A STUDY OF FUZZY CONTINUOUS MAPPINGS

A thesis is submitted in partial fulfillment of the requirement for the degree

of

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by

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Under the supervision
of

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Declaration

I hereby certify that the work which is being presented in the report entitled “A Study of Fuzzy Continuous Mappings” in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted to the Department of Mathematics, National Institute of Technology Rourkela is a review work carried out under the supervision of Dr. Akrur Behera. The matter embodied in this report has not been submitted by me for the award of any other degree.

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CERTIFICATE

This is to certify that the thesis entitled "A Study of Fuzzy Continuous Mappings" submitted by Ashutosh agrawal (Roll No: 411MA5054) in partial fulfilment of the requirements for the degree of Master of Science in Mathematics at the National Institute of Technology Rourkela is an authentic work carried out by him during final year Project under my supervision and guidance.

Date: May, 2016

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Abstract

The paper deals with the concept of semi-compactness in the generalized setting of a fuzzy topological space. We achieve a number of characterizations of a fuzzy semi-compact space. The notion of semi-compactness is further extended to arbitrary fuzzy topological sets. Such fuzzy sets are formulated in different ways and a few pertinent properties are discussed. Finally we compare semi-compact fuzzy sets with some of the existing types of compact-like fuzzy sets. We ultimately show that so far as the mutual relationships among different existing allied classes of fuzzy sets are concerned, the class of semi-compact fuzzy sets occupies a natural position in the hierarchy. The purpose of this paper is to introduce the concepts of semi*-connected spaces, semi*-compact spaces. We investigate their basic properties. We also discuss their relationship with already existing concepts.

Introduction

Barring para-compactness, there exists in the literature, a number of allied forms of compactness studied in a classical fuzzy topological space. Among these, the most widely studied compact-like covering properties are almost compactness or quasi H-closed-ness, near compactness, S-closed-ness, and semi-compactness. The thorough investigations and the applicational aspects of these covering properties have prompted topologists to generalize these concepts (with the exception of semi-compactness) to fuzzy setting. In this paper, some of interesting properties of fuzzy semi-compactness are investigated. Our intention here is to go into some details towards characterizations of semi-compactness for a fts. These characterizations are effected with the help of fuzzy sets, pre-filter-bases and similar other concepts, which comprise the deliberation in the next section. Compactness is one of the most important, useful and fundamental concepts in fuzzy topology. The purpose of this paper is to introduce the concepts of semi*-connected spaces, semi*-compact spaces. We investigate their basic properties. We also discuss their relationship with already existing concepts.

Contents

Chapter 1

1. Fuzzy Topological spaces

1.1 A topological space is an ordered pair (X , τ), where X is a set and τ is a collection of subsets of X , satisfying the following axioms

1. The empty set and X it self belongs to τ.
2. Any (finite or infinite) union of members of τ still belongs to τ.
3. The intersection of any finite number of members of τ still belongs to τ.

Definition: A fuzzy topology on a set X is a collection δ of fuzzy sets in X such that:

1. $0,1 \in \delta$,
2. $\mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$
3. $\forall\ (\mu_i)_{i \in I} \in \delta \Rightarrow \vee_{i \in I} (\mu_i) \in \delta$;

δ is called as a fuzzy topology for X , and the pair (X, δ) is a fuzzy topological space, or FTS in short. Every member of δ is called a T-open fuzzy set. Fuzzy sets of the form $1 - \mu$, where μ is an open fuzzy set, are called closed fuzzy sets.

Examples of fuzzy topologies:

- Any topology on a set X (subsets are identified with their characteristic functions).
- The indiscrete fuzzy topology {0, 1} on a set X (= indiscrete topology on X).
- The discrete fuzzy topology on X containing all fuzzy sets in X .
- The collection of all crisp fuzzy sets in X (= discrete topology on X).
- The collection of all constant fuzzy sets in X .
- The intersections of any family of fuzzy topologies on a set X .

