HARDY WAVELET INDUCED ISOMORPHISM

A THESIS

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BY

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Declaration

I hereby declare that the work which is being presented in the report entitled "Hardy wavelet induced isomorphism" in partial fulfilment of the requirement for the award of the degree of Master of Science, submitted to the Department of Mathematics, National Institute of Technology Rourkela is a review work carried out under the supervision of Dr. Divya Singh. The matter embodied in this report has not been submitted by me for the award of any other degree.

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This is to certify that the above statement made by the candidate is true to the best of my knowledge.

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Preface

The present thesis titled "Hardy Wavelet Induced Isomorphism "consists of three chapters. The first chapter is the introductory chapter about Hardy space, Hardy wavelets and MRA. The second and third chapters consist of definition and examples of Hardy wavelet induced isomorphisms and their fixed point sets in case of two-interval Hardy wavelet sets.

Acknowledgements

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I owe a gratitude to God and my family members for their unconditional love and support. They have supported me in every situation. I am grateful for their blessings and inspiration.

Contents

Chapter 1

Introduction

1.1. Preliminaries

Definition 1.1.1:[3] The set of all square integrable function is known as L^2-space;

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \middle| \int_{\mathbb{R}} |f(x)|^2 dx < +\infty \right\}$$

Definition 1.1.2:[3] For $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx, \qquad x \in \mathbb{R}$$

$\hat{f}(\xi)$ is known as the Fourier Transform of $f(x)$.

Definition 1.1.3:[3] For $g \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$, $\check{g}$ is defind as,

$$\check{g}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} g(\xi) e^{ix\xi} d\xi,$$

$\check{g}(x)$ is called the inverse Fourier Transform of $g(x)$.

Definition 1.1.4:[3] Hardy Space $H^2(\mathbb{R})$ is the collection of all square integrable functions whose Fourier transform is supported in $\mathbb{R}^+ = (0, \infty)$;

$$H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \leq 0 \}.$$

$H^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$.

1.2. Hardy Wavelet and Hardy MRA

Definition 1.2.1:[3] A function $\psi \in H^2(\mathbb{R})$ is said to be a Hardy wavelet, if the system of functions $\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $H^2(\mathbb{R})$.

Theorem 1.2.2:[3] *A function $\psi \in H^2(\mathbb{R})$, with $||\psi||_2 = 1$, is an orthonormal wavelet for $H^2(\mathbb{R})$ if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}^+}(\xi) \text{ for a.e. } \xi \in \mathbb{R}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2k\pi))} = 0 \text{ for a.e. } \xi \in \mathbb{R}, k \in 2\mathbb{Z}+1$$

Similar to MRA for wavelets in $L^2(\mathbb{R})$ MRA for Hardy wavelets are also defined called H^2-MRA.

Definition 1.2.3:[3] A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $H^2(\mathbb{R})$ is a multiresolution analysis for $H^2(\mathbb{R})$ if the following properties are satisfied:

1. $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
2. $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
4. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = H^2(\mathbb{R})$;
5. There exists a $\varphi \in V_0$, such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function φ is called a **scaling function** of the given MRA.

Associated with each H^2-MRA there is a Hardy wavelet. The construction of Hardy wavelet from an H^2-MRA follows on the same lines as that of an orthonormal wavelet in $L^2(\mathbb{R})$, given in [3].

