On Adams Completion and Cocompletion

Mitali Routaray

Department of Mathematics
National Institute of Technology Rourkela

On Adams Completion and Cocompletion

Dissertation submitted in partial fulfillment

of the requirements of the degree of

Doctor of Philosophy

in

Mathematics

by

Mitali Routaray

(Roll Number: 512ma302)

based on research carried out

under the supervision of

Prof. Akrur Behera [MA]

July, 2016

Department of Mathematics

National Institute of Technology Rourkela

Department of Mathematics
National Institute of Technology Rourkela

July 25, 2016

Certificate of Examination

Roll Number: *512ma302*
Name: *Mitali Routaray*
Title of Dissertation: *On Adams Completion and Cocompletion*

We the below signed, after checking the dissertation mentioned above and the official record book (s) of the student, hereby state our approval of the dissertation submitted in partial fulfillment of the requirements of the degree of *Doctor of Philosophy* in *Mathematics* at *National Institute of Technology Rourkela*. We are satisfied with the volume, quality, correctness, and originality of the work.

Prof. Akrur Behera [MA]
Principal Supervisor

Prof. D.P Mohapatra [CS]
Member, DSC

Prof. J. Mohapatra [MA]
Member, DSC

Prof. K.C. Pati [MA]
Member, DSC

External Examiner

Prof. S. Chakraverty [MA]
Chairperson, DSC

Prof. K.C Pati [MA]
Head of the Department

Department of Mathematics
National Institute of Technology Rourkela

Prof. Akrur Behera [MA]

July 25, 2016

Supervisor's Certificate

This is to certify that the work presented in the dissertation entitled *On Adams Completion and Cocompletion* submitted by *Mitali Routaray*, Roll Number 512ma302, is a record of original research carried out by her under my supervision in partial fulfillment of the requirements of the degree of *Doctor of Philosophy* in *Mathematics*. Neither this dissertation nor any part of it has been submitted earlier for any degree or diploma to any institute or university in India or abroad.

Prof. Akrur Behera [MA]

Declaration of Originality

I, *Mitali Routaray*, Roll Number *512ma302* hereby declare that this dissertation entitled *On Adams Completion and Cocompletion* presents my original work carried out as a doctoral student of NIT Rourkela and to the best of my knowledge, contains no material previously published or written by another person, nor any material presented by me for the award of any degree or diploma of NIT Rourkela or any other institution. Any contribution made to this research by others, with whom I have worked at NIT Rourkela or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the sections "Reference" or "Bibliography". I have also submitted my original research records to the scrutiny committee for evaluation of my dissertation.

I am fully aware that in case of any non-compliance detected in future, the Senate of NIT Rourkela may withdraw the degree awarded to me on the basis of the present dissertation.

July 25, 2016
NIT Rourkela

Mitali Routaray

Acknowledgment

First and foremost, I would like to express my sincere gratitude to my supervisor Prof. Akrur Behera for his continuous support and insightful guidance throughout my thesis work. I have greatly benefited from his patience and many a times he has listened to my naive ideas carefully and corrected my mistakes. In all my needs, I always found him as a kind and helpful person. I am indebted to him for all the care and help that I got from him during my Ph.D. time.

I thank the Director, National Institute of Technology, Rourkela, for permitting me to avail the necessary facilities of the Institute for the completion of this work. I would like to thank my institute "National Institute of Technology, Rourkela" for providing me a vibrant research environment. I have enjoyed excellent computer facility, a well managed library and a clean and well furnished hostel at my institute.

A special word of thanks to my friends Snigdha Bharati Choudhury and Prakash Kumar Sahu, Ph.D. Scholars, Department of Mathematics, National Institute of Technology Rourkela for their moral support, helpful spirit and encouragements which rejuvenated my vigor for research and motivated me to have achievements beyond my own expectations.

Last, but not the least, I feel pleased and privileged to fulfill my parents' and uncle's (Bidhubhusan Routaray) ambition and I am greatly indebted to them for bearing the inconvenience during my thesis work. I thank my brother (Chanakya Routaray) and sister (Punyatoya Routaray) for their unconditional love and emotional support, especially during the times of difficulties.

July 25, 2016 *Mitali Routaray*

NIT Rourkela Roll Number: 512ma302

Abstract

The minimal model of a 1-connected differential graded Lie algebra is obtained as the Adams cocompletion of the differential graded Lie algebra with respect to a chosen set of morphisms in the category of 1-connected differential graded Lie algebras (d.g.l.a.'s) over the field of rationals and d.g.l.a.-homomorphisms. The Postnikov-like approimation of a module is obtained as the Adams completions of the space with the help of a suitable set of morphisms in the category of some specific modules and module homomorphisms. The Cartan-Whitehead decomposition of topological G-module is obtained as the Adams cocompletion of the space with respect to suitable sets of morphisms. Postnikov-like approximation is obtained for a topological G-module, in terms of Adams completion with respect to a suitable sets of morphisms, using cohomology theory of topological G-modules. The ring of fractions of the algebra of all bounded linear operators on a separable infinite dimensional Banach space is isomorphic to the Adams completion of the algebra with respect to a carefully chosen set of morphisms in the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1. The nth tensor algebra and symmetric algebra are each isomorphic to the Adams completions of the algebras. The exterior algebra and Clifford algebra are each isomorphic to the Adams completions of the algebra with respect to a chosen set of morphisms in the category of modules and module homomorphisms.

***Keywords*: *Grothendieck universe*; *Adams completion*; *Adams cocompletion*; *Minimal model*; *G-module*; *Tensor algebra*; *Symmetric algebra*; *Exterior algebra*; *Clifford algebra*.**

Contents

Chapter 0

Introduction

The concept of the Adams completion was proposed by J. F. Adams [1–4]; in fact this idea first arose with respect to the problem of stability. Its characterization and properties were clearly categorical in nature. However, only in later works by Deleanu, Frei and Hilton the theory was freed from its topological bounds. The greatest difficulty, in dealing with the Adams completion from the categorical point of view (hence in general), lies in its set theoretical aspect. In fact category of fractions, which plays a basic role here, is not always well defined, since there is no guarantee that the collection of morphisms between any two of its objects is a set.

It is well known that the usual set theory, as described by Zermelo and Fraenkel [5, 6], when used without extreme rigor leads very easily to some incoherent results. The most famous of those is the Russell paradox, which implies that the set of all the sets is not a set. To avoid those difficulties we will work in the logical framework of "universes" of Grothendieck. The first step in this direction is to forget the existence of "primitive", i.e., indivisible, elements and to consider any set as a collection of other sets, where the collection can even be empty or consists of a single element. With this agreement Grothendieck universe is defined in [7]. This thesis does not attempt to make a study of set theory; however the concept of universes is essential since their use seems to be unavaoidable in some categorical constructions, in particular in the construction of category of fractions.

It is a firmly established fact that the collection of objects of a category need not be a set, but the logical contradiction which is at the basis of the Russell paradox works also in this case, so that the category of all categories cannot be considered as a category. Nevertheless many times it is very useful to consider this or other kinds of structures which present the same difficulty. These difficulties may be overcome by making some mild hypotheses and using Grothendieck universes [7].

Precisely speaking if we start with a category belonging to a certain Grothendieck universe then the category of fractions with respect to a set of morphisms of the category belongs to a higher universe [7]. We note that the cases in which we are interested, will not present such difficulty. However, Nanda [8] has proved that if the set of morphisms admits a calculus of left (right) fractions then the category of fractions with respect to the set of morphisms of the category belongs to the same universe as to the universe that the category belongs. Also if the set of morphisms of the category admits a calculus of left (right) fractions then the category of fractions can be described nicely; this explicit construction is given in [7].

The central idea of this thesis is to investigate some cases showing how some algebraic and geometrical constructions are characterized in terms of Adams completions or cocompletions. We will deal with such cases involving the concepts of calculus of left (right) fractions. In fact in each of the characterizations that we have undertaken in our study, the set of morphisms of the category has to admit either calculus of left fractions or calculus of right fractions.

In Chapter 1, we recall the definitions of Grothendieck universe, category of fractions, calculus of left (right) fractions [7] and generalized Adams completions (cocompletions) [9]. We state some results on the existence of global Adams completions (cocompletions) of an object in a cocomplete (complete) category with respect to a set of morphisms in the category [9]. Deleanu, Frei and Hilton [9] have shown that if the set of morphisms in the category is saturated then the Adams completion (cocompletion) of an object is characterized by a certain couniversal property. We state a stronger version of this result proved by Behera and Nanda [10] where the saturation assumption on the set of morphisms is dropped. We also state Behera and Nanda's result [10] that the canonical map from an object to its Adams completion (from Adams cocompletion to the object) is an element of the set of morphisms under very moderate assumption. These two results are fairly general in nature and applicable to most cases of interest.

The concept of rational homotopy theory was first characterized by Quillen. In fact in rational homotopy theory Sullivan introduced the concept of minimal model. In Chapter 2, a categorical construction of minimal model of lie algebra is presented. In fact we prove that the minimal model of a 1-connected differential graded Lie algebra can be expressed as the Adams cocompletion of the differential graded Lie algebra with respect to a chosen set of morphisms in the category of 1-connected differential graded Lie algebras (in short d.g.l.a.'s) over the field of rationals and d.g.l.a.-homomorphisms.

Behera and Nanda have studied Postnikov approximation of a space, by introducing a Serre class $\mathcal{C}$ of abelian groups. They have obtained the mod-$\mathcal{C}$ Postnikov approximation of a 1-connected based CW-complex, with the help of a suitable set of morphisms in the category of 1-connected based CW-complexes and based maps. In Chapter 3, we have obtained the Postnikov-like approimation of a module, where the different stages of the approximation are shown to be the Adams completions of the module, with the help of a suitable set of morphisms in the category of some specific modules and module homomorphisms.

It is known that the different stages of the Cartan-Whitehead decomposition of a 0-connected space can be obtained as the Adams cocompletion of the space with respect to suitable set of morphisms [10]. In Chapter 4, Cartan-Whitehead decomposition is obtained for topological G-module.

In Chapter 5, we study the dual of the decomposition of a topological G-module obtained in Chapter 4. In fact, the central idea of this chapter is to obtain a Postnikov-like tower of a topological G-module, using the cohomology theory of topological G-module.

In Chapter 6, it is shown that ring of fractions of $B(H)$, the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space H is isomorphic to the Adams completion of $B(H)$ with respect to a chosen set of morphisms in a suitable category. In this chapter, we show that the ring of fractions of the algebra of all bounded linear operators on a separable infinite dimensional Banach space is isomorphic to the Adams completion of the algebra with respect to a carefully chosen set of morphisms in the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norm at most 1.

Chapter 7 is devoted to categorical study of tensor algebra and symmetric algebra. The purpose here is to obtain the tensor algebra and symmetric algebra in terms of Adams completion. Under some reasonable assumption, we show that given an algebra, its nth tensor algebra and symmetric algebra are each isomorphic to the Adams completion of the algebra.

In Chapter 8, we obtain that given an algebra, its exterior algebra and Clifford algebra are each isomorphic to the Adams completion of the algebra with respect to a chosen set of morphisms in the category of modules and module homomorphisms.

Chapter 1

Pre-Requisites

In this chapter we recall the definition of Adams completion (cocompletion) and some known results on the existence of global Adams completion (cocompletion) of an object in a category $\mathscr{C}$ with respect to a family of morphisms S in $\mathscr{C}$. A characterization of Adams completion (cocompletion) in terms of its couniversal property proved by Deleanu, Frei and Hilton is recalled. We also describe a stronger version of this result proved by Behera and Nanda [11]. We also state Behera and Nanda's result [11] that the canonical map from an object to its Adams completion is an element of the set of morphisms under very moderate assumption. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required.

1.1 Category of fractions

In this section we recall the abstract definition of category of fractions and some other related definitions. We start with universe.

Definition 1.1.1. ([7], p. 266) A *Grothendeick universe* (or simply *universe*) is a collection $\mathscr{U}$ of sets such that the following axioms are satisfied:

U(1): If $\{X_i : i \in I\}$ is a family of sets belonging to $\mathscr{U}$ then $\underset{i\in I}{\cup} X_i$ is an element of $\mathscr{U}$.

U(2): If $x \in \mathscr{U}$ then $\{x\} \in \mathscr{U}$.

U(3): If $x \in X$ and $X \in \mathscr{U}$ then $x \in \mathscr{U}$.

U(4): If X is a set belonging to $\mathscr{U}$ then $P(X)$, the power set of X is an element of $\mathscr{U}$.

U(5): If X and Y are elements of $\mathscr{U}$ then $\{X, Y\}$, the ordered pair (X, Y) and $X \times Y$ are elements of $\mathscr{U}$.

We fix a universe $\mathscr{U}$ that contains $\mathbb{N}$, the set of natural numbers (and so $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

Definition 1.1.2. ([7], p. 267) A category $\mathscr{C}$ is said to be a *small $\mathscr{U}$-category*, $\mathscr{U}$ being a fixed Grothendeick universe, if the following conditions hold:

S(1): The objects of $\mathscr{C}$ form a set which is an element of $\mathscr{U}$.

S(2): For each pair (X, Y) of objects of $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an element of $\mathscr{U}$.

Definition 1.1.3. ([7], p. 269) Let $\mathscr{C}$ be any arbitrary category and S a set of morphisms of $\mathscr{C}$. A *category of fractions* of $\mathscr{C}$ with respect to S is a category denoted by $\mathscr{C}[S^{-1}]$ together with a functor

$$F_S : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

having the following properties:

CF(1): For each $s \in S, F_S(s)$ is an isomorphism in $\mathscr{C}[S^{-1}]$.

CF(2): F_S is universal with respect to this property : if $G : \mathscr{C} \to \mathscr{D}$ is a functor such that $G(s)$ is an isomorphism in $\mathscr{D}$, for each $s \in S$, then there exists a unique functor $H : \mathscr{C}[S^{-1}] \to \mathscr{D}$ such that $G = HF_S$. Thus we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F_S"] \arrow[d, "G"'] & \mathscr{C}[S^{-1}] \arrow[dl, dashed, "H"] \\ \mathscr{D} & \end{tikzcd}$$

Reamrk 1.1.4. For the explicit construction of the category $\mathscr{C}[S^{-1}]$, we refer to [7]. We content ourselves merely with the observation that the objects of $\mathscr{C}[S^{-1}]$ are same as those of $\mathscr{C}$ and in the case when S admits a calculus of left (right) fractions, the category $\mathscr{C}[S^{-1}]$ can be described very nicely [7, 12].

1.2 Calculus of left (right) fractions

As discussed in [7], for constructing the category of fractions, the notion of calculus of left (right) fractions plays a very crucial role.

Definition 1.2.1. ([7], p. 258) A family of morphisms S in the category $\mathscr{C}$ is said to admit a *calculus of left fractions* if

(a) S is closed under finite compositions and contains identities of $\mathscr{C}$,

(b) any diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \\ Z & \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "t"'] & W \end{tikzcd}$$

with $t \in S$ and $tf = gs$,

(c) given

$$\begin{tikzcd} X \arrow[r, "s"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \arrow[r, dashed, "t"] & W \end{tikzcd}$$

with $s \in S$ and $fs = gs$, there is a morphism $t : Z \to W$ in S such that $tf = tg$.

A simple characterization for a family of morphisms S to admit a calculus of left fractions is the following.

Theorem 1.2.2. ([9], Theorem 1.3, p. 67) *Let S be a closed family of morphisms of $\mathscr{C}$ satisfying*

(a) *if $uv \in S$ and $v \in S$, then $u \in S$,*

(b) *every diagram*

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \\ \bullet & \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be embedded in a weak push-out diagram

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "t"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S admits a calculus of left fractions.

The notion of a set of morphisms admitting a calculus of right fractions is defined dually.

Definition 1.2.3. ([7], p. 267) A family S of morphisms in a category $\mathscr{C}$ is said to admit a *calculus of right fractions* if

(a) S is closed under finite compositions and contains identities of $\mathscr{C}$,

(b) any diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$ can be completed to a diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd}$$

with $t \in S$ and $ft = sg$,

(c) given

$$\begin{tikzcd} W \arrow[r, dashed, "t"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Z \end{tikzcd}$$

with $s \in S$ and $sf = sg$, there is a morphism $t : W \to X$ in S such that $ft = gt$.

The analog of Theorem 1.2.2 follows immediately by duality.

Theorem 1.2.4. ([9], Theorem 1.3^*, p. 70) *Let S be a closed family of morphisms of $\mathscr{C}$ satisfying*

(a) *if $vu \in S$ and $v \in S$, then $u \in S$,*

(b) *any diagram*

$$\begin{tikzcd} & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$, can be embedded in a weak pull-back diagram

$$\begin{tikzcd} \bullet \arrow[r, "t"] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "s"'] & \bullet \end{tikzcd}$$

with $t \in S$.

Then S admits a calculus of right fractions.

Reamrk 1.2.5. There are some set-theoretic difficulties in constructing the category $\mathscr{C}[S^{-1}]$; these difficulties may be overcome by making some mild hypotheses and using Grothendeick universe. Precisely speaking, the main logical difficulty involved in the construction of a category of fractions and its use, arises from the fact that if the category $\mathscr{C}$ belongs to a particular universe, the category $\mathscr{C}[S^{-1}]$ would, in general belongs to a higher universe ([7], Proposition 19.1.2). In most applications, however, it is necessary that we remain within the given initial universe. This logical difficulty can be overcome by making some kind of assumptions which would ensure that the category of fractions remains within the same universe [13–15]. Also the following theorem (Theorem 1.2.6) shows that if S admits a calculus of left (right) fractions, then the category of fractions $\mathscr{C}[S^{-1}]$ remains within the same universe as to the universe to which the category $\mathscr{C}$ belongs.

The following result will be used in our study.

Theorem 1.2.6. [8] *Let $\mathscr{C}$ be a small $\mathscr{U}$-category and S a set of morphisms of $\mathscr{C}$ that admits a calculus of left (right) fractions. Then $\mathscr{C}[S^{-1}]$ is a small $\mathscr{U}$-category.*

1.3 Adams completion and cocompletion

Sullivan introduced the concept of localizations [16]. Bousfield introduced the concepts of localizations in categories [17]. Both the constructions are applicable to many cases of intersts. Sullivan's construction is neat and concrete. Bousfield construction is general and categorical. Several authors have worked on both the constructions [18]. The notion of generalized completion (Adams completion) arose from a categorical completion process suggested by Adams [1, 2]. Originally this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [9], where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover they have also suggested the dual notion, namely the cocompletion (Adams cocompletion) of an object in a category. We recall the definitions of Adams completion and cocompletion.

Definition 1.3.1. [9] Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and

$$F : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](\text{-}, Y) : \mathscr{C} \to \mathscr{S}$$

defines a contravariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e.,

$$\mathscr{C}[S^{-1}](\text{-}, Y) \cong \mathscr{C}(\text{-}, Y_S)$$

then Y_S is called the *(generalized) Adams completion* of Y with respect to the set of morphisms S or simply the S*-completion* of Y. We shall often refer to Y_S as the *completion* of Y.

The above definition can be dualized as follows:

Definition 1.3.2. [9] Let $\mathscr{C}$ be an arbitrary category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect S and

$$F : \mathscr{C} \to \mathscr{C}[S^{-1}]$$

be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$,

$$\mathscr{C}[S^{-1}](Y, \text{-}) : \mathscr{C} \to \mathscr{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e.,

$$\mathscr{C}[S^{-1}](Y, \text{-}) \cong \mathscr{C}(Y_S, \text{-})$$

then Y_S is called the *(generalized) Adams cocompletion* of Y with respect to the set of morphisms S or simply the S*-cocompletion* of Y. We shall often refer to Y_S as the *cocompletion* of Y.

1.4 Existence theorems

We recall some results on the existence of Adams completion and cocompletion. We state Deleanu's theorem [15] that under certain conditions, global Adams completion of an object always exists.

Theorem 1.4.1. ([15], Theorem 1; [8], Theorem 1) *Let $\mathscr{C}$ be a cocomplete small $\mathscr{U}$-category* (*$\mathscr{U}$ is a fixed Grothendeick universe*) *and S a set of morphisms of $\mathscr{C}$ that admits a calculus of left fractions. Suppose that the following compatibility condition with coproduct is satisfied.*

(C) *If each $s_i : X_i \to Y_i$, $i \in I$, is an element of S, where the index set I is an element of $\mathscr{U}$, then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} X_i \to \bigvee_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathscr{C}$ has an Adams completion X_S with respect to the set of morphisms S.

Reamrk 1.4.2. Deleanu's theorem quoted above has an extra condition to ensure that $\mathscr{C}[S^{-1}]$ is again a small $\mathscr{U}$-category; in view of Theorem 1.2.6 the extra condition is not necessary.

Theorem 1.4.1 can be dualized as follows.

Theorem 1.4.3. ([8], Theorem 2) *Let $\mathscr{C}$ be a complete small $\mathscr{U}$-category ($\mathscr{U}$ is a fixed Grothendeick universe) and S a set of morphisms of $\mathscr{C}$ that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied.*

(P) *If each $s_i : X_i \to Y_i$, $i \in I$, is an element of S, where the index set I is an element of $\mathscr{U}$, then*

$$\bigwedge_{i\in I} s_i : \bigwedge_{i\in I} X_i \to \bigwedge_{i\in I} Y_i$$

is an element of S.

Then every object X of $\mathscr{C}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

We will recall some more results on the existence of Adams completion and cocompletion in the relevant chapters.

1.5 Couniversal property

Deleanu, Frei and Hilton have developed characterization of Adams completion and cocompletion in terms of a couniversal property.

Definition 1.5.1. [9] Given a set S of morphisms of $\mathscr{C}$, we define $\bar{S}$, the *saturation* of S as the set of all morphisms u in $\mathscr{C}$ such that $F(u)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. S is said to be *saturated* if $S = \bar{S}$.

