HOMOLOGY THEORY FOR *CW*-COMPLEXES

by

Manasi Kumari Sahukar

M.Sc. Student in Mathematics
Roll No. 413MA2072
National Institute of Technology
Rourkela - 769 008 (India)

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CERTIFICATE

This is to certify that this review work entitled "**Homology Theory of $\boldsymbol{CW}$-complexes**" which is being submitted by *Manasi Kumari Sahukar*, a M.Sc. Student in Mathematics, Roll No. 413MA2072, National Institute of Technology, Rourkela - 769008 (India), for the award of the Degree of Master of Science in Mathematics from National Institute of Technology, Rourkela is a record of review work done by her under my advice. The results embodied in the dissertation are known results and the dissertation in the present form has not been submitted to any other University or Institution for the award of any Degree or Diploma.

To the best of my knowledge Ms. Manasi Kumari Sahukar bears a good moral character and is eligible to get the degree.

Professor A. Behera

Adviser
Department of Mathematics
National Institue of Technology
Rourkela (India) -769 008

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Manasi Kumari Sahukar

M.Sc. Student in Mathematics
Roll No. 413MA2072
National Institute of Technology
Rourkela - 769 008 (India)

ABSTRACT

In this thesis we will have a study on homology theory of CW- complexes with an emphasis on finite-dimensional CW-complexes. We will first give a brief introduction on basic definitions and basic preliminaries of topological space and definition of CW-complexes and brief discussion on some important keywords in CW-complexes. Then certain definitions on singular homology theory of CW-complexes will be discussed. Then, we will give a brief discussion on axioms of homology theory for topological spaces and axioms of homology theory for CW-complexes. Finally, we will discuss Whitehead theorem and its proof.

Contents

Chapter 1

INTRODUCTION

For homology theory the most tractable family of topological spaces seems to be the family of CW-complex. A CW-complex is made of basic building blocks called cells.

In this dissertation,we have done a review of the homology theory and CW-complexes. For this study in Chapter 2, we have done the preliminary results of topological spaces, Hausdorff spaces, continuous function and so on.

In chapter 3, we have recalled the definition of CW-complexes. For this study firstly we have gone through quotient space, adjuction spaces, pushout, attaching maps. We have given vivid description of CW-complexes with examples.

In chapter 4, we have studied singular homology theory of topological spaces.For this study first we interact free abelian group and an important concept from linear algebra, namely, affinely independent.This content is required to define standard n-simplex. These seven homology theories are in algebric topology. The most important one is singular homology theory. This homology theory has been applied to CW-complexes.

In chapter 5, we have shown that Singular homology theory of topological spaces for CW-complexes. The main purpose of this theorem is to study Whitehead Theorem, which is the main intention of our work.

In all the result, definition and examples the appropriate reference have been added. In case, In any event, if the appropriate reference is missing, then the author renders her sincere apology for this.

Chapter 2

TOPOLOGICAL PRELIMINARIES

In this chapter we recall the general topology, some definition and results. Some more definitions and results are included in the relevant chapters which serve as the base and background for the subsequent chapters and when required, we shall keep on referring back to it. For further details,refer [4].

2.1. Topological spaces

A topology on a set X is a collection $\Im$ of subsets of X have the following properties.

1. $\emptyset, X \in \mathscr{T}$.
2. The union of elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.
3. The intersection of elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

A set X with a topology $\mathscr{T}$ is called a topological space$(X, \mathscr{T})$. If X is a topological space with topology $\mathscr{T}$, a subset U of X is called an open set of X if $U \in \mathscr{T}$.

Basis

If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X such that

1. For each $x \in X$,
   there is at least one basis element B. containing x.

2. If $x \in B_1 \bigcap B_2$,
   then there exists a basis element B_3 containing x such that $B_3 \subset B_1 \bigcap B_2$.

The topology generated by B is defined as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in B$ such that $x \in B$ and $B \subset U$.

2.2. Types of topologies

There are some other topologies for a set X which are defined in the following.

1. **Discrete topology :** If X be any set, the collection of all subsets of X is called as discete toplogy.

2. **Indiscrete topology :** Let X be any set, the set $\emptyset, X$ is called trivial topology or indiscrete topology .

3. **Standard topology :** The topology generated by $\mathbf{B} = \{(a,b)|a,b \in R, a < b\}$ is called standard topology on the real line.

4. **Product Topology :**Let X be defined as $X := \prod_{i \in I} X_i$, then the Cartesian product of the topological spaces X_i, $i \in I$, and the canonical projections $p_i : X \to X_i$, the product topology on X is defined to be the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous.

5. **Subspace topology :** Let X be a topological space with topology $\mathscr{T}$.If Y is a subset of X, the collection $\mathscr{T}_Y = \{Y \cap U | U \in \mathscr{T}\}$ is a topology, called the Subspace topology and Y is called as a Subspace of X.

2.3. Hausdorff space

Consider,the space $\mathbb{R}$ and $\mathbb{R}^2$, where all one point sets are closed. But if we consider the topology on three point set $\{a,b,c\}$,the point set $\{b\}$ is not closed. Since neighborhood of b intersecting both neighborhood of a and c which are not in b. If we consider $x_n = b$ for all n, converges not only to the point b, but also to the point a and to the point c which misleading the conception that the properties of convergent sequence in $\mathbb{R}$ and $\mathbb{R}^2$. Hence, a new topology arised to overcome the problems which is discussed below.

Definition 2.3.1. A topological space X is called Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, U_2 of x_1, x_2 respectively.

In Hausdorff space X, every finite point set is closed and sequence of points of X converges to at most one point of X.

2.4. Continuous function

Let function is defined between topological spaces X and Y as $f : X \to Y$ and $\mathscr{T}$ and $\mathscr{T}^{'}$ be the topologies on X and Y respectively.Then both are equivalent.
(a) f is called continuous if for every $U \in \mathscr{T}^{'}$, $\exists\ f^{-1}(U) \in \mathscr{T}$
(b) f is continuous at $x \in X$ if for every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V$.

Example 2.4.1. Let X be a non-empty set and let P_1 and P_2 be two partitions on X and let T_1 and T_2 be the two associated partition topologies on X. Let $f : X \to X$ be the identity function $f(x) = x$ whose domain is equipped with T_1 and codomain with T_2. Then f is continuous if and only if every element in P_2 is a union of elements from P_1.

Chapter 3

CW-COMPLEXES

The purpose of this chapter is to introduce the definition of CW-complexes of an arbitrary topological space.