1.2 Base and subbase for FTS:

Definition: A base for a fuzzy topological space (X,τ) is a sub collection β of τ such that each member A of τ can be written as $V_{j\in A}A_j$, where each $A_j \in \beta$.

Definition: A subbase for a fuzzy topological space (X,τ) is a sub collection S of τ such that the collection of infimum of finite subfamilies of S forms a base for (X,τ).

Definition: Let (X,τ) be an FTS. Suppose A is any subset of X . Then (A, τ_A) is called a fuzzy subspace of (X,τ),

Where,

1. $\tau_A = \{B_A : B \in \tau\}$,
2. $B = \{(x, \mu_B(x)) : x \in X$,
3. $B_A = \{(x, \mu_{B/A}(x)) : x \in A\}$.

Definition: A fuzzy point L in X is a special fuzzy set with membership function defined by

$$L(x) = \begin{cases} \mu & \forall \quad x = y \\ 0 & \forall \quad x \neq y \end{cases}$$

Where, $0 < \mu \leq 1$.

L is said to have support y, value μ and is denoted by P_y^μ or $P(y, \mu)$.

Let A be a fuzzy set in X , then $P_y^\alpha \subset A \Leftrightarrow \alpha \leq A(y)$.

In particular,

$P_y^\alpha \subset P_z^\beta \Leftrightarrow y = z, \alpha \leq \beta$.

A fuzzy point P_y^α is said to be in A, denoted by

$P_y^\alpha \in A \Leftrightarrow \alpha \leq A(y)$.

The complement of the fuzzy point P_x^λ is denoted either by $P_x^{\lambda-1}$ or by $(P_x^\lambda)^c$.

Definition: The fuzzy point P_x^λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $P_x^\lambda \in A$ if and only if $\lambda < A(x)$.

Every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A. That is, if A(x) is not zero for $x \in X$, then $A(x) = \sup \{\lambda: P_x^\lambda , 0 < \lambda \leq A(x)\}$.

Definition: Two fuzzy sets A, B in X are said to be intersecting if and only if there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$. For such a case, we say that A and B intersect at x. Let A, B $\in$ I^X . Then A = B if and only if $P \in A \Leftrightarrow P \in B$ for every fuzzy point P in X .

1.3 Closure and Interior of fuzzy sets

Definition: The closure $\overline{A}$ and the interior A^0 of a fuzzy set A of X are defined as

$$\overline{A} = \inf\{K : A \leq K, K^c \in \tau\}$$

$$A^0 = \sup\{O : O \leq A, O \in \tau\}$$

1.4 Neighborhood

Definition: A fuzzy point P_x^λ is said to be quasi-coincident with A, denoted by $P_x^\lambda qA$, if and only if $\lambda > A^c(x)$, or λ + A(x) > 1.

Proposition: Let f be a function from X to Y. Let P be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then we have:

1. If $f(P)_q B$, then $P_q f^{-1}(B)$.
2. If $P_q A$, then $f(P)_q f(A)$.
3. $P \in f^{-1}(B)$, if $f(P) \in B$.
4. $f(P) \in f(A)$, if $P \in A$.

1.5 Fuzzy Continuous Map

Definition: Given fuzzy topological space (X,τ) and (Y,γ), a function $f:X\to Y$ is fuzzy continuous if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X; that is if $f^{-1}(\nu)\in\tau$ whenever $\nu\in\gamma$.

Proposition:

(a) The identity $id_X:(X,\tau)\to(X,\tau)$ on a fuzzy topological space (X,τ) is fuzzy continuous.

(b) A composition of fuzzy continuous functions is fuzzy continuous.

