Finite Element Static Analysis of Slabs on Elastic Foundation

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

BACHELOR OF TECHNOLOGY

In

Civil Engineering

By

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CERTIFICATE

This is to certify that the thesis entitled **"Finite Element Static Analysis of Slabs on Elastic Foundation"** submitted by **Prakhar Gupta (111CE0035),** in partial fulfilment of the requirement for the degree of **Bachelor of Technology** in **Civil Engineering**, National Institute of Technology, Rourkela, is an authentic work carried out by him under my supervision.

To the best of my knowledge the matter embodied in the thesis has not been submitted to any other university/institute for the award of any degree or diploma.

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ACKNOWLEDGEMENT

This project would not have been successful without the help, support and guidance of those people who were directly or indirectly involved in this work. First and foremost I wish to express my profound sense of deepest gratitude to my guide and motivator Prof. M. R. Barik, Civil Engineering Department, National Institute of Technology, Rourkela for his valuable guidance, and co-operation and finally help for providing necessary facilities and sources during the entire period of this project.

I wish to convey my sincere gratitude to all the faculties of Civil Engineering Department who have enlightened me during my studies. The facilities and co-operation received from the technical staff of the department is thankfully acknowledged.

Last, but not least, I would like to thank the authors of various research articles and book that referred to.

Prakhar Gupta

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CONTENTS

Abstract

The Finite Element Method (FEM) is a numerical technique for finding approximate solutions to boundary value problems for partial differential equations. It uses subdivision of whole problem domain into simpler parts, called finite elements, and variational methods from the calculus of variations to solve the problem by minimizing the associated error function. Analogous to the idea that connecting many tiny straight lines can approximate a larger circle, FEM encompasses methods for connecting many simple element equations over many small subdomains, named finite elements, to approximate a more complex equation over a larger domain.

Concrete building slabs (plates), upheld directly by the soil medium, is a common construction form. It is utilized as a part of private, business, mechanical, and institutional structures. In some of these structures, substantial slab loads occur, for example, in libraries, grain stockpiling structures, distribution centres, and so forth. A mat foundation, which is usually utilized as a part of the supporting of multi-story building sections, is another illustration of a vigorously loaded concrete slab supported directly by the soil medium. In every one of these structures, it is vital to compute slab displacements and consequent stresses with a worthy level of precision so as to guarantee a sheltered and practical configuration.

This project presents a finite element static analysis for estimating the structural behaviour of plates resting on elastic foundations, described by the Winkler's Model. A Matlab program computing the displacement and stresses for slabs on elastic foundation has been presented in the appendix.

List of figures

List of Tables

Chapter 1

INTRODUCTION

Soil Behaviour

Objective

Scope

Soil Behaviour

Concrete building slabs (plates), upheld directly by the soil medium, is a common construction form. It is utilized as a part of private, business, mechanical, and institutional structures. In some of these structures, substantial slab loads occur, for example, in libraries, grain stockpiling structures, distribution centres, and so forth. A mat foundation, which is usually utilized as a part of the supporting of multi-story building sections, is another illustration of a vigorously loaded concrete slab supported directly by the soil medium. In every one of these structures, it is vital to compute slab displacements and consequent stresses with a worthy level of precision so as to guarantee a sheltered and practical configuration.

Effective uses of the principles of structural engineering are unpredictably connected to the capacity of the designer to model the structure and its support conditions to perform an accurate analysis and an accordingly "correct" design. Landing at a reasonable model is entangled in foundation analysis by the great trouble of demonstrating the soil structure interaction.

Figure 1: Construction of Raft Foundation

Ultimately, all structure loads must be transferred to the soil continuum, and both the soil and the structure act together to resist and support the loads. The integral nature of the foundation and soils action is further complicated by the complexity of the soil medium itself. Soil is truly a non-homogeneous and an anisotropic medium that behaves in a non-linear manner, while concrete and steel structures can be adequately modelled and analysed, assuming isotropic and linear behaviour. In addition, the structural behaviour are well known so that the stiffness of the structure may be readily determined, given member sizing.

On the other hand, soil properties are very difficult to determine because in addition to the previously mentioned characteristics, it is a “soft” material, which makes it very difficult to obtain samples for testing that will produce laboratory results paralleling its actual “in ground” behaviour. Among other problems, the type of soil affects the ability to obtain representative samples (for example, stiff clay is more difficult to sample than soft clay). Variations in sampling techniques among laboratories further complicate the problem. Two additional complicating factors are that soil material properties are stress dependent and the soil continuum will in practice consists of layers of materials with different constitutive relations and material properties. Because of these factors, the time properties and constitutive relations of the soil continuum are essentially unknown and indeterminable. As a result, it is necessary to make a number of simplifying assumptions to analyse the soil-structure interaction.

Objective

- The objective of this research is to develop a workable approach for the analysis of slabs (plates) elastic foundation using finite element method.