3.1. Quotient space

Let $(X, \mathscr{T})$ be the topological space and $\sim$ be an equivalent relation on X.Then $X/\sim = X^*$ is the set of all equivalent classes in X, such that $X^* = \{[x] | x \in X\}$ and the function $p : X \longrightarrow X^*$ is called natural projection function defined as $p(x) = [x]$, then $\mathscr{T}^* = \{O \subset X^* | p^{-1}(O) \in \mathscr{T}\}$ is called as quotient topology and the mapping p is called as quotient map and X^* is called as quotient space.

Example 3.1.1. (Quotienting out by a subset). Let $(X, \mathscr{T}_X)$ be a topological space and let $A \subset X$ be a subset of X. Let Y be the set $Y = (X - A) \cup \{a\}$ where a is some abstract element not in X. Define the function $p : X \to Y$ by

$$\pi(x) = \begin{cases} x, & x \in X - A; \\ a, & x \in A. \end{cases}$$

and note that it is surjective. The space $(Y, \mathscr{T}_{X/\pi})$ is typically denoted by $(X/A, \mathscr{T}_{X/A})$ and referred to as the quotient of X by A. Note that it is the quotient space X/P_A associated to the partition $P_A = \{A, \{x\} | x \in X - A\}$ of X.

3.2. Adjunction space

Let X,Y be Hausdorff spaces and $A \underset{closed}{\subset} X$. Let $g : A \to Y$. Define an equivalence relation $\sim$ on $X \coprod Y$ as $a \sim g(a)$, $\forall\, a \in A$ and $z \sim z$, for all $z \in ((X - A) \bigcup (Y - g(A))$, then $X \coprod Y / \sim \cong Y \bigcup_g X$.

Example 3.2.1. Let $X = D^1 = \{x \in R : |x| \leq 1\} = [-1, 1]$, $A = \{-1, 1\}$, $Y = \{y_0\}$. Define $g : A \to Y$ by $g(-1) = g(1) = y_0$, then $Y \bigcup_g X \cong S^1$.

3.3. Pushout

A diagram consisting of two morphisms $f : A \to B$ and $s : A \to C$

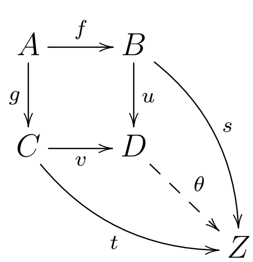
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

with a common domain is said to be a push-out diagram if and only if

1. the diagram can be completely be a commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow[v]{} & D \end{array}$$

2. for any commutative diagram, i.e., uf=vg there exist a unique morphism $\theta : D \to Z$ such that



such that $\theta u = s$ and $\theta v = t$.

Proposition 3.3.1. If

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & Y\bigcup_g X \\ & {\scriptstyle i_X}\searrow \qquad \swarrow{\scriptstyle p} & \\ & X\coprod Y & \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{\quad i \quad} & Y\bigcup_g X \\ & {\scriptstyle i_Y}\searrow \qquad \swarrow{\scriptstyle p} & \\ & X\coprod Y & \end{array}$$

then the following are true.

1. $i(X-A) \underset{open}{\subset} Y\bigcup_g X$.
2. $j(Y) \underset{closed}{\subset} Y\bigcup_g X$.
3. $i|X-A : X-A \xrightarrow[\text{onto its image}]{\text{homeomorphism}} Y\bigcup_g X$.
4. $j : Y \xrightarrow[\text{onto its image}]{\text{homeomorphism}} Y\bigcup_g X$.
5. X and Y are compact $\Rightarrow Y\bigcup_g X$ is compact.

3.4. Attaching maps

Let $X = D^n$ $A = S^{n-1}$ Define $g : S^{n-1} \to Y$, then $Y\bigcup_g D^n$ is said to obtained b attaching n-cells to Y. Then $g : S^{n-1} \to Y$ is called an attaching map and $f : (D^n, S^{n-1}) \to (Y\bigcup e^n_\alpha, Y)$ is called characteristic maps.

Example 3.4.1. Let $X = D^1 = [-1,1]$, $S = S^0 = \{-1,1\}$. Define $g : S^0 \to Y$ by $g(-1) = y_0, g(1) = y_1,\ \ y_0 \neq y_1$.

3.5. CW-complexes

A CW-complex X consists of

1. X is a Hausdorff topological space.

2. X has the structure of a cell complex.

   (a) A cell complex on X is a collection $\{e^n_\alpha : \alpha \in J_n,\ J_n$ is an indexing set of an non-negative integers $\}$ of subsets of X.

   (b) $\{e^0_\alpha,\ \alpha \in J_0$, an indexing set of non-negative integers$\}$, are called 0-cells.
   $\{e^1_\beta, \beta \in J_1$, an indexing set of non-negative integers $\}$, are called 1-cells.

$$\vdots$$

   $\{e^n_\delta, \delta \in J_n$, an indexing set of non-negative integers$\}$, are called n-cells.

   (c) X^0 is called as 0-skeleton of X,defined as the collection of all 0-cells i.e., $X^0 = \{e^0_\alpha : \alpha \in J_0, \text{an indexing set}\}$ X^1 is called as 1-skeleton of X,defined as the collection of all 0-cells and 1-cells i.e., $X^1 = X^0 \bigcup \{e^1_\beta : \beta \in J_1, \text{an indexing set of non-negative integers}\} \vdots$ X^n is called as n-skeleton of X defined as the collection of all 0-cells and 1-cells and $\cdots$ n-cells i.e., $X^n = X^0 \bigcup X^1 \bigcup \cdots \bigcup X^{n-1} \bigcup \{e^n_\delta : \delta \in J_n, \text{an indexing set of non-negative integers}\}$

   (d)

$$|X^0| = \bigcup_{\alpha \in J_0} e^0_\alpha \underset{subsapces}{\subset} X$$

$$|X^1| = \bigcup_{\alpha \in J_0} e^0_\alpha \bigcup \bigcup_{\beta \in J_1} e^1_\beta \underset{subsapces}{\subset} X$$

$$\vdots$$

$$|X^n| = \bigcup_{\alpha \in J_0} e^0_\alpha \bigcup \bigcup_{\beta \in J_1} e^1_\beta \cdots \bigcup \bigcup_{\delta \in J_n}$$

$$e^n_\delta \underset{subsapces}{\subset} X$$

$$\bigcup_{\substack{\alpha \in J_r \\ 0 \le r < \infty}} e^r_\alpha \underset{subsapces}{\subset} X$$