Theorem 1.5.2. ([9], Proposition 1.1, p. 63) *A family S of morphisms of $\mathscr{C}$ is saturated if and only if there exists a functor $F : \mathscr{C} \to \mathscr{D}$ such that S is the collection of all morphisms f such that $F(f)$ is invertible.*

Deleanu, Frei and Hilton have shown that if the set of morphisms S is saturated then the Adams completion of a space is characterized by a certain couniversal property.

Theorem 1.5.3. ([9], Theorem 1.2, p. 63) *Let S be a saturated family of morphisms of $\mathscr{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathscr{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in S which is couniversal with respect to morphisms of S : given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in S such that $ts = e$. In other words, the following diagram is commutative*:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.3 can be dualized as follows.

Theorem 1.5.4. ([9], Theorem 1.4, p. 68) *Let S be a saturated family of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathscr{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in S which is couniversal with respect to morphisms of S : given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in S such that $st = e$. In other words, the following diagram is commutative* :

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

In most of applications, however, the set of morphisms S is not saturated. The following is a stronger version of Deleanu, Frei and Hilton's characterization of Adams completion in terms of a couniversal property.

Theorem 1.5.5. ([11], Theorem 1.2, p. 528) *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathscr{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in $\bar{S}$ which is couniversal with respect to morphisms of S: given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in $\bar{S}$ such that $ts = e$. In other words, the following diagram is commutative* :

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "s"'] & Y_S \\ Z \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 1.5.5 can be dualized as follows.

Theorem 1.5.6. ([10], Proposition 1.1, p. 224) *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathscr{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in $\bar{S}$ which is couniversal with respect to morphisms of S : given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in $\bar{S}$ such that $st = e$. In other words, the following diagram is commutative* :

$$\begin{tikzcd} Y_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & Y \\ Z \arrow[ur, "s"'] & \end{tikzcd}$$

For most of the application it is essential that the morphism $e : Y \to Y_S$ $(e : Y_S \to Y)$ has to be in S; this is the case when S is saturated and the results are as follows:

Theorem 1.5.7. ([9], Theorem 2.9, p. 76) *Let S be a saturated family of morphisms of $\mathscr{C}$ and let every object of $\mathscr{C}$ admit an S-completion. Then the morphism $e : Y \to Y_S$ belongs to S and is universal for morphisms to S-complete objects and couniversal for morphisms in S.*

The above result can be dualized as follows.

Theorem 1.5.8. ([9], dual of Theorem 2.9, p. 76) *Let S be a saturated family of morphisms of $\mathscr{C}$ and let every object of $\mathscr{C}$ admit an S-cocompletion. Then the morphism $e : Y_S \to Y$ belongs to S and is universal for morphisms to S-cocomplete objects and couniversal for morphisms in S.*

However, in many cases of practical interest S is not saturated. The following result shows that under some extra conditions on S, the morphism $e : Y \to Y_S$ $(e : Y_S \to Y)$ always belongs to S.

Theorem 1.5.9. ([11], Theorem 1.3, p. 533) *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem* 1.5.5, *where Y_S is the S-completion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr{C}$ which have the following properties* :

(a) *S_1 and S_2 are closed under composition,*

(b) *$fg \in S_1$ implies that $g \in S_1$,*

(c) *$fg \in S_2$ implies that $f \in S_2$,*

(d) *$S = S_1 \cap S_2$.*

Then $e \in S$.

Theorem 1.5.9 can be dualized as follows:

Theorem 1.5.10. ([11], dual of Theorem 1.3, p. 533) *Let* S *be a set of morphisms in a category* $\mathscr{C}$ *admitting a calculus of right fractions. Let* $e : Y_S \to Y$ *be the canonical morphism as defined in Theorem* 1.5.6*, where* Y_S *is the* S*-cocompletion of* Y*. Furthermore, let* S_1 *and* S_2 *be sets of morphisms in the category* $\mathscr{C}$ *which have the following properties* :

(a) S_1 *and* S_2 *are closed under composition,*

(b) $fg \in S_1$ *implies that* $g \in S_1$*,*

(c) $fg \in S_2$ *implies that* $f \in S_2$*,*

(d) $S = S_1 \cap S_2$.

Then $e \in S$.

1.6 A Serre class $\mathcal{C}$ of modules

We collect some relevant definitions and theorems involving Serre classes of modules [19].

Definition 1.6.1. [19] A nonempty class $\mathcal{C}$ of modules is called a *Serre class of modules* if and only if whenever the three-term sequence $A \to B \to C$ of modules is exact and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.

Definition 1.6.2. [19] Let $\mathcal{C}$ be a Serrre class of modules and $A, B \in \mathcal{C}$. A homomorphism $f : A \to B$

(a) is a $\mathcal{C}$-*monomorphism* if $\ker f \in \mathcal{C}$.

(b) is a $\mathcal{C}$-*epimorphism* if $\operatorname{coker} f \in \mathcal{C}$.

(c) is a $\mathcal{C}$-*isomorphism* if it is both a $\mathcal{C}$-monomorphism and $\mathcal{C}$-epimorphism.

Theorem 1.6.3. [20] *Let* $A, B \in \mathcal{C}$ *and* $f : A \to B$ *and* $g : B \to C$ *be two homomorphisms. Then the following statements are true.*

(a) *If* gf *is* $\mathcal{C}$*-monic, then so is* f.

(b) *If* gf *is* $\mathcal{C}$*-epic, then so is* g.

Theorem 1.6.4. (Five lemma of modules) [21] *Let*

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"'] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[r] \arrow[d, "\delta"] & M_5 \arrow[d, "\epsilon"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

be a row exact commutative diagram of R-modules and R-module homomorphism where R is a ring. Then the following hold :

(a) *If α is an epimorphism and β, and δ are monomorphisms, then γ is a monomorphism.*

(b) *If ϵ is a monomorphism and β, and δ are epimorphisms, then γ is an epimorphism.*

(c) *If α, β, δ and ϵ are isomorphisms, then γ is an isomorphism.*

Chapter 2

A Categorical Construction of Minimal Model of Lie Algebra

Quillen has established algebraic models for rational homotopy theory [22]. In fact Sullivan introduced the concept of minimal model [23] in rational homotopy theory. It may be noted that there are three basic constructions in literature which relate a 1-connected topological space X to a differential graded algebra. Adams and Hilton [24] constructed a chain algebra (with integer coefficients) for the loop space ΩX, a special version of which is Adams cobra construction [25]. Later Quillen [22] associated a differential graded rational Lie algebra $L(X)$ to the space X and Sullivan [23, 26] using simplical differential forms with rational coefficients, obtained a differential graded commutative cochain algebra for X. For these cochain algebras, Sullivan introduced the notion of minimal model, which corresponds to the Postnikov decomposition of a space. The aim of this chapter is to investigate a case showing how this algebraic construction is characterized in terms of Adams cocompletion.

Baues and Lemaire [27] have constructed minimal models for chain algebras (over any field) and for rational differential graded Lie algebras. In this chapter, we prove that the minimal model of a simply connected differential graded Lie algebra is characterized in terms of Adams cocompletion.

2.1 Minimal model

We recall the following algebraic preliminaries.

Definition 2.1.1. [28, 29] Let $\mathbb{Q}$ be the set of rational numbers. By a *graded algebra A* over $\mathbb{Q}$ we mean a graded $\mathbb{Q}$ vector space

$$A = \underset{n \geq 0}{\oplus} A_n$$

together with

(a) an associative multiplication

$$\mu : A \otimes A \to A$$

which is graded ($\mu(A_n \otimes A_m) \subset A_{n+m}$) and

(b) graded commutative $a \cdot b = (-1)^{nm} b \cdot a$, $a \in A_n$ and $b \in B_m$.

We also assume, unless otherwise stated, that A has an identity element $1 \in A_0$. The elements of A_n are said to be *homogeneous of degree* n (or *dimension* n).

Definition 2.1.2. [28, 29] A *differential graded algebra* (or d.g.a) is a graded algebra A, together with a differential d, of degree $+1$, which is a derivation. This means that for each n there is a vector space homomorphism

$$d = d_n : A_n \to A_{n+1}$$

satisfying

(a) $d \circ d = 0$ (*differential*) and

(b) $d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b)$ for $a \in A_n$ (*derivation*).

Definition 2.1.3. [29] Let (A, d) be a d.g.a. Let

$$Z_n(A) = \mathrm{Ker}\{d_n : A_n \to A_{n+1}\} = \text{subalgebra of } \mathit{cycles} \text{ of } A_n,$$

$$B_n = \mathrm{Im}\{d_{n-1} : A_{n-1} \to A_n\} = \text{subalgebra of } \mathit{boundaries} \text{ of } A_n,$$

$$Z_*(A) = \underset{n \geq 0}{\oplus} Z_n(A),$$

$$B_*(A) = \underset{n \geq 0}{\oplus} B_n(A).$$

As $d^2 = 0$, we have $B_n(A) \subset Z_n(A)$. The nth *homology space* of A is defined to be the quotient vector space

$$H_n(A) = Z_n(A)/B_n(A)$$

$$H_*(A) = \underset{n \geq 0}{\oplus} H_n(A) = Z_*(A)/B_*(A)$$

is a graded algebra, called the *homology algebra* of A.

Definition 2.1.4. [29] A d.g.a. A is said to be *connected* if

$$H_0(A) \cong \mathbb{Q}$$

and that A is *simply connected* (1-connected) if it is connected and

$$H_1(A) = 0.$$

A is called *n-connected* if

(a) $H_0(A_0) \cong \mathbb{Q}$ and

(b) $H_p(A_n) = 0, 1 \leq p \leq n.$

A is said to be of *finite type* if for each n, $H_n(A)$ is a vector space of finite dimensional over $\mathbb{Q}$.

Definition 2.1.5. [29, 30] Let A and B be graded algebras over $\mathbb{Q}$. A function $f : A \to B$ is called a *homomorphism* if it preserves all algebraic structures, that is,

(a) $f(A_n) \subset B_n$,

(b) $f(a + b) = f(a) + f(b)$,

(c) $f(a \cdot b) = f(a) \cdot f(b)$.

We assume that $f(1) = 1$. If A and B are d.g.a.'s it is required also that f_n commute with differentials, i.e., $f_{n+1}d_n^A = d_n^B f_n$

$$\begin{tikzcd} A_n \arrow[r, "d_n^A"] \arrow[d, "f_n"'] & A_{n+1} \arrow[d, "f_{n+1}"] \\ B_n \arrow[r, "d_n^B"'] & B_{n+1} \end{tikzcd}$$

If $f : A \to B$ is a d.g.a. homomorphism then f induces a map

$$f_* : H_*(A) \to H_*(B)$$

defined by the rule $f_*([z]) = [f(z)]$, where $[z]$ denotes the homology class of the element $z \in Z_*(A)$. Clearly f_* is a homomorphism of graded algebras.

Let $\mathscr{DGA}$ denote the category of differential graded algebra and differential graded algebras homomorphisms.

Definition 2.1.6. [29] Let $h : A \to A'$ be a map of Lie algebra. Then h is a weak isomorphism if and only if $H_*(h) : H_*(A) \to H_*(A')$ is an isomorphism.

Proposition 2.1.7. [27] *Let $H_*(M)$ be the homology of the differential graded vector space M. A weak isomorphism $f : M \to N$ is differential graded map such that*

$$H_*(f) : H_*(M) \to H_*(N)$$

is an isomorphism.

Definition 2.1.8. [28, 29] A differential graded algebra M is called a *minimal algebra* if and only if it satisfies the following proprieties :

(a) M is free as a graded algebra,

(b) M has a decomposable differential,

(c) $M_0 = \mathbb{Q}$, $M_1 = 0$,

(d) M has homology of finite type, that is, for each n, $H_n(M)$ is a finite dimensional vector space over $\mathbb{Q}$.

Let $\mathscr{M}$ be the full subcategory of the category $\mathscr{DGA}$ consisting of all minimal algebras and all differential graded algebra maps between them.

Definition 2.1.9. [28, 29] If A is simple connected differential graded algebra. A differential graded algebra $M = M_A$ is called a *minimal model* of A if the following conditions hold :

(a) $M \in \mathscr{M}$,

(b) there is a d.g.a. map $\rho : M \to A$ which induces an isomorphism on homology, i.e., $\rho_* : H_*(M) \overset{\cong}{\to} H_*(A)$.

Definition 2.1.10. [31] A *differential graded Lie algebra* is the data of a differential graded vector space (L, d) together a with a bilinear map

$$[\text{-}, \text{-}] : L \times L \to L$$

satisfying the following properties:

(a) [-, -] is homogeneous skew symmetric; this means $[L^i, L^j] \subset L^{i+j}$ and

$$[a, b] + (-1)^{\bar{a}\bar{b}+1}[b, a] = 0$$

for every a, b homogeneous.

(b) Every a, b, c homogeneous satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]].$$

(c) $d(L^i) \subset L^{i+1}$, $d \circ d = 0$
and

$$d[a, b] = [d(a), b] + (-1)^{\bar{a}}[a, d(b)].$$

The map d is called the *differential* of d.

Definition 2.1.11. [30, 31] Given two Lie algebra L and L', their direct sum is the Lie algebra consisting of the vector space $L \oplus L'$, of the pair $(x, x'), x \in L, x' \in L'$ with the operation

$$[(x, x'), (y, y')] = ([x, y], [x', y']),$$

$x, y \in L$ and $x', y' \in L'$.

Definition 2.1.12. [31] A *morphism* of differential graded Lie algebra is a graded linear map $f : L \to L'$ that commutes with bracket and the differential, i.e.,

$$f[x, y]_L = [f(x), f(y)]_{L'}$$

and

$$f(d_L x) = d_{L'} f(x).$$

Theorem 2.1.13. ([27], p.226, Proposition 1.4) *Let L be a simply connected differential graded Lie algebra over $\mathbb{Q}$ and $M = M_L$ be a minimal model for L. Then the map*

$$h : M \to L$$

induces weak isomorphism and h has the following couniversal property: for any 1-connected differential graded Lie algebra L' and differential graded Lie algebra map

$$f : L' \to L$$

which induces a weak isomorphism, there exists a differential graded Lie algebra map

$$g : M \to L'$$

such that the diagram commutes up to Lie algebra homotopy $fg \simeq h$ and g is unique up to Lie algebra homotopy.

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, dashed, "g"'] & L \\ L' \arrow[ur, "f"'] & \end{tikzcd}$$

2.2 The category $\mathscr{A}$

Let $\mathscr{U}$ be a fixed Grothendieck universe. Let $\mathscr{A}$ be the category of 1-connected differential graded Lie algebras over $\mathbb{Q}$ (in short d.g.l.a.'s) and differential graded Lie algebra homomorphisms where every element of $\mathscr{A}$ is an element of $\mathscr{U}$.

Let S denote the set of all differential graded Lie algebra homomorphisms in $\mathscr{A}$ which induce weak isomorphisms in all dimensions.

We prove the following results.

Proposition 2.2.1. *S is saturated.*

Proof. The proof is evident from Theorem 1.5.2. □

Next we show that the set of morphisms S of the category $\mathscr{A}$ admits a calculus of right fractions.

Proposition 2.2.2. *S admits a calculus of right fractions.*

Proof. Clearly, S is a closed family of morphisms of the category $\mathscr{A}$. We shall verify conditions (a) and (b) of Theorem 1.2.4. Let $u, v \in S$. We show that if $vu \in S$ and $v \in S$, then $u \in S$. We have $(vu)_* = v_* u_*$ and v_* are both homology isomorphisms implying u_* is a homology isomorphism. Thus $u \in S$. Hence condition (a) of Theorem 1.2.4 holds.

To prove condition (b) of Theorem 1.2.4 consider the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

with $s \in S$. We assert that the above diagram can be completed to a weak pull-back diagram

$$\begin{tikzcd} D \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

with $t \in S$. Since A, B and C are in $\mathscr{A}$ we write

$$A = \bigoplus_{n \geq 0} A_n, B = \bigoplus_{n \geq 0} B_n, C = \bigoplus_{n \geq 0} C_n,$$

and

$$f = \bigoplus_{n \geq 0} f_n, s = \bigoplus_{n \geq 0} s_n$$

where

$$f_n : A_n \to B_n, s_n : C_n \to B_n$$

are differential graded Lie algebras homomorphisms. Let

$$D_n = \{[(a,c),(a',c')] = ([a,a'],[c,c']) \in A_n \times C_n : f_n[a,a'] = s_n[c,c']\} \subset A_n \times C_n$$

where $a, a' \in A_n$ and $c, c' \in C_n$. We have to show that $D = \underset{n\geq 0}{\oplus} D_n$ is a differential graded Lie algebra. Let $t_n : D_n \to A_n$ be defined by

$$t_n([a,a'],[c,c']) = [a,a']$$

and $g_n : D_n \to C_n$ be defined by

$$g_n([a,a'],[c,c']) = [c,c'].$$

Clearly, t_n and g_n are differential graded Lie algebra homomorphisms and the above diagram is commutative. Let $(a,c) \in D_n, (a',c') \in D_m$,

$$d_n^A : A_n \to A_{n+1},\ d^A = \underset{n\geq 0}{\oplus} d_n^A$$

and

$$d_n^C : C_n \to C_{n+1},\ d^C = \underset{n\geq 0}{\oplus} d_n^C.$$

Define $d_n^D : D_n \to D_{n+1}$ by the rule

$$\begin{aligned} d_n^D[(a,c),(a',c')] &= d_n^D([a,a'],[c,c']) \\ &= (d_n^A[a,a'], d_n^C[c,c']). \end{aligned}$$

Let $d^D = \underset{n\geq 0}{\oplus} d_n^D$. For $(a,c) \in D_n, (a',c') \in D_m$,

$$\begin{aligned} [(a,c),(a',c')] &= ([a,a'],[c,c']) \\ &= \big(-(-1)^{nm+1}[a',a], -(-1)^{nm+1}[c',c]\big) \\ &= -(-1)^{nm+1}([a',a],[c',c]) \\ &= -(-1)^{nm+1}[(a',c'),(a,c)]. \end{aligned}$$

Thus we get

$$[(a,c),(a',c')] + (-1)^{nm+1}[(a',c'),(a,c)] = 0.$$

Next we have to show that

$$[a,[b,c]] = [[a,b],c] + (-1)^{\bar{a}\bar{b}}[b,[a,c]].$$

Since D_{n+m} is a Lie algebra, it satisfies the Jacobi property, i.e.,

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0.$$

where $[a,[b,c]]$, $[b,[c,a]]$ and $[c,[a,b]] \in D_{n+m}$. We have

$$\begin{aligned}[a,[b,c]] &= -[b,[c,a]]-[c,[a,b]]\\ &= -[b,[c,a]]+[[a,b],c]\\ &= [[a,b],c]+(-1)^{nm}[b,[a,c]].\end{aligned}$$

We show that d^D is a differential. We have

$$\begin{aligned}d^D[(a,c),(a',c')] &= d^D([a,a'],[c,c'])\\ &= (d^A[a,a'],d^C[c,c'])\\ &= ([d^Aa,a']+(-1)^n[a,d^Aa'],[d^Cc,c']+(-1)^n[c,d^Cc'])\\ &= ([d^Aa,a'],[d^Cc,c'])+(-1)^n([a,d^Aa'],[c,d^Cc'])\\ &= [(d^Aa,d^Cc),(a',c')]+(-1)^n[(a,c)(d^Aa',d^Cc')]\\ &= [d^D(a,c),(a',c')]+(-1)^n[(a,c),d^D(a',c')].\end{aligned}$$

Thus D becomes a differential graded Lie algebra.

Next we have to show that D is 1-connected, i.e., $H_0(D)=\mathbb{Q}$ and $H_1(D)=0$. First we show that $H_0(D)=\mathbb{Q}$. We have

$$\begin{aligned}H_0(D) &= Z_0(D)/B_0(D)\\ &= Z_0(D)\\ &= \{([a,a'],[c,c'])\in Z_0(A)\times Z_0(C) : f_0[a,a']=s_0[c,c']\}.\end{aligned}$$

Let $[1_A,1_{A'}]\in A_0, [1_C,1_{C'}]\in C_0$. Then

$$d_0^D([1_A,1'_A],[1_C,1'_C])=(d_0^A[1_A,1'_A],d_0^C[1_C,1'_C])=(0,0)$$

implies that $([1_A,1'_A],[1_C,1'_C])\in Z_0(D)$.

Since A and C are 1-connected, we have

$$H_0(A)=Z_0(A)=\mathbb{Q}=\mathbb{Q}[1_A,1_{A'}]$$

and

$$H_0(C) = Z_0(C) = \mathbb{Q} = \mathbb{Q}[1_C, 1_{C'}].$$

Thus

$$([a, a'], [c, c']) \in H_0(D) = Z_0(D) \subset Z_0(A) \times Z_0(C)$$

if and only if

$$[a, a'] = r[1_A, 1_{A'}]$$

and

$$[c, c'] = r[1_C, 1_{C'}]$$

for some $r \in \mathbb{Q}$. Now we get $H_0(D) = \mathbb{Q}$.