From the definition of MRA, $V_0 \subset V_1$. Let $V_1 = V_0 \oplus W_0$, where W_0 is the orthogonal complement of V_0 in V_1. Now the elements of W_0 can be dilated by 2^j to obtain a closed subspace W_j of V_{j+1} such that

$$V_{j+1} = V_j \oplus W_j \quad \text{for each} \quad j \in \mathbb{Z}.$$

From property (4) of MRA,

$$H^2(\mathbb{R}) = \oplus_{j=-\infty}^{\infty} W_j.$$

We have to find a $\psi \in W_0$ such that $\{2^{j/2}\psi(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_j, for all $j \in \mathbb{Z}$. Then $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ will form an orthonormal basis for $H^2(\mathbb{R})$. Since $\varphi \in V_0 \subset V_1$, therefore $\{\sqrt{2}\varphi(2 \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_1, and hence

$$\varphi(\xi) = \sum_{k \in \mathbb{Z}} a_k \sqrt{2}\varphi(2\xi - k)$$

where the coefficients a_k are given by $a_k = \langle \varphi(\xi), \sqrt{2}\varphi(2\xi - k)\rangle$.

$$\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi)$$

for a.e. $\xi \in \mathbb{R}$, where m_0 is a 2π-periodic function called the low-pass filter associated with the scaling function φ.

The filter $m_0(\xi)$ associated with φ satisfies

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$$

for a.e. $\xi \in \mathbb{R}$.

$\psi \in H^2(\mathbb{R})$ is an orthonormal wavelet associated with the given MRA iff

$$\hat{\psi}(2\xi) = e^{i\xi}\nu(2\xi)\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi)$$

a.e. on $\mathbb{R}$, for some 2π-periodic measurable function ν such that

$$|\nu(\xi)| = 1 \quad \text{a.e.} \quad \text{on} \quad \mathbb{T}.$$

Relation between $|\hat{\varphi}|$ and $|\hat{\psi}|$ [3]

From previous results we obtain

$$\begin{aligned} |\hat{\varphi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 &= |\hat{\varphi}(\xi)|^2|m_0(\xi)|^2 + |\hat{\varphi}(\xi)|^2|m_0(\xi+\pi)|^2 \\ &= |\hat{\varphi}(\xi)|^2(|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2) \\ &= |\hat{\varphi}(\xi)|^2 \end{aligned}$$

By repeating this result and applying the Fatou's lemma, we obtain

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 \text{ for a.e. } \xi \in \mathbb{R}.$$

Proposition 1.2.4:[3] *If ψ is an H^2-wavelet associated with an H^2-MRA, then*

$$D_\psi(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2 = 1 \textit{ for a.e. } \xi \in \mathbb{R}.$$

Proof. From the relation given above it follows that

$$
\begin{aligned}
&|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2 \text{ for a.e. } \xi \in \mathbb{R} \\
\Rightarrow &\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{\infty} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \\
\Rightarrow &1 = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 \\
\Rightarrow &D_\psi(\xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}.
\end{aligned}
$$

Chapter 2

Hardy wavelet induced isomorphism

2.1. Hardy wavelet Set

Definition 2.1.1:[3] Hardy wavelet set is a measurable subset E of $\mathbb{R}$ such that the inverse Fourier transform of χ_E is an orthonormal wavelet in $H^2(\mathbb{R})$.

In [2] all the one and two interval Hardy wavelet sets were characterized. $[2\pi, 4\pi]$ is the only one interval Hardy wavelet set and the two-interval Hardy wavelet sets are given by

$$\left[\frac{2(k+1)}{2^{j+1}-1}\pi, \frac{2k}{2^j-1}\pi\right] \cup \left[\frac{2^{j+1}k}{2^j-1}\pi, \frac{2^{j+2}(k+1)}{2^{j+1}-1}\pi\right]$$

with $j > 0$ and $0 < k < 2(2^j - 1)$.

The following result gives a characterization of H^2 wavelet sets.

Proposition 2.1.2:[1] *Let $K \in \mathbb{R}^+$ be a measurable set. Then K is a Hardy wavelet set if and only if the following two conditions hold:*

(i) $\{K + 2k\pi : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.

(ii) $\{2^j K : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^+$.