(e)

$$\bigcup_{\substack{\alpha\in J_r\\ 0\leq r<\infty}} e^r_\alpha = X$$

$$\dot{e}^r_\alpha = e^r_\alpha \bigcap |X^{n-1}| = \text{ boundary of} \dot{e}^n_\alpha$$

$$\mathring{e}^n_\alpha = \bar{e}^n_\alpha - \dot{e}^n_\alpha$$

$$\mathring{e}^n_\alpha \bigcap \mathring{e}^m_\beta \neq \emptyset \Rightarrow n = m, \alpha = \beta$$

$$X = \bigcup_{\substack{\alpha\in J_r\\ 0\leq r<\infty}} \mathring{e}^r_\alpha$$

The map $f : (D^n, S^{n-1}) \to (e^n, \mathring{e}^m)$ is surjective and maps $D^n - S^{n-1} = \mathring{D}^n$ homeomorphically into $e^n - \mathring{e}^n = \dot{e}^n$
The cells e^n is compact and hence closed in X.
$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots \subset X$
X^n is discrete space.
X^1 is obtained from X^0 by attaching 1-cells by the characteristic map $f : (D^1, S^0) \to (X^0, \emptyset)$, X^2 is obtained from X^1 by attaching 2-cells by the characteristic map $f : (D^2, S^1) \to (X^1, X^0)$ $\cdots$ X^n is obtained from X^{n-1} by attaching n-cells by the characteristic map $f : (D^n, S^{n-1}) \to (X^{n-1}, X^{n-2})$

3. **Closure Finite Property :** For each cell e^n_α,its closure $\bar{e^n_\alpha}$ intersects only a finite number of cells.

4. **Weak Topology:** A set B is open in X iff $B \bigcap e^n_\alpha$ is open in e^n_α for each n,α.

Chapter 4

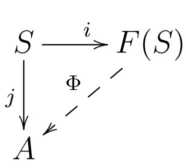
SINGULAR HOMOLOGY THEORY OF TOPOLOGICAL SPACES

The purpose of this chapter is to introduce the singular homology theory of an arbitrary topological space. The essential computational tool is stated by following the definitions and proof of homotopy invariance. The results discussed in this chapter are applied to prove number of classical theorem : Whitehead theorem. For further details, refer to [5] and [6].

4.1. Free abelian group

Let S be a non-empty set. Free abelian group generated by S is an abelian group $F(S)$ satisfying following properties.

- There exists a function $i : S \to F(S)$
- For any abelian group A and a function $j : S \to A$. Then there exists a unique homomorphism $\varphi : F(S) \to A$ such that $j = \varphi i$ i.e., the following diagram commutes.

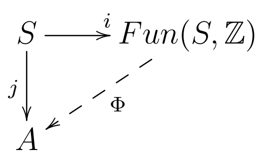


This is called the universal property of F(S). The free abelian group is written as $(F(S), i)$ or simply $F(S)$.

Proof. Let $fun(S, Z) = \{f : S \to Z : f$ takes non-zero values only a finite subset of $S\ \}$ and $f, g \in fun(S, Z)$ such that $(f + g)(s) = f(s) + g(s), s \in S$ $(-f)(s) = -f(s)$ $0(s) = 0$ for $\ $ all $s \in S$. Then fun(S, Z) is an abelian group. Define a function $s : S \to Z$ by the following.

$$s(x) = \delta_{sx} = \begin{cases} 1, & x = s; \\ 0, & \text{otherwise.} \end{cases}$$

Let $f \in fun(S, Z)$ be arbitrary. Let $f(s_1) = n_1$,$f(s_2) = n_2$,$\cdots$,$f(s_k) = n_k$, where $s_1, s_2, \ldots, s_k \in S$. Clearly $f = n_1s_1 + n_2s_2 + \cdots + n_ks_k$ Define a function $i : S \to fun(S, Z)$ by $i(s) = s,$ for all$s \in S$. Let A be any abelian group and $j : S \to A$ be any function.
Define a function $\varphi : fun(S, Z) \to A$ by $\varphi(f) = n_1j(s_1) + n_2j(s_2) + \cdots + n_kj(s_k)$ Thus the diagram



commutes and φ is unique.

4.2. Affinely independent

A subset $S \subset \mathbb{R}^n$ is called affinely independent if and only if for every finite subset $s_0, s_1, \ldots, s_k \subset S$, the objects $s_1 - s_0, \ldots, s_k - s_0$ are linearly independent.
Proposition 4.2.1. Let $S \subset \mathbb{R}^n$.the following are equivalent.

1. S is affinely independent
2. For every finite subset $s_0, s_1, \ldots, s_k \subset S$, $\sum_{i=0}^{k} t_is_i = 0$ such that $\sum_{i=0}^{k} t_i = 0$, that implies $t_i = 0$ for each i

Proof. (1) ⇒ (2) Let $s_0, s_1, \ldots, s_k \subset S$, then by the definition of affinely independent, $\sum_{i=0}^{k} t_i s_i = 0, \sum_{i=0}^{k} t_i = 0$

$$0 = \sum_{i=0}^{k} t_i s_i = \sum_{i=0}^{k} t_i s_i - (\sum_{i=0}^{k} t_i) s_0 = \sum_{i=1}^{k} (s_i - s_0) t_i$$

Now since $s_1 - s_0, \ldots, s_k - s_0$ are L.I, we have $s_i = 0, i = 0, \ldots, k$. Hence $s_0 = 0$
(2) ⇒ (1) Let $s_0, s_1, \ldots, s_k \subset S$, then $\sum_{i=0}^{k} c_i(s_i - s_0) = 0$.

$$0 = \sum_{i=0}^{k} c_i(s_i - s_0) = \sum_{i=0}^{k} c_i s_i + (-\sum_{i=1}^{k} c_i) s_0$$

Let $t_0 = -\sum_{i=1}^{k} c_i t_i = c_i, i = 1, \ldots, k$. Hence $\sum_{i=0}^{k} t_i s_i = 0$, $\sum_{i=0}^{k} t_i = 0$. Thus $t_i = 0$ for each i and $c_i = 0$ for each i □

4.3. Standard *n*-simplex

Let $\mathbb{R}^\infty = \{x = (x_i)_{i=0}^{\infty} : x_i \in \mathbb{R}$, with only a finite number of non-zero entries $\}$ i.e., $e_n = \{0, 0, \ldots, 1, 0, \ldots\}$, $e_0 = \{1, 0, \ldots\}$, $e_1 = \{0, 1, 0 \ldots\}$ and so on. Then the convex set generated by $\{e_0, e_1, \ldots, e_n\}$ is called as standard n-simplex and denoted by Δ_n

i.e., $\Delta_0 = e_0$ Let Δ_1 be the convex set generated by $\{e_0, e_1\} = \{t_0(1, 0, \ldots) + t_1(0, 1, 0, \ldots)\}$ for each $t_0, t_1 \in I$ such that $t_0 + t_1 = 1$. Thus $\Delta_1 = \{(t_0, t_1, 0, \cdots) : t_0, t_1 \in I, t_0 + t_1 = 1\}$

Properties of Δ_n

- Δ_n is path connected(hence connected).
- Δ_n is compact.
- the set of vertices $e_0, e_1, \ldots, e_n$ is affinely independent.