- To develop a Matlab program that will provide the designer with realistic stress values and displacement values for use in the design of the slabs.

Scope

The current study is involved only with the use of rectangular and skew slabs because these slabs are widely used and are appropriate for practicing engineers due to its simplicity.

- The study uses Winker’s model as its soil-structure interaction model for the analysis.

- In this study, several types of slab loading are considered, including the uniformly distributed load, concentrated loads, and combinations of these loading systems.

- The results obtained from the program output are compared with that obtained from theoretical and analytical calculations.

Chapter 2

Literature Review

Soil-Structure Interaction Models

Computational Approaches

Literature Review

Soil Structure Interaction Model

Concrete building slabs (plates), upheld directly by the soil medium, is an exceptionally regular development framework. It is utilized as a part of private, business, mechanical, and institutional structures. In some of these structures, substantial slab loads occur, for example, in libraries, grain stockpiling structures, distribution centres, and so forth. A mat foundation, which is usually utilized as a part of the supporting of multi-story building sections, is another illustration of a vigorously loaded concrete slab supported directly by the soil medium. In every one of these structures, it is vital to compute slab displacements and consequent stresses with a worthy level of precision so as to guarantee a sheltered and practical configuration.

In the past, numerous scientists have taken a shot at this issue, which is referred to as "beams and slabs on elastic foundations." In numerous practical design problems of this sort, the soil continuum is layered and may be resting over rigid rock or a generally stronger soil. Most of the past work started with the well understood Winkler model, which was initially created for the examination of railroad tracks. The utilization of the Winkler model includes one noteworthy issue and one huge behavioural irregularity. The issue includes the need for deciding the modulus of subgrade response, "k," and the behavioural irregularity is that an examination of plates conveying a uniformly distributed load will create a rigid body displacement.

At last, all structure loads must be transferred to the soil continuum, and the soil and the structure act together to oppose and support the loads. The fundamental way of the foundation and soil activities is further muddled by the multifaceted nature of soil medium itself. Soil is genuinely a non-homogeneous and an anisotropic medium that acts in a nonlinear way, while cement and steel structures can be adequately demonstrated and analysed, accepting isotropic and linear behaviour. Moreover, the properties of basic building materials are surely understood so that the firmness of the structure may be promptly determined, given member measuring and structure geometry.

Two additional convoluting elements are that soil material properties are stress dependent, and the soil continuum will comprise of layers of materials with diverse constitutive relations and material properties. Due to these elements, the properties and constitutive relations of the soil continuum are basically obscure and indeterminable. Thus, it is important to make various rearranging suppositions to examine the soil structure interaction.

Winkler's Model

One exceptionally mainstream system for displaying the soil structure association has its inceptions in the work done by Winkler in 1867, where the vertical movement of the soil, w, at a point is expected to depend just upon the contact pressure, p, acting by then in the idealized elastic foundation and a proportionality constant, k.

$P = k.w$

The proportionality constant, k, is generally called the modulus of subgrade reaction or the coefficient of subgrade response. This model was initially used to investigate the deflections of and resultant stresses in railroad tracks. In the interceding years, it has been connected to a wide range of soil-structure association issues, and it is known as the Winkler model

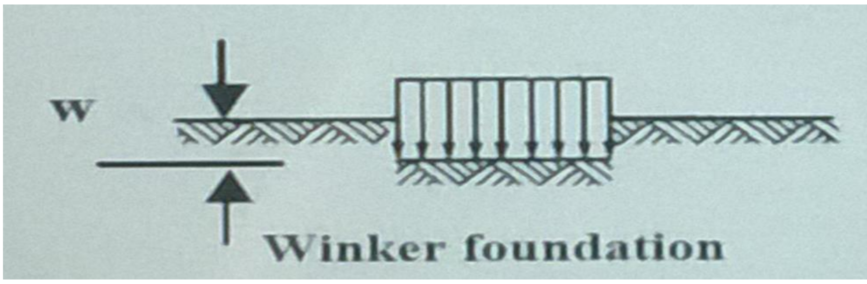

Figure 2: Winkler's Model for Soil-Structure Interaction

Description of the Model

Application of the Winkler model involves the solution of a fourth-order differential equation.
The model consists of linearly elastic springs with a stiffness of "k," placed at discrete intervals below the plate, where k is the modulus of subgrade reaction of the soil. The model is also frequently referred to as a "one-parameter model"

Elastic Continuum Model

In elastic continuum model demonstrate the continuous behavior of soil is idealized as three dimensional continuous elastic solid.

For this situation the soil surface deflections because of loading will happen under and around the loaded region.

This methodology gives considerably more reasonable results on the stresses and distortions inside soil mass than Winkler model.

Utilization of this technique is constrained to elastic and viscoelastic sorts of foundations.

