Applications of Functional Analysis to a class of elliptic PDEs

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Integrated Masters of Science

by

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CERTIFICATE

This is to certify that the review work contained in this report titling 'Applications of Functional Analysis to a class of elliptic PDEs' is a brief summary of $p-$Laplacians and their eigenvalue problems. This report is submitted by Rayaprolu Satya Chandra Mouli (411MA5046) to the Department of Mathematics, towards the course 'Research Project (MA592)' in partial fulfilment of requirements of the degree Integrated Masters of Science in Mathematics at National Institute of Technology Rourkela. The whole review is carried out by him under my constant guidance and supervision.

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Abstract

In this report we discuss about the eigensolutions of an eigenvalue problem for the $p-$Laplace operator by investigating the underlying variational problem. Mostly the discussion is restricted to $p = 1$ case however it can be extended to other values of p also. When we have constrained minimizers subject to L^1- norm, it is considered as the eigenvalue problem of $1-$Laplace operator. Several theorems are stated in the report which strongly support the existence of solutions to the variational problem, hence eigensolutions and also the sequence of eigensolutions. Solutions are treated as critical points (for the variational problem) in the sense of weak slope. Finally an additional necessary condition is introduced which is derived using inner variations to filter the non-eigensolutions of the problem. Basic concepts of functional analysis and necessary prerequisites to understand the variational problem and minimizers are also included.

Contents

1 Prerequisites

We warm up with few definitions and results before discussing the problem.

Weak derivative: If v is a function in $L^2(\Omega)$, then u is said to be a weak derivative of v if,

$$\int_a^b v(t)\phi^{'}(t)dt = -\int_a^b u(t)\phi(t)dt$$

$\forall\ \phi \in C_0^\infty(a,b)$ with ϕ vanishing on the boundary. This is a generalisation of derivative and precisely useful for functions which have no derivative in classical sense.

Sobolev space: A Sobolev space $W^{l,p}(X)$ consists of the locally integrable functions $f : X \to \mathbb{R}$, $D^\beta f \in L^p(X)$ in the weak sense, for every $|\beta| \leq l$.

LAX-MILGRAM theorem: Let $M : V \times V \to \mathbb{R}$ be a bi-linear form where V is a Hilbert space and let $F \in V^*$ be a continuous linear functional. Then $\exists$ a unique $v \in V$ such that $M(v,u) = F(u)$, $\forall u \in V$.
Lax-Milgram theorem is a generalisation of Riesz-representation theorem. Also, it gives a notion of the existence of weak solutions.

Gagliardo-Nirenberg-Sobolev inequality: Let $1 \leq q < n$, $\exists$ a constant C which depends on q and n explicitly, such that,

$$\|v\|_{L^{q^*}(\mathbb{R}^n)} \leq C\|Dv\|_{L^q(\mathbb{R}^n)}, \qquad \forall v \in C_c^1(\mathbb{R}^n)$$

where q^* is the Sobolev conjugate of q.

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}, \quad q^* > q$$

Poincare's inequality: Let X be a bounded open subset of $\mathbb{R}^n$. Suppose, $v \in W_0^{1,p}(X)$ for some $1 \leq p \leq n$. Then,

$$\|v\|_{L^q(X)} \leq C\|Dv\|_{L^p(X)}$$

for any $q \in [1, p^*]$, the constant C depends on p, q, n and X only.

1.1 Energy estimates

Theorem 1. *There exist constants* $C, K > 0$ *and* $N \geq 0$ *such that* $M(v,u)$ *is a bilinear form then,*

(i) $|M(v,u)| \leq C\|v\|_{H_0^1(X)}\|v\|_{H_0^1(X)}$

(ii) $K\|v\|^2_{H_0^1(X)} \leq M(v,v) + N\|v\|^2_{L^2(X)}$

1.2 Existence of weak solutions

First existence theorem of weak solutions: We can always find a number $a \geq 0$ such that $\forall b \geq a$ and for each function $f \in L^2(X)$ $\exists$ a unique weak solution $v \in H_0^1(X)$ of the BVP

$$\begin{cases} Lv + bv = f & \text{in X} \\ v = 0, & \text{on } \partial X \end{cases}$$

