) = & -i\mathcal{H}_{SM}[\rho_{SM}(s)] \\ & + \Gamma[\text{Tr}_M\{\tilde{\rho}_{SM}(s)\}\eta_M - \tilde{\rho}_{SM}(s)], \end{aligned} \quad (13)$$

which can be viewed as a special case, for  $\Gamma_1 = \Gamma$ , of the following, more general, equation:

$$(s + \Gamma + i\mathcal{H}_{SM})[\tilde{\rho}_{SM}(s)] - \rho_{SM}(0) = \Gamma_1 \text{Tr}_M\{\rho_{SM}(s)\}\eta_M \quad (14)$$

under the same initial condition (3). The solution of Eq. (14), which depends on both  $\Gamma$  and  $\Gamma_1$ , for a given value of  $\Gamma$ , can be expanded in powers of  $\Gamma_1$  as

$$\tilde{\rho}_{SM}(s) = \sum_{k=1}^{\infty} \Gamma_1^{k-1} \tilde{\rho}_{SM}^{(k)}(s) \quad (15)$$

where each  $\tilde{\rho}_{SM}^{(k)}(s)$  parametrically depends on  $\Gamma$ . The solution of our Eq. (13) can thus be obtained by evaluating the  $\tilde{\rho}_{SM}^{(k)}(s)$  and then setting  $\Gamma_1 = \Gamma$  in Eq. (15).

To determine  $\tilde{\rho}_{SM}^{(k)}(s)$ , we replace expansion (15) in Eq. (14) so as to end up with the set of equations (one for each power of  $\Gamma_1$ )

$$\begin{aligned} (s + \Gamma + i\mathcal{H}_{SM})[\tilde{\rho}_{SM}^{(1)}(s)] &= \rho_{SM}(0) \\ (s + \Gamma + i\mathcal{H}_{SM})[\tilde{\rho}_{SM}^{(k)}(s)] &= \text{Tr}_M\{\tilde{\rho}_{SM}^{(k-1)}(s)\}\eta_M \quad (k \geq 2). \end{aligned}$$

The first equation (corresponding to the zeroth power in  $\Gamma_1$ ) immediately yields

$$\tilde{\rho}_{SM}^{(1)}(s) = \tilde{\mathcal{U}}_{SM}(s + \Gamma)[\rho_{SM}(0)], \quad (16)$$

where

$$\tilde{\mathcal{U}}_{SM}(s) = (s + i\mathcal{H}_{SM})^{-1}. \quad (17)$$

Note that  $\tilde{\mathcal{U}}_{SM}(s)$  is the LT of the quantum map

$$\mathcal{U}_{SM}(t) = e^{-i\mathcal{H}_{SM}t}, \quad (18)$$

namely, the unitary dynamical map on  $S$ - $M$  corresponding to the ME (1) [or equivalently Eq. (11)] for  $\Gamma = 0$ , i.e., in the absence of interaction with the reservoir. Furthermore, for  $k \geq 2$ , we have

$$\tilde{\rho}_{SM}^{(k)}(s) = \tilde{\mathcal{U}}(s + \Gamma)[\text{Tr}_M\{\tilde{\rho}_{SM}^{(k-1)}(s)\}\eta_M], \quad (19)$$

which is a recurrence relation allowing us to determine each  $\tilde{\rho}_{SM}^{(k)}(s)$ . Replacing these in Eq. (15) and setting  $\Gamma_1 = \Gamma$ , we thus get the solution of ME (13) in the Laplace space.

#### B. The reduced dynamics of $S$

We now derive the reduced dynamics of  $S$ , i.e., we evaluate  $\rho(t) = \text{Tr}_M\{\rho_{SM}(t)\}$ . Let  $\tilde{\rho}(s)$  be the LT of  $\rho(t)$ ; from Eq. (15), it follows that

$$\tilde{\rho}(s) = \sum_{k=1}^{\infty} \Gamma^{k-1} \text{Tr}_M\{\tilde{\rho}_{SM}^{(k)}(s)\} \quad (20)$$

with  $\tilde{\rho}_{SM}^{(k)}(s)$  given by Eqs. (16) and (19).

We will additionally need the LTs of maps  $\mathcal{E}(t)$  and  $\Phi(t)$  on  $S$ , which in the light of Eqs. (7) and (8) can be expressed in terms of the joint map (17) as

$$\tilde{\mathcal{E}}(s)[\rho] = \text{Tr}_M\{\tilde{\mathcal{U}}_{SM}(s)[\rho \otimes \eta_M]\}, \quad (21)$$

$$\tilde{\Phi}(s)[\rho] = \text{Tr}_M\{\tilde{\mathcal{U}}_{SM}(s)[\rho \otimes \bar{\eta}_M]\}. \quad (22)$$

For  $k = 1$ , recalling the initial condition Eq. (3), we thus find from Eq. (16) that

$$\text{Tr}_M\{\tilde{\rho}_{SM}^{(1)}(s)\} = \tilde{\Phi}(s + \Gamma)[\rho_0].$$

For  $k = 2$ , from Eqs. (19) and (21) it immediately follows that

$$\text{Tr}_M\{\tilde{\rho}_{SM}^{(2)}(s)\} = \tilde{\mathcal{E}}(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma)[\rho_0].$$