Proof. (a) $\forall\nu\in\tau, id_X^{-1}(\nu)=\nu\circ id_X=\nu$

(b) Let $f:(X,\tau)\to(Y,\gamma)$ & $g:(Y,\gamma)\to(Z,\beta)$ be fuzzy continuous. For $\eta\in\beta,(g\circ f)^{-1}(\eta)=\eta\circ(g\circ f)=(\eta\circ g)\circ f = f^{-1}(\eta\circ g)= f^{-1}(g^{-1}(\eta))$. $g^{-1}(\eta)\in\gamma$ since g is fuzzy continuous, and so $(g\circ f)^{-1}(\eta)= f^{-1}(g^{-1}(\eta))\in\tau$ = f −1 (g −1 (η)) ∈ τ since f is fuzzy continuous.

Chapter 2

2. Generalized locally closed sets and GLC-continuous function

2.1 Fuzzy G-Closed sets

$S \in (X, \tau)$ is Fuzzy G-closed,

$\Leftrightarrow cl(S) \subset G$,

$S \subset G$,

G is open in (X , τ).

2.2 Fuzzy G-open Sets

$S \in (X, \tau)$ is fuzzy G-open,

$\Leftrightarrow (X - S)$ is fuzzy g-closed.

2.3 Fuzzy Locally Closed sets

$S \in$ is fuzzy locally closed

$\Leftrightarrow S = G \cap F$,

Where, $G \in \tau$ and F is closed in (X, τ)

2.4 Fuzzy G-Locally closed sets

$S \in (X, \tau)$ is fuzzy G-locally closed

$\Leftrightarrow S = G \cap F,$

Where, G is fuzzy g-open in (X, τ)

F is fuzzy g-closed in (X, τ).

2.5 Fuzzy GLC*

$$S \in (X, \tau)$$

$S \in fuzzy\ GLC^*(X, \tau)$

$\Leftrightarrow S = G \cap F$

Where, G is fuzzy g-open set of (X, τ)

F is fuzzy-closed set of (X, τ)

2.6 Fuzzy GLC**

$$S \in (X, \tau)$$

$S \in fuzzy\ GLC^{**}(X, \tau)$

$\Leftrightarrow S = G \cap F$

Where, G is fuzzy-open set of (X, τ)

F is fuzzy g-closed set of (X, τ)

Theorem:

$$S \in (X, \tau)$$

1. $S \in fuzzy\ GLC^{*}(X, \tau)$
2. $S = P \cap cl(S)\ \forall\ fuzzy$ g-open $set\ P$
3. $cl(S) - S\ is\ fuzzy$ g-closed
4. $S \cup cl\big(X - cl(S)\big) is\ fuzzy$ g-open

Proposition

$$A, Z\ \in (X, \tau)$$

$$A \subset Z$$

1. Z is fuzzy g-open in (X, τ)

$A \in GLC^{*}\ \big(Z,\ \tau \,\big|\, Z\big)$

$\Rightarrow A \in GLC^{*}(X, \tau)$

2. Z is fuzzy g-closed in (X, τ)

$A \in GLC^{**}\big(Z,\ \tau \,\big|\, Z\big)$

$\Rightarrow A \in GLC^{**}(X, \tau)$

3. Z is fuzzy g-closed and fuzzy g-open in (X, τ)

$A \in GLC\big(Z, \tau \,\big|\, Z\big)$

$\Rightarrow A \in GLC(X, \tau)$

2.7 Fuzzy Generalized Locally Closed Functions:

Fuzzy GLC-irresolute:

$$f:\ (\mathrm{X},\ \tau) \rightarrow (Y, \sigma)$$

$$\Leftrightarrow f^{-1}(V) \in GLC\ (\mathrm{X},\ \tau)\ \forall\ V \in GLC\ (Y, \sigma).$$

Fuzzy GLC-continuous:

$$f:\ (\mathrm{X},\ \tau) \rightarrow (Y, \sigma)$$

$$\Leftrightarrow f^{-1}(V) \in GLC\ (\mathrm{X},\ \tau)\ \forall\ V \in \sigma.$$

Chapter 3

3. Fuzzy semi-compact spaces

Definition: A FTS X is said to be a fuzzy semi-compact space if every fuzzy cover of X by fuzzy semi-open sets (such a cover will be called a fuzzy semi-open cover of X) has a finite sub-cover.