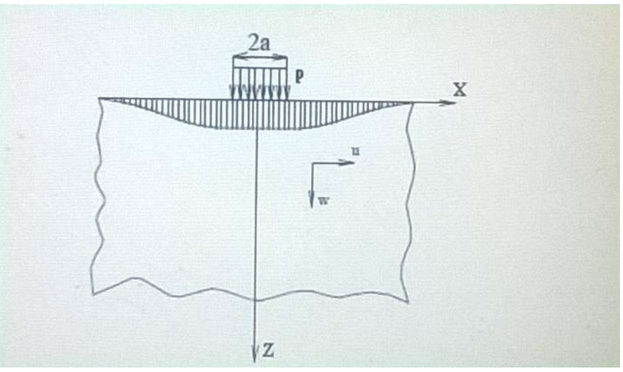

Figure 3: Elastic Continuum Model

Computational Approaches

One of the essential objectives of the study is to develop not just a reasonable methodology for the examination of plates on an elastic foundation, but a useful and effortlessly connected strategy as well nearly, the solution of this kind of soil engineering problems, which includes equilibrium equations together with constitutive relations, compatibility considerations, and complex boundary conditions, would require such an effort, to the point that a simply mathematical methodology is quite unreasonable. Another option is to utilize a numerical analysis technique that will give surmised solutions as near to the precise solutions as needed for practical engineering design problems.

Analysis of footings on Winkler foundation model using analytical and numerical methods has been carried out by several pioneers in this area. Some important contributions are highlighted in this section.

Analytical Solutions

The earliest classical works on the subject were due to Winkler, Hertz, Zimmermann, Reissner, Hetenyi, Gorbunov-Posadov, Seely and Smith, Timoshenko and Krieger, Vlasov and Leontov, and several others . Vlasov and Leontiev [9] also gave solutions to a large number of problems of beams, plates and shells on elastic foundations, idealizing the soil medium as a two parameter model which ignores the horizontal displacements in the medium. Kameswara Rao [7] presented general solutions to beams and plates on elastic foundations using a discrete continuum model for soil, which incorporates horizontal displacements also as a modification to Vlasov's model. They presented the solutions using the versatile method of initial parameters. Butterfield and Banerjee [1] gave solutions for settlement and contact pressure for rigid rectangular rafts. Brown obtained solutions for contact pressure and bending moment in rigid, square and rectangular rafts subjected to various combinations of concentrated loads.

Chan and Cheung [3] gave values of contact pressure for rectangular and circular rigid footings due to concentric load and eccentric loading. These solutions enable an estimate to be made of the bending moment in a rigid footing.

Fertis G. Demeter [5] solved the problems related to analysis of slab on elastic foundation using potential energy approach.

The governing equation for the slab-structure interaction is

$$EI\frac{\partial^4 y}{\partial x^4} + kw = p(x)$$

Where,
w =vertical deflection at the interface of the beam foundation system
EI = flexural rigidity of the beam
K = ksb = spring constant of the soil idealizing it as Winkler's single parameter model
Ks = modulus of subgrade reaction
Kw = contact pressure/soil reaction
B = width of the beam
H = depth of the beam
p(x) = vertical load applied on the beam.

Numerical Methods and Finite Difference Method

Several solutions have been presented using numerical methods such as the finite difference method (FDM), the Runge–Kutta method and iterative methods to take care of the problems not solvable by exact methods. Of these the most popular is FDM. Malter gave solutions of beams on elastic foundations using FDM. Wang (1964) [10] worked out several examples using FDM. Rijhsinghani presented detailed solutions for plates on elastic foundations (PEF) using FDM. There are a very large number of books and publications on FDM and its applications in soil–structure interaction analysis (Wang, 1964)[10]. Andrea R.D. Silva et al. [8] presented detailed solution of plates on tensionless elastic foundation using different numerical analysis techniques.

Glyn Jones presented a detailed analysis of beams on Winkler's elastic foundations using finite difference theory. He also gave a number of references on the subject. He developed a software package for slabs on elastic foundations.

Finite Element Method

In mathematics, the finite element method (FEM) is a numerical procedure for finding approximate solutions for boundary value problems for partial differential equations. It utilizes subdivision of an entire problem space into smaller parts, called finite elements, and variational methods from the math of varieties to tackle the issue by minimizing a related mistake capacity. Practically equivalent to the thought that joining numerous modest straight lines can surmised a bigger circle, FEM includes methods for associating numerous basic element equations over numerous little subdomains, named finite elements, to inexact a more intricate mathematical statement over a larger domain.

The analysis of beams and plates on elastic foundations was also analysed by various authors using the finite element method (FEM) as summarized below.

Carl T. F. Ross [2] utilized finite element method to solve static and dynamic problems of slab analysis.