4.4. Face maps

For $0 \leq i \leq n$, define $\partial_n^i : \Delta_{n-1} \to \Delta_n$ by

$$\partial_n^i(e_k) = \begin{cases} e_k, & k < i; \\ e_k + 1, & k \geq i. \end{cases}$$

Since Δ_{n-1} is the convex set generated by $e_0, e_1, \ldots, e_{n-1}$, each $x \in \Delta_{n-1}$ can be written as $x = \sum_{k=0}^{n-1} t_k e_k$, $\partial_n^i(x) = \sum_{k=0}^{n-1} \partial_n^i(e_k)$ For $n = 1$ $\partial_1^i : \Delta_0 \to \Delta_1, i = 0, 1,$ $\partial_1^0, \partial_1^1 : \Delta_0 \to \Delta_1$, $\partial_1^0(e_0) = e_1 \partial_1^1(e_1) = e_0$. For $n = 2$ $\partial_2^i : \Delta_1 \to \Delta_2, i = 0, 1, 2,$ $\partial_2^0, \partial_2^1, \partial_2^2 : \Delta_1 \to \Delta_2$, $\partial_2^0(e_0) = e_1, \partial_2^1(e_1) = e_2$, $\partial_2^1(e_0) = e_0, \partial_2^1(e_1) = e_2$, $\partial_2^2(e_0) = e_0, \partial_2^2(e_1) = e_1$.

4.5. Singular *n*-simplex

Let X be a topological space, then the map $\sigma_n : \Delta_n \to X$ is called a singular n-simplex of X and $S_n(X)$ is called as free abelian group generated by singular n-simplices σ_n and the element of $S_n(X)$ is called an n-chain of X.

For $n \geq 0$ and for an n-chain $c \in S_n(X)$, let

$$c = n_1\sigma_1 + n_2\sigma_2 + \ldots + n_k\sigma_k$$

Then for $i = 0, 1, \ldots, n$, $\sigma \circ \partial_n^i$ is a singular *(n-1)*-simplex.

Define

$$d_n : S_n(X) \longrightarrow S_{n-1}(X)$$

such that

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^n \sigma \circ \partial_n^i.$$

Proposition 4.5.1. For a singular n-simplex σ in X, $d^2 = 0$.

Proposition 4.5.2. Let

$$S_n(X) \xrightarrow{d_n} S_{n-1}(X) \xrightarrow{d_{n-1}} S_{n-2}(X)$$

We prove that $d_{n-1}d_n = 0$.

Proof.

$$\begin{aligned}
d_{n-1}d_n(\sigma) &= d_{n-1}\sum_{j=0}^{n}(-1)^j\sigma\circ\partial_n^j \\
&= \sum_{j=0}^{n}(-1)^j d_{n-1}(\sigma\circ\partial_n^j) \\
&= \sum_{j=0}^{n}(-1)^j\sum_{i=0}^{n-1}(-1)^i\sigma \circ \partial_n^j \circ \partial_{n-1}^i \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i \\
&= \sum_{0\le i<j\le n}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i + \sum_{0\le j\le i\le n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i \\
&= \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i + \sum_{0\le j\le i\le n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i
\end{aligned}$$

Let $i = j'$ and $j - 1 = i'$

$$\begin{aligned}
d_{n-1}d_n(\sigma) = &\sum_{0\le j'\le i'\le n-1}(-1)^{i'+j'+1}\sigma \circ \partial_n^i \circ \partial_{n-1}^{j-1} + \sum_{0\le j\le i\le n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i \\
&+ \sum_{0\le j'\le i'\le n-1}(-1)^{i'+j'+1}\sigma \circ \partial_n^{j'} \circ \partial_{n-1}^{i'} + \sum_{0\le j\le i\le n-1}(-1)^{i+j}\sigma \circ \partial_n^j \circ \partial_{n-1}^i \\
&= 0
\end{aligned}$$

□

4.6. Singular homology

The chain complex is defined as

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{d_{n+1}} S_n(X) \xrightarrow{d_n} S_{n-1}(X) \xrightarrow{d_{n-1}} \cdots \longrightarrow S_1(X) \xrightarrow{d_1} S_0(X) \longrightarrow 0$$

Definition 4.6.1. Group of n-cycles is defined as

$$\begin{aligned} Z_n(X) &= ker(d_n) \\ &= \{\sigma \in S_n(X) : d_n(\sigma) = 0\} \end{aligned}$$

Definition 4.6.2. Group of n-boundaries is defined as

$$\begin{aligned} B_n(X) &= Im(d_{n+1}) \\ &= \{d_{n+1}(\sigma) : \sigma \in S_{n+1}(X)\} \end{aligned}$$

Proposition 4.6.3.

$$B_m(X) \subset Z_n(X)$$

Proof. Since

$$\begin{aligned} &d_n \circ d_{n+1}(\sigma) = 0 \\ &\Rightarrow B_m(X) \subset Z_n(X) \end{aligned}$$

□

Let $f : X \to Y$, then there exists an induced homomorphism $f_\sharp : S_*(X) \to S_*(Y)$ such that $f_\sharp(\sigma) = f \circ \sigma : \Delta_n \longrightarrow X \longrightarrow Y$.

Proposition 4.6.4.