A direct consequence of the above definition yields the following alternative formulation of a fuzzy semi-compact space.

Theorem: A FTS X is fuzzy semi-compact $\Leftrightarrow$ each family U of fuzzy semi-closed sets in X with finite intersection property (i.e., for every finite sub-collection U_0 of U , $\cap U_0 \neq 0_X$) has a non-null intersection.

Theorem: A FTS X is fuzzy semi-compact $\Leftrightarrow$ every pre-filter base on X has a fuzzy semi-cluster point.

Proof: Let X be fuzzy semi-compact and let $E = \{F_\alpha : \alpha \in \Lambda\}$ be a pre-filter base on X having no fuzzy semi-cluster point. Let $x \in X$. Corresponding to each $n \in N$ (here and hereafter N denotes the set of natural numbers), there exists a semi-q-nbd U_x^n of the fuzzy point $x_{1/n}$ and an $F_x^n \in \mathrm{E}$ such that $U_x^n \bar{q} F_x^n$. Since $U_x^n(X) > 1 - 1/n,$ we have $U_x(x) = 1,$ where $U_x = \cup\{U_x^n : n \in N\}.$ Thus $U = \{U_x^n : n \in N, x \in X\}$ is a fuzzy semi-open cover of X . Since X is fuzzy semi-compact, there exist finitely many members $U_{x1}^{n1}, U_{x2}^{n2}, ..., U_{xk}^{nk}$ of U such that $\cup_{i=1}^{k} U_{xi}^{ni} = 1_X$. If $F \in E$ such that $F \leq F_{x1}^{n1} \cap F_{x2}^{n2} \cap ... \cap F_{xk}^{nk},$ then $F \bar{q} 1_X$. Consequently, $F = 0_X$ and this contradicts the definition of a pre-filter base.

Definition: A fuzzy point x_α in a FTS X is called a complete semi accumulation point of a fuzzy set A in X if and only if for each semi-q-nbd U of x_α, $|\sup A| = |\{y \in X : A(y) + U(y) > 1\}|$, where for a subset B of X, by $|B|$ we mean, the cardinality of B.

Theorem: A necessary condition for a FTS X to be fuzzy semi-compact is that every fuzzy set A in X with $|\sup A| \geq N_0$ has a complete semi accumulation point.

Proof: Let A be a fuzzy set in a fuzzy semi-compact space X such that

$|\sup A| \geq N_0$,

And if possible, suppose A has no complete semi accumulation point in Y. Then for each $x \in X$ and $n \in N$, there is a semi-q-nbd U_x^n of the fuzzy point x_1 / n such that

$|\{x \in X : A(x) + U_x^n(x) > 1\}| < |\sup A|$.

Now, since $U_x^n(x) + 1/n > 1$, it follows that

$\{U_x^n : x \in X, n \in N\}$ is a fuzzy cover of X by fuzzy semi-open sets. As X is fuzzy semi-compact, there exist a finite subset $\{x_1, x_2, ..., x_n\}$ of X and finitely many positive integers $n_1, n_2, ..., n_m$ such that $U_{i=1}^m U_{xi}^{ni} = 1_X$.

Now, $x \in \sup A \Rightarrow U_{x_k}^{n_k} = 1$, for some K(1<K<m)

$$\Rightarrow \quad U_{x_k}^{n_k}(x) + A(x) > 1$$

$$\Rightarrow \quad x \in \{y \in X : A(y) + U_{x_k}^{n_k}(y) > 1\} = A \cup n_k$$

As, $A \cup n_k \subseteq \cup_{i=1}^{m} AU_{n_i}$, we have $\sup A \subseteq \cup_{i=1}^{m} AU_{n_i}$

But, $| AU_{n_k} | < | \sup A |$ for i=1,2,…,m. Thus,

$$| U_{i=1}^{m} = AU_{n_i} | = \max_{1 \le i \le m} | AU_i | < | \sup A |$$

Hence, we get

$$| \sup A | \le | U_{i=1}^{m} AU_{n_i} | < | \sup A |$$

It is a contradiction. This proves our theorem.