Next we have to show that $H_1(D) = 0$. Let

$$([a, a'], [c, c']) \in Z_1(D).$$

This implies that

$$[a, a'] \in Z_1(A), \quad [c, c'] \in Z_1(C)$$

and

$$f_1[a, a'] = s_1[c, c'].$$

Since A is 1-connected, we have

$$H_1(A) = 0, \quad \text{i.e.,} \quad Z_1(A)/B_1(A) = B_1(A);$$

hence

$$[a, a'] = d_0^A[\tilde{a}, \hat{a}]$$

where $[\tilde{a}, \hat{a}] \in A_0$. Similarly since C is 1-connected we have

$$H_1(C) = 0, \quad \text{i.e.,}\, Z_1(C)/B_1(C) = B_1(C);$$

hence

$$[c, c'] = d_0^C[\tilde{c}, \hat{c}]$$

where $[\tilde{c}, \hat{c}] \in C_0$. Now

$$f_1[a, a'] = s_1[c, c'],$$

i.e.,

$$f_1 d_0^A[\tilde{a}, \hat{a}] = s_1 d_0^C[\tilde{c}, \hat{c}].$$

This gives

$$d_0^B f_0[\tilde{a}, \hat{a}] = d_0^B s_0[\tilde{c}, \hat{c}],$$

i.e.,

$$d_0^B(f_0[\tilde{a}, \hat{a}] - s_0[\tilde{c}, \hat{c}]) = 0$$

showing

$$f_0[\tilde{a}, \hat{a}] - s_0[\tilde{c}, \hat{c}]) \in \text{ ker } d_0^B.$$

Thus

$$[f_0[\tilde{a}, \hat{a}] - s_0[\tilde{c}, \hat{c}]] \in H_0(B).$$

But $s_0 \in S$. Hence $s_{0_*} : H_0(C) \to H_0(B)$ is an isomorphism. Hence there exists an element $[\bar{c}, \ddot{c}] \in H_0(C)$ such that

$$s_0[\bar{c}, \ddot{c}] = f_0[\tilde{a}, \hat{a}] - s_0[\tilde{c}, \hat{c}]$$

Thus

$$s_0[\bar{c}, \ddot{c}] - f_0[\tilde{a}, \hat{a}] - s_0[\tilde{c}, \hat{c}] \in B_0(B) = 0,$$

i.e.,

$$f_0[\tilde{a}, \hat{a}] = s_0([\tilde{c}, \hat{c}] + [\bar{c}, \ddot{c}]).$$

So

$$([\tilde{a}, \hat{a}], [\tilde{c}, \hat{c}] + [\bar{c}, \ddot{c}]) \in D_0.$$

Moreover

$$\begin{aligned}
d_0^D(([a', \hat{a}], [\bar{c}, \bar{\bar{c}}]) + [c', \tilde{c}]) &= (d_0^A[a', \hat{a}], d_0^C([\bar{c}, \ddot{c}] + [c', \hat{c}])) \\
&= (d_0^A[a', \hat{a}], (d_0^C[\bar{c}, \ddot{c}] + d_0^C[c', \hat{c}])) \\
&= (d_0^A[a', \hat{a}], 0 + d_0^C[c', \hat{c}]) \\
&= (d_0^A[a', \hat{a}], d_0^C[c', \hat{c}]) \\
&= ([a, a'], [c, c'])
\end{aligned}$$

showing that $([a, a'], [c, c']) \in B_1(D)$. Thus $H_1(D) = 0$.

Next we have to shown that $t \in S$, i.e., $t_* : H_*(D) \to H_*(A)$ is an isomorphism. Let

F= ker g. Then we have the following commutative diagram [32]

$$
\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] \\
D \arrow[r, "t"'] \arrow[d, "g"'] & A \arrow[d, "f"] \\
C \arrow[r, "s"'] & B
\end{tikzcd}
$$

We consider the exact homology sequences

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{n-1}(C) \arrow[r] \arrow[d, "s_*"'] & H_n(F) \arrow[r] \arrow[d, equal] & H_n(D) \arrow[r] \arrow[d, "t_*"'] & {} \\
\cdots \arrow[r] & H_{n-1}(B) \arrow[r] & H_n(F) \arrow[r] & H_n(A) \arrow[r] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_n(C) \arrow[r] \arrow[d, "s_*"'] & H_{n+1}(F) \arrow[r] \arrow[d, equal] & \cdots \\
H_n(B) \arrow[r] & H_{n+1}(F) \arrow[r] & \cdots
\end{tikzcd}
$$

Clearly, the above diagram is commutative. By Five lemma t_* is an isomorphism. So $t \in S$.

Next for any differential graded Lie algebra $E = \bigoplus\limits_{n\geq 0} E_n$ and differential graded Lie algebra homomorphism

$$u = \bigoplus_{n\geq 0} u_n : E \to A$$

and

$$v = \bigoplus_{n\geq 0} v_n : E \to C$$

in $\mathscr{A}$, let the following diagram

$$
\begin{tikzcd}
E \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[d, "f"] \\
C \arrow[r, "s"'] & B
\end{tikzcd}
$$

commute, i.e., $fu = sv$. For any d.g.l.a E and d.g.l.a. homomorphisms $u : E \to A$ and

$v : E \to C$ with $fu = sv$. Consider the diagram

$$\begin{tikzcd}
E \arrow[dr, dashed, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& D \arrow[r, "t"] \arrow[d, "g"'] & A \arrow[d, "f"] \\
& C \arrow[r, "s"'] & B
\end{tikzcd}$$

Define $h : E \to D$ by the rule

$$h[x, y] = ([ux, uy], [vx, vy])$$

for $[x, y] \in E$. Clearly, h is well defined and also differential graded Lie algebra homomorphism. Next we show that the two triangles are commutative, i.e., $th = u$ and $gh = v$. For any $[x, y] \in E$,

$$th[x, y] = t([ux, uy], [vx, vy]) = [ux, uy] = u[x, y]$$

and

$$gh[x, y] = g([ux, uy], [vx, vy]) = [vx, vy] = v[x, y],$$

i.e., $th = u$ and $gh = v$. □

Proposition 2.2.3. *Let $\{s_i : L_i \to L'_i, i \in I\}$ be a subset of S; then*

$$\bigwedge_{i \in I} s_i : \bigwedge_{i \in I} L_i \to \bigwedge_{i \in I} L'_i$$

is an element of S, where the index set I is an element of $\mathscr{U}$.

Proof. The proof is trivial. □

The following result can be obtained from the above discussion.

Proposition 2.2.4. *The category $\mathscr{A}$ is complete.*

From Propositions 2.2.2, 2.2.3 and 2.2.4, it follows that the conditions of Theorem 1.4.3 are fulfilled and by the use of Theorem 1.5.4, we obtain the following result.

Theorem 2.2.5. *Every object L of the category $\mathscr{A}$ has an Adams cocompletion L_S with respect to the set of morphisms S and there exists a morphism $e : L_S \to L$ in S which is couniversal with respect to the morphisms in S, that is, given a morphism $s : L' \to L$ in*

S there exists a unique morphism $t : L_S \to L'$ in S such that $st = e$. In other words the following diagram is commutative:

$$\begin{tikzcd} L_S \arrow[r, "e"] \arrow[d, dashed, "t"'] & L \\ L' \arrow[ur, "s"'] & \end{tikzcd}$$

2.3 The main result

We show that the minimal model of a 1-connected differential graded Lie algebra can be expressed as the Adams cocompletion of the d.g.l.a. with respect to the chosen set of differential graded Lie algebra maps.

Theorem 2.3.1. $M_L \simeq L_S$.

Proof. Let $e : L_S \to L$ be the map as in Theorem 2.2.5 and $h : M_L \to M$ be a differential graded Lie algebra map as in Theorem 2.1.13. By the couniversal property of e there exists a d.g.l.a map $\theta : L_S \to M_L$ such that $e = h\theta$

$$\begin{tikzcd} L_S \arrow[r, "e"] \arrow[d, dashed, "\theta"'] & L \\ M_L \arrow[ur, "h"'] & \end{tikzcd}$$

By the couniversal property of h (Theorem 2.1.13) there exists a differential graded Lie algebra map $\varphi : M_L \to L_S$ such that $e\varphi = h$.

$$\begin{tikzcd} M_L \arrow[r, "h"] \arrow[d, dashed, "\varphi"'] & L \\ L_S \arrow[ur, "e"'] & \end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
L_S \arrow[r, "e"] \arrow[d, "\theta"'] \arrow[dd, bend right=60, "1_{L_S}"'] & L \\
M_L \arrow[d, "\varphi"'] & \\
L_S \arrow[uur, "e"'] &
\end{tikzcd}$$

We have $e\varphi\theta = h\theta = e$. By the uniqueness condition of the couniversal property of e (Theorem 2.2.5), we conclude that $\varphi\theta = 1_{L_S}$.

Next consider the following diagram:

$$\begin{tikzcd}
M_L \arrow[r, "h"] \arrow[d, "\varphi"'] \arrow[dd, bend right=60, "1_{M_L}"'] & L \\
L_S \arrow[d, "\theta"'] & \\
M_L \arrow[uur, "h"'] &
\end{tikzcd}$$

We have $h\theta\varphi = e\varphi = h$. By the uniqueness condition of the property of h (Theorem 2.1.13), we conclude that $\theta\varphi \simeq 1_{M_L}$. Thus $M_L \simeq L_S$. $\square$

Chapter 3

Homotopy Approximation of Modules

Behera and Nanda [11] have obtained the Postnikov approximation of a 1-connected based CW-complex with the help of a suitable set of morphisms. They have obtained the said decomposition by introducing a Serre class $\mathcal{C}$ of abelian groups. This chapter contains a Postnikov-like decomposition of a module over a ring with unity with respect to a Serre class.

The relative homotopy theory of modules, including the (module) homotopy exact sequence was introduced by Peter Hilton ([33], Chapter 13). In fact he has developed homotopy theory in module theory, parallel to the existing homotopy theory in topology. Unlike homotopy theory in topology, there are two types of homotopy theory in module theory, the injective theory and projective theory. They are dual but not isomorphic [34–36]. Using injective theory we have obtained, by considering a Serre class $\mathcal{C}$ of modules, the Postnikov-like factorzation of a module.

3.1 Homotopy of Modules

The narrative may be recalled from [4, 33]. We briefly describe some of the concepts towards notational view-points.

Definition 3.1.1. [33] Let Λ be a ring with unity. Let $\mathscr{M}$ be the category of Λ-modules and Λ-module homomorphisms. Let A and B be right Λ-modules and $f : A \to B$ Λ-homomorphism in the category $\mathscr{M}$. The map f is *i-nullhomotopic* denoted $f \simeq_i 0$, if f can be extended to some injective module $\bar{A}$ containing A. Also if $g : A \to B$ then $f \simeq_i g$ if $f - g \simeq_i 0$. The i-homotopy class of f is denoted by $[f]_i$.

Definition 3.1.2. [18, 33] Let A and B be right Λ modules and $f : A \to B$. A *mapping cylinder* of f is the module $\bar{A} \oplus B$ together with maps $\lambda : A \to \bar{A} \oplus B$, given by $\lambda(a) = i(a) + f(a)$ where $i : A \to \bar{A}$ is the inclusion and $\kappa : \bar{A} \oplus B \to B$ is defined by $\kappa(\bar{a} + b) = b$.

Definition 3.1.3. We now move towards a definition of *suspension*. Consider the short exact sequence

$$0 \to A \to \bar{A} \to \bar{A}/A \to 0$$

where $\bar{A}$ is injective. We define a *suspension* of A, $S(A)$ as $\bar{A}/A$ and $S^2(A)$ is defined as $\overline{S(A)}/S(A)$. Then repeating this process we get a sequence, namely, $SA, S^2A, \cdots, S^nA, \cdots$. This enables us to consider unambiguously the group $\bar{\pi}(SA, B)$ or more generally $\bar{\pi}_n(A, B)$. Notice that these groups have effectively been defined by means of an injective resolution of A, namely

$$A \to \bar{A} \to \overline{SA} \to \cdots \to \overline{S^nA} \to \cdots$$

with successive cokernels $SA, S^2A, \cdots S^nA, \cdots$. Then $\bar{\pi}_n(A, B)$ is the nth homology group of the complex obtained by applying the functor $Hom(, B)$ to this resolution. we may describe the (injective) procedure for defining $Ext^n(B, A)$ in similar terms.

$$Hom(B, A) \to Hom(B, \bar{A}) \to Hom(B, \overline{SA}) \to \cdots \to Hom(B, \overline{S^nA}) \to \cdots$$

Then C^n, the group of n-cochains, is $Hom(B, \overline{S^nA})$ and δ_n, the coboundary operator, is the map $C^n \to C^{n+1}$ induced by $i_{n+1}p_n$

$$\overline{S^nA} \xrightarrow{p_n} S^{n+1}A \xrightarrow{i_{n+1}} \overline{S^{n+1}A},$$

where p_n is the natural projection and i_{n+1} the injection. Also, Z^n, the group of n-cocycles, may be identified with $Hom(B, S^nA)$. For $f : B \to \overline{S^nA}$ is a cocycle if and only if $i_{n+1} \circ p_n \circ f = 0$, that is, $p_n \circ f = 0$. This means, by exactness, that $f(B) \subset i_n(S^nA)$. Thus f may be regarded as a map $B \to S^n$. Then B^n, the group of n-coboundaries, will be identified with $p_{n-1*}Hom(B, \overline{S^{n-1}A})$. For, in order that $f : B \to \overline{S^nA}$ be a coboundary, f must equal $i_n \circ p_{n-1} \circ g$ for some $g : B \to \overline{S^{n-1}A}$. Thus we see that

$$Ext^n(B, A) = Z^n/B^n = Hom(B, S^nA)/p_{n-1*}Hom(B, \overline{S^{n-1}A})$$

where $n \geq 1$.

Proposition 3.1.4. [33] *Let A and B be right Λ-modules. The following statements are equivalent, for $n \geq 0$*

(a) $S^n(A)$ *is injective.*

(b) $Ext^{n+1}(B, A) = 0$ *for all B.*

(c) $\bar{\pi}_n(A, B) = 0$ *for all B.*

3.2 The category $\tilde{\mathscr{M}}$

Let $\mathscr{U}$ be a fixed Grothendieck universe. Let $\mathscr{M}$ denote the category of all Λ-modules and Λ-homomorphisms and let $\tilde{\mathscr{M}}$ be the corresponding i-homotopy category, that is, the objects of $\tilde{\mathscr{M}}$ are all Λ-modules and the morphisms of $\tilde{\mathscr{M}}$ are i-homotopy classes of Λ-homomorphisms. We assume that the underlying sets of the elements of $\mathscr{M}$ are elements of $\mathscr{U}$.

We fix a suitable set of morphisms in $\tilde{\mathscr{M}}$. Let S_n denote the set of all maps $\alpha : A \to B$ such that for any module M,

$$\alpha_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$$

is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism for $m = n + 1$.

We will show that the set of morphisms S_n of the category $\tilde{\mathscr{M}}$ admits a calculus of left fractions.

Proposition 3.2.1. *S_n admits a calculus of left fractions.*

Proof. Clearly S_n is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.2. Let $\beta\alpha \in S_n$ and $\alpha \in S_n$ where $\alpha : A \to B$ and $\beta : B \to C$. Since $\beta\alpha, \alpha \in S_n$, for any module M in $\tilde{\mathscr{M}}$,

$$(\beta\alpha)_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, C)$$

and

$$\alpha_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$$

are $\mathcal{C}$-isomorphisms for $m \leq n$ and $\mathcal{C}$-epimorphisms for $m = n + 1$. It is to be shown that $\beta_* : \bar{\pi}_m(M, B) \to \bar{\pi}_m(M, C)$ is $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$. Since $\beta_*\alpha_*$ and α_* are $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$, it follows that

$$\beta_* : \bar{\pi}_m(M, B) \to \bar{\pi}_m(M, C)$$

is a $\mathcal{C}$-epimorphism for $m = n + 1$.

It is enough to show that β_* is a $\mathcal{C}$-monomorphism for $m \leq n$. For any $[b], [\tilde{b}] \in \bar{\pi}_m(M, B)$ with $\beta_* [b] = \beta_*[\tilde{b}]$ there exist $[a], [\tilde{b}] \in \bar{\pi}_m(M, A)$ such that $\alpha_* [a] = [b]$ and $\alpha_*[\tilde{a}] = [\tilde{b}]$, since α_* is a $\mathcal{C}$-isomorphism for $m \leq n$; hence

$$(\beta\alpha)_* [a] = \beta_*\alpha_* [a] = \beta_* [b] = \beta_*[\tilde{b}] = \beta_*\alpha_*[\tilde{a}] = (\beta\alpha)_* [\tilde{a}]$$

giving $[a] = [\tilde{a}]$ as $(\beta_*\alpha_*)$ is a $\mathcal{C}$-isomorphism for $m \leq n$. Hence $\beta \in S_n$.

In order to prove condition (b) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \\ C & \end{tikzcd}$$

in $\tilde{\mathscr{M}}$ with $\gamma \in S_n$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, dashed, "\delta"] \\ C \arrow[r, dashed, "\beta"'] & D \end{tikzcd}$$

in $\tilde{\mathscr{M}}$ with $\delta \in S_n$. Let $\alpha = [f]_i$ and $\gamma = [s]_i$. Let $\bar{A}$ be an injective module containing A and $\iota : A \to \bar{A}$ be the inclusion. The map

$$\iota_f : A \to \bar{A} \oplus B$$

is defined by

$$\iota_f(a) = i(a) + f(a)$$

and

$$r : \bar{A} \oplus B \to B$$

is defined by

$$r(\bar{a} + b) = b.$$

Clearly, $r\iota_f = f$; ι_f is cofibration [37]. Let

$$j : B \to \bar{A} \oplus B$$

be defined by

$$j(b) = 0 + b = b.$$

Clearly, $rj = 1_B$. We need to show that $jr \simeq 1_{\bar{A}\oplus B}$, i.e., $1_{\bar{A}\oplus B} - jr \simeq_i 0$. We have

$$jr(\bar{a} + b) = j(b) = b$$

and

$$(1_{\bar{A}\oplus B} - jr)\,(\bar{a} + b) - jr(\bar{a} + b) = \bar{a}.$$

Let
$$t : \bar{A} \oplus B \to \bar{A}$$
be defined by
$$t(\bar{a} + b) = \bar{a}$$
and
$$s : \bar{A} \to \bar{A} \oplus B$$
be defined by
$$s(\bar{a}) = \bar{a}.$$
We have
$$st : \bar{A} \oplus B \to \bar{A} \oplus B$$
and
$$st(\bar{a} + b) = s(t(\bar{a} + b)) = s(\bar{a}) = \bar{a}.$$
Clearly, $1_{\bar{A}\oplus B} - jr = st$. Since $\bar{A}$ is injective it follows that $1_{\bar{A}\oplus B} - jr \simeq_i 0$. Thus $1_{\bar{A}\oplus B} \simeq_i jr$. We consider the diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "\iota_f"'] & C \arrow[d, dashed, "u"] \\ \bar{A} \oplus B \arrow[r, dashed, "v"'] & Q \end{tikzcd}$$

and form its push-out in $\mathscr{M}$ where $Q = \left(\bar{A} \oplus B \oplus C\right)/L$ is the factor module and

$$L = \{i(a) + f(a) + s(a) : a \in A\}$$

is a Λ-submodule of $\bar{A} \oplus B \oplus C$. Define

$$u : C \to Q$$

by
$$u(c) = (0 + 0 + c) + L$$
and
$$v : \bar{A} \oplus B \to Q$$
by
$$v(\bar{a} + b) = (\bar{a} + b + 0) + L.$$

Clearly, the two maps are well defined and Λ-module homomorphisms. For any $a \in A$,

$$us(a) = u(s(a)) = (0 + 0 + s(a)) + L = (s(a)) + L = L.$$

On the other hand

$$v\iota_f(a) = v\,(\iota(a) + f(a)) = (\iota(a) + f(a) + 0) + L = L.$$

Thus $us = v\iota_f$. Hence the above diagram is commutative. Since ι_f is cofibration, so is u [37, 38], we therefore have the following diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_f"] \arrow[d, "s"'] & \bar{A} \oplus B \arrow[r, "p"] \arrow[d, "v"] & X \arrow[d, equal] \\
C \arrow[r, "u"'] & Q \arrow[r, "q"'] & X
\end{tikzcd}$$

where X is the cokernel of ι_f, as well as of u; p and q are the usual projections. We consider the exact homotopy sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{\pi}_{m+1}(M, X) \arrow[r] \arrow[d, equal] & \bar{\pi}_m(M, A) \arrow[r] \arrow[d, "s_*"'] & \bar{\pi}_m(M, \bar{A} \oplus B) \arrow[r] \arrow[d, "v_*"'] & {} \\
\cdots \arrow[r] & \bar{\pi}_{m+1}(M, X) \arrow[r] & \bar{\pi}_m(M, C) \arrow[r] & \bar{\pi}_m(M, Q) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{\pi}_m(M, X) \arrow[r] \arrow[d, equal] & \bar{\pi}_{m-1}(M, A) \arrow[r] \arrow[d, "s_*"'] & \cdots \\
\bar{\pi}_m(M, X) \arrow[r] & \bar{\pi}_{m-1}(M, C) \arrow[r] & \cdots
\end{tikzcd}$$

From Five Lemma it follows that

$$v_* : \bar{\pi}_m(M, \bar{A} \oplus B) \to \bar{\pi}_m(M, Q)$$

is $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$. Since j is a i-null homotopy equivalence, $(vj)_* : \bar{\pi}_m\,(M, B) \to \bar{\pi}_m\,(M, Q)$ is a $\mathcal{C}$-isomorphism for $m \leq n$

and a $\mathcal{C}$-epimorphism for $m = n + 1$. We consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & C \arrow[d, "u"] \\ B \arrow[r, "vj"'] & Q \end{tikzcd}$$

Let $\beta = [u]_i$ and $\delta = [vj]_i$. Taking the corresponding i-homotopy classes, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "r = [s]_i"] \arrow[d, "\alpha = [f]_i"'] & C \arrow[d, "\beta = [u]_i"] \\ B \arrow[r, "\delta = [vj]_i"'] & Q \end{tikzcd}$$

in $\tilde{\mathscr{M}}$ with $\delta \in S_n$. This indeed is a weak push-out diagram in $\tilde{\mathscr{M}}$. $\square$

Proposition 3.2.2. *Let* $\{s_j : A_j \to B_j, j \in J\}$ *be a subset of* S_n*; then*

$$\bigvee_{j \in J} s_j : \bigvee_{j \in J} A_j \to \bigvee_{j \in J} B_j$$

is an element of S_n*, where the index set J is in* $\mathscr{U}$.