1. If $I_X : X \to X$,then there exists a induced homomorphism $I_{X^\sharp} : S_n(X) \to S_n(X)$ called as identity homomorphism.
2. If $f : X \to Y$ and $g : Y \to Z$,then there exists $f_\sharp : S_n(X) \to S_n(Y)$ and $g_\sharp : S_n(Y) \to S_n(Z)$,such that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$

Proof. (1) $I_{X^\sharp}(\sigma) = I_X \circ \sigma = \sigma$. Since σ is arbitrary,$I_{X^\sharp}$ is a identity homomorphism.
(2)

$$\begin{aligned}(g \circ f)_\sharp(\sigma) &= [gf\sigma] \\ &= g_\sharp(f\sigma) \\ &= g_\sharp \circ f_\sharp(\sigma)\end{aligned}$$

Since σ is arbitrary,this implies $(g \circ f)_\sharp = (g)_\sharp \circ (f)_\sharp$ □

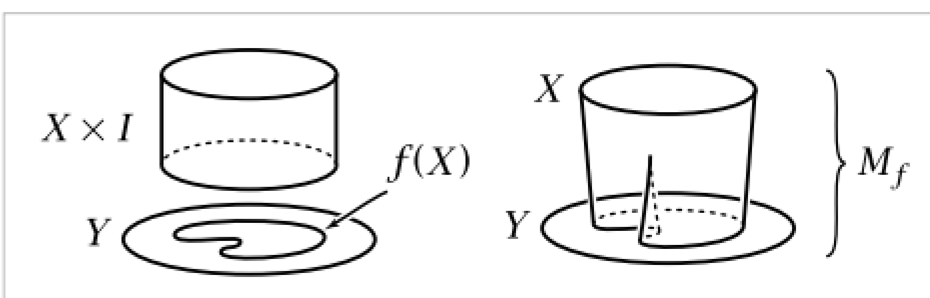
4.7. Mapping cylinder

Let $f : X \to Y$ be a map of spaces. Then the mapping cylinder M_f is obtained by gluing a cylinder $X * I$ on Y by identifying points $(x, 1)$ equivalent to $f(x)$and is defined by the following pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X * I & \xrightarrow[k]{} & M_f \end{array}$$

i.e

$$M_f = \frac{(X * I) \coprod Y}{(x, 1) \sim f(x)}$$



Chapter 5

SINGULAR HOMOLOGY THEORY FOR CW-COMPLEXES

5.1. Homology theory for topological space

By a homology theory $\mathscr{H}$ on $\mathscr{C}$, H be a function assign to each topological space (X, A) in a catagory $\mathscr{C}$.For each integer q, an abelian group there exists a q-dimensional homology group $H_q(X, A)$ of topological pair (X, A). $*$ is assigned to each map $f : (X, A) \to (Y, B)$ in $\mathcal{C}$ as $f_* : H_q(X, A) \to H_q(Y, B)$ called as the homomorphism induced by the map f in the homology theory $\mathscr{H}$.
Let

$$\partial = \partial(X, A, q) : H_q(X, A) \to H_{q-1}(A)$$

be the boundary operator on the group $H_q(X, A)$ in $\mathcal{H}$
(a) **Axiom-1: Commutativity axiom**
If $f : (X, A) \longrightarrow (Y, B)$ and $g : A \longrightarrow B$ such that $f(x) = g(x)$ $\forall x \in A$, then $\partial o f_* = g_* \circ \partial$ i.e.

$$\begin{array}{ccc} H_q(X, A) & \xrightarrow{f_*} & H_q(Y, B) \\ \partial \downarrow & & \downarrow \partial \\ H_{q-1}(A) & \xrightarrow[g_* = (f|A)_*]{} & H_{q-1}(B) \end{array}$$

(b)**Axiom-2: Homotopy axiom**
If $f, g : (X, A) \longrightarrow (Y, B)$ such that $f \simeq g$, then there exist induced homomorphisms $f_*, g_* : H_q(X, A) \longrightarrow H_q(Y, B)$ such that $f_* = g_*$.

Proof. To show $f_* = g_*$, it is sufficient to show that : $f_\sharp, g_\sharp : S(X) \to S(Y)$ are chain homotopic i.e. there exists $T_1 : S(X) \to S(Y)$ such that $\partial T_1 + T_1 \partial = f_\sharp - g_\sharp$
$f \simeq g$ implies that there exists a homotopy

$$F : (X * I, A * I) \to (Y, B)$$

such that

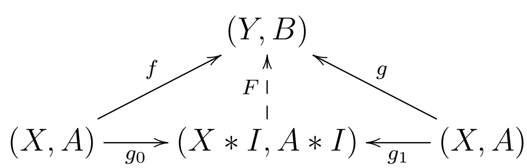
$$F(x, 0) = f(x), F(x, 1) = g(x)\ \forall x \in X$$

Define

$$g_0, g_1 : (X, A) \to (X * I, A * I)$$

by

$$g_0(x) = (x, 0), g_1(x) = (x, 1) \forall x \in X$$



such that

$$f = F \circ g_0$$

$$g = F \circ g_1$$

Let

$$g_{0\sharp}, g_{1\sharp} : S(X) \to S(X * I)$$

such that there exists a homomorphism

$$T : S(X) \to S(X * I)$$

which satisfy the following.

$$\partial T + T\partial = g_{0\sharp} - g_{1\sharp}$$

That implies

$$\Rightarrow F_\sharp(\partial T + T\partial) = F_\sharp(g_{0\sharp} - g_{1\sharp})$$
$$\Rightarrow F_\sharp(\partial T) + F_\sharp(T\partial) = F_\sharp(g_{0\sharp}) - F_\sharp g_{1\sharp}$$
$$\Rightarrow \partial(F_{\sharp)T} + (F_\sharp T)\partial = f_\sharp - g_\sharp.$$

Then

$$F_\sharp T : S(X) \to S(Y)$$

is chain homotopy between $f_\sharp$ and $g_\sharp$.
Let

$$\tau_n \in S_n(A_n)$$

For any

$$\sigma_n : \Delta_n \to X$$

$$\sigma_\sharp : S_n(\Delta_n) \to S_n(X)$$

such that

$$\sigma_\sharp(\tau_n) = \sigma.$$

Define

$$T : S_i(X) \to S_{i+1}(X * I)$$

for all X, $n > 0$ and $i < n$ such that

$$\partial T + T\partial = g_{0\sharp} - g_{1\sharp}.$$

Assume that for any

$$h : X \to W$$

$$\begin{array}{ccc} S_i(X) & \xrightarrow{T_X} & S_i(X * I) \\ {\scriptstyle h_\sharp}\downarrow & & \downarrow{\scriptstyle (h*I)_\sharp} \\ S_i(W) & \xrightarrow[T_W]{} & S_{i+1}(W * I) \end{array}$$

commutes for all $i < n$

$$T_X(\sigma) = T_X(\sigma_\sharp(\tau_n)) = (\sigma * I)_\sharp(T_{\Delta_n}(\tau_n))$$