Likewise, for  $k = 3$

$$\text{Tr}_M\{\tilde{\rho}_{SM}^{(3)}(s)\} = [\mathcal{E}(s + \Gamma)]^2 \circ \tilde{\Phi}(s + \Gamma)[\rho_0]$$

with  $[\tilde{\mathcal{E}}(s + \Gamma)]^2 = \tilde{\mathcal{E}}(s + \Gamma) \circ \tilde{\mathcal{E}}(s + \Gamma)$ . By induction, for arbitrary  $k \geq 1$

$$\text{Tr}_M\{\tilde{\rho}_{SM}^{(k)}(s)\} = \tilde{\mathcal{E}}^{k-1}(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma)[\rho_0]$$

with  $\tilde{\mathcal{E}}^0 = \mathbb{I}_S$ . Substituting into Eq. (20), the solution for  $\rho(s)$  thus reads

$$\begin{aligned} \tilde{\rho}(s) &= \sum_{k=1}^{\infty} \Gamma^{k-1} \tilde{\mathcal{E}}^{k-1}(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma)[\rho_0] \\ &= [\mathbb{I}_S - \Gamma \tilde{\mathcal{E}}(s + \Gamma)]^{-1} \circ \tilde{\Phi}(s + \Gamma)[\rho_0]. \end{aligned} \quad (23)$$

It is straightforward to show (see Appendix A) that in the Laplace representation this precisely coincides with the solution of ME (6) under the initial condition (3). This shows that MEs (1) and (6) correspond to the same open dynamics of  $S$ , which completes our proof.

In the remainder of this paper, we will show how the steps leading from MEs (1) to (6) can be described and understood in terms of CMs in the continuous-time limit.

#### IV. A COLLISION MODEL WITH INTERNAL MEMORY

In collision models the reservoir  $R$  consists of a large collection of identical ancillas—here labeled with a positive integer  $n = 1, 2, \dots$ —all initially in the same state  $\eta$ . The interaction between a system  $S$  and  $R$  is described in terms of a sequence of collisions, each lasting a short time  $\tau$ , between  $S$  and the  $n$ th ancilla. Each collision is described by a unitary operator  $\hat{U}_{S_n} = e^{-i\hat{H}_{S_n}\tau}$  where  $\hat{H}_{S_n}$  is the interaction Hamiltonian. The Hilbert space dimension of both  $S$  and a generic ancilla can be arbitrary. The initial—product—state of  $S$ - $R$  reads

$$\sigma_0 = \rho_0 \otimes \eta_1 \otimes \eta_2 \otimes \dots \quad (24)$$

where  $\rho_0$  is the initial state of  $S$ . In the absence of any other type of dynamical processes, at the  $n$ th step the joint  $S$ - $R$  state is

$$\begin{aligned} \sigma_n &= (\hat{U}_{S_n} \dots \hat{U}_{S_2} \hat{U}_{S_1}) \rho_0 \otimes \eta_1 \otimes \eta_2 \otimes \dots \otimes \eta_n \\ &\quad \times (\hat{U}_{S_n} \dots \hat{U}_{S_2} \hat{U}_{S_1})^\dagger. \end{aligned} \quad (25)$$

Provided the system collides only once with the same ancilla this leads to a fully Markovian open dynamics for  $S$ . Indeed, if  $\rho_n = \text{Tr}_R\{\sigma_n\} \equiv \text{Tr}_R\{\sigma_n\}$ , at the  $n$ th step,  $\rho_n = \mathcal{E}_\tau[\rho_{n-1}] = \mathcal{E}_\tau^n[\rho_0]$ , where map  $\mathcal{E}_\tau = \mathcal{E}(\tau)$  coincides with Eq. (7) for  $t = \tau$  and  $\eta \equiv \eta_M$ . The above would give rise to a standard *memoryless* CM [7,8] (note that the discrete dynamical map  $\mathcal{E}_\tau^n$  fulfils the semigroup property), which in the continuous-time limit gives rise to a Lindblad-type ME for  $S$  [9].

A way to endow the open dynamics of  $S$  with memory is to introduce *interancillary* pairwise collisions occurring between consecutive system-ancilla (SA) interactions. In other words, the collision between  $S$  and the  $n$ th ancilla is followed by a collision between the  $n$ th and  $(n+1)$ th ancillas, which is in turn followed by a new collision involving  $S$  and the  $(n+1)$ th ancilla, then by a new collision between ancillas  $n+1$  and  $n+2$ , and so on. A pictorial sketch of such dynamics is given in Fig. 1(a). In line with Ref. [5] the interancillary collision between ancillas  $n$  and  $n+1$  is modeled as a probabilistic swap operation, which is a nonunitary process with an associated quantum map that transforms the joint  $S$ - $R$  state  $\sigma$  according to

$$\mathcal{K}_p^{(n,n+1)}[\sigma] = q\sigma + p \hat{S}_{n,n+1} \sigma \hat{S}_{n,n+1} \quad (q = 1 - p), \quad (26)$$

where  $\hat{S}_{i,j}$  is the swap unitary operator exchanging the states of ancillas  $i$  and  $j$ . In other words, the states of the two involved ancillas are swapped with probability  $p$  or left unchanged with probability  $q$ . We define as “step” the product of an interancillary and of the following system-ancilla collisions. Besides introducing interancillary collisions, we also slightly