Proof. We consider the commutative diagram

$$\begin{tikzcd} \bigoplus_{j \in J} \bar{\pi}(M, A_j) \arrow[r, "\{\alpha_{j*}\}", "\simeq"'] \arrow[d, "\bigoplus_{j \in J} s_{j*}"', "\simeq"] & \bar{\pi}(M, \bigvee_{j \in J} A_j) \arrow[d, "(\bigvee_{j \in J} s_j)_*"] \\ \bigoplus_{j \in J} \bar{\pi}(M, B_j) \arrow[r, "\simeq", "\{\beta_{j*}\}"'] & \bar{\pi}(M, \bigvee_{j \in J} A_j) \end{tikzcd}$$

where

$$\alpha_j : A_j \to \bigvee_{j \in J} A_j \text{ and } \beta_j : B_j \to \bigvee_{j \in J} B_j$$

are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a $\mathcal{C}$-isomorphism. Since each s_{j_*} is a $\mathcal{C}$-isomorphism in dim $\leq n$ and a $\mathcal{C}$-epimorphism in dimension $n + 1$, so is $\bigoplus_{j \in J} s_{j_*}$ and from the commutative diagram it follows that $(\bigvee_{j \in J} s_j)_*$ is also a $\mathcal{C}$-isomorphism in dim $\leq n$ and a $\mathcal{C}$-epimorphism in dim $n + 1$. Thus $\bigvee_{j \in J} s_j \in S_n$. $\square$

The following result is well known.

Proposition 3.2.3. *The category $\tilde{\mathscr{M}}$ is cocomplete.*

3.3 Existence of Adams completion in $\tilde{\mathscr{M}}$

From Propositions 3.2.1, 3.2.2 and 3.2.3 we see that all the conditions of the Theorem 1.5.5 are satisfied and hence we have the following result.

Theorem 3.3.1. *Every object M of the category $\tilde{\mathscr{M}}$ has an Adams completion M_{S_n} with respect to the set S_n of Λ-module homomorphisms. Furthermore, there exists a Λ-module homomorphism $e_n : M \to M_{S_n}$ in $\bar{S}_n$ which is couniversal with respect to the Λ-module homomorphisms in S_n : given a Λ- module homomorphism $s : M \to N$ in S_n there exists a unique Λ-module homomorphism $t_n : N \to M_{S_n}$ in $\bar{S}_n$ such that $t_n s = e_n$. In other words the following diagram is commutative* :

$$\begin{tikzcd} M \arrow[r, "e_n"] \arrow[d, "s"'] & M_{S_n} \\ N \arrow[ur, dashed, "t_n"'] & \end{tikzcd}$$

Theorem 3.3.2. *The Λ-module homomorphism $e_n : M \to M_{S_n}$ is in S_n.*

Proof. Let S_n^1 be the set of all morphisms $f : A \to B$ in the category $\tilde{\mathscr{M}}$ such that

$$f_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$$

is a $\mathcal{C}$-monomorphism for $m \leq n$ and S_n^2 be the set of all morphisms $f : A \to B$ in the category $\tilde{\mathscr{M}}$ such that

$$f_* : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, B)$$

is a $\mathcal{C}$-epimorphism for $m \leq n + 1$. Clearly

(i) $S_n = S_n^1 \cap S_n^2$,

(ii) S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.5.9.

Therefore $e_n \in S_n$. □

3.4 A Postnikov-like approximation

We can obtain a decomposition of a module with the help of the sets of morphisms S_n.

Theorem 3.4.1. *For any Λ-module A, for $n \geq 0$, there exist modules A_n, maps $e_n : A \to A_n$ and maps $p_{n+1} : A_{n+1} \to A_n$ such that*

(a) $e_{n*} : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, A_n)$ *is* $\mathcal{C}$*-isomorphism for* $m \leq n$ *and* $\bar{\pi}_m(M, A_n) = 0$*, for* $m > n$*,*

(b) $e_n = p_{n+1} \circ e_{n+1}$.

Proof. For each integer $n \geq 0$, let A_n be the S_n-completion of A and $e_n : A \to A_n$ be the canonical map as stated in Theorem 3.3.2. Since $e_{n+1} \in S_{n+1}$, it follows that $e_{n+1} \in S_n$. Hence by the couniversal property of e_{n+1}, there exists a Λ-homomorphism $p_{n+1} : A_{n+1} \to A_n$ making the following diagram commutative, i.e., $p_{n+1} \circ e_{n+1} = e_n$

$$\begin{tikzcd} A \arrow[r, "e_{n+1}"] \arrow[d, "e_n"'] & A_{n+1} \arrow[dl, dashed, "p_{n+1}"] \\ A_n & \end{tikzcd}$$

Since $e_n \in S_n$,

$$e_{n*} : \bar{\pi}_m(M, A) \to \bar{\pi}_m(M, A_n)$$

is a $\mathcal{C}$-isomorphism for $m \leq n$. We show that $\bar{\pi}_m(M, A_n) = 0 \;\; m > n$. Every Λ-module M has an injective resolution [21]. So we can take an injective resolution of M as

$$M \to \overline{M} \to \overline{SM} \to \cdots \to \overline{S^m M} \to \cdots$$

with successive cokernels $SM, S^2M, \cdots, S^{m+1}M, \cdots$. We break the exact sequence into short exact sequences:

$$\begin{aligned} &0 \to M \to \overline{M} \to SM \to 0, \\ &0 \to SM \to \overline{SM} \to S^2M \to 0, \\ &\qquad\quad \vdots \\ &0 \to S^{m-1}M \to \overline{S^{m-1}M} \to S^m M \to 0. \\ &\qquad\quad \vdots \end{aligned}$$

Applying $\mathrm{Ext}^j_\Lambda(M, -)$ to the short exact sequence

$$0 \to S^{m-1}M \to \overline{S^{m-1}M} \to S^m M \to 0$$

of Λ-modules, we get the exact sequence

$$0 \to \mathrm{Ext}^j_\Lambda\left(M, S^{m-1}M\right) \to \mathrm{Ext}^j_\Lambda\left(M, \overline{S^{m-1}M}\right) \to \mathrm{Ext}^j_\Lambda\left(M, S^m M\right) \to 0$$

for any $j > 0$. Since $\overline{S^{m-1}M}$ is injective, $\mathrm{Ext}^j_\Lambda(M, \overline{S^{m-1}M}) = 0$ for each $j > 0$ [21]. It is clear that $\mathrm{Ext}^j_\Lambda(M, S^m M) = 0$ and $S^m M$ is injective [33]. Hence $\bar{\pi}_m(M, A_n) = 0$ for all $m > n$.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & A_{n+1} \arrow[d, "\theta_{n+1}"] \\
 & A_n \arrow[d] \\
 & \vdots \arrow[d] \\
 & A_2 \arrow[d, "\theta_2"] \\
A \arrow[r, "e_1"'] \arrow[ur, "e_2"'] \arrow[uuur, "e_n"] \arrow[uuuur, "e_{n+1}"] & A_1
\end{tikzcd}$$

Thus we get Postnikov-like approximation of a module in $\tilde{\mathscr{M}}$. $\square$

Chapter 4

Topological G-Module and Adams cocompletion

Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also studied the dual notion, namely the Adams cocompletion of an object in a category [9]. Behera and Nanda [10] have obtained the Cartan-Whitehead decomposition of a 0-connected based CW-complex with the help of a suitable set of morphisms. They have obtained the said decomposition by introducing a Serre class $\mathcal{C}$ of groups. In this chapter, following the arguments in [10] and using the cohomology theory of topological G-modules, we characterize a topological G-module in terms of Adams cocompletions. In fact, the central idea of this chapter is to obtain a mod-$\mathcal{C}$ Whitehead-like tower of a topological G-module using cohomology theory of topological G-modules, where $\mathcal{C}$ is a Serre class of modules [19].

4.1 Topological G-modules

We need the following preliminaries.

Definition 4.1.1. [39, 40] Let G be a group. A *G-module* is an abelian group A together with a G-action on A that is compatible with the structure of A as an abelian group, i.e., a map $\cdot : G \times A \to A$ satisfying the following properties:

(a) $1 \cdot a = a$ for all $a \in A$.

(b) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $a \in A$ and $g_1, g_2 \in G$.

(c) $g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2$ for all a_1, a_2 and $g \in G$.

Definition 4.1.2. [39, 40] Let G be a topological group. A *topological G-module* is an abelian topological group A together with a continuous map $\cdot : G \times A \to A$ satisfying the following properties:

(a) $1 \cdot a = a$ for all $a \in A$.

(b) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $a \in A$ and $g_1, g_2 \in G$.

(c) $g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2$ for all a_1, a_2 and $g \in G$.

Definition 4.1.3. [39] A *homomorphism* $\alpha : A \to B$ of G-modules is just a homomorphism of abelian groups that satisfies $\alpha(ga) = g\alpha(a)$ for all $a \in A$ and $g \in G$.

We recall the group cohomology via cochains from [39, 41].

Definition 4.1.4. [39] Let A be a G-module and let $n \geq 0$.

(a) The group of n-*cochains* of G with coefficients in A is the set of all functions from G^n to $A : C^n(G, A) = \{f : G^n \to A\}$.

(b) The n*th differential*

$$d^n = d^n_A : C^n(G, A) \to C^{n+1}(G, A)$$

is the map

$$\begin{aligned} d^n(f)(g_0, g_1, \cdots, g_n) = g_0 \cdot f(g_1, \cdots, g_n) \\ + \sum_{j=1}^{n} (-1)^j f(g_0, g_1, \cdots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \cdots g_n) \\ + (-1)^{n+1} f(g_1, g_2, \cdots g_{n-1}) \end{aligned}$$

It can be checked that for any $n \geq 0$, $d^{n+1} \circ d^n = 0$ and

$$C(G, A) = (C^n(G, A), d^n)$$

is a cochain complex.

Definition 4.1.5. [39, 41] Let A be a G-module and d^n denotes the nth differential, $n \geq 1$. Let

$Z^n(G, A) = \text{Ker } d^n$, the group of n-cocycles of G with cofficients in A.

$B^n(G, A) = \text{Im } d^{n-1}$, for $n \geq 1$ the group of n-coboundaries of G with cofficients in A.

The n*th cohomology group* of G with coefficients in A is defined to be

$$H^n(G, A) = Z^n(G, A)/B^n(G, A).$$

Theorem 4.1.6. ([39] Lemma 1.2.7, p. 8) *If $\alpha : A \to B$ is a G-module homomorphism, then for each $n \geq 0$, there is an induced homomorphism of groups*

$$\alpha^n : C^n(G, A) \to C^n(G, B)$$

taking f to αf and compatible with the differentials in the sense that

$$d_B^n \circ \alpha^n = \alpha^{n+1} \circ d_A^n$$

Theorem 4.1.7. ([39], Corollary 1.2.9, p. 8) *A G-module homomorphism $\alpha : A \to B$ induces maps*

$$\alpha^* : H^n(G, A) \to H^n(G, B).$$

on cohomology.

4.2 The category $\mathscr{G}$

Let $\mathscr{U}$ be a fixed Grothendieck universe. Let $\mathscr{G}$ denote the category of all topological G-modules and continuous G-homomorphisms. We assume that the underlying sets of the elements of $\mathscr{G}$ are elements of $\mathscr{U}$. We fix a suitable set of morhisms in $\mathscr{G}$.

For $n \geq 1$, let S_n denote the set of all maps $\alpha : A \to B$ such that for any topological group G, $\alpha^* : H^m(G, A) \to H^m(G, B)$ is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$.

We will show that the set of morphisms S_n of the category $\mathscr{G}$ admits a calculus of right fractions.

Proposition 4.2.1. *S_n admits a calculus of right fractions.*

Proof. Clearly, S_n is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.4. Let $\beta\alpha \in S_n$ and $\beta \in S_n$,where $\alpha : A \to B$ and $\beta : B \to C$. Since $\beta^*\alpha^*$ and α^* are $\mathcal{C}$-isomorphism for $m > n$ and $\mathcal{C}$-monomorphism for $m = n$. it follows that $\alpha^* : H^m(G, B) \to H^m(G, C)$ is a $\mathcal{C}$-monomorphism for $m \geq n$. It is enough to show that α^* is a $\mathcal{C}$-epimorphism for $m > n$. We have

$$\beta^*\alpha^*(H^m(G, A)) = H^m(G, C)$$

for $m > n$, i.e.,

$$\beta^*(\alpha^*(H^m(G, A))) = \beta^*(H^m(G, B))$$

for $m > n$. From this we conclude that

$$\alpha^*(H^m(G, A)) = H^m(G, B)$$

for $m > n$, that is, α^* is a $\mathcal{C}$- epimorhism for $m > n$ and $\mathcal{C}$-monomorphism for $m = n$. Therefore α^* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$. Hence condition (a) of Theorem 1.2.4 holds.

To prove condition (b) of Theorem 1.2.4 consider the diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

with $s \in S_n$. We assert that the above diagram can be completed to a weak pull-back diagram

$$\begin{tikzcd} P \arrow[r, dashed, "t"] \arrow[d, dashed, "g"'] & A \arrow[d, "f"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

with $t \in S_n$. Let

$$P = \{(a, c) \in A \times C : f(a) = s(c)\} \subseteq A \times C.$$

Define $t : P \to A$ by the rule

$$t(a, c) = a$$

for $a \in A$ and $g : P \to C$ by the rule

$$g(a, c) = c$$

for $c \in \mathcal{C}$. Clearly, the two maps are well defined and continuous G-homomorphisms. Next we show the above diagram is commutative, i.e., $ft = sg$. For any $(a, c) \in P$, $ft(a, c) = f(a)$ and $sg(a, c) = s(c)$. Since $f(a) = s(c)$, hence $ft = sg$, i.e., the diagram is commutative.

Next let $u : X \to A$ and $v : X \to C$ in $\mathscr{G}$ be two morphisms such that $fu = sv$.

$$\begin{tikzcd} X \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\ & P \arrow[r, "t"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ & C \arrow[r, "s"'] & B \end{tikzcd}$$

Define $\theta : X \to P$ by the rule

$$\theta(x) = (u(x), v(x))$$

for $x \in X$. Clearly, θ is well defined and also a continuous G-homomorphism. Next we show that the two triangles are commutative. Now

$$t\theta(x) = t(u(x), v(x)) = u(x)$$

and

$$g\theta(x) = g(u(x), v(x)) = v(x).$$

So $t\theta = u$ and $g\theta = v$.

We need to show that $t \in S_n$. Let F= ker f=ker g and from the commutative diagram

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] \\
P \arrow[r, "t"'] \arrow[d, "g"'] & A \arrow[d, "f"] \\
C \arrow[r, "s"'] & B
\end{tikzcd}$$

in $\mathscr{G}$ we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{m+1}(G, C) \arrow[r] \arrow[d, "s^*"'] & H^m(G, F) \arrow[r] \arrow[d, equal] & H^m(G, P) \arrow[r] \arrow[d, "t^*"'] & {} \\
\cdots \arrow[r] & H^{m+1}(G, B) \arrow[r] & H^m(G, F) \arrow[r] & H^m(G, A) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^m(G, C) \arrow[r] \arrow[d, "s^*"'] & H^{m-1}(G, F) \arrow[r] \arrow[d, equal] & \cdots \\
H^m(G, B) \arrow[r] & H^{m-1}(G, F) \arrow[r] & \cdots
\end{tikzcd}$$

From Five Lemma it follows that t^* is $\mathcal{C}$-isomorphism for $m > n$ and $\mathcal{C}$-monomorphism for $m = n$. $\square$

Proposition 4.2.2. *Let $s_j : A_j \to B_j$ lie in S_n, for each $j \in J$, where the index set J is an* *element of $\mathscr{U}$. Then*

$$\underset{j\in J}{\wedge} s_j : \underset{j\in J}{\wedge} A_j \to \underset{j\in J}{\wedge} B_j$$

lies in S_n.

Proof. Let $s = \underset{j\in J}{\wedge} s_j$, $A = \underset{j\in J}{\wedge} A_j$ and $B = \underset{j\in J}{\wedge} B_j$. Define a map $s : A \to B$ by the rule

$$s(a) = (s_j(a_j))_{j\in J}$$

where $a = (a_j)_{j\in J}$. Clearly, s is well defined and is also a G-morphism in $\mathscr{G}$. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "p_j"'] & B \arrow[d, "q_j"] \\ A_j \arrow[r, "s_j"'] & B_j \end{tikzcd}$$

where p_j and q_j are the projections. Let F = ker p_j and from the commutative diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, "k"'] & F \arrow[d, "l"] \\ A \arrow[r, "s"'] \arrow[d, "p_j"'] & B \arrow[d, "q_j"] \\ A_j \arrow[r, "s_j"'] & B_j \end{tikzcd}$$

we have the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^{m+1}(G, A_j) \arrow[r] \arrow[d, "s_j^*"'] & H^m(G, F) \arrow[r] \arrow[d, equal] & H^m(G, A) \arrow[r] \arrow[d, "s^*"'] & {} \\ \cdots \arrow[r] & H^{m+1}(G, B_j) \arrow[r] & H^m(G, F) \arrow[r] & H^m(G, B) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} H^m(G, A_j) \arrow[r] \arrow[d, "s_j^*"'] & H^{m-1}(G, F) \arrow[r] \arrow[d, equal] & \cdots \\ H^m(G, B_j) \arrow[r] & H^{m-1}(G, F) \arrow[r] & \cdots \end{tikzcd}$$

By Five Lemma, s^* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$, that is, $s \in S_n$. $\square$

The following result is well known.

Proposition 4.2.3. *The category $\mathscr{G}$ is complete.*

4.3 Existence of Adams completion in $\mathscr{G}$

Using Propositions 4.2.1, 4.2.2 and 4.2.3, we draw the following result.

Theorem 4.3.1. *Every object A in the category $\mathscr{G}$ has an Adams cocompletion A_{S_n} with respect to the set of morphisms S_n.*

Since every object in the category $\mathscr{G}$ has Adams cocompletion with respect to the set of morphisms S_n, from Theorem 1.5.6 we will have the following result.

Theorem 4.3.2. *Every object A of the category $\mathscr{G}$ has a S_n cocompletion with respect to the set of morphisms S_n if and only if there exists a morphism $e_n : A_{S_n} \to A$ in $\bar{S}_n$ which is couniversal with respect to the morphisms in S_n: given a morphism $s : B \to A$ in S_n there exists a unique morphism $t_n : A_{S_n} \to B$ in $\bar{S}_n$ such that $st_n = e_n$. In other words the following diagram is commutative*:

$$
\begin{tikzcd}
A_{S_n} \arrow[r, "e_n"] \arrow[d, dashed, "t_n"'] & A \\
B \arrow[ur, "s"'] &
\end{tikzcd}
$$

Theorem 4.3.3. *The topological G-module homomorphism $e_n : A_{S_n} \to A$ is in S_n.*

Proof. Let S_n^1 be the set of all morphisms $f : A \to B$ in the category $\mathscr{G}$ such that

$$\alpha^* : H^m(G, A) \to H^m(G, B)$$

is a $\mathcal{C}$-monomorphism for $m \geq n$ and S_n^2 be the set of all morphisms $\alpha : A \to B$ in the category $\mathcal{G}$ such that

$$\alpha^* : H^m(G, A) \to H^m(G, B)$$

is a $\mathcal{C}$-epimorphism for $m \geq n + 1$. Clearly,

(i) $S_n = S_n^1 \cap S_n^2$;

(ii) S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.5.10.

Therefore, $e_n \in S_n$. $\square$

4.4 Cartan-Whitehead-like tower

We obtain a decomposition of a module with the help of the set of morphisms S_n.

Theorem 4.4.1. *For a topological G-module A, for $n \geq 1$, there exist topological G-modules A_n, maps $e_n : A_n \to A$ and maps $\theta_{n+1} : A_{n+1} \to A_n$ such that*

(a) $e_n^* : H^m(G, A_n) \to H^m(G, A)$ *is $\mathcal{C}$-isomorphism for $m > n$ and $H^m(G, A_n) = 0$ for $m \leq n$,*

(b) $e_{n+1} = \theta_{n+1} \circ e_n$.

Proof. For each integer $n \geq 1$, let A_n be the S_n-cocompletion of A and $e_n : A_n \to A$ be the canonical map as stated in Theorem 4.3.3. Since $e_n \in S_n$, it follows that $e_{n+1} \in S_n$; hence by the couniversal property of e_{n+1}, there exists a map

$$\theta_{n+1} : A_{n+1} \to A_n$$

making the following diagram commutative, i.e., $\theta_{n+1} \circ e_{n+1} = e_n$.

$$\begin{tikzcd} A \arrow[r, "e_{n+1}"] \arrow[d, "e_n"'] & A_{n+1} \arrow[dl, dashed, "\theta_{n+1}"] \\ A_n & \end{tikzcd}$$

Since $e_n \in S_n$,

$$e_{n^*} : H^m(G, A_n) \to H^m(G, A)$$

is a $\mathcal{C}$-isomorphism for $m > n$. Let A^n be the cokernel of

$$e_n : A_n \to A.$$

Consider the exact cohomology sequence of

$$A_n \xrightarrow{e_n} A \longrightarrow A^n$$

We conclude that

$$H^m(G, A_n) = 0$$

for $m \leq n$. Since

$$e_n \in S_n \subset S_{n+1},$$

it follows from the couniversal property of e_{n+1} that there exists a unique map

$$\theta_{n+1} : A_{n+1} \to A_n$$

such that the following

$$\begin{tikzcd}
A_{n+1} \arrow[r, "e_{n+1}"] \arrow[d, dashed, "\theta_{n+1}"'] & A \\
A_n \arrow[ur, "e_n"'] &
\end{tikzcd}$$

diagram is commutative, i.e., $e_{n+1} = e_n \theta_{n+1}$.

$$\begin{tikzcd}
\vdots \arrow[d] & \\
A_{n+1} \arrow[d, "\theta_{n+1}"'] \arrow[dddddr, "e_{n+1}"] & \\
A_n \arrow[d] \arrow[ddddr, "e_n"'] & \\
\vdots \arrow[d] & \\
A_2 \arrow[d, "\theta_2"'] \arrow[dr, "e_2"'] & \\
A_1 \arrow[r, "e_1"'] & A
\end{tikzcd}$$

Thus we have a tower of topological G-modules. $\square$

Chapter 5

Cohomology Decomposition of Topological *G*-Module

In this chapter we study the dual construction of the previous chapter. Behera and Nanda [11] have obtained the Postnikov approximation of a 1-connected based CW-complex with the help of a suitable set of morphisms in the category of 1-connected based CW-complexes. They have obtained this decomposition by introducing a Serre class $\mathcal{C}$ of abelian groups; we follow the techniques of their study to the case of cohomology of G-modules [41]. This chapter contains a Postnikov-like decomposition of a topological G-module by using the cohomology theory of topological G-modules and considering a Serre class $\mathcal{C}$ of abelian groups.