So to define T_X it is sufficient to define T_{Δ_n} on $S_n(\Delta_n)$.
Let d be the singular n-simplex on Δ_n.
Let

$$c = g_{0\sharp}(d) - g_{1\sharp}(d) - T_{\Delta_n}(\partial d).$$

Then

$$\begin{aligned}
\partial c &= \partial g_{0\sharp}(d) - \partial g_{1\sharp}(d) - \partial T_{\Delta_n}(\partial d) \\
&= g_{0\sharp}(\partial d) - g_{1\sharp}(\partial d) - (g_{0\sharp}(\partial d) - g_{1\sharp}(\partial d) - T_{\Delta_n}\partial(\partial d)) \\
&= 0
\end{aligned}$$

Thus c is a cycle of dimension n in the convex set $\sigma_n * I$. Hence c is the boundary. Let $b \in S_{n+1}(\Delta_n * I)$ with

$$\partial b = c$$

Define

$$T_{\Delta_n}(d) = b$$

$$\partial T(d) + T\partial = g_{0\sharp}(d) - g_{1\sharp}(d)$$

By definition for T_X on n-chains of X

$$\partial T_X + T_X \partial = g_{0\sharp} - g_{1\sharp}$$

$$g_{0\sharp}(\sigma) = g_{0\sharp}\sigma_{\sharp}(\tau_n) = (\sigma * I)_{\sharp} g_{0\sharp}(\tau_n)$$

and similarly

$$g_{1\sharp}(\sigma) = g_{1\sharp}\sigma_{\sharp}(\tau_n) = (\sigma * I)_{\sharp} g_{1\sharp}(\tau_n)$$

now

$$\begin{aligned}
\partial T(d) + T\partial(d) &= \partial T\sigma_{\sharp}(\tau_n) + T\partial\sigma_{\sharp}(\tau_n) \\
&= \partial(\sigma * I)_{\sharp} T(\tau_n) + \partial(\sigma)_{\sharp} T(\tau_n) \\
&= (\sigma * I)_{\sharp}\partial T(\tau_n) + (\sigma * I)_{\sharp} T\partial(\tau_n) \\
&= (\sigma * I)_{\sharp}(g_{0\sharp}(\tau_n) - g_{1\sharp}(\tau_n)) \\
&= g_{0\sharp}(\sigma) - g_{1\sharp}(\sigma)
\end{aligned}$$

□

(c) **Axiom-3: Composition axiom**

If

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad g \circ f : X \to Z$$

,then

$$H_q(X,A) \xrightarrow{f_*} H_q(Y,B) \xrightarrow{g_*} H_q(Z,C), \quad g_* \circ f_* : H_q(X,A) \to H_q(Z,C)$$

Proof. Let $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(Z,C)$, then for any $[z]\in H_q(X,A)$,

$$\begin{aligned}(g\circ f)_*[z] =& [(g\circ f)_\sharp(z)]\\ &= [g_\sharp f_\sharp(z)]\\ &= g_*[f_\sharp(z)]\\ &= g_* f_*[z]\end{aligned}$$

Since $[z]$ is arbitrary, $(fg)_* = f_* g_*$ for all $[z]\in h_n(X,A)$. □

(d)**Axiom-4: Excision axiom**
U be a open set of a topological space X such that $U\subset \overline{U}\subset A^o\subset A\subset X$ and $e:(X\backslash U, A\backslash U)\hookrightarrow (X,A)$, then $e_{*q}: H_q(X\backslash U, A\backslash U)\cong H_q(X,A)$(isomorphic), where e is called as excision of U and e_{*q} is q -dimensional excision isomorphism.

Proof. Refer to Theorem 2.20 in [1] □

(e) **Axiom-5: Exactness axiom**
If $i:A\hookrightarrow X$ and $j:X\hookrightarrow(X,A)$, then

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \cdots$$

is exact.i.e.

1. $im(i_*)\subset ker(j_*)$
2. $im(j_*)\subset ker(\partial)$
3. $im(\partial)\subset ker(i_*)$
4. $ker(j_*)\subset im(i_*)$
5. $ker(\partial)\subset im(j_*)$
6. $ker(i_*)\subset im(\partial)$

Proof. **of (1)** Let

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \cdots$$

Let α be any element of $H_n(X)$ in the image $Im(i_*)$ of the induced homomorphism i_*. Then, by definition of $Im(i_*)$, there exists an element $\beta \in H_n(A)$ with

$$i_*(\beta) = \alpha$$

Consider a singular cycle

$$z \in \beta \subset C_n(A)$$

By the definition of i_*,

$$[C_n(i)](z) \in \alpha \subset C_n(X)$$

. Then by definition of j_*, we have

$$C_n(j)[C_n(i)(z)] \in j_*(\alpha) \subset C_n(X,A)$$

Now,since

$$C_n(i) : C_n(A) \hookrightarrow C_n(X)$$

is obviously the inclusion homomorphism and

$$C_n(j) : C_n(X) \hookrightarrow C_n(X,A)$$

is obviously the natural projection, it follows that

$$C_n(j)[C_n(i)(z)] = 0 \in C_n(X,A)$$

This implies

$$j_*(\alpha) = 0 \in H_n(X,A)$$

This implies

$$\alpha \in ker(j_*)$$

Since α is arbitrary element of $Im(i_*)$,this proves (1) □

Proof. **of (2)** Since

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \cdots$$

Let α be any element of $H_n(X,A)$ in the image $Im(i_*)$ of the induced homomorphism i_*. Then by definition of $Im(j_*)$, there exists an element $\beta \in H_n(X)$ with

$$j_*(\beta) = \alpha$$

Consider a singular cycle

$$z \in \beta \subset C_n(X)$$

By the definition of j_*,

$$[C_n(j)(z)] \in \alpha \subset C_n(X,A)$$

Then by definition of j_*,we have

$$C_n(j)[C_n(i)(z)] \in j_*(\alpha) \subset C_n(X,A)$$

Since $C_n(j)$ is the natural projection of $C_n(X)$ onto $C_n(X,A)$,it follows from the definition of boundary operator