Cheung and Zienkiewicz [4] obtained the solutions for square rafts of arbitrary flexibility. The stiffness of the soil was gotten from Boussinesq's equation and joined with plate bending finite elements to form a stiffness matrix for the whole system. Madhujit Mukhopadhyay and Hamid Sheikh Abdul [6] solved problems related to beam and slab analysis using FEM. The displacements were solved utilizing the FEM technique. The strategy is fit for taking care of both isotropic and orthotropic plates on elastic media with general loading utilizing either a semi-infinte elastic continuum model or a linear Winkler model for the soil medium. Cheung and Zienkiewicz [4] examined plates and beams on a elastic continuum utilizing the FEM. The horizontal contact pressures at the interface in the middle of structure and foundations were incorporated in the examination. The impacts because of separation of contact surfaces and because of uplift were likewise explored.

Chapter 3

RESEARCH METHODLOGY

Literature Review

Selection of Soil-Structure Interaction Model

Selection of Computational Approach

Mathematical Modelling

Application of the Model using FEM

Research Methodology

1. LITERATURE REVIEW

To acquaint with the theoretical part various publication and research articles were investigated on the effect of various loadings on a slab which is supported on elastic foundation. In addition to this various books and design codes were studied. The motivation of literature review was to obtain the vague knowledge on the methods of studies adopted so that it can be used as guide lines for the present work. The investigation of past studies help in modelling soil-structure and analysis.

2. SELECTION OF SOIL- STRUCTURE INTERACTION MODEL

Soils are not linearly elastic and perfectly plastic for the entire range of loading. Truth be told, actual behaviour of soil is very complicated and it demonstrates a great variety of behaviour when subjected to different conditions.

Different constitutive models have been suggested to describe different aspects of soil behaviour in detail. The simplest type of idealized soil response is to assume the behaviour of supporting soil medium as a linear elastic continuum. The basic elastic model is Winkler's model.

In Winkler model, soil is accepted as an arrangement of indistinguishable yet commonly autonomous, nearly divided, discrete, linearly elastic springs. The trademark highlights of this representation of soil medium are the discontinuous behaviour of the surface displacement. As indicated by the idealizing, deformity of the soil medium because of the applied load is bound to the stacked area only. The surface displacement of the soil medium at each point is specifically corresponding to the stress connected to it by then and totally autonomous of the stresses or displacements at other or even immediately neighbouring point of the soil-structure interface.

3) SELECTION OF COMPUTATIONAL APPROACH

One of the essential objectives of the study is to develop not just a reasonable methodology for the examination of plates on an elastic foundation, but a useful and effortlessly connected strategy as well nearly, the solution of this kind of soil engineering problems, which includes equilibrium equations together with constitutive relations, compatibility considerations, and complex boundary conditions, would require such an effort, to the point that a simply mathematical methodology is quite unreasonable. Another option is to utilize a numerical analysis technique that will give surmised solutions as near to the precise solutions as needed for practical engineering design problems.

Both the finite-element method and the method of finite-differences can be utilized. Every method will produce and oblige solutions for an arrangement of equations; however the utilization of the finite-element method will create a coefficient matrix (K-matrix) that can further be utilized to compute out the displacement qualities and the stresses in the slab.

In mathematics, the finite element method (FEM) is a numerical procedure for finding approximate solutions for boundary value problems for partial differential equations. It utilizes subdivision of an entire problem space into smaller parts, called finite elements, and variational methods from the math of varieties to tackle the issue by minimizing a related mistake capacity. Practically equivalent to the thought that joining numerous modest straight lines can surmised a bigger circle, FEM includes methods for associating numerous basic element equations over numerous little subdomains, named finite elements, to inexact a more intricate mathematical statement over a larger domain.