2.2. Hardy wavelet induced isomorphism

Ionascu in his paper [4] defined wavelet induced isomorphisms for wavelet sets in $L^2(\mathbb{R})$. By using the similar idea we defined Hardy wavelet induced isomorphisms (HWII) for

Hardy wavelet sets. In [4] the set E was taken as $[-2\pi, -\pi) \cup [\pi, 2\pi)$, but since the dilates of Hardy wavelet sets partition only the positive real axis therefore for Hardy wavelet sets we used $E = [2\pi, 4\pi)$. The translation map τ is same and the dilation map has domain $\mathbb{R}^+$ instead of $\mathbb{R}$.

The translation map $\tau : \mathbb{R} \longrightarrow E$ is given by $\tau(x) = x + 2\pi j$, where j is a unique integer satisfying $x + 2\pi j \in E$. Similarly the dilation map $\delta : \mathbb{R}^+ \longrightarrow E$ is defined by $\delta(x) = 2^k x$, where k is a unique integer satisfying $2^k x \in E$.

Let W be a Hardy wavelet set, then the map h_W from E to E is defined by $h_W = \tau_{|W} \circ (\delta_{|W})^{-1}$. Further, similar to [4], the Hardy wavelet induced isomorphism (HWII) is $\tilde{h}_W = \xi \circ h_W \circ \xi^{-1} : [0, 1) \longrightarrow [0, 1)$, where $\xi : E \longrightarrow [0, 1)$ is defined as

$$\xi(x) = \frac{x}{2\pi} - 1, \; x \in E,$$

and

$$\xi^{-1}(x) = 2\pi(x + 1), \; x \in [0, 1).$$

Theorem 2.2.1: *Let W be a Hardy wavelet set and $\tilde{h}_W$ be defined as above. Then the map $\tilde{h}_W$ has the following properties:*

(i) $\tilde{h}_W$ is a measurable bijection of $[0, 1)$,

(ii) For each $x \in [0, 1)$,

$$\tilde{h}_W = \lfloor 2^l (x + 1) \rfloor,$$

where $l \in \mathbb{Z}$ and $\lfloor x \rfloor$ denotes the fractional part of the real number x.

(iii) if h is a map satisfying (i) and (ii) then there exists a wavelet set W such $h = \tilde{h}_W$.

Proof: (i) By definition it follows that $\tau_{|W}, \delta_{|W} : W \longrightarrow E$ are measurable bijections. Hence $(\delta_{|W})^{-1}$ is also a measurable bijection. We can easily see that ξ and ξ^{-1} are measurable bijections and therefore $\tilde{h}_W$ is a measurable bijection of $[0, 1)$.

(ii) Now, $\tau(t) = \xi^{-1}(\lfloor \frac{t}{2\pi} \rfloor)$, for every $t \in \mathbb{R}$. For if, $\lfloor \frac{t}{2\pi} \rfloor = n + d$, where n=integral part and d=fractional part, then $\xi^{-1}(\lfloor \frac{t}{2\pi} \rfloor) = 2\pi \lfloor \frac{t}{2\pi} \rfloor + 2\pi = 2\pi(d) + 2\pi = 2\pi(d + 1)$. Further,

$$\begin{aligned}
\tau(t) &= \tau(2n\pi + 2d\pi) \\
&= (2n\pi + 2d\pi) + 2m\pi, \quad m \in \mathbb{Z} \\
&= 2\pi d + 2\pi(n + m) \\
&= 2\pi(d + n + m) \in [2\pi, 4\pi) \\
\Rightarrow & 1 \leq d + n + m \leq 2 \\
\Rightarrow & n + m = 1 \\
\Rightarrow & \tau(t) = 2\pi(d + 1)
\end{aligned}$$