$$\partial : H_n(X,A) \longrightarrow H_n(A)$$

that we have

$$\partial(z) \in \partial(\alpha) \subset C_{n-1}(A)$$

Since $z \in Z_n(X)$,we have $\partial(z) = 0$.This implies

$$\partial(\alpha) = 0 \in H_{n-1}(A)$$

Hence

$$\alpha \in ker(\partial)$$

Since α is arbitrary element of $Im(j_*)$,this proves (2) □

Proof. **of (3)** Since

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \cdots$$

Let α be any element of $H_{n-1}(A)$ in the image $Im(\partial)$ of the boundary operator ∂. Then by definition of $Im(\partial)$, there exists an element $\beta \in H_n(X, A)$ with

$$\partial(\beta) = \alpha$$

Consider a singular cycle

$$z \in \beta \subset C_n(X, A)$$

Now,since

$$C_n(j) : C_n(X) \to C_n(X, A)$$

is an epimorphism, there exists $u \in C_n(X)$ such that

$$C_n(j)(u) = z$$

By the definition of the boundary operator ∂,

$$\partial(u) \in \alpha \subset C_{n-1}(A)$$

From the definition of i_*,

$$\partial(u) \in i_*(\alpha) \subset C_{n-1}(X)$$

Since $\partial(u) \in B_{n-1}(X)$,this implies

$$i_*(\alpha) = 0 \in H_{n-1}(X)$$

Hence $\alpha \in ker(i_*)$ Since α is arbitrary element of $Im(\partial)$,this proves (3) □

Proof. **of (4)** Since

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} \cdots$$

Let α be any element of $H_n(X)$ in the ker $Ker(j_*)$ of the induced homomorphism j_*.Consider a singular cycle

$$z \in \alpha \subset C_n(X)$$

Since $j_*(\alpha) = 0$, we have

$$C_n(j)(z) \in B_n(X, A)$$

Hence there exists $y \in C_{n+1}(X, A)$ such that

$$\partial_{n+1}(y) = C_n(A)(z)$$

Since

$$C_{n+1}(j) : C_{n+1}(X) \to C_{n+1}(X, A)$$

is an epimorphism,there exists $x \in C_{n+1}(X)$ such that

$$C_{n+1}(j)(x) = y$$

Then we have

$$C_n[z - \partial(x)] = C_n(z) - C_n[\partial(x)] \quad = C_n(z) - \partial[C_{n+1}(x)] = C_n(z) - \partial(y) \quad = 0$$

This implies

$$z - \partial(x) \in C_n(A)$$

Since $\partial[z - \partial(x)] = \partial(z) - \partial^2(x) = 0$,we have

$$z - \partial(x) \in Z_n(A)$$

Let $\beta \in H_n(A)$ which contains the singular cycle $z - \partial(x)$. Since $z \in \alpha$ and $\partial(x) \in B_n(X)$,

$$z - \partial(x) \in \alpha \subset C_n(X, G)$$

.This implies

$$i_*(\beta) = \alpha$$

Hence

$$\alpha \in im(i_*)$$

Since α is arbitrary element of $Ker(j_*)$,this proves (4) □

Proof. **of (5)** Since

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} \cdots$$

Let α be any element of $H_n(X, A)$ in the ker $Ker(\partial)$ of the boundary operator

∂.Consider a singular cycle

$$z \in \alpha \subset C_n(X, A)$$

Since,

$$C_n(j) : C_n(X) \to C_n(X, A)$$

is an epimorphism,there exists $u \in C_n(X)$ such that

$$C_n(j)(u) = z$$

By definition of boundary operator ∂,

$$\partial_n \in \partial_n(\alpha) \subset C_{n-1}(A)$$

Since

$$\partial_n(\alpha) = 0 \in H_{n-1}(A)$$

there exists $v \in C_n(A)$ such that

$$\partial_n(u) = \partial_n(v)$$

Let $y = u - v \in C_n(X)$,we have

$$\partial_n(y) = \partial_n(u) = \partial_n(v) = 0$$

This implies

$$y \in Z_n(X)$$

Let $\beta \in H_n(X)$ which contains the singular cycle y. Since $v \in C_n(A)$,we have

$$C_n(j)(y) = C_n(j)(u) - C_n(j)(v) = C_n(j)(u) = z$$

This implies

$$j_*(\beta) = \alpha$$

Hence

$$\alpha \in im(j_*)$$

Since α is arbitrary element of $Ker(\partial)$,this proves (5) □

Proof. **of (6)** Since

$$\cdots \longrightarrow H_{q+1}(A) \xrightarrow{i_*} H_{q+1}(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \cdots$$

Let $\alpha \in ker(i_*)$.Choose a singular cycle

$$z \in \alpha \subset C_{n-1}(A)$$

Since $i_*(\alpha) = 0$ there exists $u \in C_n(X)$ such that

$$\partial(u) = z$$

Let

$$y = C_n(j)(u) \in C_n(X,A)$$

then

$$\partial(y) = \partial[C_n(j)(u)] \qquad = C_{n+1}[\partial(u)] = C_{n+1}(j)(z) \qquad = 0$$

This implies

$$y \in Z_n(X,A)$$

Let $\beta \in H_n(X,A)$ which contains the singular cycle y.Since

$$C_n(j)(u) = y$$

it follows from the definition of $\partial(\beta)$ we have

$$z = \partial(u) \in \partial(\beta) \subset C_{n-1}(A)$$

This implies

$$\partial(\beta) = \alpha$$

Hence

$$\alpha \in im(\partial)$$

Since α is arbitrary element of $Ker(\partial)$,this proves (5)
For further details,refer theorem 7.1 of [2] □

(f) **Axiom-6: Dimension axiom**

If $H_q(A)$ be a q dimensional homology group of a singleton space$A$,then $H_q(A) = 0$ $\forall\ q \neq 0$.

Proof. Let the chain homotopy be

$$\cdots \longrightarrow H_{n+1}(\{*\}) \xrightarrow{d_{n+1}} H_n(\{*\}) \xrightarrow{d_n} \cdots \longrightarrow H_2(\{*\}) \xrightarrow{d_2} H_1(\{*\}) \xrightarrow{d_1} H_0(\{*\})$$

and

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

Let

$$\sigma_n : \Delta_n \to \{*\}$$

such that

$$\begin{aligned} d_n(\sigma_n) &= \sum_{i=0}^{n} (-1)^i \sigma_n \partial_n^i \\ &= \begin{cases} 0, & \text{n odd;} \\ \sigma_{n-1}, & \text{n even.} \end{cases} \end{aligned}$$

This implies

$$\begin{aligned} H_n(\{*\}) &= Ker(d_n)/Im(d_{n+1}) \\ &= \begin{cases} 0, & \text{n even;} \\ C_n/C_n = 0, & \text{n odd.} \end{cases} \end{aligned}$$

Hence $H_n(\{*\}) = 0$ □

5.2. Homology theory for CW-complexes

For each non-empty CW pair (X, A), there exists a sequence of abelian group $h_n(X, A)$. If $f : (X, A) \to (Y, B)$, then $f_* : h_n(X, A) \to h_n(Y, B)$ is called as a sequence of induced homomorphism and the function defined on $h_n(X, A), \partial(n, X, A) : h_n(X, A) \to h_{n-1}(A)$ is called as boundary operator and any CW pair (X, A) the following axioms are satisfied.