4) Mathematical Modelling

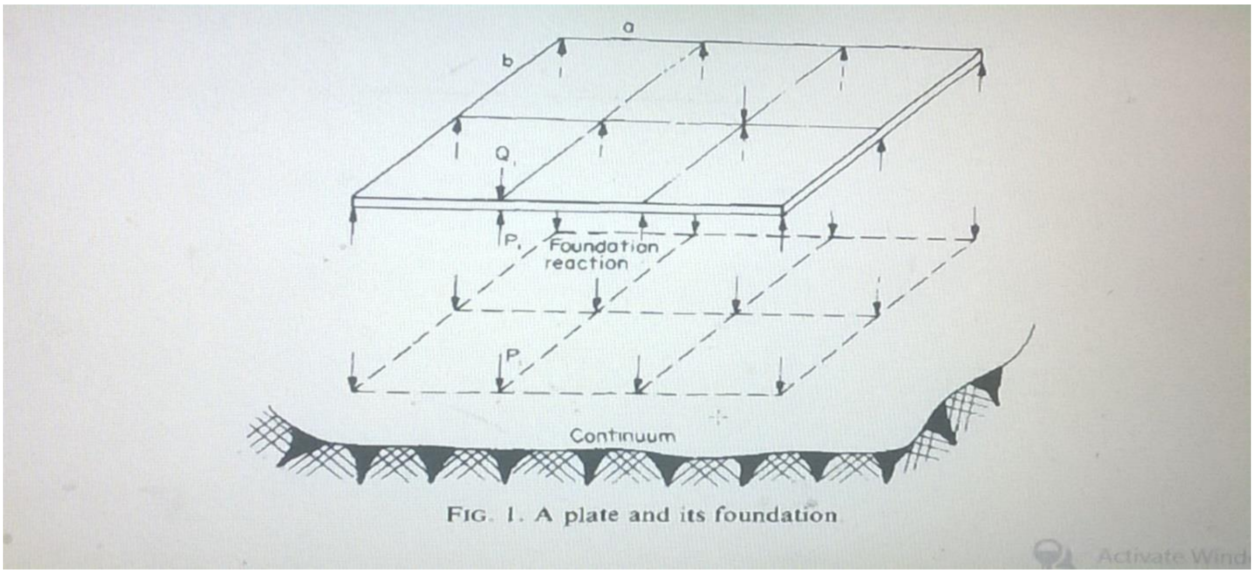

Figure 4: A Slab and its foundation

The Slab is initially partitioned into various little elements which are then joined at a discrete number of nodal points where continuity and equilibrium conditions are secured. From the subsequent mathematical equations, the deformations can be found out, the contact pressures and the plate moments can be worked out effortlessly by simple matrix operations.

In the problems of slab on elastic foundation, diverse assumptions have been introduced to simplify the mathematical formulation.

- No partition happens when negative responses are available.
- No cooperation exists between neighbouring points of the foundation and this responds as a series of disconnected springs.

In Winkler foundation, the contact pressure p is regarded as being directly proportional to the deflection w,

$$P = K.w$$

Where, K is the modulus of subgrade reaction

For a division into a rectangular finite element mesh with sides a and b, equation can be written as:

$$P_i = \alpha_i . a. b. k_i w_i$$

Where,

P_i is the normal force at node

α_i is a coefficient which takes value of 0.25 at corners,0.5 at sides and 1 at interior nodes

k_i is the modulus of subgrade reaction at the node i

w_i is the displacement occurring at node i.

In matrix form, this can be written as:

$$[P] = a. b. k[\alpha]\{w\}$$

Where,

[α] is purely a diagonal matrix

Complete Stiffness Formulation

$$\{N\} = [S].\{U\}$$

For each force N_i and displacement $\{U_i\}$, three components are present. These correspond to lateral displacement w_i and two rotations θxi and θyi.

Noting that if Qi represents an external applied load to anode, Qi-Pi is the effective force acting on that node and we can write, for an isotropic plate:

$$\{Q\} - \{P\} = \frac{D}{(15.a.b).[K_p]\{w\}}$$

Where D is the rigidity of the plate;

$$D = \frac{(Ep.t^3)}{12.(1-\nu^2)}$$

Eliminating P, $\{Q\} = \frac{D}{(15.a.b)}\left([K_p] + \left(\frac{15.a.b}{D}\right)k.a.b\right)\{w\}$

5) Application of the model using Finite Element Method

The rectangular elements can be effectively used for plates having rectangular edges. Rectangular elements can also be employed for irregular plates in conjunction with the other types of elements (e.g. triangular elements). A node of the plate bending element will have three degrees of freedom – the transverse deflection and orthogonal rotations.

The rectangular plate bending element along with their dimensions, coordinate system and node numbering as shown in the figure below. The positive directions of rotations are indicated by right hand screw rule.

The displacement at node 1 are $\{w_1, \theta_{x1}, \theta_{y1}\}$ and the corresponding forces are $\{P_1, M_{x1}, M_{y1}\}$.

Therefore, complete displacement vectors for this element are

$$\{X\}_e^T = \{w_1\ \theta_{x1}\ \theta_{y1}\ w_2\ \theta_{x2}\ \theta_{y2}\ w_3\ \theta_{x3}\ \theta_{y3}\ w_4\ \theta_{x4}\ \theta_{y4}\} \quad (1)$$

$$\{P\}_e^T = \{P_1\ M_{x1}\ M_{y1}\ P_2\ M_{x2}\ M_{y2}\ P_3\ M_{x3}\ M_{y3}\ P_4\ M_{x4}\ M_{y4}\} \quad (2)$$