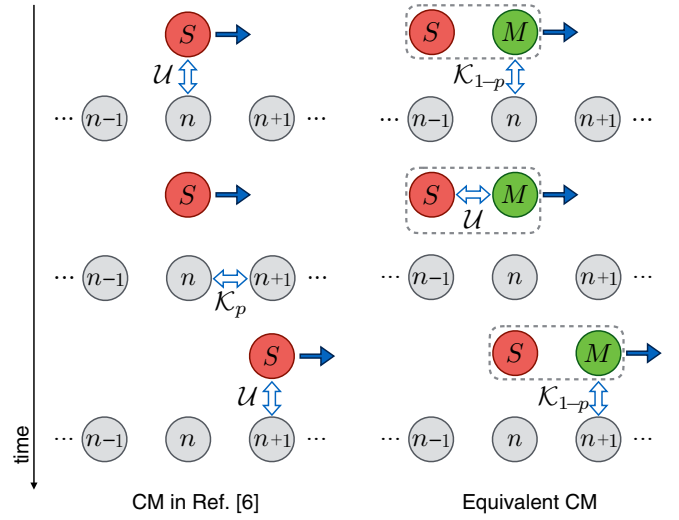


FIG. 1. (a) CM with internal memory: interancillary collisions are interspersed with system-ancilla collisions. (b) Equivalent Markovian collision model for  $S$ - $M$ : only the auxiliary system  $M$  undergoes successive interactions with the bath ancillas, which are interspersed with unitary  $S$ - $M$  collisions. In the latter model no interancillary collisions occur.

generalize the initial state (24) as

$$\sigma_0 = \rho_0 \otimes \bar{\eta}_1 \otimes \eta_2 \otimes \eta_3 \otimes \dots, \quad (27)$$

where the initial state of ancilla 1,  $\bar{\eta}_1$ , is in general different from the common state of ancillas  $\eta_n = \eta$  with  $n \geq 2$ . The reason for considering this more general state will become clear later.

After the first  $S$ -1 collision, the initial joint state  $\sigma_0$  [cf. Eq. (27)] is transformed into  $\sigma_1 = \hat{U}_{S_1}^\dagger \sigma_0 \hat{U}_{S_1}$ . Next, the interancillary 1-2 collision occurs followed by the collision  $S$ -2, which yields, at the end of the second step,  $\sigma_2 = q(\hat{U}_{S_2} \sigma_1 \hat{U}_{S_2}^\dagger) + p(\hat{U}_{S_2} \hat{S}_{12} \sigma_1 \hat{S}_{12} \hat{U}_{S_2}^\dagger)$ . This can equivalently be expressed in the form

$$\sigma_2 = q(\hat{U}_{S_2} \sigma_1 \hat{U}_{S_2}^\dagger) + p \hat{U}_{S_2}^2 (\rho_0 \otimes \eta_1 \otimes \bar{\eta}_2 \otimes \eta_3 \otimes \dots) \hat{U}_{S_2}^{\dagger 2}, \quad (28)$$

where in the second term we replaced  $\sigma_1 = \hat{U}_{S_1} \sigma_0 \hat{U}_{S_1}^\dagger$  and used the identity  $\hat{S}_{12} \hat{U}_{S_1} = \hat{U}_{S_2} \hat{S}_{12}$ . Accordingly, the state between brackets in the second term of Eq. (28) is the initial state (27) where ancillas 1 and 2 have been swapped (note that state  $\bar{\eta}$  is now the state of ancilla 2). By iteration (see Appendix B), the state at the  $n$ th step will read

$$\begin{aligned} \sigma_n &= q \sum_{j=1}^{n-1} p^{j-1} \hat{U}_{S_n}^j \sigma_{n-j} \hat{U}_{S_n}^{j\dagger} \\ &\quad + p^{n-1} \hat{U}_{S_n}^n (\rho_0 \otimes \eta_1 \otimes \dots \otimes \eta_{n-1} \otimes \bar{\eta}_n \otimes \eta_{n+1} \otimes \dots) \hat{U}_{S_n}^{n\dagger}. \end{aligned} \quad (29)$$

In the state between brackets in the second term, the  $n$ th ancilla is in state  $\bar{\eta}$ , while all the remaining ones are in  $\eta$ . Note that in Eq. (29) only the unitary operator  $\hat{U}_{S_n}$  associated with the  $n$ th SA collision appears. This remarkable property allows us to write the corresponding equation for the *reduced*  $S$  density

operator  $\rho_n = \text{Tr}_R \sigma_n$  as

$$\rho_n = q \sum_{j=1}^{n-1} p^{j-1} \mathcal{E}_j[\rho_{n-j}] + p^{n-1} \Phi_n[\rho_0], \quad (30)$$

where [cf. Eqs. (7) and (8)]  $\mathcal{E}_j \equiv \mathcal{E}(j\tau)$  and  $\Phi_j = \Phi(j\tau)$ . Correspondingly, the variation of  $\rho_n$  between the  $(n-1)$ th and  $n$ th steps, i.e.,  $\Delta\rho_n = (\rho_n - \rho_{n-1})$ , is given by

$$\begin{aligned} \Delta\rho_n &= q \sum_{j=1}^{n-2} p^{j-1} \mathcal{E}_j[\Delta\rho_{n-j}] + q p^{n-1} \mathcal{E}_{n-1}[\rho_1] \\ &+ \Delta(p^{n-1} \Phi_n)[\rho_0]. \end{aligned} \quad (31)$$

In the continuous-time limit, Eq. (31) can be cast in the form of an exact ME. In this limit  $n\tau \rightarrow t$  and  $j\tau \rightarrow t'$  so that  $p^j = (p^{\frac{1}{\tau}})^{j\tau} = e^{-\Gamma t'}$ , where the memory rate  $\Gamma$  is defined in terms of  $p$  [cf. Eq. (26)] and  $\tau$  as

$$p = \exp[-\Gamma\tau]. \quad (32)$$

Furthermore, in the same limit,  $\tau$  must be far shorter than any characteristic time, in particular  $\Gamma^{-1}$ . Hence,  $\Gamma\tau \ll 1$  and thus  $q = 1 - p = 1 - e^{-\Gamma\tau} \simeq \Gamma\tau$ . When this is used in Eq. (31), we end up with the memory-kernel ME (6) as we show in detail in Appendix C.