(a) **Axiom-1: Identity axiom**

If $Id : (X, A) \to (X, A)$, then there exists a induced homomorphism $Id_* : h_n(X, A) \to h_n(X, A)$ such that $Id_* = Id$.

Proof. Suppose $[z] \in h_n(X, A)$.
Then $Id_*[z] = [Id_\sharp(z)] = [z]$ Since $[z]$ is arbitrary, $Id_* = Id$ for all elements in $h_n(X, A)$. □

(b)**Axiom-2: Composition axiom**
If

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \quad g \circ f : X \to Z$$

,then

$$h_q(X, A) \xrightarrow{f_*} h_q(Y, B) \xrightarrow{g_*} h_q(Z, C), \quad g_* \circ f_* : h_q(X, A) \to h_q(Z, C)$$

Proof. Refer to Axiom-4 of 5.1. □

(c) **Axiom-3: Homotopy axiom**
If $f \simeq g : (X, A) \longrightarrow (Y, B)$,then $f_* = g_* : h_q(X, A) \longrightarrow h_q(Y, B)$

Proof. Refer to axiom-2 of 5.1. □

(d) **Axiom-4: Commutativity axiom**
If $f : (X, A) \longrightarrow (Y, B)$ and $g : A \longrightarrow B$ are such that $f(x) = g(x)\ \forall x \in A$, then $\partial o f_* = g_* o \partial$ i.e.

$$\begin{array}{ccc} h_q(X, A) & \xrightarrow{f_*} & h_q(Y, B) \\ \downarrow \partial & & \downarrow \partial \\ h_{q-1}(A) & \xrightarrow[g_* = (f|A)_*]{} & h_{q-1}(B) \end{array}$$

(e) **Axiom-5: Exactness axiom**
If $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$, then

$$\cdots \longrightarrow h_{q+1}(A) \xrightarrow{i_*} h_{q+1}(X) \xrightarrow{j_*} h_q(X, A) \xrightarrow{\partial} \cdots$$

is exact.

Proof. Refer to Axiom-4 of 5.1. □

(f) **axiom-6 :**
For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions

$$i_\alpha : X_\alpha \hookrightarrow X,$$

the direct sum map $\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$ is an isomorphism for each n.
(g). **Axiom-7: Dimension axiom**
For a single point space $\{*\}$,the n dimensional homology group $h_n(\{*\}) = 0$ for $n \neq 0$.

Proof. Refer to Axiom-5 of 5.1. □

Definition 5.2.1. A pair (X, A) is **0-connected** if every path component of X meets A i.e. path connected.

Definition 5.2.2. A pair (X, A) is called as n**-connected** iff

1. (X, A) is 0-conneted.

2. $\pi_r(X, A, a) = 0$ for all $1 \leq r \leq n$ for all $a \in A$

Proposition 5.2.3. For any pair (X, A) is n-connected, $n \geq 0$ iff there exists a function $i_* : \pi_r(A, x_0) \longrightarrow \pi_r(X, x_0)$

1. bijective for $r < n$

2. surjective for $r = n$ for all $x_0 \in A$

Definition 5.2.3 . f is called as n**-equivalence** if and only if (M_f, X) is n-connected.

5.3. Whitehead Theorem

Theorem 5.3.1. *If (X, A) is an (n-1)-connected pair, for $n \geq 2$ and A is 1-connected, then*

$$h : \pi_q(X, A, x_0) \to H_q(X, A, z)$$

is an isomorphism for $q \leq n$ and epimorphism for $q = n + 1$.

Theorem 5.3.2 .Whitehead Theorem
Let $f : X \to Y$ be a map of spaces which are 0-connected (path connected). Then the followings are true.

1. *If f is an n-equivalence ($n = \inf$ allowed) then $f_* : \widetilde{H}_q(X, Z) \to \widetilde{H}_q(Y, Z)$ is an isomorphism for $q < n$ and epimorphism for $q = n$.*

2. *If X, Y are 1-connected and $f_* : \widetilde{H}_q(X, Z) \to \widetilde{H}_q(Y, Z)$ is an isomorphism for $q < n$ and epimorphism for $q = n$ then f is an n-equivalence.*

Proof. (1) By the definition of n-equivalence, f is an n-equivalence if and only if (M_f, X) is n-connected. Since

$$\cdots \longrightarrow \tilde{H}_n(X, Z) \xrightarrow{f_*} \tilde{H}_n(Y, Z) \longrightarrow H_n(M_f, X; Z) \xrightarrow{\partial} \tilde{H}_n(X, Z) \xrightarrow{f_*} \cdots$$

is exact.
This implies $f_* : \widetilde{H}_q(X, Z) \to \widetilde{H}_q(Y, Z)$ is an isomorphism for $q < n$ and epimorphism for $q = n$ iff

$$H_q(M_f, X, Z) = 0, \text{ for all } q \leq n.$$

Suppose f is a n-equivalence. Then $\pi_q(M_f, X, *) = 0$ for all $q \leq n$. By theorem 5.3.1, since there exists a function such that

$$h : \pi_q(M_f, X, *) \longrightarrow H_q(M_f, X, Z)$$

is an isomorphism, $H_q(M_f, X, Z) = 0$, for all $q \leq n$

(2) If

$$H_q(M_f, X, Z) = 0 \text{ for all } q \leq n$$

and X, Y are 1-connected this implies (M_f, X) is 1-connected. For $n = 2$, by theorem 5.3.1,

$$h : \pi_2(M_f, X, *) \to H_2(M_f, X; Z)$$

is an isomorphism.

If $n \geq 2$, then (M_f, X) is 2-connected. If we continue by using mathematical induction, we find (M_f, X) is n-connected i.e. f is n-equivalence. □

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Manasi Kumari Sahukar

M.Sc. Student in Mathematics
Roll No. 413MA2072
National Institute of Technology
Rourkela - 769 008 (India)