Second existence theorem for weak solutions: One of these statements holds, either

$$\begin{cases} \forall f \in L^2(X)\ \exists \text{ a unique weak solution } v \text{ of the BVP} \\ \begin{cases} Lv = f & \text{in X} \\ v = 0, & \text{on } \partial X \end{cases} \end{cases}$$

or

$$\begin{cases} \exists \text{ a weak solution } v \not\equiv 0 \text{ of the homogeneous problem} \\ \begin{cases} Lv = 0 & \text{in X} \\ v = 0, & \text{on } \partial X \end{cases} \end{cases}$$

2 Weak Formulations

2.1 p-Laplacian

The Laplace equation $\Delta v = 0$ is the Euler Lagrange equation of the Dirichlet integral

$$D(v) = \int_\Omega |\nabla v|^2 dx$$

Extending this to p^{th} power, the integral or the variational integral will be

$$I(v) = \int_\Omega |\nabla v|^p dx$$

and the pertaining Euler Lagrange equation is,

$$div(|\nabla v|^{p-2}\nabla v) = 0$$

which is our p-Laplacian operator symbolised as $\Delta_p v$,

$$\Delta_p v = div(|\nabla v|^{p-2}\nabla v)$$

Let us review $p = 1$ case

$$\Delta_1 v = \nabla \cdot \left(\frac{\nabla v}{|\nabla v|}\right) = -H$$

where, H is called the mean curvature operator.

Dirichlet problem and weak solutions:

Theorem 2. *The below mentioned statements are equivalent* $\forall v \in W^{1,p}(\Omega)$
(i) v is minimizing:

$$\int |\nabla v|^p dx \leq \int |\nabla u|^p dx, when\ u - v \in W_0^{1,p}(\Omega).$$

(ii) first variation of the functional vanishes:

$$\int < |\nabla v|^{p-2}\nabla v, \nabla \mu > dx = 0,\ when\ \mu \in W_0^{1,p}(\Omega).$$

Moreover, if $\Delta_p v$ is continuous, then these statements are equivalent to $\Delta_p v = 0$ over Ω.

Eigenvalue problem:

$$-div\left(\frac{Dv}{|Dv|}\right) = \lambda\frac{v}{|v|}$$

$$OR$$

$$-div|Dv|^{p-2}Dv = \lambda|v|^{p-2}v \ \ on \ \ \Omega$$
$$v = 0 \ \ on \ \ \partial\Omega$$

This is the general form of eigenvalue problem for p-Laplace operator.

2.2 Variational problem

We try to put up analysis of general class of eigenvalue problems and in particular our eigenvalue problem. It is general intuition to study these problems in $BV(\Omega)$, they are studied in suitable subspaces $BV(\Omega) \cap L^p(\Omega)$. Let $\Omega \subset \mathbb{R}^n$ be open bounded having a Lipschitz boundary.
Consider the energy function

$$E(v) := \int_\Omega d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega} - v_0^{\partial\Omega}|d\mathcal{H}^{n-1} - \int_\Omega fvdx + a\int_\Omega |v-g|^r dx$$

Here the variational problem is $E(v)$ attaining its minimum over the considered domain, i.e.

$$Min\,(E(v))\,,\ where\ \ v \in BV(\Omega) \cap L^p(\Omega),$$

$$\int_\Omega |v-h|^q dx = 1$$

and suppose

$$v_0^{\partial\Omega} \in L^1(\partial\Omega),\ f \in L^n(\Omega),\ g \in L^r(\Omega),\ h \in L^q(\Omega),$$

$$\frac{n}{n-1} \leq p < \infty,\ 1 \leq r < \infty,\ 1 \leq q \leq p,\ a \geq 0$$