This collision model generalizes the one in Ref. [5], the latter being retrieved in the special case  $\bar{\eta} \equiv \eta$ . Such an extension is indeed necessary to ensure that the closed memory-kernel ME (6) corresponds to a bipartite ME for  $S$ - $M$  where the memory  $M$  is initially in an *arbitrary* state  $\bar{\eta}$  (see Sec. II). In the CM discussed above, this requirement simply translates into allowing the first and the remaining reservoir ancillas to be initially in different states.

## V. A MEMORYLESS COLLISION MODEL

We now show that, as anticipated in the introduction, our memory-kernel ME can be derived by a second memoryless CM describing a subsystem  $S$  coupled to a fully Markovian environment via an auxiliary system  $M$ . In this CM the reservoir  $R$  consists again of a large collection of identical ancillas  $n = 1, 2, \dots$ , which, however, are now noninteracting (no interancillary collisions occur). The “system” relaxing into the reservoir  $R$  is now *bipartite*, with its subsystems  $S$  and  $M$  mutually interacting according to a Hamiltonian  $\hat{H}_{SM}$ . By hypothesis, the Hilbert-space dimensions of  $M$  and each of the ancillas are assumed to be equal. The initial  $S$ - $M$ - $R$  joint state is assumed to be

$$\sigma_0 = [\rho_0 \otimes \bar{\eta}_M] \otimes \eta_1 \otimes \eta_2 \otimes \dots \quad (33)$$

As the reservoir ancillas do not interact with each other (in contrast to the CM of the previous section) here only system-ancilla collisions take place. These collisions, which we assume to involve only  $M$  (i.e.,  $S$  is not in direct contact with  $R$ ) are described by the *nonunitary* probabilistic swap  $\mathcal{K}_{1-p}^{(M,n)}$ . The definition of such a map, acting on  $M$  and the  $n$ th ancilla, is the same as in Eq. (26) *but the replacement*  $p \rightarrow 1 - p$  (as highlighted by the subscript). In other words, at each collision, the current state of  $M$  is either swapped with the ancillary state  $\eta$  with probability  $1 - p$  or left unchanged

with probability  $p$ . In addition to such collisions a unitary dynamics internal to the bipartite  $S$ - $M$  system, generated by the Hamiltonian  $\hat{H}_{SM}$ , takes place. Such dynamics has the form of  $S$ - $M$  unitary collisions that are interspersed with collisions between  $M$  and the reservoir ancillas. A sketch of the CM dynamics is shown in Fig. 1(b). Note that the joint  $S$ - $M$  system undergoes a fully Markovian dynamics (while in general this is not the case for  $S$ ).

It is convenient to define a map  $\mathcal{S}$  on  $S$ - $M$  as

$$\mathcal{S}(\rho_{SM}) = \text{Tr}_n \{ \hat{S}_{Mn}(\rho_{SM} \eta_n) \hat{S}_{Mn}^\dagger \} = \text{Tr}_M \{ \rho_{SM} \} \eta_M, \quad (34)$$

which describes how an arbitrary  $S$ - $M$  state  $\rho_{SM}$  is changed after a (unitary) swap operation is applied on  $M$  and a generic ancilla initially in state  $\eta$  (we recall that this has the same dimension as  $M$ ). A proof of the last identity in Eq. (34) is given in Appendix D. Using Eq. (34), from Eq. (26) we get

$$\text{Tr}_n \{ \mathcal{K}_{1-p}^{(M,n)}[\rho_{SM} \eta_n] \} = p \rho_{SM} + q \mathcal{S}[\rho_{SM}]. \quad (35)$$

Initially (the zeroth step),  $S$  and  $M$  are in the state  $\rho_{SM}^{(0)} = \rho_0 \bar{\eta}_M$ , each of the ancillas being in state  $\eta$  [see Eq. (33)]. They then collide with each other, hence their state after the first step reads  $\rho_{SM}^{(1)} = \mathcal{U}_\tau[\rho_{SM}^{(0)}]$ . We have set for brevity  $\mathcal{U}_\tau = \mathcal{U}_{SM}(\tau)$  [cf. Eq. (18)]. Next, an  $M$ -1 collision described by map  $\mathcal{K}_{1-p}^{(M,1)}$  takes place followed by a new  $S$ - $M$  unitary collision. Hence, at the second step

$$\rho_{SM}^{(2)} = p \mathcal{U}_\tau[\rho_{SM}^{(1)}] + q \mathcal{U}_\tau \circ \mathcal{S}[\rho_{SM}^{(1)}] = p \mathcal{U}_\tau^2[\rho_{SM}^{(0)}] + q \mathcal{U}_\tau \mathcal{S}[\rho_{SM}^{(1)}] \quad (36)$$

(the map composition symbol “ $\circ$ ” will be always omitted henceforth).