Thus, $\tau(t) = \xi^{-1}(\lfloor \frac{t}{2\pi} \rfloor)$. Let $u = \xi^{-1}(x) = 2\pi(x + 1)$ and $\delta^{-1}(u) = 2^l u$ with $l \in \mathbb{Z}$. Then we have $\tau(2^l u) = \xi^{-1}(\lfloor \frac{2^l u}{2\pi} \rfloor) = \xi^{-1}(\lfloor 2^l(x+1) \rfloor)$. We know that $h_W(x) = \tau_{|W} \circ (\delta_{|W})^{-1}(x) = \tau(2^l x)$. Now

$$
\begin{aligned}
\tilde{h}_W(x) &= \xi \circ h_W \circ \xi^{-1}(x) \\
&= \xi \circ h_W(2\pi(x+1)) \\
&= \xi(\tau(2^l(x+1)2\pi)) \\
&= \xi(\tau(2^l u)) \\
&= \xi\xi^{-1}\lfloor 2^l(x+1)\rfloor \\
&= \lfloor 2^l(x+1)\rfloor \\
\Rightarrow \tilde{h}_W(x) &= \lfloor 2^l(x+1)\rfloor
\end{aligned}
$$

(iii) Next suppose that h has the properties (i) and (ii), and denote $h_1 = \xi^{-1} \circ h \circ \xi : E \to E$.

$$
\begin{aligned}
h_1(x) &= \xi^{-1} \circ h \circ \xi(x) \\
&= \xi^{-1} \circ h(\frac{x}{2\pi} - 1) \\
&= \xi^{-1}(\lfloor 2^k(\frac{x}{2\pi} - 1 + 1)\rfloor) \\
&= \xi^{-1}(\lfloor \frac{2^k x}{2\pi}\rfloor) \\
&= 2^k x + 2l\pi \\
&= 2^{k(x)}x + 2l(x)\pi, k(x), l(x) \in \mathbb{Z}
\end{aligned}
$$

The map $x \to k(x)$ and $x \to l(x)$ are measurable. Let $\psi : E \to \mathbb{R}$ be defined by $\psi(x) = 2^{k(x)}x, x \in E$, and $W = \psi(E)$. Then W is measurable. ψ is bijective as it is both onto and one to one. If we take any point from W then we can find a pre-image in E.

Now let $x_1 \neq x_2$ be two points in E and $\psi(x_1) = 2^{k_1}x_1$, $\psi(x_2) = 2^{k_2}x_2$. We want to prove that $2^{k_1}x_1 \neq 2^{k_2}x_2$. On the contrary suppose that $2^{k_1}x_1 = 2^{k_2}x_2$.

$k_1 = k_2 = k \Rightarrow 2^k(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. Therefore k_1 can not be equal to k_2.

Further if, $k_1 > k_2$, then $2^{k_1}x_1 = 2^{k_2}x_2$ implies that $2^{k_1-k_2}x_1 = x_2$. Since $x_1, x_2 \in E$, therefore even for $k_1 - k_2 = 1$, $\Rightarrow 2^{k_1-k_2}x_1 \neq x_2 \Rightarrow \psi(x_1) \neq \psi(x_2)$. Hence ψ is one-one.

Since $\delta_{|W}(W) = E$, therefore $(\delta_{|W})^{-1}(E) = W$ and $\psi(E) = W$ implies that $\psi = (\delta_{|W})^{-1}$.

Next we define $\varphi : W \to E$ by $\varphi(y) = y + 2l(\psi^{-1}(y))\pi$.

$$\begin{aligned}
\varphi \circ \psi(x) &= \varphi(\psi(x)) \\
&= \varphi(2^{k(x)}x) \\
&= 2^{k(x)}x + 2l(2^{-k(x)}2^{k(x)}x)\pi \\
&= 2^{k(x)}x + 2l(x)\pi \\
&= h_1(x) \\
\Rightarrow \varphi \circ \psi &= h_1
\end{aligned}$$

Now as defined above $\varphi(W) = E$. Since $\tau_{|W}(W) = E$, therefore $\varphi(y) = \tau_{|W}(y)$, for $y \in W$.