We know that, $BV(\Omega)$ is continuous embedded into $L^{\frac{n}{n-1}}(\Omega)$ and that $L^n(\Omega)$ is the dual of $L^{\frac{n}{n-1}}(\Omega)$. We can clearly observe that minimisers of the functional which grow linearly belong to $BV(\Omega)$ and need not necessarily belong to $W^{1,1}(\Omega)$ which is the exact reason to study this problem in $BV(\Omega)$. In general variational problems have boundary conditions, the operator $v \to v^{\partial\Omega}$

on $BV(\Omega)$ has several properties showcasing weak continuity. It will be very much restrictive while working with the usual notation of the trace operator $u^{\partial\Omega}$ in $BV(\Omega)$. Let us consider an open ball $B \subset \mathbb{R}^n$ such that it covers the closure of Ω and let the extension of each $v \in BV(\Omega)$ be

$$\overline{v}(x) := \begin{cases} v(x) & \text{on } \Omega, \\ 0 & \text{on } B \setminus \Omega \end{cases}$$

Very clearly the extension is in $BV(\Omega)$ and

$$D\underline{v}_0 = Dv - v^{\partial\Omega}\nu\mathcal{H}^{n-1}\partial\Omega, \; |D\overline{v}|(B) = |Dv|(\Omega) + \int_{\partial\Omega} |v^{\partial\Omega}| d\mathcal{H}^{n-1}$$

where ν is the outward unit normal to the surface Ω. $\overline{v} \in L^p(B)$ since $v \in L^p(\Omega)$, here the trace operator maps $BV(B \setminus \overline{\Omega})$ onto $L^1(\partial B \cup \partial\Omega)$, therefore we find our v_0 in this domain whose trace is u_0^{Ω} on $\partial\Omega$, hence the extension which also belongs to $BV(B)$

$$\underline{v}_0(x) := \begin{cases} 0 & \text{on } \Omega, \\ v_0(x) & \text{on } B \setminus \Omega \end{cases}$$

and

$$|D(\underline{v}_0 + \overline{v})|(B) = |Dv|(\Omega) + |Dv_0|(B \setminus \overline{\Omega}) + \int_{\partial\Omega} |u^{\partial\Omega} - u_0^{\partial\Omega}| d\mathcal{H}^{n-1}.$$

Let us define

$$G(u) := \int_B d|Du| - \int_\Omega fu dx + a \int_\Omega |u - g|^r dx$$

for every $u \in BV(B) \cap L^p(B|_\Omega)$ in which,

$$L^p(B|_\Omega) := \{u \in L^1(B) | u_{|\Omega} \in L^p(\Omega)\}.$$

As an immediate effect we can see

$$G(\underline{v}_0 + \overline{v}) = E(v) + |Dv_0|(B \setminus \overline{\Omega}) \;\; \forall \, v \in BV(\Omega) \cap L^p(\Omega).$$

As a consequence our variational problem boils down to

$$Min\,(G(u))\,,\; where \;\; u \in BV(B) \cap L^p(B|_\Omega),$$
$$\int_\Omega |u - h|^q dx = 1,$$
$$u = u_0 \;\; a.e. \; on \; B \setminus \Omega$$

This can be interpreted as a boundary condition in weaker sense.

2.3 Existence of minimizers

Eigenvalue problem: The variational problem

$$Min\left(\int_{\Omega}|Dv|^{p}dx\right),\ where\ v\in W^{1,p}(\Omega),$$

$$\int_{\Omega}|v|^{p}dx=1,\quad v=0\ on\ \partial\Omega$$

is rigorously studied for $1<p<\infty$. If we use Lagrange multiplier technique we will be directed to following Euler-Lagrange equation

$$-div(|Dv|^{p-2}Dv)=\lambda|v|^{p-2}v\quad on\ \Omega,\quad v=0\ on\ \partial\Omega$$

and this is exactly the eigenvalue problem(defined in section 2.1) for p-Laplace operator. We know that the $p=1$ case is highly singular. From the earlier discussion, we understood that the solutions for this variational problem $p=1$ case reside in $BV(\Omega)$ and not in $W^{1,1}(\Omega)$. Similarly, there were other discrepancies in Dirichlet boundary conditions as well. Besides this, if we can visualise that the solutions may be piecewise constant we can claim that this eigenvalue problem for $p=1$ case is not well defined. So, it will be rational to consider our variational problem

$$Min\left(\int_{\Omega}d|Dv|+\int_{\partial\Omega}|v^{\partial\Omega}|d\mathcal{H}^{n-1}\right),\ where\ v\in BV(\Omega),$$

$$\int_{\Omega}|v|dx=1$$

So, for the case $p=1$ in which the surface integral compensates the Dirichlet data in a generalized way. Hence follows the existence theorem for any p. Let us once again state Poincare's inequality in the useful sense,
if there is a constant $c>0\ni$

$$\|u\|_{L^{\frac{n}{n-1}}(B)}\leq c|Du|(B)+\int_{\partial B}|u^{\partial B}|d\mathcal{H}^{n-1}$$