At the third step,

$$\begin{aligned} \rho_{SM}^{(3)} &= p \mathcal{U}_\tau[\rho_{SM}^{(2)}] + q \mathcal{U}_\tau \mathcal{S}[\rho_{SM}^{(2)}] \\ &= p \mathcal{U}_\tau [p \mathcal{U}_\tau^2[\rho_{SM}^{(0)}] + q \mathcal{U}_\tau \mathcal{S}[\rho_{SM}^{(1)}]] + q \mathcal{U}_\tau \mathcal{S}[\rho_{SM}^{(2)}] \\ &= p^2 \mathcal{U}_\tau^3[\rho_{SM}^{(0)}] + q p \mathcal{U}_\tau^2 \mathcal{S}[\rho_{SM}^{(1)}] + q \mathcal{U}_\tau \mathcal{S}[\rho_{SM}^{(2)}], \end{aligned}$$

where to obtain the second identity we have used Eq. (36).

By induction (see Appendix E), the  $n$ th-step state can be arranged as

$$\rho_{SM}^{(n)} = q \sum_{j=1}^{n-1} p^{j-1} \mathcal{U}_\tau^j \mathcal{S}[\rho_{SM}^{(n-j)}] + p^{n-1} \mathcal{U}_\tau^n[\rho_{SM}^{(0)}]. \quad (37)$$

Due to Eq. (34),  $\mathcal{S}[\rho_{SM}^{(n-j)}] = \rho_{n-j} \eta_M$ . Replacing this and  $\rho_{SM}^{(0)} = \rho_0 \bar{\eta}_M$  in Eq. (37), upon trace over  $M$  we thus find

$$\rho_{SM}^{(n)} = q \sum_{j=1}^{n-1} p^{j-1} \mathcal{U}_\tau^j[\rho_{n-j} \eta_M] + p^{n-1} \mathcal{U}_\tau^n[\rho_0 \bar{\eta}_M]. \quad (38)$$

Upon trace over  $M$ , recalling the definition of maps  $\mathcal{E}(t)$  and  $\Phi(t)$  [cf. Eqs. (7) and (8)], we thus end up with Eq. (30). This shows that, as far as the reduced dynamics of  $S$  is concerned, the present CM, in which the joint  $S$ - $M$  system undergoes incoherent binary collisions between  $M$  and each of the noninteracting ancillas of an infinitely large reservoir interspersed with “internal”  $S$ - $M$  coherent unitary collisions, is equivalent to the discrete CM described in Sec. V.

### Microscopic derivation of ME (1)

We now show that, in the continuous-time limit, the bipartite CM discussed above gives rise to ME (1). To this end, we use an approach similar to the one adopted for composite CMs [18] (these differ from the present CM in that, unlike here, the system-ancilla collisions are unitary). At step  $n$ , the  $S$ - $M$  joint state is

$$\begin{aligned} \rho_{SM}^{(n)} &= p \text{Tr}_n \{ e^{-i\hat{H}_{SM}\tau} \rho_{SM}^{(n-1)} \eta_n e^{i\hat{H}_{SM}\tau} \} \\ &\quad + q \text{Tr}_n \{ e^{-i\hat{H}_{SM}\tau} \hat{S}_{Mn} \rho_{SM}^{(n-1)} \eta_n \hat{S}_{Mn} e^{i\hat{H}_{SM}\tau} \}, \end{aligned} \quad (39)$$

where  $p$  is given by Eq. (32) and we have replaced the explicit forms of maps (18) and (34).

In the continuous-time limit [see also the discussion following Eq. (32)],  $\tau \simeq 0$  in a way that  $p \simeq 1 - \Gamma\tau$  and  $q \simeq \Gamma\tau$ . Hence, up to first order in  $\tau$ ,  $e^{-i\hat{H}_{SM}\tau} \rho_{SM}^{(n-1)} \eta_n e^{i\hat{H}_{SM}\tau} \simeq \rho_{SM}^{(n-1)} \eta_n - i\tau [\hat{H}_{SM}, \rho_{SM}^{(n-1)} \eta_n]$ . An analogous result is obtained [see the second line of Eq. (39)] under the replacement  $\rho_{SM}^{(n-1)} \eta_n \rightarrow \hat{S}_{Mn} \rho_{SM}^{(n-1)} \eta_n \hat{S}_{Mn}$ . Thereby, the  $n$ th-step state change  $\Delta\rho_{SM}^{(n)} = \rho_{SM}^{(n)} - \rho_{SM}^{(n-1)}$ , up to first order in  $\tau$ , takes the form

$$\begin{aligned} \Delta\rho_{SM}^{(n)} &= -i\tau [\hat{H}_{SM}, \rho_{SM}^{(n)}] - \Gamma\tau \rho_{SM}^{(n-1)} \\ &\quad + \Gamma\tau \text{Tr}_n \{ \hat{S}_{Mn} \rho_{SM}^{(n-1)} \eta_n \hat{S}_{Mn} \}. \end{aligned} \quad (40)$$

In the continuous-time limit,  $\Delta\rho_{SM}^{(n)}/\tau \rightarrow \dot{\rho}_{SM}$ . Using this and the last identity in Eq. (34) we thus end up with ME (1) [we recall that the Lindbladian  $\mathcal{L}_M$  defined by Eq. (5) can be equivalently expressed in the form (12)].