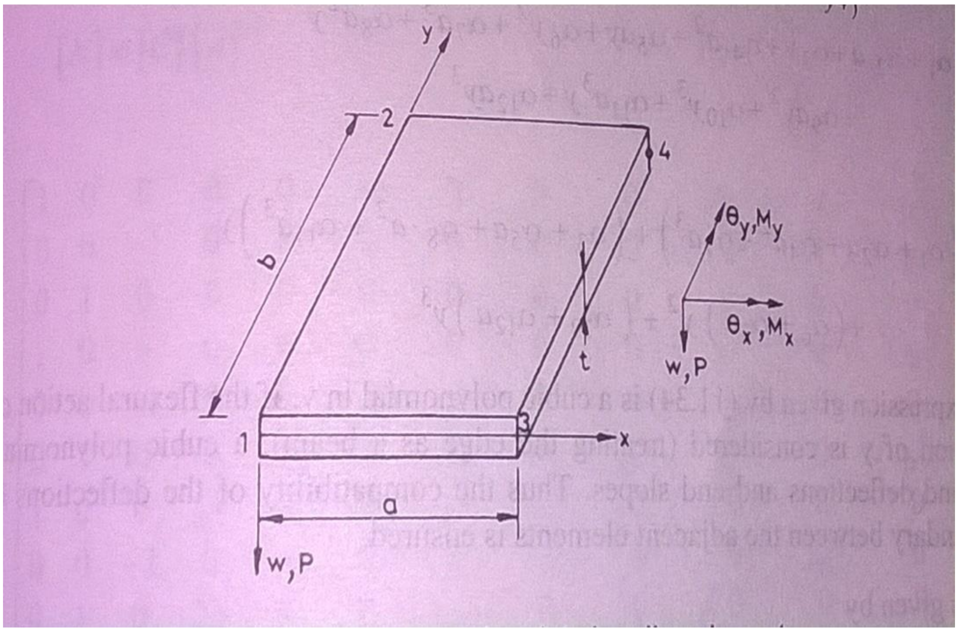

Figure 6: Rectangular Plate Bending Element

Displacement Function

There are three degrees of freedom associated with each node. So for a four-noded rectangle, there are in all twelve degrees of freedom. The polynomial expression to be chosen should have twelve terms. A suitable functions is given by

$$\{f\} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \quad \text{...(3)}$$

Or $\{f\}=[1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2 \;\; x^3 \;\; x^2 \;\; y \;\; xy^2 \;\; y^3 \;\; x^3 \;\; y \;\; xy^3]\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_{12} \end{Bmatrix}$ …(4)

Or, $\{f\} = [C]\{\alpha\}$ …(5)

The displacement function of eqn. (3) gives the following expression for rotataions.

$$\theta_x = -\frac{\partial w}{\partial y} = -(\alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_8 x^2 + 2\alpha_9 xy + 3\alpha_{10} y^2 + \alpha_{11} x^3 + 3\alpha_{12} xy^2) \quad \text{…(6)}$$

And

$$\theta_y = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 x + 3\alpha_7 x^2 + 2\alpha_8 xy + \alpha_9 y^2 + 3\alpha_{11} x^2 y + \alpha_{12} y^3 \quad \text{…(7)}$$

Displacement Function Expressed in Terms of Nodal Displacements

The coordinates of nodes 1, 2, 3 and 4 are (0, 0), (0, b), (a, 0) and (a, b) respectively. Substituting the values of the nodal coordinates in eqns. (3), (6) and (7) respectively, the following equation results.

$\{X\}_e = [A]\{\alpha\}$ …(8)

Inverting eqn. (8)

$\{\alpha\} = [A]^{-1}\{X\}_e$ …(9)

Combining eqns. (5) and (9) yields

$\{f\} = w = [C]\,[A]^{-1}\{X\}_e$

Or

$\{f\} = w = [N]\,\{X\}_e$ …(10)

Where

$[N] = [C]\,[A]^{-1}$ …(11)

Strain-Nodal Parameter Relationship

The 'strains' in plate bending problem are the curvatures. The strain matrix is given by

$$\{\varepsilon\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \qquad \ldots(12)$$

By directly differentiating w given in eqn. (3) with respect to the quantities indicated in eqn. (12), we get

$$\{\varepsilon\} = \begin{Bmatrix} -(2\alpha_4 + 6\alpha_7 x + 2\alpha_8 y + 6\alpha_{11} xy) \\ -(2\alpha_6 + 2\alpha_9 x + 2\alpha_{10} y + 6\alpha_{12} xy) \\ 2\alpha_5 + 4\alpha_8 x + 4\alpha_9 y + 6\alpha_{11} x^2 + 6\alpha_{12} x^2 \end{Bmatrix} \qquad \ldots(13)$$

Or

$\{\varepsilon\} = [Q]\{\alpha\}$ …(14)

Where

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2x & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6xy & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix}$$

Substituting $\{\alpha\}$ from eqn. (9) into eqn. (14), gives

$\{\varepsilon\} = [Q][A]^{-1}\{X\}_e$

Or $\{\varepsilon\} = [B]\{X\}$ …(16)

Eqns. (14) and (15) reveal that the displacement function of eqn. (3) satisfies one of the requirements of convergence, as it contains constant strain (curvature) terms.