$\forall u\in BV(B)$ and c is the constant from Gagliardo-Nirenberg-Sobolev inequality. Whenever the boundary conditions in the weaker sense are satified by u then $u_0^{\partial B}$ can be considered instead of $u^{\partial B}$ in the integral.

Theorem 3. *Let the conditions mentioned in the initial variational problem be satisfied and let* $c\|f\|_{L^n} < 1$ *with* $c > 0$ *else let* $a > 0$, $r > \frac{n}{n-1}$.
(1) If $q < \frac{n}{n-1} = p$, *then the variational problem has a solution* $v \in BV(\Omega)$.
(2) If $\frac{n}{n-1} \le p \le r$, $q < r$, $a > 0$, *then the variational problem has a solution* $v \in BV(\Omega) \cap L^p(\Omega)$.
(3) If $\frac{n}{n-1} = p$, *then the variational problem(without side condition) has a solution* $v \in BV(\Omega)$.
(4) If $p \le r$, $a > 0$, *then the variational problem(without side condition) has a solution* $v \in BV(\Omega) \cap L^p(\Omega)$.

3 Necessary results to analyse the eigen value problem

3.1 Necessary conditions for minimizers

Euler-Lagrange condition is the fundamental condition for minimizers of variational problem. Likewise, infinitely many Euler-Lagrange conditions ought to be fulfilled for minimizers of a vast subclass of variational problems containing the eigenvalue problem. Agian consider the energy functional,

$$E(v) := \int_{\Omega} d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}| d\mathcal{H}^{n-1} - \int_{\Omega} f v dx + a \int_{\Omega} |v-g|^r dx$$

also consider our variational problem

$$Min\left(E(v)\right),\ where\ \ v \in BV(\Omega) \cap L^p(\Omega),$$

$$\int_{\Omega} |v-h|^q dx = 1.$$

along with these assume the conditions

$$f \in L^{p'},\ \ g \in L^r(\Omega),\ \ h \in BV(\Omega) \cap L^p(\Omega),$$

$$\frac{n}{n-1} \le p < \infty,\ \ 1 \le q, r \le p,\ \ a \in \mathbb{R}.$$

Theorem 4. *Let $v \in BV(\Omega) \cap L^p(\Omega)$ be a minimizer of the above variational problem with E as defined, satisfying the above conditions. Then $\exists$ some $\lambda \in \mathbb{R}$ and some $z \in L^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{L^\infty} \le 1,\ \nabla \cdot z \in L^{p'}(\Omega),\ \ \int_{\Omega} d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}| d\mathcal{H}^{n-1} = -\int_{\Omega} u(\nabla \cdot z) dx,$$

in which $\|z\|_{L^\infty} = 1$ if $u \neq 0$, such that:
(i) if $q = r = 1$, then

$$0 \in -\nabla \cdot z - f + aQ(v-g) - \lambda Q(v-h)\ \ almost\ everywhere\ on\ \Omega,$$

where $Q(y)$ is set valued sign function

$$Q(y) := \begin{cases} 1 & if\ y > 0, \\ [-1, 1] & if\ y = 0, \\ -1 & if\ y < 0. \end{cases}$$

(ii) if either of q or r is greater than 1, then above implication from (i) holds with $|v-g|^{r-2}(v-g)$ or $|v-h|^{q-2}(v-h)$ in place of $Q(v-q)$ or $Q(v-h)$, respectively.
(iii) if $h=0$ and $a=0$, then $\lambda = E(v)$, i.e. λ can be replaced by $E(v)$ in both (i) and (ii).