Chapter 4

4. Fuzzy semi-compact sets

Definition: A fuzzy set A in a FTS X is said to be:

1. A fuzzy compact set, if every fuzzy open cover of A has a finite sub-cover for A.

2. A fuzzy nearly compact set, if every fuzzy regular open cover of A has a finite sub-cover for A.

3. A fuzzy s-closed set, if every fuzzy semi-open cover of A has a semi-proximate sub-cover for A.

4. A fuzzy almost compact set, if every fuzzy open cover of A has a finite proximate sub-cover for A.

5. A fuzzy θ-rigid set, if for every fuzzy open cover U of A, there exists a finite subfamily U_0 of U such that $A \leq \operatorname{int} cl(\cup U_0)$.

6. A fuzzy θ^*-rigid, if for every semi-open cover U of A, there exists a finite subfamily U_0 of U such that $A \leq scl(\cup\{sclU : U \in U_0\})$.

Theorem: If A is a Fs*C-set in a FTS X and $f: X \to Y$ is fuzzy irresolute then $f(A)$ is a Fs*C-set in the FTS Y.

Proof: For each fuzzy semi-open cover $\{V_\alpha : \alpha \in \Lambda\}$ of $f(A)$ in Y, $\{f^{-1}(V_\alpha : \alpha \in \Lambda\}$ is a fuzzy semi-open cover of A in X. Hence,

$$A \leq \cup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha), \text{ for some finite subset } \Lambda_0 \text{ of } \Lambda.$$

Then,

$$f(A) \leq f(\cup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)) \leq ff^{-1}(\cup_{\alpha \in \Lambda_0} V_\alpha) \leq \cup_{\alpha \in \Lambda_0} V_\alpha.$$

Thus $f(A)$ is a Fs*C-set in Y.

Chapter 5

5. Semi*-connectedness in Fuzzy Topological Spaces:-

Definition 5.1: Let A be a subset of a fuzzy topological space X. The **generalized closure** of A is defined as the intersection of all g-closed sets containing A and is denoted by $Cl*(A)$.

A subset B of a fuzzy topological space X is called **g-closed**, if $Cl(B) \subseteq U$ whenever $B \subseteq U$ and U is open in X.

Definition 5.2: A subset A of a fuzzy topological space X is called **semi*-open** if $A \subseteq Cl^*(Int(A))$.

Definition 5.3: A subset A of a fuzzy topological space X is called **semi*-regular** if it is both semi*-open and semi*closed.

Definition 5.4: Let A be a subset of X. Then the **semi*-closure** of A is defined as the intersection of all semi*-closed sets containing A and is denoted by $s*Cl(A)$.

Definition 5.5: A subset A of a fuzzy topological spaces X, the **semi*-frontier** of A is defined by $s*Fr(A) = s*Cl(A) \setminus s*Int(A)$.

Definition 5.6: A function $f: X \to Y$ is said to be

(i) **semi*-continuous** if $f^{-1}(V)$ is semi*-open in X for every open set V in Y.

(ii) **semi*-irresolute** if $f^{-1}(V)$ is semi*-open in X for every semi*-open set V in Y

Theorem 5.7: Let $f : X \to Y$ be a function. Then

(i) f is **semi*-continuous** if and only if $f^{-1}(V)$ is semi*-closed in X for every closed set V in Y.

(ii) f is **semi*-irresolute** if and only if $f^{-1}(V)$ is semi*-closed in X for every semi*-closed set V in Y.

Definition 5.8: A fuzzy topological space X is said to be **semi*-connected** if X cannot be expressed as the union of two disjoint non-empty semi*-open sets in X .

Theorem 5.9: Every open set is semi*-open.