5.1 The category $\mathscr{M}$

Let $\mathscr{U}$ be a fixed Grothendieck universe. Let $\mathscr{M}$ denote the category of all topological G-modules and continuous G-homomorphisms. We assume that the underlying sets of the elements of $\mathscr{M}$ are elements of $\mathscr{U}$. We follow the narrative of topological a G-module A and the cohomology group $H^n(G, A)$ as given in Chapter 3. We fix a suitable set of morhisms in $\mathscr{M}$.

For $n \geq 1$, let S_n denote the set of all maps $\alpha : A \to B$ such that for any topological group G, $\alpha^* : H^m(G, A) \to H^m(G, B)$ is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism for $m = n + 1$.

We will show that the set of morphisms S_n of the category $\mathscr{M}$ admits a calculus of left fractions.

Proposition 5.1.1. *S_n admits a calculus of left fractions.*

Proof. Clearly, S_n is closed under composition. We shall verify conditions (a) and (b) of Theorem 1.2.2. Let $\beta\alpha \in S_n$ and $\alpha \in S_n$ where $\alpha : A \to B$ and $\beta : B \to C$. Since

$\beta\alpha, \in S_n$, for any topological G-module G in $\mathscr{M}$,

$$(\beta\alpha)^* = \beta^*\alpha^* : H^m(G, A) \to H^m(G, C)$$

is $\mathcal{C}$ isomorphisms for $m \leq n$ and

$$\alpha^* : H^m(G, A) \to H^m(G, B)$$

is $\mathcal{C}$-epimorphisms for $m = n + 1$. It is to be shown that

$$\beta^* : H^m(G, B) \to H^m(G, C)$$

is $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n + 1$. Since $\beta^*\alpha^*$ and α^* are $\mathcal{C}$-isomorphisms for $m \leq n$ and $\mathcal{C}$-epimorphisms for $m = n + 1$,

$$\beta^* : H^m(G, B) \to H^m(G, C)$$

is a $\mathcal{C}$-epimorphism for $m \leq n + 1$. It is enough to show that β^* is a $\mathcal{C}$-monomorphism for $m \leq n$. This is obvious for any $b, \tilde{b} \in H^m(G, B)$ let $\beta^*(b) = \beta^*(\tilde{b})$. Since α^* is a $\mathcal{C}$-isomorphism for $m \leq n$ there exist $a, \tilde{a} \in H^m(G, A)$ such that $\alpha^*(a) = b$ and $\alpha^*(\tilde{a}) = (\tilde{b})$; hence

$$(\beta\alpha)^*(a) = \beta^*\alpha^*(a) = \beta^*(b) = \beta^*(\tilde{b}) = \beta^*\alpha^*(\tilde{a}) = (\beta\alpha)^*(\tilde{a})$$

giving $a = \tilde{a}$. Thus β is a $\mathcal{C}$-isomorphism for $m \leq n$. Hence $\beta \in S_n$.

In order to prove condition (b) of Theorem1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{M}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{M}$ with $t \in S$. Let
$$D = (B \oplus C)/N$$
where N is a submodule of $B \oplus C$ generated by
$$\{(f(a), -s(a) : a \in A\}.$$
Define $t : B \to D$ by the rule
$$t(b) = (b, 0) + N$$
and $g : C \to D$ by the rule
$$g(c) = (0, c) + N.$$
Clearly, the two maps are well defined and continuous topological G-module homomorphisms. For any $a \in A$,
$$tf(a) = (f(a), 0) + N = (0, s(a)) + N = gs(a),$$
implying that $tf = gs$. Hence the diagram is commutative.

Next let $u : B \to X$ and $v : C \to X$ be in category $\mathscr{M}$ such that $uf = vs$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & Q \arrow[dr, dashed, "\theta"] & \\
 & & X
\end{tikzcd}$$

Define $\theta : Q \to X$, by the rule $\theta((b, c) + N) = u(b) + v(c)$. It is easy to show that θ is well defined and continuous topological G-module homomorphism. Next we show the two triangles are commutative. For any $b \in B$,
$$\theta t(b) = \theta((b, 0) + N) = u(b)$$
and for any $c \in C$,
$$\theta g(c) = \theta((0, c) + N)) = v(c).$$
So $\theta t = u$ and $\theta g = v$.

We consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[r, "p"] \arrow[d, "t"] & K \arrow[d, equal] \\ C \arrow[r, "g"'] & Q \arrow[r, "q"'] & K \end{tikzcd}$$

where K is the cokernel of f as well as of g, p and q are the usual projections. We consider the exact cohomology sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^{m+1}(G,K) \arrow[r] \arrow[d, equal] & H^m(G,A) \arrow[r] \arrow[d, "s^*"'] & H^m(G,B) \arrow[r] \arrow[d, "t^*"'] & {} \\ \cdots \arrow[r] & H^m(G,K) \arrow[r] & H^m(G,C) \arrow[r] & H^m(G,Q) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} H^{m+1}(G,K) \arrow[r] \arrow[d, equal] & H^{m-1}(G,A) \arrow[r] \arrow[d, "s^*"'] & \cdots \\ H^m(G,K) \arrow[r] & H^{m-1}(G,C) \arrow[r] & \cdots \end{tikzcd}$$

Since $s^* : H^m(G,A) \to H^m(G,C)$ is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism for $m = n+1$, it follows from the Five lemma that

$$t^* : H^m(G,B) \to H^m(G,Q)$$

is a $\mathcal{C}$-isomorphism for $m \leq n$ and $\mathcal{C}$-epimorphism for $m = n+1$. Thus $t \in S_n$. $\square$

Proposition 5.1.2. *Let $s_j : A_j \to B_j$ lie in S_n, for each $j \in J$, where the index set J is an element of $\mathscr{U}$. Then*

$$\bigvee_{j\in J} s_j : \bigvee_{j\in J} A_j \to \bigvee_{j\in J} B_j$$

lies in S_n.

Proof. We consider the commutative diagram

$$\begin{tikzcd} \bigoplus\limits_{j\in J} H^*(G,A_j) \arrow[r, "\{\alpha_j^*\}", "\simeq"'] \arrow[d, "\bigoplus\limits_{j\in J} s_j^*"', "\simeq"] & H^*(G, \bigvee\limits_{j\in J} A_j) \arrow[d, "(\bigvee\limits_{j\in J} s_j)^*"] \\ \bigoplus\limits_{j\in J} H^*(G,B_j) \arrow[r, "\simeq", "\{\beta_j^*\}"'] & H^*(G, \bigvee\limits_{j\in J} A_j) \end{tikzcd}$$

where

$$\alpha_j : A_j \to \bigvee_{j\in J} A_j$$

and

$$\beta_j : B_j \to \bigvee_{j\in J} B_j$$

are the canonical inclusions. Note that each horizontal row is an isomorphism, hence a $\mathcal{C}$-isomorphism. Since each s_j^* is a $\mathcal{C}$-isomorphism in dim $\leq n$ and a $\mathcal{C}$-epimorphism in dimension $n+1$, so is $\bigoplus_{j\in J} s_j^*$, and from the commutative diagram it follows that $(\bigvee_{j\in J} s_j)^*$ is also a $\mathcal{C}$-isomorphism in dimension $\leq n$ and a $\mathcal{C}$-epimorphism in dimension $n+1$. Thus $\bigvee_{j\in J} s_j \in S_n$. □

The following result is well known.

Proposition 5.1.3. *The category $\mathscr{M}$ is cocomplete.*

From Propositions 5.1.1, 5.1.2 and 5.1.3 we see that all the conditions of Theorem 1.4.1 are satisfied and hence we have the following result.

Theorem 5.1.4. *Every object M of the category $\mathscr{M}$ has an Adams completion M_{S_n} with respect to the set S_n of continuous topological G-module homomorphisms. Furthermore, there exists a continuous topological G-module homomorphisms $e_n : M \to M_{S_n}$ in $\bar{S}_n$ which is couniversal with respect to the continuous topological G-module homomorphisms in S_n : given a continuous topological G-module homomorphism $s : M \to N$ in S_n there exists a unique continuous topological G-module homomorphism $t : N \to M_{S_n}$ in $\bar{S}_n$ such that $ts = e_n$. In other words the following diagram is commutative*:

$$\begin{tikzcd} M \arrow[r, "e_n"] \arrow[d, "s"'] & M_{S_n} \\ N \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

We show that the G-module homorphism $e_n : M \to M_{S_n}$ as constructed in Theorem 5.1.4 belongs to S_n.

Theorem 5.1.5. *The topological G-module homomorphism $e_n : M \to M_{S_n}$ is in S_n.*

Proof. Let S_n^1 be the set of all morphisms $\alpha : A \to B$ in the category $\mathscr{M}$ such that

$$\alpha^* : H^m(G, A) \to H^m(G, B)$$

is a $\mathcal{C}$-monomorphism for $m \leq n$ and S_n^2 be the set of all morphisms

$$\alpha : A \to B$$

in the category $\mathscr{M}$ such that

$$\alpha^* : H^m(G, A) \to H^m(G, B)$$

is a $\mathcal{C}$-epimorphism for $m \leq n + 1$. Clearly

(a) $S_n = S_n^1 \cap S_n^2$

(b) S_n^1 and S_n^2

satisfy all the conditions of Theorem 1.5.9. Therefore, $e_n \in S_n$. □

5.2 Existence of Adams Completion in $\mathscr{M}$

We obtain a decomposition of a G-module with the help of the sets of morphisms S_n.

Theorem 5.2.1. *Let A be a topological G-module. Then for $n \geq 1$, there exist topological G-modules A_n, continuous G-homomorphism $e_n : A \to A_n$ and $\theta_n^m : A_m \to A_n$ for each pair of indices m, n satisfying $n \leq m$ such that*

(a) $e_n^* : H^m(G, A) \to H^m(G, A_n)$ *is $\mathcal{C}$-isomorphism for $i \leq n$ and a $\mathcal{C}$-epimorphism for $i = n + 1$,*

(b) $e_n = \theta_n^m \circ e_m$, $n \leq m$,

(c) $\theta_n^m = 1_{A_n}$.

Proof. For each integer $n \geq 1$, let A_n be the S_n-completion of A and $e_n : A \to A_n$ be the canonical G-homomorphism as stated in Proposition 5.1.5. Since $e_n \in S_n$, it follows that $e_m \in S_n$, for $n \leq m$ and hence by the couniversal property of e_n, there exists a continuous G-homomorphism $\theta_n^m : A_m \to A_n$ making the following diagram commutative, i.e., $\theta_n^m \circ e_m = e_n$, $n \leq m$

$$\begin{tikzcd}
A \arrow[r, "e_n"] \arrow[d, "e_n"'] & A_n \arrow[dl, dashed, "\theta_n^m"] \\
A_m &
\end{tikzcd}$$

It follows that $\theta_n^n = 1_{A_n}$. Since $e_n \in S_n$,

$$e_n^* : H^m(G, A) \to H^m(G, A_n)$$

is a $\mathcal{C}$-isomorphism for $m \leq n$ and a $\mathcal{C}$-epimorphism $m = n + 1$. Thus we get a tower of topological G-modules.

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & A_{n+1} \arrow[d, "\theta_{n+1}"] \\
 & A_n \arrow[d] \\
 & \vdots \arrow[d] \\
 & A_2 \arrow[d, "\theta_2"] \\
A \arrow[r, "e_1"'] \arrow[ur, "e_2"'] \arrow[uuur, "e_n"'] \arrow[uuuur, "e_{n+1}"] & A_1
\end{tikzcd}
$$

Thus we have a tower of topological G-modules. $\square$

Chapter 6

Ring of Fractions as Adams Completion

The formation of ring of fractions is one of the most important technical tools in commutative algebra. It corresponds in the algebraic-geometric picture to concentrating attention on an open set or near a point. This chapter describes the categorical construction of ring of fractions in Banach space (and Hilbert space). It is shown that, the ring of fractions of the algebra of all bounded linear operators on a separable infinite dimensional Banach space is isomorphic to the Adams completion of the algebra with respect to a chosen set of morphisms in the category of separable infinite dimensional Banach spaces and bounded linear norm preserving operators of norms at most 1.

6.1 Ring of fractions

We briefly recall the ring of fractions.

Definition 6.1.1. [42] A subset S of a ring A with unit 1 is called a (*right*) *denominator* set if S satisfies the following conditions:

(a) If $s, t \in S$ then $st \in S$ and $1 \in S$.

(b) If $s \in S$ and $a \in A$ then there exist $t \in S$ and $b \in A$ such that $sb = at$.

(c) If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.

(d) S does not contain 0 (to avoid triviality).

Definition 6.1.2. [42] Let A be a ring and let S be a multiplicatively closed subset of A, i.e., $s, t \in S$ implies $st \in S$ and $1 \in S$. A *ring of fractions* (right) of A with respect to S is defined as a ring $A[S^{-1}]$ together with a ring homomorphism

$$u : A \to A[S^{-1}]$$

satisfying:

(a) $u(s)$ is invertible for every $s \in S$.

(b) Every element in $A[S^{-1}]$ has the form $u(a)u(s)^{-1}$ with $s \in S$.

(c) $u(a) = 0$ if and only if $as = 0$ for some $s \in S$.

Proposition 6.1.3. [42] *Let S be a multiplicatively closed subset of A. Then $A[S^{-1}]$ exists, if and only if S satisfies*

(a) *If $s \in S$ and $a \in A$ then there exists $t \in S$ and $b \in A$ such that $sb = at$,*

(b) *If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.*

Reamrk 6.1.4. For the detailed construction of $A[S^{-1}]$ we refer to [42]. However there is a universal property in $A[S^{-1}]$.

Proposition 6.1.5. [42] *When $A[S^{-1}]$ exists, it has the following universal property: for every ring homomorphism*

$$g : A \to B$$

such that $g(s)$ is invertible in B for every $s \in S$, then there exists a unique ring homomorphism

$$h : A[S^{-1}] \to B$$

such that $g = hu$, i.e., the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & A[S^{-1}] \arrow[dl, dashed, "h"] \\ B & \end{tikzcd}$$

We shall use the following result in the sequel.

Proposition 6.1.6. *Let A and B be the algebras of all bounded linear operators on a separable infinite dimensional Banach space. Let $g : A \to B$ be a surjective bounded linear homomorphism such that*

(a) *$g(s)$ is a unit in B for every $s \in S$.*

(b) *$g(a) = 0$ implies $as = 0$ for some $s \in S$.*

Then there exists a unique ring homomorphism $\theta : B \to A[S^{-1}]$ such that $\theta g = u$, i.e., the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & A[S^{-1}] \\ B \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

Proof. By Proposition 6.1.5 there exists a unique isomorphism

$$h : A[S^{-1}] \to B$$

such that $g = hu$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & A[S^{-1}] \arrow[dl, dashed, "h"] \\ B & \end{tikzcd}$$

Let $\theta = h^{-1}$. For any $a \in A$,

$$\theta g(a) = h^{-1}g(a) = h^{-1}hu(a) = u(a)$$

implying $\theta g = u$, i.e., the above diagram is commutative.

We show that θ is unique. Let there exist another $\theta' : B \to A[S^{-1}]$ such that $\theta' g = u$. For any $b \in B$, we have

$$\theta'(b) = \theta'(g(b')) = u(b) = \theta g(b),$$

i.e., $\theta = \theta'$. $\square$

6.2 The category $\mathscr{B}$

Let $\mathscr{B}$ denote the category of separable infinite dimensional Banach spaces and linear norm preserving operators of norm at most 1. We assume that the underlying sets of the elements of $\mathscr{B}$ are elements of $\mathscr{U}$. We fix a set of morphisms in the category $\mathscr{B}$.

Let S be the set of those bounded linear norm preserving operators of norm at most 1 which are surjective in $\mathscr{B}$.

We prove the following results.

Proposition 6.2.1. *Let $s_i : P_i \to Q_i$ lie in S for each $i \in I$, where the index set I is an element of $\mathscr{U}$. Then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} P_i \to \bigvee_{i\in I} Q_i$$

lies in S.

Proof. Coproducts in $\mathscr{B}$ are l_1 sums. Let

$$P = \underset{i\in I}{\vee} P_i$$

and

$$Q = \underset{i\in I}{\vee} Q_i.$$

Define $s = \underset{i\in I}{\vee} s_i : P \to Q$ by the rule

$$s(p) = (s_i(p_i))_{i\in I}.$$

Clearly, s is well defined bounded linear homomorphism. For any $(q_i)_{i\in I} \in Q_i$, since s_i is surjective we have $s_i(p_i) = q_i$ and

$$(q_i)_{i\in I} = (s_i(p_i))_{i\in I} = s(p).$$

That s is a bounded linear norm preserving operator of norm at most 1, can be proved easily. Hence $s = \underset{i\in I}{\vee} s_i$ lies in S. □

Proposition 6.2.2. *S admits a calculus of left fractions.*

Proof. Let A and B be any two objects of category $\mathscr{B}$. Clearly, S is a closed family of morphisms of the category $\mathscr{B}$. We shall verify conditions (a) and (b) of Theorem 1.2.2. Let $s : A \to B$ and $t : B \to C$ be two morphisms of the category $\mathscr{B}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$. Clearly, $t \in S$. Hence the condition (a) of Theorem 1.2.2 holds.

In order to prove condition (b) of Theorem 1.2.2, consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\mathscr{B}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\mathscr{B}$ with $t \in S$. Let

$$D = (B \oplus_{l_1} C)/\bar{\triangle}$$

is the quotient of the direct sum $(B \oplus_{l_1} D)$ endowed with the l_1 norm [43] and $\bar{\triangle}$ is the closer of the subspace

$$\triangle = \{(f(a), -s(a)) : a \in A\}.$$

Define $g : C \to D$ by the rule, for $c \in C$

$$g(c) = (0, c) + \bar{\triangle}$$

and $t : B \to D$ by the rule, for $b \in B$

$$t(b) = (b, 0) + \bar{\triangle}.$$

Clearly, the two maps are well defined and bounded linear homomorphisms. For any $a \in A$,

$$tf(a) = t(f(a)) = (f(a), 0) + \bar{\triangle} = (0, s(a)) + \bar{\triangle} = g(s(a)) = gs(a)$$

implying that $tf = gs$. Hence the diagram is commutative in $\mathscr{B}$ with $t \in S$.

In order to show that t is surjective, take an element

$$(b, c) + \bar{\triangle} \in D.$$

Then

$$\begin{aligned}
(b, c) + \bar{\triangle} &= (b, 0) + (0, c) + \bar{\triangle} \\
&= t(b) + g(c) \\
&= t(b) + g(s(a)) \\
&= t(b) + tf(a) \\
&= t(b + f(a))
\end{aligned}$$

(since t is linear), implying t is surjective, i.e., $t \in S$.

Next let $u : B \to Z$ and $v : C \to Z$ be in category $\mathscr{B}$ such that $uf = vs$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\
& & Z
\end{tikzcd}
$$

Define $\theta : D \to Z$ by the rule

$$\theta((b, c) + \bar{\triangle}) = u(b) + v(c).$$

It is easy to show that θ is well defined and bounded linear homomorphism (since $\|\theta\| \leq \max\{\|u\|, \|v\|\}$). It follows that θ is bounded.

Next we show the two triangles are commutative. For any $b \in B$,

$$\theta t(b) = \theta(b, 0) = u(b),$$

showing that $\theta t = u$. Similarly $\theta g = v$. Thus the two triangles are commutative. □

The following result is well known.

Proposition 6.2.3. *The category $\mathscr{B}$ is cocomplete.*

From Propositions 6.2.1, 6.2.2, and 6.2.3 we see that all the conditions of Theorem 1.4.1 are satisfied and hence we have the following result.

Theorem 6.2.4. *Every object A of the category $\mathscr{B}$ has an Adams completion A_S with respect to the set S of homomorphisms. Furthermore, there exists a homomorphism $e : A \to A_S$ in $\bar{S}$ which is couniversal with respect to the homomorphisms in S: given a homomorphism $s : A \to B$ in S there exists a unique homomorphism $t : B \to A_S$ in $\bar{S}$ such that $ts = e$. In other words the following diagram is commutative*:

$$
\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "s"'] & A_S \\
B \arrow[ur, dashed, "t"'] &
\end{tikzcd}
$$

Theorem 6.2.5. *The homomorphism $e : A \to A_S$ (as obtained in Theorem* 6.2.4 *) is in S.*

Proof. Let

$$S_1 = \{f : X \to Y \text{ in } \mathscr{B} \mid f \text{ is a surjective homorphism}\}$$

and

$$S_2 = \{f : X \to Y \text{in } \mathscr{B} \mid f \text{ is a bounded linear norm preserving operator}\}.$$

Clearly,

(a) $S = S_1 \cap S_2$ and

(b) S_1 and S_2 satisfy all the conditions of Theorem 1.5.9. Therefore $e \in S$.

□

6.3 The main result

We show that the ring of fractions $A[S^{-1}]$ of the algebra A of all bounded linear operators on a separable infinite dimensional Banach space is precisely the Adams completion A_S of A.

Theorem 6.3.1. $A[S^{-1}] \cong A_S$.

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & A[S^{-1}] \\ A_S \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

By Proposition 6.1.6, there exists a unique homomorphism $\varphi : A_S \to A[S^{-1}]$ in S such that $\varphi e = u$.