We conclude by pointing out that upon decomposition of the ancillary state  $\eta_n$  in its eigenstates [see Eq. (4) for  $M \rightarrow n$ ] the partial trace in Eq. (40) can be expressed as

$$\begin{aligned} \text{Tr}_n [\hat{S}_{Mn} \rho_{SM}^{(n-1)} \eta_n \hat{S}_{Mn}] &= \sum_{\mu} \langle \mu | \hat{S}_{Mn} \rho_{SM}^{(n-1)} \eta_n \hat{S}_{Mn} | \mu \rangle_n \\ &= \sum_{\mu\nu} p_{\nu} \langle \mu | \hat{S}_{Mn} | \nu \rangle_n \rho_{SM}^{(n-1)} \langle \nu | \hat{S}_{Mn} | \mu \rangle_n \\ &= \sum_{\mu\nu} \hat{L}_{\mu\nu} \rho_{SM}^{(n-1)} \hat{L}_{\mu\nu}^{\dagger}, \end{aligned}$$

where  $\{\hat{L}_{\mu\nu}\}$  indeed coincide with the jump operators (5). This illustrates that the form of the Lindbladian  $\mathcal{L}_M$  introduced in Sec. II can be interpreted as stemming from swaplike interactions between  $M$  and the reservoir  $R$ .

Note that the above can be regarded as an ‘‘indirect’’ demonstration of the fact that ME (6) arises from ME (1) upon trace over  $M$ .

## VI. CONCLUSIONS

In this work, we addressed the problem of an open quantum system  $S$  coherently coupled to a memory  $M$ , which in turn is in contact with a Markovian bath, the bipartite  $S$ - $M$  system being governed by a Lindblad-type ME. In contrast to the typical case where tracing out the degrees of freedom of  $M$  does not yield a closed ME, we have found a class of MEs of the above form where this partial trace does give rise to a closed, exact ME for  $S$ . This can be viewed as a generalization of a ME originally derived via a CM. This leads us to interpret

the link between the memory-kernel ME for  $S$  and the  $S$ - $M$  Lindbladian ME in terms of suitably defined CMs, hence providing in fact a comprehensive microscopic framework underlying our central result. In particular, we have shown that our ME can be derived as the continuous-time limit of two distinct collision models each describing a different physical scenario.

We note (see also footnote [3]) that in the case of the dissipative Jaynes-Cummins model, which is also governed by a bipartite Lindbladian ME where an atom (mode) embodies  $S$  ( $M$ ), it is known that a reduced ME for the atom can be obtained [1] but this is in fact formulated in terms of its solution (as if it were *a priori* known), at variance with our case. One could object that even the memory-kernel ME (6) is expressed in terms of maps  $\mathcal{E}(t)$  and  $\Phi(t)$ , which are assumed to be known. Note, though, that these can be regarded as the solutions of the problem (for two different initial conditions) where  $S$  and  $M$  undergo a joint *unitary* evolution, fully dependent on their total Hamiltonian  $\hat{H}_{SM}$ , and one aims at working out the corresponding dynamical map of  $S$ . Such a problem is often amenable to analytical solution.

We also comment on the relationship with the ME in Ref. [5] and, accordingly, the associated CM. In the case  $\mathcal{E}(t) \equiv \Phi(t)$ ,  $\eta_M$  would coincide with  $\bar{\eta}_M$ . This would bring about that, for a given Lindbladian (2) specified by the jump operators (5), there would be only a single possible initial state  $\bar{\eta}_M$  of  $M$  such that the partial trace of ME (1) leads to ME (6). Alternatively, given an arbitrary  $M$  initial state  $\bar{\eta}_m$ , only the Lindbladian specified by the jump operators (5) with  $\eta_M \equiv \bar{\eta}_M$  would entail Eq. (6) for  $S$ . Instead, the presence of the two maps  $\mathcal{E}(t)$  and  $\Phi(t)$  in ME (6) ensures the existence of a class of bipartite MEs such that, for any element of this, the partial trace over  $M$  leads to ME (6) for *any* initial state of  $M$  entering Eq. (3). Furthermore, it is easy to show [19] that, as mentioned in the Introduction, the sufficient conditions found in Ref. [4] for a well-behaved Nakajima-Zwanzig ME are satisfied in our case Eq. (6).

The special property of the class of MEs specified by Lindbladian (5), which enables us to work out a closed ME for  $S$ , is that the  $M$ -reservoir coupling is based on swaplike interactions (the properties of the swap operator somehow enter all of our proofs). While this is a peculiar model, yet not academic [see Eq. (10)], we envisage that it can be exploited as an advantageous theoretical testbed for investigating quantum non-Markovian concepts.

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## APPENDIX A: LAPLACE TRANSFORM OF ME (6)

Here, we derive the solution of ME (6) in the Laplace space. The proof is based on calculations that are similar to those in Refs. [5,20]. Let  $\Lambda(t)$  be the dynamical map describing the nonunitary dynamics of  $S$  corresponding to ME (6) according to  $\rho(t) = \Lambda(t)[\rho_0]$  with  $\Lambda(0) = \mathbb{I}_S$ . Map  $\Lambda(t)$  obeys Eq. (6) under the formal replacement  $\rho \rightarrow \Lambda$ .