Theorem 5.10: Every semi*-open set is semi-open.

Theorem 5.11: Let A be a subset of a fuzzy topological space X. Then A is semi*-regular if and only if $s*Fr(A) = \Phi$.

Theorem 5.12: If a fuzzy topological space X is semi*-connected, than it is connected.

Proof: Let X be semi*-connected. Suppose, X is not connected. Then by definition of connected space, we can say that $\exists A, B \neq \Phi$ & $A \cap B = \Phi$, such that $X = A \cup B$. Where, A and B are open sets. By Theorem 5.9, we can say that A and B are semi*-open sets. This is a contradiction to X is semi*-connected. Hence, the fuzzy topological space X is connected.

Theorem 5.13: If a fuzzy topological space X is semi-connected, than it is semi*-connected.

Proof: Let the fuzzy topological space X be semi-connected. Let X is not semi*-connected. Then by Definition 5.8, we can say that $\exists A, B \neq \Phi$ & $A \cap B = \Phi$, such that $X = A \cup B$. Where A and B are semi*-open sets. By Theorem 5.10, we can say that A and B are semi-open sets.

This is a contradiction to X is semi-connected. Hence, fuzzy topological space X is semi*-connected.

Theorem 5.14: A fuzzy topological space is a semi*-connected space if and only if the only semi*-regular subsets of X are Φ and X .

Necessity: Suppose the fuzzy topological space X is semi*-connected. Let A be a non-empty proper subset of X that is semi*-regular. Then A and $X \setminus A$ are non-empty semi*-open sets and $X = A \cup (X \setminus A)$.

This is a contradiction to our assumption that X is semi*-connected.

Sufficiency: Suppose $X = A \cup B.\ A, B \neq \Phi$ & $A \cap B = \Phi$. A and B are semi*-open sets.

Then, $A = X \setminus B$ is semi*-closed. Thus, A is non-empty proper subset that is semi*-regular. This is a contradiction to our assumption. Hence, our theorem is proved.

Theorem 5.15: A fuzzy topological space X is semi*-connected if every semi*-continuous function of X into a discrete space Y with at least two points is a constant function.

Proof: Let f be a semi*-continuous function of the semi*-connected space into the discrete space Y . Then for each $y \in Y, f^{-1}(\{y\})$ is a semi*-regular set of X . Since X is semi*-connected $f^{-1}(\{y\}) = \Phi or X$. If $f^{-1}(\{y\}) = \Phi \forall y \in Y$, then f ceases to be a function. Therefore $f^{-1}(\{y\}) = X$ for a unique $y_0 \in Y$. This implies $f(x) = \{y_0\}$ and hence f is a constant function.

Theorem 5.16: A fuzzy topological space X is semi*-connected if and only if every non-empty proper subset of X has non-empty semi*-frontier.

Proof: Suppose that the fuzzy topological space X is semi*-connected. Let A be a non-empty proper subset of X. We claim that $s*Fr(A) \neq \Phi$. On the contrary, let $s*Fr(A) = \Phi$.

Then by Theorem 5.11, A is semi*-regular subset of X .By Theorem 5.14, X is not semi*-connected, which is a contradiction.

Conversely, suppose that every non-empty proper subset of X has a non-empty semi*-frontier. We claim that X is semi*-connected. On the contrary, suppose X is not semi*-connected. By Theorem 5.14, X has a non-empty proper subset A, which is semi*-regular. By Theorem 5.11, $s*Fr(A) = \Phi$, which is a contradiction to our assumption. Hence, the fuzzy topological space X is semi*-connected.

Theorem 5.17: Let $f : X \rightarrow Y$ be a semi*-continuous surjection and the fuzzy topological space X be semi*-connected. Then Y is connected.