Consider a situation where $h,\ f=0,\ a=0,\ q=1,\ p=\frac{n}{n-1}$ and if we apply this theorem to the eigenvalue problem we end up at our variational problem

$$Min\left(\int_\Omega d|Dv| + \int_{\partial\Omega} |v^{\partial\Omega}| d\mathcal{H}^{n-1}\right),\ where\ v \in BV(\Omega),$$

$$\int_\Omega |v|dx = 1$$

and the theorem facilitates a necessary condition that

$$-\nabla \cdot z \in \lambda Q(v) \quad a.e.\ on\ \Omega, \quad \lambda = E(v) > 0$$

where $z \in L^\infty(\Omega, \mathbb{R}^n)$, here the vector space z need not be unique, some z satisfying the following conditions

$$\|z\|_{L^\infty} = 1, \quad \nabla \cdot z \in L^n(\Omega), \quad E(v) = -\int_\Omega v(\nabla \cdot z)dx.$$

Our necessary condition gives an implication that there exists a function $w(measurable)$ of $Q(v)$ i.e., $w(x) \in Q(v(x))$ a.e. on Ω, such that

$$-\nabla \cdot z = \lambda w \quad a.e.\ on\ \Omega$$

It's trivial to claim that this relation is exactly in the form of our eigenvalue problem for the 1−Laplace operator, in-fact this is a generalisation, recall

$$-div\left(\frac{Dv}{|Dv|}\right) = \lambda \frac{v}{|v|} \quad on\ \Omega$$

3.2 Existence of a vector field z

Generally expressions in this eigenvalue problem are not well defined since these minimizers v are either constant or will vanish on a set of non zero measure. Theorem below ascertains a suitable substitute for eigenvalue problem.

Theorem 5. *Suppose $v \in BV(\Omega)$ is a minimizer of the constrained variational problem*

$$E(v) := Min\left(\int_{\Omega} d|Dv| + \int_{\partial\Omega} |v| d\mathcal{H}^{n-1}\right), \quad v \in BV(\Omega),$$

$$\int_{\Omega} |u| dx = 1.$$

Then for every measurable selection $w(x) \in Q(v(x))$ a.e. on Ω the exists a vector field $z \in L^{\infty}(\Omega, \mathbb{R}^n)$ with

$$\|z\|_{L^{\infty}} = 1, \quad \nabla \cdot z \in L^n(\Omega),$$

$$\int_{\Omega} d|Dv| + \int_{\partial\Omega} |v| d\mathcal{H}^n - 1 = -\int_{\Omega} v(\nabla \cdot z) dx$$

so that

$$-\nabla \cdot z = \lambda w \quad a.e.\ on\ \Omega, \quad \lambda = E(v).$$

We have already discussed in detail that, there are infinitely many Euler-Lagrange equations as necessary conditions for the minimizer of v which can be visualised here in the above equation too. And these conditions should be satisfied for any arbitrary selection of w(*measurable*). This discussion gives us an idea about the eigenvalue problem for the 1-Laplace operator. Any solution v is called the eigenfunction corresponding to the eigenvalue λ.

Theorem 5 assures the existence of vector fields z which satisfies the stated equations (coupling conditions) in the theorem. It would have been very useful if we had more or any details about these vector fields and appositeness of the coupling conditions for this vector field. We are only fortunate to an extent of establishing their existence as of now. Even for a particular v and when w is given we have no concrete method to determine z.

3.3 Weak slope

Definition 1. *Let Y be a metric space with metric m and let $h : Y \to \mathbb{R}$ be a continuous function, and suppose $v \in Y$. Denote by $|dh|(v)$ the supremum of the $\gamma \in [0, +\infty)$ such that $\exists$ an $\epsilon > 0$ and a continuous map*

$$\mathcal{H} : B(v, \epsilon) \times [0, \epsilon] \to Y$$

such that, for every

$$u \in B(v, \epsilon),\ \forall s \in [0, \epsilon]$$

it leads to

$$m(\mathcal{H}(u, s), u) \leq s, \quad h(\mathcal{H}(u, s)) \leq h(u) - \gamma s.$$

The extended real number $|dh|(v)$ is called the weak slope of h at v.