Indeed, replacing  $\rho(t) = \Lambda(t)[\rho_0]$  in Eq. (6) and using that  $\rho_0$  is arbitrary, we find that  $\Lambda(t)$  is governed by the equation

$$\begin{aligned} \dot{\Lambda}(t) &= \Gamma \int_0^t dt' e^{-\Gamma t'} \mathcal{E}(t') [\dot{\Lambda}(t-t')] \\ &\quad + \Gamma e^{-\Gamma t} (\mathcal{E}(t) - \Phi(t)) + e^{-\Gamma t} \dot{\Phi}(t). \end{aligned} \quad (\text{A1})$$

Upon LT, the equation becomes

$$\tilde{\Lambda}(s) = \Gamma \tilde{\mathcal{E}}(s + \Gamma) \tilde{\Lambda}(s) + \tilde{\Phi}(s + \Gamma) \quad (\text{A2})$$

where for  $s$  complex the LT is defined as

$$\tilde{F}(s) = L[F(t)](s) = \int_0^\infty dt e^{-st} F(t). \quad (\text{A3})$$

By rearranging terms in Eq. (A2) as

$$[\mathcal{I} - \Gamma \tilde{\mathcal{E}}(s + \Gamma)] \tilde{\Lambda}(s) = \tilde{\Phi}(s + \Gamma) \quad (\text{A4})$$

and introducing the inverse of map  $\mathcal{I} - \Gamma \tilde{\mathcal{E}}(s + \Gamma)$  we end up with [cf. Eq. (23)]

$$\tilde{\Lambda}(s) = [\mathcal{I} - \Gamma \tilde{\mathcal{E}}(s + \Gamma)]^{-1} \circ \tilde{\Phi}(s + \Gamma). \quad (\text{A5})$$

Expanding Eq. (A5) in powers of  $\Gamma$  gives

$$\tilde{\Lambda}(s) = \sum_{k=1}^{\infty} \Gamma^{k-1} \tilde{\mathcal{E}}^{k-1}(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma) \quad (\text{A6})$$

the inverse LT of which is

$$\Lambda(t) = \sum_{k=1}^{\infty} \Gamma^{k-1} L^{-1}[\tilde{\mathcal{E}}^{k-1}(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma)](t). \quad (\text{A7})$$

Basic properties of LT allow us to immediately calculate the inverse LT within brackets as

$$\begin{aligned} L^{-1}[\tilde{\mathcal{E}}^j(s + \Gamma) \circ \tilde{\Phi}(s + \Gamma)] \\ &= e^{-\Gamma t} \int_0^t dt_1 \cdots \int_0^{t_{j-1}} dt_j \mathcal{E}(t_1) \circ \cdots \\ &\quad \times \circ \mathcal{E}(t_j) \circ \Phi(t_{j-1} - t_j). \end{aligned} \quad (\text{A8})$$

The integrand in Eq. (A8) is evidently a composition of CPTP quantum maps, hence it is CPTP itself [we recall that both  $\mathcal{E}(t)$  and  $\Phi(t)$  are CPTP maps, see Eqs. (7) and (8)]. Therefore, we see that the dynamical map Eq. (A7) is in fact a combination of CPTP maps with *positive* weights. This proves that the dynamical map  $\Lambda(t)$  corresponding to ME (6) is completely positive and trace preserving.

#### APPENDIX B: INDUCTION PROOF OF EQ. (29)

At the third step, the overall state is given by  $\sigma_3 = q(\hat{U}_{S_3} \sigma_2 \hat{U}_{S_3}^\dagger) + p(\hat{U}_{S_3} \hat{S}_{23} \sigma_2 \hat{S}_{23} \hat{U}_{S_3}^\dagger)$ , which with the help of Eq. (28) can be arranged as

$$\begin{aligned} \sigma_3 &= q \sum_{j=1}^2 p^{j-1} \hat{U}_{S_3}^j \sigma_{3-j} \hat{U}_{S_3}^{j\dagger} \\ &\quad + p^2 \hat{U}_{S_3}^3 (\rho_0 \otimes \eta_1 \otimes \eta_2 \otimes \bar{\eta}_3 \otimes \cdots) \hat{U}_{S_3}^{3\dagger}, \end{aligned}$$

where we used  $\hat{S}_{23} \hat{U}_{S_2} = \hat{U}_{S_3} \hat{S}_{23}$  and  $\hat{S}_{23} \sigma_1 \hat{S}_{23} = \sigma_1$  (the latter identity follows from the fact that, at the end of the first step, ancillas 2 and 3 are still in the initial state  $\eta$ ). Equation (29)

thus holds for  $n \leq 3$ . To prove that it is valid for arbitrary  $n$ , let us consider the state after the  $(n+1)$ th step, which reads

$$\begin{aligned} \sigma_{n+1} &= q(\hat{U}_{S,n+1} \sigma_n \hat{U}_{S,n+1}^\dagger) \\ &\quad + p(\hat{U}_{S,n+1} \hat{S}_{n,n+1} \sigma_n \hat{S}_{n,n+1} \hat{U}_{S,n+1}^\dagger). \end{aligned}$$

Substituting Eq. (29) in the second term yields

$$\begin{aligned} \sigma_{n+1} &= q \sum_{j=0}^{n-1} p^j \hat{U}_{S_n}^{j+1} \sigma_{n-j} \hat{U}_{S_n}^{j+1\dagger} \\ &\quad + p^n \hat{U}_{S,n+1}^{n+1} (\rho_0 \otimes \eta_1 \otimes \cdots \otimes \eta_n \otimes \bar{\eta}_{n+1} \otimes \eta_{n+2} \otimes \cdots) \\ &\quad \times \hat{U}_{S,n+1}^{n+1\dagger}. \end{aligned}$$

Rearranging index  $j$  in the above expression we end up with Eq. (29) for  $n \rightarrow n+1$ . This proves that Eq. (29) holds.