Proof: Let $f : X \rightarrow Y$ be semi*-continuous surjection and the topological space X be a semi*-connected. Let V be a clopen subset of Y. By Definition 5.6 (i) and Theorem 5.7 (i), $f^{-1}(V)$ is semi*-regular in X. Since X is semi*-connected $f^{-1}(V) = \Phi or X$. Hence $V = \Phi or Y$. This proves Y is connected.

Theorem 5.18: Let $f : X \rightarrow Y$ be a fuzzy semi*-irresolute surjection. If X is a fuzzy semi*-connected, then Y is so.

Proof: Let $f : X \rightarrow Y$ be fuzzy semi*-irresolute surjection and X be a fuzzy semi*-connected. Let V be a subset of Y that is semi*-regular in Y. By definition 5.6 (ii) and Theorem 5.7 (ii), $f^{-1}(V)$ is semi*-regular in X. Since X is fuzzy semi*-connected, $f^{-1}(V) = \Phi or X$. Hence, $V = \Phi or Y$. This proves Y is fuzzy semi*-connected.

Chapter 6

6. FUZZY WEAKLY-COMPACT SPACES

In this chapter we define sets fuzzy weakly-compact relative to a topological space and investigate the relationship between such sets and fuzzy weakly-compact subspaces.

6.1 Definition. A fuzzy subset S is said to be *fuzzy regular open* (resp. *fuzzy regular closed*) if int(cl(S))= S (resp. cl(int(S))=S).

6.2 Definition. A fuzzy open cover $\{V_\alpha : \alpha \in L\}$ of an fts is said to be *fuzzy regular* if for each $\alpha \in L$ there exists a nonempty fuzzy regular closed set F_α in X such that $F_\alpha \subset V_\alpha$ and $X = \text{U}\{\text{int}\,(F_\alpha) : \alpha \in L\}$

6.3 Definition. An fts X is said to be *fuzzy weakly-compact* (resp. *fuzzy alomost-compact*) if every fuzzy regular (resp. fuzzy open) cover of X has a finite subfamily whose fuzzy closures cover X. It is clear that every fuzzy almost-compat space is fuzzy weakly-compact.

A fuzzy subset S of the fts X is said to be *fuzzy weakly-compact* if S is fuzzy weakly-compact as a fuzzy subspace of X.

6.4 Definition. A fuzzy subset S of an fts X is said to be *fuzzy weakly-compact relative to X* if for each cover $\{V_\alpha : \alpha \in L\}$ of S by fuzzy open sets of X satisfying the condition (∗) :

(∗) For each $\alpha \in L$, there exists a nonempty fuzzy regular closed set F_α of X such that $F_\alpha \subset V_\alpha$ and $S \subset \text{U}\{\text{int}\,(F_\alpha) : \alpha \in L\}$.

there exists a finite subset L_o of L such that $S \subset \{\text{cl}(V_\alpha) : \alpha \in L_o\}$.

6.5 Definition. An fts X is said to be *fuzzy nearly compact* if every regular fuzzy open cover of X has a finite fuzzy subcover.

Let A be a fuzzy subspace of an fts X and S be any fuzzy subset of A. In this section $\text{cl}_A(S)$ (resp. $\text{int}_A(S)$) denotes the fuzzy closure (resp. fuzzy interior) of S in the subspace A.