Next consider the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & A_S \\ A[S^{-1}] \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

By Theorem 6.2.4, there exists a unique homomorphism $\psi : A[S^{-1}] \to A_S$ in S such that $\psi u = e$.

In the following diagram

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dd, "e"'] & & A_S \\
& A[S^{-1}] \arrow[ur, "\psi"'] & \\
A_S \arrow[ur, "\varphi"'] \arrow[uurr, "1_{A_S}"'] & &
\end{tikzcd}$$

we have $\psi\varphi e = \psi u = e$. By the uniqueness condition of the couniversal property of e, we conclude that $\psi\varphi = 1_{A_S}$.

Also in the diagram

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dd, "u"'] & & A[S^{-1}] \\
& A_S \arrow[ur, "\varphi"'] & \\
A[S^{-1}] \arrow[ur, "\psi"'] \arrow[uurr, "1_{A[S^{-1}]}"'] & &
\end{tikzcd}$$

we have $\varphi\psi u = \varphi e = u$. By the uniqueness condition of the couniversal property of u, we conclude that $\varphi\psi = 1_{A[S^{-1}]}$. Thus $A[S^{-1}] \cong A_S$. $\square$

Note 6.3.2. The above results can recasted with minor changes to show that the ring of fractions $A[S^{-1}]$ of the algebra A of all bounded linear operators on a separable infinite dimensional Hilbert space [44] is precisely the Adams completion A_S of A. We omit the details in order to avoid the repeatedness of the proofs of the results.

Chapter 7

A Categorical Study of Symmetric and Tensor Algebras

In this chapter we present a categorical construction of tensor and symmetric algebras. In fact we obtain these algebras as Adams completion by choosing different sets of morphisms in appropriate categories.

First we make categorical study of tensor algebra. For this study we work in the category of all K-modules and module homomorphisms where K is a commutative ring with unit 1. We recall tensor algebra.

7.1 Tensor algebra

The tensor algebra of a module V, denoted as $T(V)$ or $T^*(V)$, is the algebra of tensors on V (of any rank) with multiplication being the tensor product. It is the free algebra on V in the sense of being left adjoint to the forgetful functor from algebras to vector spaces.

In this chapter, it is shown that given an algebra, its nth tensor algebra is isomorphic to the Adams completion of the given algebra.

Definition 7.1.1. [45, 46] Let K be a commutative ring. Let V be a K-module. The *tensor algebra* $T(V)$ of V over K is defined to be the K- algebra formed by the K-module

$$T(V) = \oplus_{i=0}^{\infty} V^{\otimes i} = K \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

equipped with a multiplication which is defined by

$$(a_i)_{i\in I} \cdot (b_i)_{i\in I} = \left(\sum_{i=0}^{n} a_i \otimes b_{n-i} \right)_{n\in N}$$

for every

$$(a_i)_{i\in I} \in \oplus_{i\in I} V^{\otimes i}$$

and

$$(b_i)_{i\in I} \in \oplus_{i\in I} V^{\otimes i}.$$

$T(V)$ is a graded K-algebra with the graded piece of degree $n \geq 0$ being the subgroup $V^{\otimes n}$, which we denote by $T^n(V)$. The map

$$V^{\otimes n} \to T^n V$$

defined by

$$m_1 \otimes \cdots \otimes m_n \mapsto (0, \cdots, m_1 \otimes \cdots \otimes m_n, 0, \cdots)$$

is a morphism of K-modules, which gives an isomorphism of K-modules of V with its image $T^n(V)$.

We prove the following couniversal property of nth term of tensor algebra.

Theorem 7.1.2. *Let $V^{\otimes n}$ and $W^{\otimes n}$ be K-modules and let $f : V^{\otimes n} \to W^{\otimes n}$ be module isomorphism of K-modules. Then f has the following property: given a module isomorphism $g : V^{\otimes n} \to T^n(V)$, there exists a unique module isomorphism such that $g = \theta f$.*

$$\begin{tikzcd}
V^{\otimes n} \arrow[r, "g"] \arrow[d, "f"'] & T^n(V) \\
W^{\otimes n} \arrow[ur, dashed, "\theta"'] &
\end{tikzcd}$$

Proof. For $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in W^{\otimes n}$, define

$$\theta : W^{\otimes n} \to T^n(V)$$

by the rule

$$\theta(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = gf^{-1}(w_1 \otimes w_2 \otimes \cdots \otimes w_n).$$

Clearly, θ is well defined and is a homomorphism. We show that θ is one-one. If

$$\theta(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = \theta(w'_1 \otimes w'_2 \otimes \cdots \otimes w'_n)$$

then

$$gf^{-1}(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = gf^{-1}(w'_1 \otimes w'_2 \otimes \cdots \otimes w'_n);$$

since f and g are isomorphisms, we have

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n = w'_1 \otimes w'_2 \otimes \cdots \otimes w'_n.$$

We show that θ is onto. Since g, f are surjective,

$$T^n(V) = g(V^{\otimes n}) = g(f^{-1}(W^{\otimes n})) = \theta(W^{\otimes n}).$$

Next we have

$$\begin{aligned}\theta f(v_1 \otimes v_2 \otimes \cdots \otimes v_n) &= \theta(f(v_1 \otimes v_2 \otimes \cdots \otimes v_n))\\ &= gf^{-1}(f(v_1 \otimes v_2 \otimes \cdots \otimes v_n))\\ &= g(v_1 \otimes v_2 \otimes \cdots \otimes v_n).\end{aligned}$$

showing $\theta f = g$. Suppose that there exists another $\theta' : W^{\otimes n} \to T^n(V)$ such that $\theta' f = g$. Then

$$\begin{aligned}\theta(w_1 \otimes w_2 \otimes \cdots \otimes w_n) &= gf^{-1}(w_1 \otimes w_2 \otimes \cdots \otimes w_n)\\ &= \theta' f f^{-1}(w_1 \otimes w_2 \otimes \cdots \otimes w_n)\\ &= \theta'(w_1 \otimes w_2 \otimes \cdots \otimes w_n).\end{aligned}$$

Hence we get θ is unique. □

7.2 The Category $\mathscr{A}$

Let $\mathscr{A}$ denote the category of all K-modules and module homomorphisms where K is a commutative ring with unit 1. We assume that the underlying sets of the elements of $\mathscr{A}$ are elements of $\mathscr{U}$. We fix a set of morphisms in $\mathscr{A}$.

Let S_n denote the set of all free K-module homomorphisms $s_n : P^{\otimes n} \to Q^{\otimes n}$ such that s_n is isomorphism.

Proposition 7.2.1. *Let $s_i : P_i^{\otimes n} \to Q_i^{\otimes n}, i \in I$ lie in S_n, where the index set I is an element of $\mathscr{U}$; then*

$$\bigvee_{i\in I} S_i : \bigvee_{i\in I} P_i^{\otimes n} \to \bigvee_{i\in I} Q_i^{\otimes n}$$

lies in S_n.

Proof. Coproducts in $\mathscr{A}$ are direct sums equipped with a collection of projection maps. Here

$$P = \bigvee_{i\in I} P_i^{\otimes n},$$

and

$$Q = \bigvee_{i\in I} Q_i^{\otimes n}.$$

Define $s = \bigvee_{i \in I} s_i : P \to Q$ by the rule $s(p) = (s_i(p_i))_{i \in I}$. Clearly, s is well defined and is also a homomorphism.

In order to show s is injective, take $p, p' \in P$ and consider $s(p) = s(p')$. Then $(s_i(p_i))_{i \in I} = (s_i(p'_i))_{i \in I}$ for each $i \in I$ (since s_i is injective for each $i \in I$) showing $p = p'$. Hence s is injective.

To show $s(P) = Q$, take $(q_i)_{i \in I} \in Q_i^{\otimes n}$. Since $s_i(P_i^{\otimes n}) = Q_i^{\otimes n}$, we have

$$(q_i)_{i \in I} = (s_i(p_i))_{i \in I} = s(p),$$

showing s is surjective. Therefore $s : P \to Q$ is an isomorphism, i.e., $s = \bigvee_{i \in I} s_i$ lies in S_n. $\square$

We will show that the set of morphisms S_n of the category $\mathscr{A}$ of K-modules and homomorphisms admit a calculus of left fractions.

Proposition 7.2.2. *S_n admits a calculus of left fractions.*

Proof. Since S_n consists of all isomorphisms in $\mathscr{A}$, clearly, S_n is a closed family of morphisms of the category $\mathscr{A}$. We shall verify conditions (a) and (b) of Theorem 1.2.2. Let $s : P^{\otimes n} \to Q^{\otimes n}$ and $t : Q^{\otimes n} \to R^{\otimes n}$ be two morphisms of the category $\mathscr{A}$. We show that if $ts \in S_n$ and $s \in S_n$, then $t \in S_n$. For any

$$q_1 \otimes q_2 \otimes \cdots \otimes q_n \in Q^{\otimes n}$$

and

$$q'_1 \otimes q'_2 \otimes \cdots \otimes q'_n \in Q^{\otimes n}$$

consider

$$t(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = t(q'_1 \otimes q'_2 \otimes \cdots \otimes q'_n).$$

Then since s is an isomorphism.

$$(q_1 \otimes q_2 \otimes \cdots \otimes q_n) = s(p_1 \otimes p_2 \otimes \cdots \otimes p_n$$

and

$$(q'_1 \otimes q'_2 \otimes \cdots \otimes q'_n) = s(p'_1 \otimes p'_2 \otimes \cdots \otimes p'_n).$$

Thus

$$t(s(p_1 \otimes p_2 \otimes \cdots \otimes p_n)) = t(s(p'_1 \otimes p'_2 \otimes \cdots \otimes p'_n)),$$

i.e.,

$$ts(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = ts(p'_1 \otimes p'_2 \otimes \cdots \otimes p'_n)$$

since ts is isomorphism we have

$$p_1 \otimes p_2 \otimes \cdots \otimes p_n = p_1' \otimes p_2' \otimes \cdots \otimes p_n'.$$

Hence

$$s(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = s(p_1' \otimes p_2' \otimes \cdots \otimes p_n')$$

implying

$$q_1 \otimes q_2 \otimes \cdots \otimes q_n = q_1' \otimes q_2' \otimes \cdots \otimes q_n',$$

i.e., t is injective. Since $ts \in S_n$ and $s \in S_n$, we have

$$ts(P^{\otimes n}) = R^{\otimes n} \text{ and } s(P^{\otimes n}) = Q^{\otimes n}.$$

Then

$$t(Q^{\otimes n}) = t(s(P^{\otimes n})) = R^{\otimes n}.$$

So t is surjective. Thus t is an isomorphism, i.e., $t \in S_n$. Hence the condition (a) of Theorem 1.2.2 holds.

In order to prove condition (b) of Theorem 1.2.2, consider the diagram

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "f"] \arrow[d, "s"'] & B^{\otimes n} \\ C^{\otimes n} & \end{tikzcd}$$

in $\mathscr{A}$ with $s \in S_n$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "f"] \arrow[d, "s"'] & B^{\otimes n} \arrow[d, dashed, "t"] \\ C^{\otimes n} \arrow[r, dashed, "g"'] & D^{\otimes n} \end{tikzcd}$$

in $\mathscr{A}$ with $t \in S_n$. Let

$$D^{\otimes n} = (B^{\otimes n} \oplus C^{\otimes n})/N^{\otimes n}$$

where $N^{\otimes n}$ is a sub module of $B^{\otimes n} \oplus C^{\otimes n}$ generated by

$$\{(f(a_1 \otimes a_2 \otimes \cdots \otimes a_n), -s(a_1 \otimes a_2 \otimes \cdots \otimes a_n)) : a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\bigotimes n}\}.$$

Define $t : B^{\otimes n} \to D^{\otimes n}$ by the rule

$$t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = (b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + N$$

and $g : C^{\otimes n} \to D^{\otimes n}$ by the rule

$$g(c_1 \otimes c_2 \otimes \cdots \otimes c_n) = (0, c_1 \otimes c_2 \otimes \cdots \otimes c_n) + N.$$

Clearly, the two maps are well defined and homomorphisms. For any $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$, we have

$$\begin{aligned} tf(a_1 \otimes a_2 \otimes \cdots \otimes a_n) &= (f(a_1 \otimes a_2 \otimes \cdots \otimes a_n), 0) + N \\ &= (0, s(a_1 \otimes a_2 \otimes \cdots \otimes a_n)) + N \\ &= gs(a_1 \otimes a_2 \otimes \cdots \otimes a_n). \end{aligned}$$

Thus $tf = gs$. Hence the above diagram is commutative.

Next we show $t \in S$, i.e., t is injective. Take

$$b_1 \otimes b_2 \otimes \cdots \otimes b_n \in B^{\otimes n}$$

with $t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = N$. This implies that

$$(b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + N = N,$$

i.e.,

$$(b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) \in N.$$

So

$$(b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) = (f(a_1 \otimes a_2 \otimes \cdots \otimes a_n), -s(a_1 \otimes a_2 \otimes \cdots \otimes a_n))$$

from which it follows that

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n = 0.$$

Now we get

$$f(0) = (b_1 \otimes b_2 \otimes \cdots \otimes b_n) = 0.$$

Thus t is injective.

In order to show t is surjective, take an element

$$d_1 \otimes d_2 \otimes \cdots \otimes d_n + N \in D^{\otimes n},$$

where

$$d_1 \otimes d_2 \otimes \cdots \otimes d_n = (b_1 \otimes b_2 \otimes \cdots \otimes b_n, c_1 \otimes c_2 \otimes \cdots \otimes c_n).$$

Then

$$\begin{aligned}
(d_1 \otimes d_2 \otimes \cdots \otimes d_n) + N &= ((b_1 \otimes b_2 \otimes \cdots \otimes b_n), (c_1 \otimes c_2 \otimes \cdots \otimes c_n)) + N \\
&= (b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + (0, c_1 \otimes c_2 \otimes \cdots \otimes c_n) + N \\
&= t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + g(c_1 \otimes c_2 \otimes \cdots \otimes c_n) \\
&= t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + g(s(a_1 \otimes a_2 \otimes \cdots \otimes a_n)) \\
&= t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + tf(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\
&= t((b_1 \otimes b_2 \otimes \cdots \otimes b_n) + f(a_1 \otimes a_2 \otimes \cdots \otimes a_n))
\end{aligned}$$

Thus t is surjective. So $t \in S_n$.

Next let $u : B^{\otimes n} \to X^{\otimes n}$ and $v : C^{\otimes n} \to X^{\otimes n}$ be in category $\mathscr{A}$ such that $uf = vs$.

$$\begin{tikzcd}
A^{\otimes n} \arrow[r, "f"] \arrow[d, "s"'] & B^{\otimes n} \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C^{\otimes n} \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D^{\otimes n} \arrow[dr, dashed, "\theta"] & \\
& & X^{\otimes n}
\end{tikzcd}$$

Define

$$\theta : D^{\otimes n} \to X^{\otimes n}$$

by the rule

$$\theta((d_1 \otimes d_2 \otimes \cdots \otimes d_n) + N) = u(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + v(c_1 \otimes c_2 \otimes \cdots \otimes c_n)$$

where

$$(d_1 \otimes d_2 \otimes \cdots \otimes d_n) = ((b_1 \otimes b_2 \otimes \cdots \otimes b_n), (c_1 \otimes c_2 \otimes \cdots \otimes c_n)).$$

It is easy to show that θ is well defined and also a homomorphism. Next we show the two triangles are commutative. For any $(b_1 \otimes b_2 \otimes \cdots \otimes b_n) \in B^{\otimes n}$, we have

$$\begin{aligned}
\theta t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) &= \theta((b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + N) \\
&= u(b_1 \otimes b_2 \otimes \cdots \otimes b_n)
\end{aligned}$$

and for any $(c_1 \otimes c_2 \otimes \cdots \otimes c_n) \in C^{\otimes n}$,

$$\begin{aligned} \theta g(c_1 \otimes c_2 \otimes \cdots \otimes c_n) &= \theta(0, (c_1 \otimes c_2 \otimes \cdots \otimes c_n) + N) \\ &= v(c_1 \otimes c_2 \otimes \cdots \otimes c_n). \end{aligned}$$

So $\theta t = u$ and $\theta g = v$. □

The following result is a well-known.

Theorem 7.2.3. *The category $\mathscr{A}$ is cocomplete.*

From Propositions 7.2.1 7.2.2 and Theorem 7.2.3 we see that all the conditions of the Theorem 1.4.1 are satisfied. So from the Theorem 1.5.5 hence we have the following result.

Theorem 7.2.4. *Every object $V^{\otimes n}$ of the category $\mathscr{A}$ has an Adams completion V_{S_n} with respect to the set of morphisms S_n. Furthermore, there exists a morphism $e : V^{\otimes n} \to V_{S_n}$ in S_n which is couniversal with respect to the morphisms in S_n: given a morphism $s : V^{\otimes n} \to U^{\otimes n}$ in S_n there exists a unique morphism $t : U^{\otimes n} \to V_{S_n}$ in S_n such that $ts = e$. In other words the following diagram is commutative*:

$$\begin{tikzcd} V^{\otimes n} \arrow[r, "e"] \arrow[d, "s"'] & V_{S_n} \\ U^{\otimes n} \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

7.3 Tensor algebra as Adams completion

We show that the nth term of tensor algebra $T^n(V)$ of a K-module V, is precisely the Adams completion V_{S_n} of $V^{\otimes n}$.

Theorem 7.3.1. $T^n(V) \cong V_{S_n}$.

Proof. Consider the following diagram:

$$\begin{tikzcd} V^{\otimes n} \arrow[r, "g"] \arrow[d, "e"'] & T^n(V) \\ V_{S_n} \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

By Theorem 7.1.2, there exists a unique morphism $\varphi : V_{S_n} \to T^n(V)$ in S_n such that $\varphi e = g$.

Next consider the following diagram:

$$\begin{tikzcd}
V^{\otimes n} \arrow[r, "e"] \arrow[d, "g"'] & V_{S_n} \\
T^n(V) \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

By Theorem 7.2.4, there exists a unique morphism $\psi : T^n(V) \to V_{S_n}$ in S_n such that $\psi g = e$.

Consider the following diagram:

$$\begin{tikzcd}
V^{\otimes n} \arrow[rr, "e"] \arrow[dd, "e"'] & & V_{S_n} \\
 & T^n(V) \arrow[ur, "\psi"'] & \\
V_{S_n} \arrow[ur, "\varphi"'] \arrow[uurr, "1_{V_{S_n}}"'] & &
\end{tikzcd}$$

We have $\psi\varphi e = \psi g = e$. By the uniqueness condition of the couniversal property of e, we conclude $\psi\varphi = 1_{V_{S_n}}$.

Next consider the following diagram :

$$\begin{tikzcd}
V^{\otimes n} \arrow[rr, "g"] \arrow[dd, "g"'] & & T^n(V) \\
 & V_{S_n} \arrow[ur, "\varphi"'] & \\
T^n(V) \arrow[ur, "\psi"'] \arrow[uurr, "1_{T^n(V)}"'] & &
\end{tikzcd}$$

We have $\varphi\psi g = \varphi e = g$. By the uniqueness condition of the couniversal property of g, we conclude $\varphi\psi = 1_{T^n(V)}$. Thus $T^n(V) \cong V_{S_n}$. $\square$

Next we extend our study to symmetric algebra using the properties of the category $\mathscr{A}$ as described in Section 7.2. In fact we do a categorical study of symmetric algebra. For this study we add some extra assumption on the category $\mathscr{A}$; in particular we work in the of category $\mathscr{M}$ of all free K-modules and free module homomorphisms where K is a commutative ring with unit 1. We recall symmetric algebra.

7.4 Symmetric algebra

We recall the the definition of symmetric algebra [47].

Definition 7.4.1. [47] Let K be a commutative ring. Let M be a K-module. Let $n \in \mathbb{N}$. Let S^n denote the nth symmetric group. Let I_n be the K-submodule

$$\left\langle m_1 \otimes m_2 \cdots m_n - m_{\sigma(1)} \otimes m_{\sigma(2)} \cdots \otimes m_{\sigma(n)} \mid \left(\left(m_1, m_2, \cdots, m_n \right), \sigma \right) \in M^n \times S^n \right\rangle$$

of the K-module $M^{\otimes n}$. The factor K-module

$$S^n(M) = M^{\otimes n} / I_n(M)$$

is called the nth symmetric power of the K-module M. We denote by p_n, the canonical projection

$$M^{\otimes n} \to M^{\otimes n} / I_n(M) = S^n(M).$$

Clearly, this map p_n is a surjective K-module homomorphism for $n \geq 2$.

Let $S^n(M)$ denote the nth symmetric algebra. The map $\rho_n : M^{\bigotimes n} \to S^n(M)$ is a surjective K-module homomorphism for $n \geq 2$. We prove the following for our need.