Lemma 1. *Let $g : Y \to \mathbb{R}$ be a continuous function. Then, for every $(v, \zeta) \in epi(g)$, we have*

$$|dG_g|(v, \zeta) = \begin{cases} \frac{|dg|(v)}{\sqrt{1+(|dg|(v))^2}}, & \textit{if } g(v) = \zeta \textit{ and } |dg|(v) \leq +\infty, \\ 1, & \textit{if } g(v) \leq \zeta \textit{ or } |dg|(v) = +\infty. \end{cases}$$

Now, the weak slope of a lower semi-continuous function g can defined by using $|dG_g|(v, g(v))$. More precisely, we have the following

Definition 2. *Let $g : Y \to \overline{\mathbb{R}}$ be a lower semi-continuous function. For every $v \in Y$ such that $g(v) \in \mathbb{R}$, let*

$$|dg|(v) = \begin{cases} \frac{|dG_g|(v,g(v))}{\sqrt{1-|dG_g|(v,g(v))^2}}, & \textit{if } |dG_g|(v, g(v)) < 1, \\ +\infty, & \textit{if } |dG_g|(v, g(v)) = 1. \end{cases}$$

From the above notion we can imply

Definition 3. *Let Y be a metric space and $g : Y \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function. Any $v \in dom(g)$ is a critical point of g if $|dg|(v) = 0$. One can claim that $b \in \mathbb{R}$ is a critical value of g if there exists a critical point $v \in dom(g)$ such that $g(v) = b$.*

3.4 Existence of a sequence of eigensolutions

Again, if we look back to Theorem 5 in which v is a solution of the 1-Laplace operator, the (v, λ) is the first eigensolution of the 1-Laplace operator. We also verified that the Euler-Lagrange equation should be satisfied for all possible selections of $w(measurable)$. We claim that there exists more critical points of the variational problem which can be considered as eigensolutions of the 1-Laplace operator. Critical points definition in the sense of weak slope may be used since it works for both continuous and some lower semi-continuous functions.

Theorem 6. *There is always a sequence* $(v_k, -v_k)$ *where* $k \in \mathbb{N}$ *and each* $v_k \in BV(\Omega)$, *these are pairs of critical points for the variational problem*

$$E(v) := \int_{\Omega} d|Dv| + \int_{\partial\Omega} |v| d\mathcal{H}^{n-1}$$

with the constraint

$$\int_{\Omega} |v| dx = 1,$$

where $E(v_k) < \infty \ \forall k \in \mathbb{N}$. *Also, for every critical point* $v_k \in BV(\Omega)$ *and for some choice of* q *such that* $1 < q < \frac{n}{n-1}$ *there certainly exists a measurable selection* $w_k(x) \in Q(v_k(x))$ *a.e.* $x \in \Omega$ *and a vector field* $z_k \in L^{\infty}(\Omega, \mathbb{R}^n)$ *satisfying*

$$\|z_k\|_{L^\infty} = 1, \quad \nabla \cdot z_k \in L^{q*}(\Omega),$$

$$E(v_k) = -\int_{\Omega} v_k(\nabla \cdot z_k) dx$$

such that

$$-\nabla \cdot z_k = \lambda_k w_k \text{ a.e. on } \Omega, \quad \lambda_k = E(v_k).$$

Also, $\lambda_k \to \infty$ *as* k *approaches infinity.*

This Theorem 6 is very much similar to Theorem 5 strongly supporting the existence of an infinite sequence of eigensolutions. Here we have to notice that unlike Theorem 5 the Euler-Lagrange equation may not be satisfied for all possible measurable selections of $w_k(x) \in Q(w_k(x))$. Here the choice of $q < \frac{n}{n-1}$ ensures the compact embedding of $BV(\Omega)$ into $L^q(\Omega)$.