#### APPENDIX C: ME (6) AS THE CONTINUOUS LIMIT OF EQ. (31)

The derivation of ME (6) from Eq. (31) is a slight generalization of the analogous task carried out in Ref. [5] (see also Ref. [20]). When Eq. (31) is divided by  $\tau$  and using the limiting expressions discussed in the main text [see Eq. (32) and related discussion], the terms on the right-hand side of Eq. (31) in the continuous-time limit take the form

$$\begin{aligned} &\frac{q \sum_{j=1}^{n-2} p^{(j-1)} \mathcal{E}_j[\rho_{n-j} - \rho_{n-1-j}]}{\tau} \\ &\simeq \Gamma \int_0^t dt' e^{-\Gamma t'} \mathcal{E}(t') \left[ \frac{d\rho(t-t')}{d(t-t')} \right], \\ &\frac{qp^{n-1} \mathcal{E}_{n-1}}{\tau} [\rho_1] \simeq \Gamma e^{-\Gamma t} \mathcal{E}(t) [\rho_0], \\ &\frac{\Delta(p^{n-1} \Phi_n)}{\tau} \\ &= \frac{(p^{n-1} \Phi_n - p^{n-2} \Phi_{n-1})}{\tau} \\ &\simeq \frac{e^{-\Gamma t} \Phi(t) - e^{-\Gamma(t-\tau)} \Phi(t-\tau)}{\tau} \simeq dt [e^{-\Gamma t} \Phi(t)]. \end{aligned}$$

On the other hand,  $\Delta\rho_n/\tau \rightarrow \dot{\rho}(t)$ . By plugging all the above expressions into Eq. (31), ME (6) is obtained.

#### APPENDIX D: PROOF OF EQ. (34)

Let  $\{|v\rangle_n\}$  be the set of eigenstates of state  $\eta_n$ , i.e.,

$$\eta_n = \sum_v p_v |v\rangle_n \langle v|.$$

The set  $\{|v\rangle_M\}$  is also a basis for the  $M$ 's Hilbert space (having the same dimension as  $M$ ). The swap unitary operator can then be expressed as

$$\hat{S}_{Mn} = \sum_{v,v'} |v, v'\rangle_{Mn} \langle v', v|. \quad (\text{D1})$$

Thereby,  $\eta_n \hat{S}_{Mn} = \sum_{v,v'} p_{v'} |v\rangle_M \langle v'| \otimes |v'\rangle_n \langle v|$ , which once plugged into Eq. (34) yields

$$\begin{aligned} \mathcal{S}(\rho_{SM}) &= \text{Tr}_n \{ \hat{S}_{Mn} \rho_{SM} \eta_n \hat{S}_{Mn} \} \\ &= \sum_{\mu,\mu'} \sum_{v,v'} p_{v'} \langle \mu' | \rho_{SM} | v \rangle_M | \mu \rangle_M \langle v' | \text{Tr}_n \{ | \mu' \rangle_n \langle v | \} \delta_{\mu,v} \\ &= \sum_{\mu,v} p_{\mu} \langle v | \rho_{SM} | v \rangle_M | \mu \rangle_M \langle \mu | = \text{Tr} \{ \rho_{SM} \} \eta_M. \quad (\text{D2}) \end{aligned}$$

#### APPENDIX E: INDUCTION PROOF OF EQ. (37)

By construction, the  $(n+1)$ th-step state is related to the  $n$ th one as

$$\rho_{SM}^{(n+1)} = p \mathcal{U}_\tau [\rho_{SM}^{(n)}] + q \mathcal{U}_\tau \mathcal{S} [\rho_{SM}^{(n)}]. \quad (\text{E1})$$

Our task is to show that if Eq. (37) holds then Eq. (E1) can be arranged in the same form as Eq. (37) for  $n \rightarrow (n+1)$ . Substituting Eq. (37) for  $\rho_{SM}^{(n)}$  in the first term on the right-hand

side of Eq. (E1) yields

$$\begin{aligned} \rho_{SM}^{(n+1)} &= p \mathcal{U}_\tau \left\{ q \sum_{j=1}^{n-1} p^{j-1} \mathcal{U}_\tau^j \mathcal{S} [\rho_{SM}^{(n-j)}] + p^{n-1} \mathcal{U}_\tau^n [\rho_{SM}^{(0)}] \right\} \\ &\quad + q \mathcal{U}_\tau \mathcal{S} [\rho_{SM}^{(n)}] \\ &= q \sum_{j=1}^{n-1} p^j \mathcal{U}_\tau^{j+1} \mathcal{S} [\rho_{SM}^{(n-j)}] + p^n \mathcal{U}_\tau^{n+1} [\rho_{SM}^{(0)}] \\ &\quad + q \mathcal{U}_\tau \mathcal{S} [\rho_{SM}^{(n)}]. \end{aligned}$$

The last term on the right-hand side can be included in the sum over  $j$  by making the index  $j$  start from  $j=0$ . Carrying out next the index change  $j+1 \rightarrow j$ , we thus end up with

$$\rho_{SM}^{(n+1)} = q \sum_{j=1}^n p^{j-1} \mathcal{U}_\tau^j \mathcal{S} [\rho_{SM}^{(n+1-j)}] + p^n \mathcal{U}_\tau^{n+1} [\rho_{SM}^{(0)}],$$

which coincides with Eq. (37) for  $n \rightarrow n+1$ . This concludes our induction proof.

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