This proves that ψ is onto and h_1 is one to one, hence φ is one to one. Since h_1 is onto, therefore φ is also onto. Thus W is a wavelet set. The map $h_1 = \varphi \circ \psi = \tau_{|W} \circ (\delta_{|W})^{-1} = h_W$ and $h = \xi \circ h \circ \xi^{-1} = \tilde{h}_W$.

Chapter 3

Fixed point sets of Hardy wavelet induced isomorphism

3.1. Examples of HWII

Consider the two interval Hardy wavelet sets W given by

$$W = \left[\frac{2(k+1)}{2^{j+1}-1}\pi, \frac{2k}{2^j-1}\pi\right) \cup \left[\frac{2^{j+1}k}{2^j-1}\pi, \frac{2^{j+2}(k+1)}{2^{j+1}-1}\pi\right)$$

with $j > 0$ and $0 < k < 2(2^j - 1)$.

Let us take the first case, $j = 1, k = 1$

$$W = \left[\frac{4}{3}\pi, 2\pi\right] \cup \left[4\pi, \frac{16}{3}\pi\right]$$

Then $\tau_{|W}, \delta_{|W} : W \longrightarrow E$ are

$$\tau_{|W}(x) = \begin{cases} x + 2\pi & x \in [\frac{4}{3}\pi, 2\pi) \\ x - 2\pi & x \in [4\pi, \frac{16}{3}\pi) \end{cases}$$

and

$$\delta_{|W}(x) = \begin{cases} 2x & x \in [\frac{4}{3}\pi, 2\pi) \\ 2^{-1}x & x \in [4\pi, \frac{16}{3}\pi) \end{cases}$$

$(\delta_{|W})^{-1} : E \longrightarrow W$

$$(\delta_{|W})^{-1}(x) = \begin{cases} 2x & x \in [2\pi, \frac{8}{3}\pi) \\ 2^{-1}x & x \in [\frac{8}{3}\pi, 4\pi) \end{cases}$$

Now $h_W = \tau_{|W} \circ (\delta_{|W})^{-1} : E \longrightarrow E$

$$h_W(x) = \begin{cases} 2x - 2\pi & x \in [2\pi, \frac{8}{3}\pi) \\ 2^{-1}x + 2\pi & x \in [\frac{8}{3}\pi, 4\pi) \end{cases}$$

Again $\tilde{h}_W = \xi \circ h_W \circ \xi^{-1} : [0, 1) \longrightarrow [0, 1)$. Here

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2(x+1) \rfloor & x \in [0, \frac{1}{3}) \\ \lfloor 2^{-1}(x+1) \rfloor & x \in [\frac{1}{3}, 1) \end{cases}$$

Similarly we can obtain $\tilde{h}_W$ for $j = 2$:

$j = 2, k = 1$

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2(x+1) \rfloor & x \in [0, \frac{1}{7}) \\ \lfloor 2^{-2}(x+1) \rfloor & x \in [\frac{1}{7}, \frac{1}{3}) \\ \lfloor 2^{0}(x+1) \rfloor & x \in [\frac{1}{3}, 1) \end{cases}$$

$j = 2, k = 2$

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2^{-1}(x+1) \rfloor & x \in [0, \frac{1}{3}) \\ \lfloor 2(x+1) \rfloor & x \in [\frac{1}{3}, \frac{1}{2}) \\ \lfloor 2(x+1) \rfloor & x \in [\frac{1}{2}, \frac{5}{7}) \\ \lfloor 2^{-2}(x+1) \rfloor & x \in [\frac{5}{7}, 1) \end{cases}$$

$j = 2, k = 3$

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2^{2}(x+1) \rfloor & x \in [0, \frac{1}{7}) \\ \lfloor 2^{2}(x+1) \rfloor & x \in [\frac{1}{7}, 1) \end{cases}$$

$j = 2, k = 4$

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2^{0}(x+1) \rfloor & x \in [0, \frac{1}{3}) \\ \lfloor 2^{2}(x+1) \rfloor & x \in [\frac{1}{3}, \frac{3}{7}) \\ \lfloor 2^{-1}(x+1) \rfloor & x \in [\frac{3}{7}, 1) \end{cases}$$