4 Eigenvalue problem for the $1-$Laplace operator

As stated earlier in this document,

$$-\nabla \cdot |Dv|^{p-2}Dv = \lambda|v|^{p-2} \quad on\,\Omega,$$

$$v = 0 \quad on\,\partial\Omega$$

is the general form of eigenvalue problem for $p-$Laplace operator in relation with the variational problem

$$Min\left(\int_\Omega |Dv|^p dx\right) \quad in\,W_0^{1,p}(\Omega)$$

with

$$\int_\Omega |v|^p dx = 1.$$

We know for the case $p = 1$

$$-\nabla \cdot \frac{Dv}{|Dv|} = \lambda\frac{v}{|v|}$$

We can observe that no solutions exist in $W_0^{1,1}(\Omega)$ due to which have to look for a better space i.e. $BV(\Omega)$ with boundary conditions having weaker notions, leading us to

$$Min\left(\int_\Omega d|Dv| + \int_{\partial\Omega} |v|d\mathcal{H}^{n-1} \;\; in\,BV(\Omega)\,subject\,to\;\; \int_\Omega |v|dx = 1\right.$$

where, $\Omega \subset \mathbb{R}^n$ is taken to be open and bounded with a Lipschitz boundary. Hence by the conventional methods existence of minimizer is followed.
The Euler-Lagrange equation

$$-\nabla \cdot \frac{Dv}{|Dv|} = \lambda\frac{v}{|v|}$$

has to be appropriately interpreted since it doesn't give required information. We use Lagrange multiplier rule to find a minimizing condition. Also, by computing the convex sub-differentials of the functions in our variational problem we obtain that for an arbitrary minimizer $v \in BV(\Omega)$

$$\exists\, w(x) \in Q(v(x))\; a.e.\; on\,\Omega\; and\, \exists\, z \in L^\infty(\Omega, \mathbb{R}^n)$$

satisfying

$$|z(x)| \leq 1\, a.e.,\ \ \nabla \cdot z \in L^{\infty}(\Omega),\ \ E(v) = -\int_{\Omega} v(\nabla \cdot z)dx$$

(The variational problem is called single equation if it is based on the above two equations)
such that

$$-\nabla \cdot z = \lambda w\ a.e.\ on\ \Omega, \quad \lambda = E(v),$$

(the variational problem is called multiple equation if it is related to this equation). (Also, refer Theorem 5)

In the Euler-Lagrange equation, $\frac{v}{|v|}$ is replaced by w and $\frac{Dv}{|Dv|}$ is replaced by z. Further analysis (shown in section 3.2) assures that for every measurable selection $w(x) \in Q(v(x))$ there exists a vector space z, (which need not be unique) satisfying the variational problem, i.e. the problem has infinitely many necessary conditions (section 3.1).

The higher eigensolutions (section 3.4) cannot be defined in terms of an eigenvalue equation, unlike the trivial cases. As we know the defined single equation has many solutions, but after interpreting the problem in terms of z and w, i.e. multiple equation, we see that minimizers are the only solutions. Hence, eigensolutions are visualised as critical points of the variational problem. Critical points are defined in the sense of weak slope (section 3.3) since the curve is not smooth.

Theorem 6 states that there exists eigenfunctions v_k, $\forall k \in \mathbb{N}$ along with corresponding eigenvalues λ_k, i.e. (v_k, λ_k) satisfy the variational problem as a single equation. As we know that single equation has too many solutions (apart from eigensolutions), a stronger necessary condition is derived using inner variations which is,

$$\int_{\Omega} \langle z, D\mu \cdot z \rangle - (\nabla \cdot \mu)\, d|Dv| = -\lambda \int_{\Omega} |v|\, (\nabla \cdot \mu)\, dx \ \ \forall \mu \in C_0^{\infty}(\Omega)$$

where $\lambda = E(v)$ and z based on the polar decomposition of the total variation measure i.e. $z|Dv| = Dv$. This additional necessary condition eliminates solutions of our single equation which are non-critical points. It is yet to be discovered if all the solutions obtained by applying both the conditions are eigensolutions of the $1-$Laplace operator.

Conclusion

The notion of eigensolutions is given by the weak slope (section 3.3). In the above discussion we considered L^1- topology, however $BV-$topology may also be used to study the above problem. The eigensolutions may also be defined in terms of strong slope (DeGiorgi) or any other alternative concepts of slope. When we consider the case where $\Omega = (0,1) \in \mathbb{R}$ then the eigensolutions are determined precisely for each choice of slope and topology, where we can observe that the set of solutions are really different for different choices. Amidst all this discussion, we can claim that the definition of eigensolution in the notion of weak slope using L^1-topology was an appropriate approach for analysing the eigensolutions of $1-$Laplace operator.

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