Theorem 7.4.2. *Let K be a commutative ring with unit 1. Let $M^{\otimes n}$ and $N^{\otimes n}$ be free K-modules and let $f : M^{\otimes n} \to N^{\otimes n}$ be a surjective free K- module homomorphism in S_n. Then i has the following property: given a surjective free K-module homorphism $i : M^{\otimes n} \to S^n M$ in $\bar{S_n}$, there exists a unique surjective free K- module homorphism $\varphi : N^{\otimes n} \to S^n M$ in $\bar{S_n}$ such that $i = f\varphi$, i.e., the following diagram is commutative :*

$$\begin{tikzcd} M^{\otimes n} \arrow[r, "f"] \arrow[d, "i"'] & N^{\otimes n} \arrow[dl, dashed, "\varphi"] \\ S^n(M) & \end{tikzcd}$$

Proof. By Theorem 7.1.2, there exists a unique K-module isomorphism $\psi : N^{\otimes n} \to T^n M$ such that $\psi f = g$

$$\begin{tikzcd} M^{\otimes n} \arrow[r, "f"] \arrow[d, "g"'] & N^{\otimes n} \arrow[dl, dashed, "\psi"] \\ T^n(M) & \end{tikzcd}$$

where $g : M^{\otimes n} \to T^n(M)$) is a K-module isomorphism. Consider the diagram

$$\begin{tikzcd}
 & T^n(M) \arrow[dr, "h"] & \\
M^{\otimes n} \arrow[ur, "g"] \arrow[r, "f"'] & N^{\otimes n} \arrow[u, "\psi"'] \arrow[r, dashed, "\varphi"'] & S^n(M)
\end{tikzcd}$$

Let $h = ig^{-1}$. For each $n \in N^{\otimes n}$, define $\varphi : N^{\otimes n} \to S^n(M)$ by the rule $\varphi(n) = h\psi(n)$. Then for each $m \in M^{\otimes n}$ we have

$$\begin{aligned}
\varphi f(m) &= h\psi(f(m)) \\
&= ig^{-1}\psi(f(m)) \\
&= if^{-1}\psi^{-1}\psi(f(m)) \\
&= if^{-1}(f(m)) \\
&= i(m)
\end{aligned}$$

showing $\varphi f = i$. Clearly, φ is surjective. For the uniqueness of φ, suppose there exists another $\varphi' : N^{\otimes n} \to S^n(M)$ such that $i = \varphi' f$. For each $n \in N^{\otimes n}$, let $n = f(m), m \in M^{\otimes n}$, thus

$$\varphi(n) = \varphi f(m) = i(m) = \varphi' f(m) = \varphi'(n),$$

showing $\varphi = \varphi'$. □

7.5 The category $\mathscr{M}$

Let $\mathscr{M}$ denote the category of all free K-modules and free K-module homomorphisms where K is a commutative ring with unit 1. We assume that the underlying sets of the elements of $\mathscr{M}$ are elements of $\mathscr{U}$.

Let S_n denote the set of all free K-module homomorphisms $f : M^{\otimes n} \to N^{\otimes n}$ such that f is surjective for $n \geq 2$ and isomorphism for $n = 1$.

Proposition 7.5.1. *Let $\{s_i : X_i^{\otimes n} \to Y_i^{\otimes n}, i \in I\}$ be a subset of S_n; where the index set I is an element of $\mathscr{U}$, then*

$$\bigvee_{i\in I} S_i : \bigvee_{i\in I} P_i^{\otimes n} \to \bigvee_{i\in I} Q_i^{\otimes n}$$

is an element of S_n.

Proof. The proof is trivial □

We will show that the set S_n of free K-module homomorphisms of the category $\mathscr{M}$ of free K-modules and free K-modules homomorphisms admits a calculus of left fractions.

Proposition 7.5.2. *S_n admits a calculus of left fractions.*

Proof. The proof is almost the proof of Theorem 7.2.2; hence it is omitted. $\square$

Proposition 7.5.3. *The category $\mathscr{M}$ is cocomplete.*

For $n = 1$, the set of morphisms S_n is saturated. When $n \geq 2$, from Propositions 7.5.1 7.5.2 and 7.5.3 we see that all the conditions of the Theorem 1.4.1 are satisfied. So from the Theorem 1.5.5 we have the following result.

Theorem 7.5.4. *Every object $M^{\otimes n}$ of the category $\mathscr{M}$ has an Adams completion M_{S_n} with respect to the set of morphisms S_n. Furthermore, there exists a morphism $e : M^{\otimes n} \to M_{S_n}$ in $\bar{S}_n$ which is couniversal with respect to the morphisms in S_n: given a morphism $s : M^{\otimes n} \to N^{\otimes n}$ in S_n there exists a unique morphism $t : N^{\otimes n} \to M_{S_n}$ in $\bar{S}_n$ such that $ts = e$. In other words the following diagram is commutative*:

$$\begin{tikzcd} M^{\otimes n} \arrow[r, "e"] \arrow[d, "s"'] & M_{S_n} \\ N^{\otimes n} \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 7.5.5. *The free K-module homomorphism $e : M^{\otimes n} \to M_{S_n}$ is in S_n.*

Proof. Let

$$S_n^1 = \{s : M^{\otimes n} \to N^{\otimes n} \text{ in } \mathscr{M} \mid s \text{ is a surjective free } K\text{-module homorphism for } n \geq 2\}$$

and

$$S_n^2 = \{s : M^{\otimes n} \to N^{\otimes n} \text{in } \mathscr{M} \mid s \text{ is a free } K\text{-module isomorphism for } n = 1\}.$$

Clearly,

(a) $S = S_n^1 \cap S_n^2$ and

(b) S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.5.9. Thus $e \in S_n$.

$\square$

7.6 Symmetric algebra as Adams completion

We show that the nth term of symmetric algebra $S^n(M)$ of a free K-module M is precisely the Adams completion M_{S_n} of $M^{\otimes n}$.

Theorem 7.6.1. $S^n(M) \cong M_{S_n}$.

Proof. The arguments of the proof of this therem are exactly that of Theorem 7.3.1. □

Chapter 8

Exterior Algebra and Clifford Algebra as Adams Completion

The concepts of exterior algebra and Clifford algebra are fundamental to the theory of differential forms used in geometry and analysis. In this chapter we express both the exterior and Clifford algebra of a module as the Adams completion of the module by choosing a suitable sets of morphisms in appropriate categories. First we express exterior algebra in terms of Adams completion

8.1 Exterior algebra

In this section, we describe a few results about exterior algebra of K-modules, where K is a commutative ring with 1.

Definition 8.1.1. [48, 49] The *exterior algebra* of a K-module M is the K-algebra obtained by taking the quotient of the tensor algebra $T(M)$ by the ideal $I(M)$ generated by all elements of the form

$$m \otimes m, \quad m \in M$$

The exterior algebra $T(M)/I(M)$ is denoted by $\wedge(M)$. The exterior algebra is graded with nth homogeneous component

$$\wedge^n(M) = T^n(M)/I^n(M).$$

We can identify K with $\wedge^0(M)$ and M with $\wedge^1(M)$ and so we consider M as a K-submodule of the K-algebra $\wedge(M)$.

Furthermore the exterior algebra $\wedge(M)$ of the K-module M has the following universal property.

Theorem 8.1.2. [48, 49] *Let A be an associative algebra with unit element e and let*

$$f : M \to A$$

be a linear map of M into A such that

$$f(m)^2 = 0$$

for all $m \in M$. Then f extends uniquely to an associative algebra homomorphism

$$\widetilde{f} : \wedge(M) \to A$$

from the exterior algebra $\wedge(M)$ into A. That is, there is a unique associative algebra homomorphism

$$\widetilde{f} : \wedge(M) \to A$$

such that

$$\widetilde{f}(1) = e \text{ and } \widetilde{f} \circ i_M = f,$$

where i_M is the natural inclusion mapping of $M = \wedge^1(M)$ into $\wedge(M)$; in otherwords the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "f"'] & \wedge(M) \arrow[dl, dashed, "\tilde{f}"] \\ A & \end{tikzcd}$$

Theorem 8.1.3. *Let K be a commutative ring with unit 1. Let M and N be free K-modules and let $f : M \to N$ be an injective free K-module homomorphism with $f(m)^2 = 0$ for all $m \in M$. Then f has the following property*: *given an injective free K-module homomorphism $i_M : M \to \wedge(M)$, there exists a unique injective free K-module homomorphism $\varphi : N \to \wedge(M)$ such that $i_M = f\varphi$, i.e., the following diagram is commutative*:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \arrow[dl, dashed, "\varphi"] \\ \wedge(M) & \end{tikzcd}$$

Proof. By Theorem 7.1.2, there exists a unique K-module isomorphism $\psi : N \to T^1(M)$ such that $\psi f = g$. Consider the diagram

$$\begin{tikzcd} & T^1(M) \arrow[dr, "g'"] & \\ M \arrow[ur, "g"] \arrow[r, "f"'] & N \arrow[u, "\psi"] \arrow[r, dashed, "\varphi"'] & \wedge(M) \end{tikzcd}$$

where $g : M \to T^1(M)$ is a K-module isomorphism. Let $g' = i_M g^{-1}$. For each $n \in N$, define $\varphi : N \to \wedge(M)$ by the rule

$$\varphi(n) = g'\psi(n).$$

For each $m \in M$, we have $\varphi(f(m)) = g'\psi(f(m))$. For $m \in M$ let $f(m) = n$, so

$$\begin{aligned} \varphi f(m) &= \varphi(n) = g'\psi(n) \\ &= i_M g^{-1}\psi(n) = i_M f^{-1}\psi^{-1}\psi(n) \\ &= i_M f^{-1}(n) = i_M(m) \end{aligned}$$

showing $\varphi f = i_M$. To show that φ is one-one, consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \arrow[d, "i_N"] \arrow[dl, dashed, "\varphi"'] \\ \wedge(M) \arrow[r, "\wedge(f)"'] & \wedge(N) \end{tikzcd}$$

Since $\wedge(f) \circ \varphi : N \to \wedge(N)$ is injective we have that φ is injective. Next for each $n \in N$, we have

$$\varphi(n)^2 = g'(\psi(n))^2 = 0.$$

For the uniqueness of φ, for each $n \in N$, let $n = f(m), m \in M$. Let there exist

$$\varphi' : N \to \wedge(M)$$

such that $\varphi'\eta = i_M$. Thus

$$\varphi(n) = \varphi f(m) = i_M(m) = \varphi' f(m) = \varphi'(n)$$

showing $\varphi = \varphi'$. □

8.2 The category $\mathscr{A}$

Let $\mathscr{A}$ denote the category of all free K-modules and free module homomorphisms where K is a commutative ring with unit 1. We assume that the underlying sets of the elements of $\mathscr{A}$ are elements of $\mathscr{U}$. We fix suitable set of morphisms in the category $\mathscr{A}$.

Let S denote the set of all free K-module homomorphisms $f : M \to N$ such that f is injective and $f(m)^2 = 0$ for all $m \in M$.

Proposition 8.2.1. *Let $s_i : P_i \to Q_i$ lie in S, for each $i \in I$, where the index set I is an element of $\mathscr{U}$. Then*

$$\bigvee_{i\in I} s_i : \bigvee_{i\in I} P \to \bigvee_{i\in I} Q$$

lies in S.

Proof. Coproducts in $\mathscr{A}$ are direct sums equipped with a collection of projection maps. Here

$$P = \bigvee_{i\in I} P_i$$

and

$$Q = \bigvee_{i\in I} Q_i.$$

Define

$$s = \bigvee_{i\in I} s_i : P \to Q$$

by the rule

$$s(p) = (s_i(p_i))_{i\in I}.$$

Clearly, s is well defined and is also a free K-module homomorphism.

In order to show s is injective, let $p, p' \in P$ with $s(p) = s(p')$. Then $(s_i(p_i))_{i\in I} = (s_i(p_i'))_{i\in I}$ for each $i \in I$ (since s_i is injective for each $i \in I$) showing $p = p'$. Hence s is injective. Now we have to show that $s(p)^2 = 0$. We see that

$$s(p)^2 = s(p) \wedge s(p) = -(s(p) \wedge s(p))$$

implying $2\,(s(p) \wedge s(p)) = 0$. Thus $(s(p) \wedge s(p)) = 0$. □

We will show that the set S of free K-module homomorphisms of the category $\mathscr{A}$ of free K-modules and free K-modules homomorphisms admit a calculus of left fractions.

Proposition 8.2.2. *S admits a calculus of left fractions.*

Proof. Let M, N and P be in $\mathscr{A}$. Let $s : M \to N$ and $t : N \to P$ be two free K-module homomorphisms of the category $\mathscr{A}$. We have to show that $ts(m)^2 = 0$ for all $m \in M$. Since s and t are in S, we have $s(m)^2 = 0$, $t(n)^2 = 0$. Consider $ts(m)^2 = t(s(m))^2 = 0$ for all $m \in M$. So S is a closed family of free K-module homomorphisms of category $\mathscr{A}$.We shall verify conditions (a) and (b) of Theorem 1.2.2. Let s, t be two free K-module homomorphisms in the category $\mathscr{A}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$, i.e., $t(n)^2 = 0$ and t is injective. Consider the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, "i_M"'] & N \arrow[r, "t"] \arrow[d, "i_N"] & P \arrow[d, "i_P"] \arrow[dl, "\varphi'"] \\
\wedge(M) \arrow[r, "\wedge(s)"'] & \wedge(N) \arrow[r, "\wedge(t)"'] & \wedge(P)
\end{tikzcd}$$

From the above diagram we have a diagram

$$\begin{tikzcd}
& \wedge(M) \arrow[dr, "\wedge(s)"] & \\
M \arrow[ur, "i_M"] \arrow[r, "ts"'] & P \arrow[u, "\varphi"'] \arrow[r, dashed, "\varphi'"'] & \wedge(N)
\end{tikzcd}$$

By Theorem 8.1.3, φ is one-one and $\varphi ts = i_M$. Again we have $\varphi' = \wedge(s)\varphi$ implying φ' is injective. From $\varphi' t = i_N$ we conclude t is injective. We have

$$t(n)^2 = t(n) \wedge t(n) = -(t(n) \wedge t(n))$$

implying

$$2(t(n) \wedge t(n)) = 0.$$

Thus

$$t(n) \wedge t(n) = 0.$$

Hence the condition (a) of Theorem 1.2.2 holds.

In order to prove condition (b) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \\
C &
\end{tikzcd}$$

in $\mathscr{A}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\
C \arrow[r, dashed, "g"'] & D
\end{tikzcd}$$

in $\mathscr{A}$ with $t \in S$. Let

$$D = (B \oplus C)/N$$

where N is a sub module of $B \oplus C$ generated by

$$\{(f(a), -s(a)) : a \in A\}.$$

Define $t : B \to D$ by the rule

$$t(b) = (b, 0) + N$$

and $g : C \to D$ by the rule

$$g(c) = (0, c) + N.$$

Clearly, the two maps are well defined and free K-module homomorphisms. For any $a \in A$,

$$tf(a) = (f(a), 0) + N = (0, s(a)) + N = gs(a),$$

impling that $tf = gs$. Hence the above diagram is commutative.

Next we show that $t \in S$, i.e., t is injective. Take $b \in B$ with $t(b) = N$; this implies $(b, 0) + N = N$, i.e., $(b, 0) \in N$. So $(b, 0) = (f(a), -s(a))$ from which it follows that $a = 0$. Now we get $f(0) = (b) = 0$. Thus t is injective. Clearly, $t(b)^2 = 0$ for all $b \in B$.

Next let $u : B \to X$ and $v : C \to X$ be in category $\mathscr{A}$ such that $uf = vs$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\
& & X
\end{tikzcd}$$

Define $\theta : D \to X$ by the rule

$$\theta((b, c) + N) = u(b) + v(c).$$

It is easy to show that θ is well defined and also a free K-module homomorphism. Next we show the two triangles are commutative. For any $b \in B$,

$$\theta t(b) = \theta((b, 0) + N) = u(b)$$

and for any $c \in C$,
$$\theta g(c) = \theta((0, c) + N) = v(c).$$
So $\theta t = u$ and $\theta g = v$.

□

The following result holds in the category of free modules and free module homomorphisms.

Proposition 8.2.3. *The category $\mathscr{A}$ is cocomplete.*

From Propositions 8.2.1, 8.2.2 and 8.2.3 we see that all the conditions of Theorem 1.4.1 are satisfied, hence we have the following result.

Theorem 8.2.4. *Every object M of the category $\mathscr{A}$ has an Adams completion M_S with respect to the set S of free K-module homomorphisms. Furthermore, there exists a free K-module homomorphism $e : M \to M_S$ in $\bar{S}$ which is couniversal with respect to the free K-module homomorphisms in S : given a free K-module homomorphism $s : M \to N$ in S there exists a unique free K-module homomorphism $t : N \to M_S$ in $\bar{S}$ such that $ts = e$. In other words the following diagram is commutative*:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "s"'] & M_S \\ N \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 8.2.5. *The free K-module homomorphism $e : M \to M_S$ is in S.*

Proof. Let
$$S_1 = \{s : M \to N \text{ in } \mathscr{A} \mid s \text{ is an injective free } K\text{-module}\}$$
and
$$S_2 = \left\{s : M \to N \text{ in } \mathscr{A} \mid s \text{ is a free } K\text{-module homomorphism such that } s(m)^2 = 0\right\}.$$
For S_1 and S_2, it easily follows that all the conditions of Theorem 1.5.9 are satisfied. Therefore, $e \in S$. □

8.3 Exterior algebra as Adams completion

We show that the exterior algebra $\wedge(M)$ of a free K-module M is precisely the Adams completion M_S of M.

Theorem 8.3.1. $\wedge(M) \cong M_S$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
M \arrow[r, "i_M"] \arrow[d, "e"'] & \wedge(M) \\
M_S \arrow[ur, dashed, "\varphi"'] &
\end{tikzcd}$$

By Theorem 8.1.3, there exists a unique free K-module homomorphism $\varphi : M_S \to \wedge(M)$ in S such that $\varphi e = i_M$.

Next consider the following diagram

$$\begin{tikzcd}
M \arrow[r, "e"] \arrow[d, "i_M"'] & M_S \\
\wedge(M) \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

By Theorem 8.2.4, there exists a unique free K-module homomorphism $\psi : \wedge(M) \to M_S$ in S such that $\psi i_M = e$.

Consider following diagram

$$\begin{tikzcd}
M \arrow[rr, "e"] \arrow[dd, "e"'] & & M_S \\
& \wedge(M) \arrow[ur, "\psi"'] & \\
M_S \arrow[ur, "\varphi"'] \arrow[uurr, "1_{M_S}"'] & &
\end{tikzcd}$$

we have $\psi\varphi e = \psi i_M = e$. By the uniqueness condition of the couniversal property of e, we conclude that $\psi\varphi = 1_{M_S}$.

Next consider the following diagram

$$\begin{tikzcd}
M \arrow[rr, "i_M"] \arrow[dd, "i_M"'] & & \wedge(M) \\
& M_S \arrow[ur, "\varphi"'] & \\
\wedge(M) \arrow[ur, "\psi"'] \arrow[uurr, "1_{\wedge(M)}"'] & &
\end{tikzcd}$$

we have $\varphi\psi i_M = \varphi e = i_M$. By the uniqueness condition of the couniversal property of i_M, we conclude that $\varphi\psi = 1_{\wedge(M)}$. Thus $\wedge(M) \cong M_S$. $\square$

8.4 Clifford algebra

Clifford algebras were introduced by Clifford in the late 19th century as part of his search for generalizations of quaternions. Clifford algebras have become a more popular tool in theoretical physics. We show that given an algebra, its Clifford algebra is isomorphic to the Adams completion of the algebra with respect to a chosen set of morphisms in the category of modules and module homomorphisms.

Theorem 8.4.1. [50] *Let K be a commutative ring. Let M be a K-module and*

$$f : M \times M \to K$$

be a bilinear form on M. Let

$$\varphi : M \to Cl(M, f)$$

be the K-module homomorphism defined by $\varphi = (proj) \circ (inj)$, where

$$inj : M \to \otimes M$$

is the canonical injection of the K-module M into its tensor algebra $\otimes M$ and where

$$proj : \otimes M \to Cl(M, f)$$

is the canonical projection of the tensor algebra $\otimes M$ onto its factor algebra

$$\otimes M / I_f = Cl(M, f),$$

where I_f is the two-sided ideal

$$(\otimes L) \cdot \langle M \otimes M - f(M, M) \mid M \in L \rangle \cdot (\otimes L)$$

of the algebra $\otimes L$. The homomorphism f is injective.

Definition 8.4.2. [51] Let M and N be two K-modules. The mapping

$$q : M \to N$$

is said to be K-*quadratic mapping* if

$$q(\lambda x) = \lambda^2 q(x)$$

for all $\lambda \in K$, $x \in M$ and the map

$$f : M \times M \to N$$

defined by

$$f(x, y) = q(x + y) - q(x) - q(y)$$

is K-bilinear, this mapping $f(x, y)$ is called the *associated bilinear mapping*. When $N = K$, q is called a quadratic form on M and (M, q) is called quadratic K-module.

Definition 8.4.3. [51, 52] The Clifford algebra of a quadratic form (M, q) is a pair $((Cl(M), q), \rho)$, where $Cl(M, q)$ is a K-quadratic module and $\rho : M \to Cl(M, q)$ is a quadratic mapping such that $\rho(m)^2 = q(m)1_M$ we assume the following universal property: for every linear mapping

$$f : M \to A$$

with

$$f(m)^2 = q(m)1_A$$

for all $m \in M$, there exists a unique algebra morphism

$$f' : Cl(M, q) \to A$$

such that the following diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "f"'] & Cl(M,q) \arrow[dl, dashed, "f'"] \\ A & \end{tikzcd}$$

is commutative, i.e., $f = f'\rho$.

Theorem 8.4.4. [51] *If $u : (M, q) \to (M', q')$ is a morphism of quadratic modules, i.e.,*

$$q'(u(x)) = q(x)$$

for all $x \in M$, then the algebra morphism $Cl(u)$ is defined as follows

$$q'(u(x))^2 = q'(x)1_{q'}$$

and the universal property of $\delta : M \to Cl(M, q)$ implies the existence of a unique morphism

$$Cl(u) : Cl(M, q) \to Cl(M', q')$$

such that the following diagram

$$\begin{tikzcd}
(M,q) \arrow[r, "\rho"] \arrow[d, "u"'] & Cl((M,q) \arrow[d, dashed, "Cl(u)"] \\
(M',q') \arrow[r, "\rho'"'] & Cl(M',q')
\end{tikzcd}$$

is commutative, i.e., $\rho' u = Cl(u)\rho$.

We prove the following result which will be used in the sequel.