$j = 2, k = 5$

$$\tilde{h}_W(x) = \begin{cases} \lfloor 2^{0}(x+1) \rfloor & x \in [0, \frac{2}{3}) \\ \lfloor 2^{2}(x+1) \rfloor & x \in [\frac{2}{3}, \frac{5}{7}) \\ \lfloor 2^{-1}(x+1) \rfloor & x \in [\frac{5}{7}, 1) \end{cases}$$

3.2. Fixed point sets of HWII

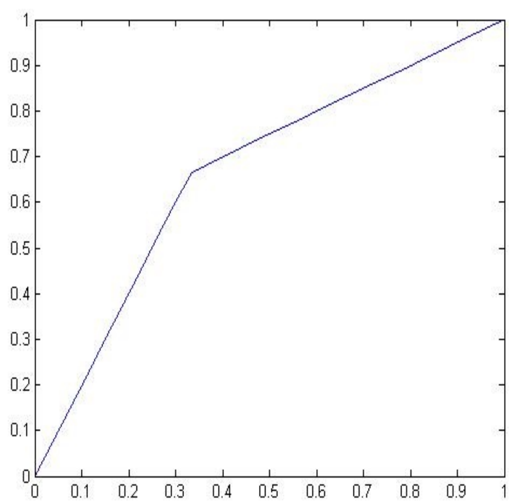
Theorem 3.2.1: *For the two-interval Hardy wavelet sets, whenever $k > 2^j - 1$, the corresponding wavelet induced isomorphism $\tilde{h}_W$ possess fixed point sets of non-zero measure.*

Proof: For the proof note that if $k > 2^j - 1$, then $\frac{k}{2^j-1} > 1$, or $\frac{2\pi k}{2^j-1} > 2\pi$.

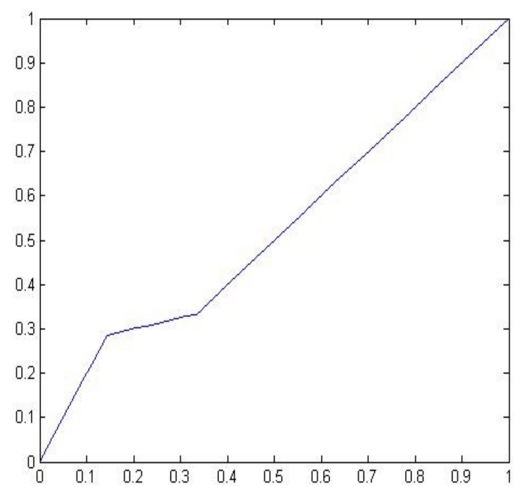
Also $k < 2(2^j - 1) = 2^{j+1} - 2$ implies that $\frac{2\pi(k+1)}{2^{j+1}-1} < 2\pi$. Further $\frac{2\pi k}{2^j-1} < 4\pi$. Thus the point 2π always lies in the first of the two intervals and

$$\left[2\pi, \frac{2k\pi}{2^j - 1}\right) \subseteq [2\pi, 4\pi).$$

Thus both the translation and inverse dilation maps τ, δ^{-1} are identity maps on the subinterval $[2\pi, \frac{2k}{2^j-1}\pi)$ of the two interval Hardy wavelet set and hence $\tilde{h}_W$ is also an identity map on this subinterval. This proves that $\tilde{h}_W$ has fixed point set of non-zero measure.