6.6 Theorem. *If A is a fuzzy weakly-compact subspace of a space X, then A is fuzzy weakly-compact relative to X.*

Proof. Let $\{U_\alpha:\alpha\in L\}$ be a fuzzy cover of A by fuzzy open subsets of X satisfying condition (*) of Definition 6.4. Then for each $\alpha\in L$ there exists a nonempty fuzzy regular closed sets F_α such that $F_\alpha\subset U_\alpha$ and $A\subset\{\text{int}(U_\alpha):\alpha\in L\}$. For each $\alpha\in L$, $\text{int}(F_\alpha)\cap A$ and $U_\alpha\cap A$ are fuzzy open in A and $(F_\alpha)\cap A$ is fuzzy closed in A. The family $\{U_\alpha\cap A:\alpha\in L\}$ is fuzzy open cover of A.
For each $\alpha\in L$ we have $\text{cl}_A(\text{int}(F_\alpha)\cap A)\subset F_\alpha\cap A\subset U_\alpha\cap A$. Moreover, we have. $A =$ $\text{U}\{\text{int}(F_\alpha)\cap A:\alpha\in L\}$ and $(\text{int}(F_\alpha)\cap A)\subset\text{int}_A(\text{cl}_A(\text{int}(F_\alpha)\cap A))$. Since $\text{cl}_A(\text{int}(F_\alpha)\cap A)$ is fuzzy regular closed in A, $\{U_\alpha\cap A:\alpha\in L\}$ is a fuzzy regular cover of the fuzzy subspace A.
There exists a finite subset L_o of L such that $A= \text{U}\{\text{cl}_A(U_\alpha\cap A):\alpha\in L_o\}$. Since $(\text{cl}_A(U_\alpha\cap A))\subset$ $\text{cl}_A(U_\alpha)$ for each $\alpha\in L_o$, we obtain $A\subset\text{U}\{\text{cl}_A(U_\alpha):\alpha\in L_o\}$. This shows that A is fuzzy weakly-compact relative to X. This completes the proof of Theorem 6.6.

6.7 Theorem. *If every proper fuzzy regular closed subset of an fts X is fuzzy weakly-compact relative to X, then X is fuzzy weakly-compact.*

Proof. Let $\{U_\alpha:\alpha\in L\}$ be a fuzzy regular cover of X. Then for each $\alpha\in L$ there exists a nonempty fuzzy regular closed set F_α in X such that $F_\alpha\subset U_\alpha$ and $X=\{\text{int}(F_\alpha):\alpha\in L\}$.

Choose and fix $\alpha_o\in L$. Let $K=X-\text{int}(F_{\alpha_o})$; then K is fuzzy regular closed in X and K $\subset U\{\text{int}.(F_\alpha):\alpha\in L-\{\alpha_o\}\}$. Therefore, $\{U_\alpha:\alpha\in L-\{\alpha_o\}\}$ is a fuzzy cover of K by fuzzy open sets of X satisying (*) of Definition 3.5.4 and hence for some finite subset L_o of L we have $K\subset U\{\text{cl}_A(U_\alpha):\alpha\in L_o\}$. Thus, we obtain $X = K\ U\text{int}(F_{\alpha_o}) = K U\text{cl}(V_{\alpha_o}) = U\text{cl}\{(V_{\alpha_o}):\alpha\in L_o U\{\alpha_o\}\}$

This shows that X is fuzzy weakly-compact. This completes the proof of Theorem 6.7.

6.8 Corollary. *If every proper fuzzy regular closed subset of a space X is fuzzy weakly-compact, then X is fuzzy weakly-compact.*

Proof. The proof follows from Theorems 6.6 and 6.7.

6.9 Theorem. *Let X be a fuzzy weakly-compact space. If A is a fuzzy clopen subset of X, then A is fuzzy weakly-compact relative to X.*

Proof. Let $\{U_\alpha:\alpha\in L\}$ be a fuzzy cover of A by fuzzy open sets of X satisfying the condition (*) of Definition 6.4. Assume that $(X-A)\neq\emptyset$. Since A is fuzzy clopen in X, $(X-A)$ is also fuzzy clopen in X. Therefore the family $\{U_\alpha:\alpha\in L\}U\{(X-A)\}$ is a fuzzy regular cover of X. Since X is fuzzy weakly-compact there exists a finite subset L_o of L such that $X U\{\text{cl}(U_\alpha):\alpha\in L_o\}\ U\text{cl}(X-A)=U\{\text{cl}(U_\alpha):\alpha\in L_o\}U\text{cl}(X-A)$ Therefore, we obtain $A\subset U\{\text{cl}(U_\alpha):\alpha\in L_o\}$. This completes the proof of Theorem 6.9.

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