Theorem 8.4.5. *Let K be a commutative ring with unit 1. Let M and N be free K-quadratic modules and let $\eta : M \to N$ be an injective free K-quadratic module homomorphism. Then η has the following property: given an injective free K-quadratic module morphism*

$$i_M : M \to Cl(M, q_M)$$

there exists a unique injective K-quadratic module morphism

$$\theta' : N \to Cl(M)$$

such that $\theta'\eta = i_M$, i.e., the following diagram is commutative

$$\begin{tikzcd}
M \arrow[r, "\eta"] \arrow[d, "i_M"'] & N \arrow[dl, dashed, "\theta'"] \\
Cl(M) &
\end{tikzcd}$$

Proof. By Theorem 7.1.2, there exists a unique K-module isomorphism $\theta : N \to T^1(M)$ such that $\theta\eta = g$

$$\begin{tikzcd}
M \arrow[r, "\eta"] \arrow[d, "g"'] & N \arrow[dl, dashed, "\theta"] \\
T^1(M) &
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
& T^1(M) \arrow[dr, "g'"] & \\
M \arrow[ur, "g"] \arrow[r, "\eta"'] & N \arrow[u, "\theta"'] \arrow[r, dashed, "\theta'"'] & Cl(M)
\end{tikzcd}$$

where $g : M \to T^1(M)$ is a K-module isomorphism. Let $g' = i_M g^{-1}$. For each $n \in N$, define $\theta' : N \to Cl(M)$ by the rule

$$\theta'(n) = g'\theta(n).$$

For each $m \in M$, we have $\theta'(\eta(m)) = g'\theta(\eta(m))$. For $m \in M$, let $\eta(m) = n$. Thus we have

$$\begin{aligned}\theta'\eta(m) &= \theta'(n) = g'\theta(n)\\ &= i_M g^{-1}\theta(n) = i_M f^{-1}\theta^{-1}\theta(n)\\ &= i_M f^{-1}(n) = i_M(m)\end{aligned}$$

showing $\theta'\eta = i_M$.

To show that θ' is one-one, consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[d, "i_M"'] & N \arrow[d, "i_N"] \arrow[dl, dashed, "\theta'"'] \\ Cl(M) \arrow[r, "Cl(\eta)"'] & Cl(N) \end{tikzcd}$$

Since $Cl(\eta) \circ \theta' : N \to Cl(N)$ is injective we have that θ' is injective. Next we show that $\theta'(n)^2 = q_N(n) \cdot 1_M$. We have $q_N\eta(m) = q_M(m)$. Since $(M, q_M), (N, q_N)$ are quadratic free modules, we have

$$\theta'(n)^2 = g'g(m)^2 = q_M(m) \cdot 1_M = q_N(n) \cdot 1_M.$$

For the uniqueness of θ', for each $n \in N$, let $n = \eta(m), m \in M$. Let there exist

$$\theta'' : (N, q_N) \to Cl(M, q_M)$$

such that $\theta''\eta = i_M$ Thus

$$\theta'(n) = \theta'\eta(m) = i_M(m) = \theta''\eta(m) = \theta''(n)$$

showing $\theta' = \theta''$. $\square$

8.5 The category $\tilde{\mathscr{A}}$

Here we modify the category $\mathscr{A}$ as chosen above, also we choose different set of morphisms as described in Section 8.2.

Let $\tilde{\mathscr{A}}$ denote the category of all free K-quadratic modules and free quadratic module homomorphisms where K is a commutative ring with unit 1. We assume that the underlying sets of the elements of $\tilde{\mathscr{A}}$ are elements of $\mathscr{U}$. We fix a suitable set of morphisms S in $\tilde{\mathscr{A}}$.

Let S denote the set of all free K-quadratic module homomorphisms

$$s : (M, q_M) \to (N, q_N)$$

such that s is injective and

$$s(m)^2 = q_M(m) \cdot 1_N$$

for all $m \in M$.

Proposition 8.5.1. *Let $s_i : (P_i, q_{P_i}) \to (Q_i, q_{Q_i})$ lie in S, for each $i \in I$, where the index set I is an element of $\mathscr{U}$. Then*

$$\bigvee_{i \in I} S_i : \bigvee_{i \in I} P_i \to \bigvee_{i \in I} Q_i$$

lies in S.

Proof. Coproducts in $\tilde{\mathscr{A}}$ are direct sums equipped with a collection of projection maps. Here

$$P = \bigvee_{i \in I} P_i$$

and

$$Q = \bigvee_{i \in I} Q_i.$$

Define

$$s = \bigvee_{i \in I} s_i : P \to Q$$

by the rule $s(p) = (s_i(p_i))_{i \in I}$. Clearly, s is well defined and is also a free K-module homomorphism.

In order to show s is injective, take $p, p' \in P$ and consider $s(p) = s(p')$. Then $(s_i(p_i))_{i \in I} = (s_i(p'_i))_{i \in I}$ for each $i \in I$ (since s_i is injective for each $i \in I$) showing $p = p'$. Hence s is injective. We consider the homomorphism

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[d, "i_M"'] & N \arrow[d, "i_N"] \arrow[dl, dashed, "\theta'"'] \\ Cl(M) \arrow[r, "Cl(\eta)"'] & Cl(N) \end{tikzcd}$$

$$(P, i_P) \xrightarrow{s} (Q, i_Q) \xrightarrow{i_Q} Cl(Q, i_Q)$$

Then

$$i_Q s(p)^2 = i_Q(s(p))^2 = q_Q(s(p)) \cdot 1_Q = q_P(p) \cdot 1_Q.$$

Thus we get $s(p)^2 = q_Q(p) \cdot 1_Q$. □

We will show that the set S of free K-quadratic module homomorphisms of the category $\tilde{\mathscr{A}}$ of free K-quadratic modules and free K-quadratic modules homomorphisms admit a calculus of left fractions.

Proposition 8.5.2. *S admits a calculus of left fractions.*

Proof. Let $(M, q_M), (N, q_N)$ and (P, q_P) be in $\tilde{\mathscr{A}}$ and

$$i_M : M \to Cl(M, q_M),$$

$$i_N : N \to Cl(N, q_N)$$

and

$$i_P : P \to Cl(P, q_P)$$

the corresponding Clifford algebras. Let

$$s : (M, q_M) \to (N, q_N)$$

and

$$t : (N, q_N) \to (P, q_P)$$

be two morphisms of the category $\tilde{\mathscr{A}}$. We have to show that

$$ts(m)^2 = q_M \cdot 1_P$$

for all $m \in M$. Since s and t are the free quadratic K-module morphisms, we have

$$q_N s(m) = q_M(m),\ q_P t(n) = q_N(n).$$

Consider

$$q_P ts(m) = q_P t(s(m)) = q_N(s(m)) = q_M(m)$$

for all $m \in M$. Composition of injective map is injective. So S is a closed family of morphisms of category $\mathscr{A}$.

We shall verify conditions (a) and (b) of Theorem 1.2.2. Let s, t be two free K-quadratic module homomorphisms in category $\tilde{\mathscr{A}}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$, i.e., $t(n)^2 = q_N 1_P$ and t is injective. Consider the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, "i_M"'] & N \arrow[r, "t"] \arrow[d, "i_N"] & P \arrow[d, "i_P"] \arrow[dl, "\varphi'"] \\
Cl(M) \arrow[r, "Cl(s)"'] & Cl(N) \arrow[r, "Cl(t)"'] & Cl(P)
\end{tikzcd}$$

From the above diagram we have a diagram

$$\begin{tikzcd}
 & Cl(M) \arrow[dr, "Cl(s)"] & \\
M \arrow[ur, "i_M"] \arrow[r, "ts"'] & P \arrow[u, "\varphi"'] \arrow[r, dashed, "\varphi'"'] & Cl(N)
\end{tikzcd}$$

By Theorem 8.4.5, φ is one-one and $\varphi ts = i_M$. Again we have $\varphi' = Cl(s)\varphi$ implying φ' is injective. From $\varphi' t = i_N$, we conclude t is injective. Now consider the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "t"] \arrow[d, "i_N"'] & P \arrow[d, "i_P"] \\
Cl(N) \arrow[r, "Cl(t)"'] & Cl(P)
\end{tikzcd}$$

with

$$Cl(t) \circ i_N = i_P \circ t \cdot i_P \circ t(n)^2 = i_P(t(n))^2 = t(n)^2.$$

Since t is quadratic homomorphism, we have $q_P t(n) = q_N(n)$ for all $n \in N$. Again

$$i_P(t(n))^2 = q_P t(n) 1_P = q_N 1_P$$

for all $n \in N$. Thus $t \in S$ and the condition (a) of Theorem 1.2.2 holds.

In order to prove condition (b) of Theorem 1.2.2 consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \\ C & \end{tikzcd}$$

in $\tilde{\mathscr{A}}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\ C \arrow[r, dashed, "g"'] & D \end{tikzcd}$$

in $\tilde{\mathscr{A}}$ with $t \in S$. Let

$$D = (B \oplus C)/N$$

where N is a sub module of $B \oplus C$ generated by

$$\{(f(a), -s(a)) : a \in A\}.$$

Define $t : B \to D$ by the rule

$$t(b) = (b, 0) + N$$

and $g : C \to D$ by the rule

$$g(c) = (0, c) + N.$$

Clearly, the two maps are well defined and free K-quadratic module homomorphisms. For any $a \in A$,

$$tf(a) = (f(a), 0) + N = (0, s(a)) + N = gs(a),$$

implying that $tf = gs$. Hence the above diagram is commutative.

Next we show that $t \in S$, i.e., t is injective. Take $b \in B$ with $t(b) = N$; this implies $(b, 0) + N = N$, i.e., $(b, 0) \in N$. So

$$(b, 0) = (f(a), -s(a))$$

from which it follows that $a = 0$. Now we get $f(0) = (b) = 0$. Thus t is injective. We consider $i_D t$:

$$(B, q_B) \xrightarrow{t} (D, q_D) \xrightarrow{i_D} Cl(D, q_D)$$

$$i_D t(b)^2 = i_D(t(b))^2 = q_D(t(b)) \cdot 1_D = q_B(b) \cdot 1_D.$$

So we get $t \in S$.

Next let $u : B \to X$ and $v : C \to X$ be in category $\tilde{\mathscr{A}}$ such that $uf = vs$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\
 & & X
\end{tikzcd}$$

Define $\theta : D \to X$, by the rule

$$\theta((b, c) + N) = u(b) + v(c).$$

It is easy to show that θ is well defined and also a free K-quadratic module homomorphism. Next we show the two triangles are commutative. For any $b \in B$,

$$\theta t(b) = \theta((b, 0) + N) = u(b)$$

and for any $c \in C$,

$$\theta g(c) = \theta(0, (c) + N) = v(c).$$

So $\theta t = u$ and $\theta g = v$. □

The following theorem is a well known result in the category of free quadratic modules and free quadratic module homomorphisms.

Proposition 8.5.3. *The category $\tilde{\mathscr{A}}$ is cocomplete.*

From propositions 8.5.1, 8.5.2 and 8.5.3 we see that all the conditions of Theorem 1.4.1 are satisfied, hence we have the following result.

Theorem 8.5.4. *Every object M of the category $\tilde{\mathscr{A}}$ has an Adams completion M_S with respect to the set S of free K-quadratic module homomorphisms if and only if there exists a morphism $e : M \to M_S$ in $\bar{S}$ which is couniversal with respect to the free K-quadratic module homomorphisms in S: given a free K-quadratic module homomorphism $s : M \to N$ in S there exists a unique free K-module homomorphism $t : N \to M_S$ in $\bar{S}$ such that $ts = e$. In other words the following diagram is commutative* :

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "s"'] & M_S \\ N \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 8.5.5. *The free K-quadratic module homomorphism $e : M \to M_S$ is in S.*

Proof. Let

$$S_1 = \{s : (M, q_M) \to (N, q_N) \text{ in } \mathscr{A} \mid s \text{ is an injective free } K\text{-quadratic module homomorphism}\}$$

and

$$S_2 = \{s : (M, q_M) \to (N, q_N) \text{ in } \mathscr{A} \mid s \text{ is a free } K\text{-quadratic module homomorphism such that } s(m)^2 = q_M \cdot 1_N\}.$$

Clearly,

(a) $S = S_1 \cap S_2$ and

(b) S_1 and S_2 satisfy all the conditions of Theorem 1.5.9.

Hence $e \in S$. □

8.6 Clifford algebra as Adams completion

We show that the Clifford algebra $Cl(M)$ of a free K-quadratic module M, is precisely the Adams completion M_S of M.

Theorem 8.6.1. $Cl(M) \cong M_S$.

Proof. The proof is same as that of Theorem 8.3.1. □

References

[1] J. F. Adams, “Idempotent functors in homotopy theory,” in *Proceedings of the International Conference on Manifolds and Related Topics in Topology, Tokyo*, pp. 247–263, 1973.

[2] J. F. Adams, “Localization and Completion,” in *Lecture Notes in Mathematics, Univ. of Chicago*, 1975.

[3] J. F. Adams, *Stable homotopy and generalised homology*. University of Chicago press, 1995.

[4] P. Selick, *Introduction to homotopy theory*, vol. 9. American Mathematical Soc., 2008.

[5] D. W. Barnes and J. M. Mack, “Zermelo-Fraenkel set theory,” in *An Algebraic Introduction to Mathematical Logic*, pp. 52–61, Springer, 1975.

[6] S. M. Srivastava, *A course on mathematical logic*. Springer Science & Business Media, 2013.

[7] H. Schubert, *Categories*. Springer-Verlag, New York, 1972.

[8] S. Nanda, “A note on the universe of a category of fractions,” *Canad. Math. Bull.*, vol. 23, no. 4, pp. 425 – 427, 2004.

[9] A. Deleanu, A. Frei, and P. Hilton, “Generalized Adams completion,” *Cahiers de topologie et géométrie différentielle catégoriques*, vol. 15, no. 1, pp. 61–82, 1974.

[10] A. Behera and S. Nanda, “Cartan-Whitehead decomposition as Adams cocompletion,” *J. Austra. Math. Soc.*, vol. 42, pp. 223–226, 1987.

[11] A. Behera and S. Nanda, “Mod-$\mathcal{C}$ Postnikov approximation of a 1-connected space,” *Can. J. Math*, vol. 33, no. 3, pp. 527–543, 1987.

[12] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, vol. 35. Springer Science & Business Media, 2012.

[13] A. Deleanu and P. Hilton, “Localization, homology and a construction of Adams,” *Transactions of the American Mathematical Society*, vol. 179, pp. 349–362, 1973.

[14] A. Deleanu, “Existence of the Adams completion for CW-complexes,” *Journal of Pure and Applied Algebra*, vol. 4, no. 3, pp. 299–308, 1974.

[15] A. Deleanu, “Existence of the Adams completion for objects of complete categories,” *Journal of Pure and Applied Algebra*, vol. 6, no. 1, pp. 31–39, 1975.

[16] D. Sullivan, *Localization, Periodicity, and Galois Symmetry*. 1970.

[17] A. K. Bousfield, “The localization of spaces with respect to homology,” *Topology*, vol. 14, no. 2, pp. 133–150, 1975.

[18] S. Deo, T. B. Singh, and R. A. Shukla, "On an extension of localization theorem and generalized Conner conjecture," *Transactions of the American Mathematical Society*, vol. 269, no. 2, pp. 395–402, 1982.

[19] S. Cormack, "Extensions relative to a Serre class," *Proceedings of the Edinburgh Mathematical Society*, vol. 19, no. 4, pp. 375–381, 1975.

[20] E. H. Spanier, *Algebraic topology*, vol. 55. Springer Science & Business Media, 1994.

[21] P. E. Bland, *Rings and their modules*. Walter de Gruyter, 2011.

[22] D. Quillen, "Rational homotopy theory," *Annals of Mathematics*, pp. 205–295, 1969.

[23] D. Sullivan, "Differential forms and the topology of manifolds," *Manifolds Tokyo*, pp. 37–49, 1973.

[24] J. F. Adams and P. J. Hilton, "On the chain algebra of a loop space," *Commentarii Mathematici Helvetici*, vol. 30, no. 1, pp. 305–330, 1956.

[25] J. F. Adams, "On the cobar construction," *Proceedings of the National Academy of Sciences*, vol. 42, no. 7, pp. 409–412, 1956.

[26] D. Sullivan, "Infinitesimal computations in topology," *Publications Mathématiques de l'IHÉS*, vol. 47, no. 1, pp. 269–331, 1977.

[27] H.-J. Baues and J.-M. Lemaire, "Minimal models in homotopy theory," *Mathematische Annalen*, vol. 225, no. 3, pp. 219–242, 1977.

[28] S. Halperin, "Lectures on minimal models," *Mémoires de la Société mathématique de France*, vol. 9, pp. 1–261, 1983.

[29] W. Wen-tsün, *Rational Homotopy Type*. LNM 1264, Springer-Verlag, Berlin, Heidelberg, 1987.

[30] B. Jessup, *Rational Lusternik-Schnirelmann category and a conjecture of Ganea*. 1989.

[31] M. Manetti, "Lectures on deformations of complex manifolds," *Rendiconti di Mathematica*, vol. 24, pp. 1–183, 2004.

[32] P. W. Michor, *Topics in differential geometry*, vol. 93. American Mathematical Soc., 2008.

[33] P. J. Hilton, *Homotopy theory and duality*, vol. 1965. Gordon & Breach Science Pub, 1966.

[34] C. J. Su, "The category of long exact sequences and the homotopy exact sequence of modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 22, pp. 1383–1395, 2003.

[35] C. J. Su, "On long exact $(\bar{\pi}, Ext\lambda)$-sequences in module theory," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 26, pp. 1347–1361, 2004.

[36] C. J. Su, "On relative homotopy groups of modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, 2007.

[37] K. Varadarajan, "Numerical invariants in homotopical algebra-I," *Can. J. Math*, vol. 27, no. 4, pp. 901–934, 1975.

[38] K. Varadarajan, "Numerical invariants in homotopical algebra, II-applications," *Canadian Journal of Mathematics*, vol. 27, no. 4, pp. 935–960, 1975.

[39] S. Romyar, *Group and Galois Cohomology*. http://math.arizona.edu/ sharifi/groupcoh.pdf.

[40] D. Wigner, "Algebraic cohomology of topological groups," *Transactions of the American Mathematical Society*, vol. 178, pp. 83–93, 1973.

[41] H. E. A. Campbell and D. Wehlau, *Modular invariant theory*, vol. 139. Springer Science & Business Media, 2011.

[42] B. Stenström, *Rings and modules of quotients*. Springer–Verlag, New York, 1975.

[43] C. J. Bose, "Invariant measures and equilibrium states for piecewise $C^{1+\alpha}$ endomorphisms of the unit interval," *Transactions of the American Mathematical Society*, vol. 315, no. 1, pp. 105–125, 1989.

[44] Y. Kato, "Rings of fractions of $B(H)$," *Publications of the Research Institute for Mathematical Sciences*, vol. 30, no. 4, pp. 689–693, 1994.

[45] D. Murfet, *Tensor, Exterior and Symmetric*. http://therisingsea.org/notes/TensorExteriorSymmetric.pdf, 2006.

[46] S. Deo, *Algebraic topoloy- A primer*. Hindustan Book Agency, New Delhi, 2003.

[47] D. Grinberg, *A few classical results on tensor, symmetric and exterior powers*. http://web.mit.edu, 2013.

[48] D. S. Dummit and R. M. Foote, *Abstract algebra*, vol. 1984. Wiley Hoboken, 2004.

[49] T. Yokonuma, *Tensor spaces and exterior algebra*. No. 108, American Mathematical Soc., 1992.

[50] D. Grinberg, "The Clifford algebra and the Chevalley map-a computational approach (summary version 1),"

[51] J. Helmstetter and A. Micali, *Quadratic mappings and Clifford algebras*. Springer Science & Business Media, 2008.

[52] D. Husemoller, "Fibre bundles, volume 20 of graduate texts in mathematics," 1994.

Dissemination

Internationally indexed journals (*Web of Science, SCI, Scopus, etc.*)[1]

1. M. Routaray and A. Behera, *Categorical construction of the ring of fractions*, Tbilisi Mathematical Journal, 9(1) 2016, 1-8.

2. M. Routaray and A. Behera, *A nonlinear complementarity-type problem and variational-type inequality*, JP Journal of Fixed Point Theory and Applications, 11(1) 2016, 1-21.

3. M. Routaray, *Generalized invex sets and preinvex functions on differentiable manifolds and multiobjective optimization problems on Riemannian manifolds*, Transactions on Mathematical Programming and Applications, 3(2) 2015, 38-51.

4. M. Routaray and A. Behera, *Categorical construction of the ring of fractions of* $B(H)$, Bulletin of the Calcutta Mathematical Society, 107(5) 2015, 377-384.

5. M. Routaray and A. Behera, *Adams completion and symmetric algebra*, Accepted in African Journal of Mathematics and Computer Science Research, 2015.

6. M. Routaray and A. Behera, *Adams Completion and Exterior Algebra*, Accepted in Caspian journal of mathematical science.

7. Mitali Routaray and A. Behera, *Homotopy Approximation of Modules*, Accepted in Journal of Algebra and Related Topics.

Conferences [1]

1. A. Behera, S.B Choudhury and M. Routaray, *A Categorical Construction of Minimal Model*, ICSER Singapore, 2015, 48-63.

[1] Articles already published, in press, or formally accepted for publication.

Article under preparation [2]

1. M. Routaray and A. Behera, *A Categorical Construction of Minimal Model of Lie Algebra*, Communicated to Topology and its Application.

2. M. Routaray, *Cohomology Decomposition of Topological -module*, Communicated to Lithuanian Mathematical Journal.

3. M.Routaray, *Topological G-module and Adams Cocompletion*, Communicated to Journal of Homotopy and Related Structure.

4. M.Routaray, *A Study on Hermity-Hardamard type Inequality*, Communicated to Kyungpook Mathematical Journal.

5. M. Routaray and A. Behera, *Adams Completion and Clifford Algebra*, Communicated to Malaysian Journal of Mathematical Sciences.

6. M. Routaray and A. Behera, *Categorical Construction of Tensor Algebra*, Communicated to Georgian Mathematical Journal.

[2] Articles under review, communicated, or to be communicated.