For j=1, k=1



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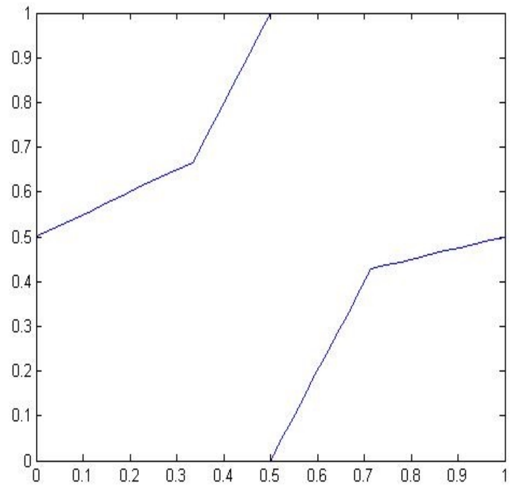

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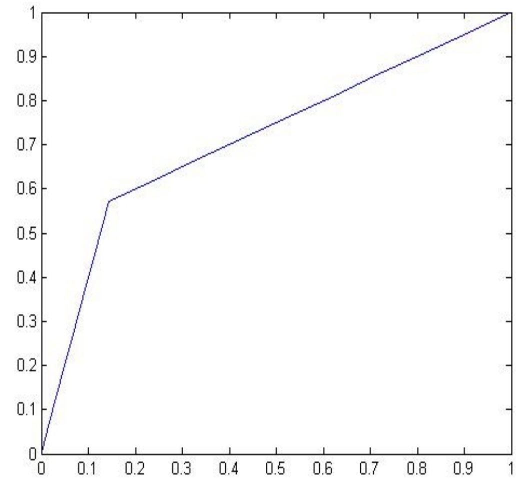

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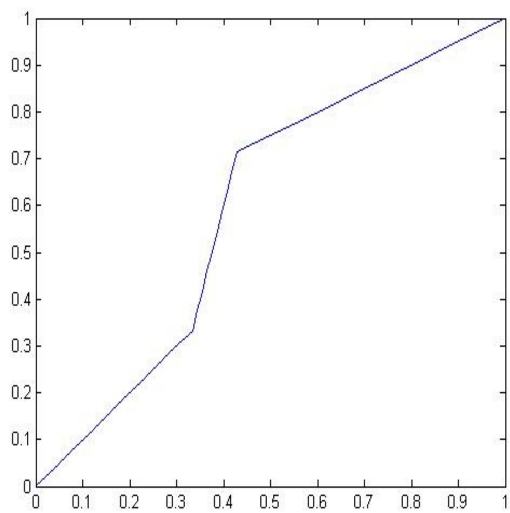

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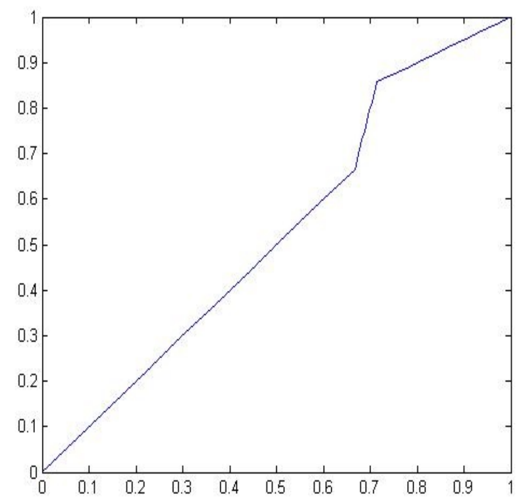

For j=2, k=5

References

[1] N. Arcozzi, B. Behera and S. Madan, *Large classes of minimally supported frequency wavelets of* $L^2(\mathbb{R})$ *and* $H^2(\mathbb{R})$, J. Geom. Anal., **13**(4) (2003), 557–579.

[2] Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee, Jun Keun Seo, *Unimodular for* L^2 *and the Hardy space* H^2, Michigan Math j. Publication, 1992, pp. 345–360.

[3] Eugenio Hernandez, Guido Weiss, *A First Course on wavelets*, CRC press, 1996.

[4] Eugen J. Ionascu, *A new construction of wavelets sets*, Real Anal. Exchange, Volume 28, Number 2,2002, pp. 593–610.