Stress (moment) – Strain (curvature) Relationship

The moment – curvature relationship for orthotropic plate has been deduced

$$\{\sigma\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \qquad \ldots(17)$$

The orthotropic constants are given as

$$D_x = \frac{(EI)_x}{1-\nu_x\nu_y}, \; D_y = \frac{(EI)_y}{1-\nu_x\nu_y}$$

$$D_1 = \nu_y D_x = \nu_x D_y$$

For an isotropic plate the constants of eqn. (17) will be

$$D_x = D_y = \mathrm{D} = \frac{Et^3}{12(1-\nu^2)}$$

$$D_1 = \nu \mathrm{D}$$

$$D_{xy} = \frac{1-\nu}{2} D$$

Eqn. (17) in compact form, becomes

$$\{\sigma\} = [\mathrm{D}]\{\varepsilon\} \qquad \ldots(18)$$

Substituting {ε} from eqn. (16) into (18), gives

$$\{\sigma\} = [\mathrm{D}][\mathrm{B}]\{X\}_e \qquad \ldots(19)$$

Derivation of the Element Stiffness Matrix

The element stiffness matrix is derived by applying the principle of minimum potential energy. The potential energy of the plate element is given by

$$\boldsymbol{\Phi} = \frac{1}{2}\int_0^a \int_0^b \left(-M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y}\right)\mathrm{dxdy} - \int_0^a \int_0^b \{f\}^T q \, dxdy \qquad ...(20)$$

Where q is any discrete loading inside the element. Based on the notations used so far for the rectangular plate-bending element, eqn. (20) can be written as

$$\boldsymbol{\Phi} = \frac{1}{2}\int_0^a \int_0^b \{\varepsilon\}^T \{\sigma\}\mathrm{dxdy} - \int_0^a \int_0^b \{f\}^T q\mathrm{dxdy} \qquad \ldots(21)$$

According to the principle of minimum potential energy –

$$\left\{\frac{\partial \Phi}{\partial \{X\}_e}\right\} = \{0\} \qquad \ldots (22)$$

Further mathematical analysis gives,

$$[k]_e \{X\}_e = \{P\}_e \qquad \ldots (23)$$

Where, $[k]_e = \int_0^a \int_0^b [B]^T [D][B]\mathrm{dxdy}$ … (24)

And

$$[P]_e = \int_0^a \int_0^b [N]^T q\mathrm{dxdy} \qquad \ldots (25)$$

Here, $[k]_e$ is the element Stiffness Matrix and $[P]_e$ is the load matrix.

Chapter 4

Results And Discussions

Problem Discussion 1

Problem Discussion 2

Absolute Mean Error

Conclusion

Problem Statement 1

Determine the nodal displacements and stresses for the square plate of sides 80.52cm. The plate is uniformly loaded with 13.79KN/m^2 .

Properties of the plate are :-

Modulus of Elasticity = 206845MPa

Poisson's Ratio = 0.3 and

Thickness = 0.635cm

The plate is assumed to be resting on soil with Modulus of Subgrade Reaction as 7.5MN/m^3

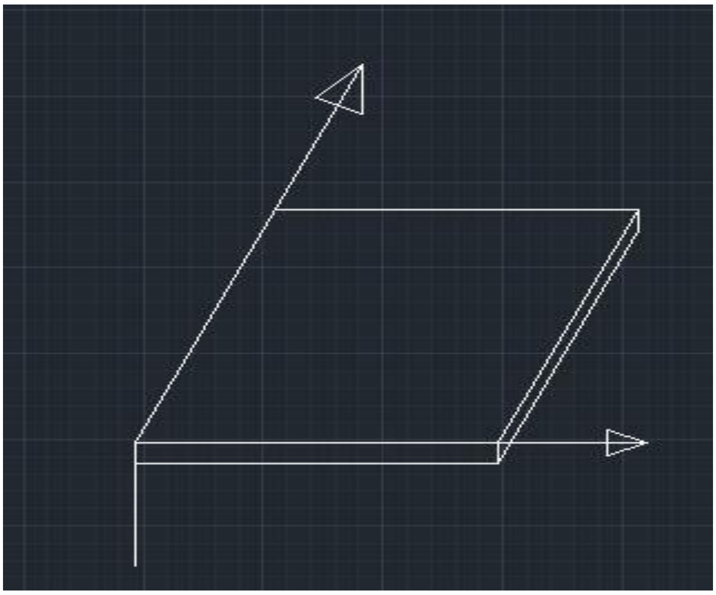

Figure 6: Plate in the given problem

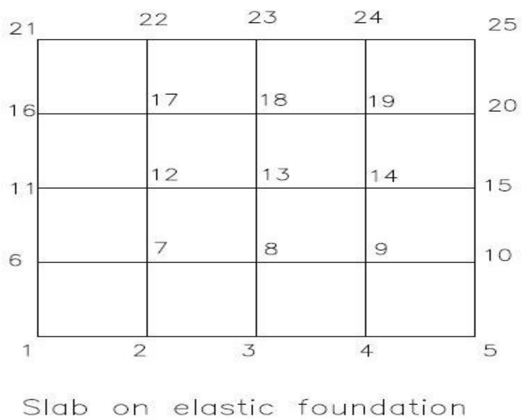

Figure 7: Slab showing nodes

Solution:

Displacements at all the nodes is given in the table

Nodes	Vertical Displacement w (in 10^{-3}cm)	Orthogonal Rotation θ_x	Orthogonal Rotation θ_y
1	4.49	0.33	-0.33
2	4.56	0.33	00
3	4.56	0.34	00
4	4.56	0.33	00
5	4.49	0.33	0.33
6	4.56	00	-0.33
7	4.63	00	00
8	4.63	00	00
9	4.63	00	00
10	4.56	00	0.33
11	4.56	00	-0.33
12	4.63	00	00
13	4.63	00	00
14	4.63	00	00
15	4.56	00	0.33
16	4.56	00	-0.33
17	4.63	00	00
18	4.63	00	00
19	4.63	00	00
20	4.56	00	0.33
21	4.49	-0.33	-0.33
22	4.56	-0.33	00
23	4.56	-0.34	00
24	4.56	-0.33	00
25	4.49	-0.33	0.33

Table 1: Displacements at each node

ε_1 = 5.51%

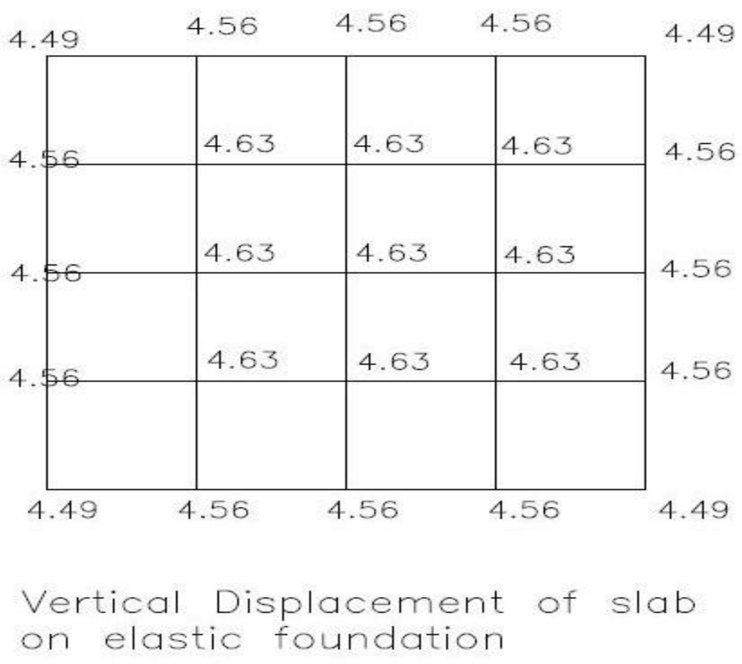

Figure 8: Displacements at each node

	Maximum Displacement	Maximum Stress
Theoretical	$4.9X10^{-3}cm$	7825.55KPa
Present	$4.63X10^{-3}cm$	7394.35KPa

Table 2: Comparing results from theory

Absolute error:

$$\varepsilon_1 = \frac{4.9X10^{-3} - 4.63X10^{-3}}{4.9X10^{-3}} \text{X100}$$

ε_1 = 5.51%

Problem Statement 2

Determine the nodal displacements and stresses for the skew slab of sides 6.42m and 10.46m as shown in the figure. The plate is loaded with 6 concentrated loads of 8.9 KN each at nodes 7,8,9,17,18 and 19 .

Properties of the plate are :-

Modulus of Elasticity = 206845MPa

Poisson's Ratio = 0.15 and

Thickness = 23cm

The plate is assumed to be resting on soil with Modulus of Subgrade Reaction as 7.5MN/m^3

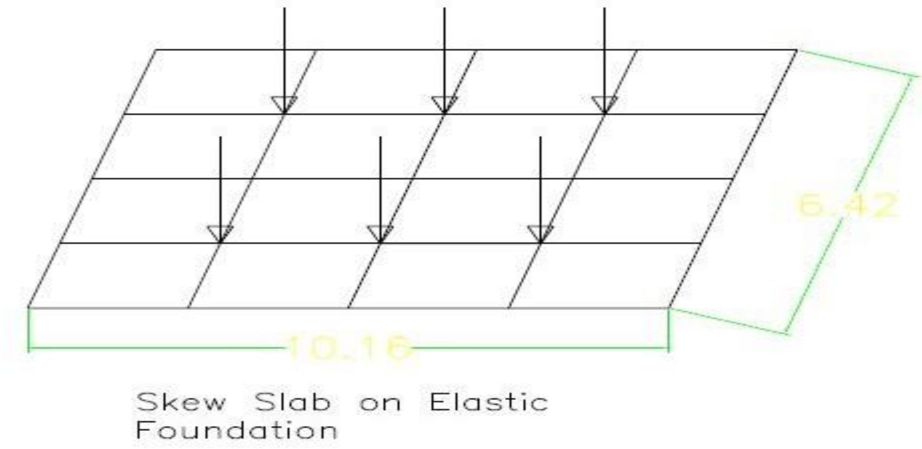

Figure 9: Loading on Skew Slab on Elastic Foundation

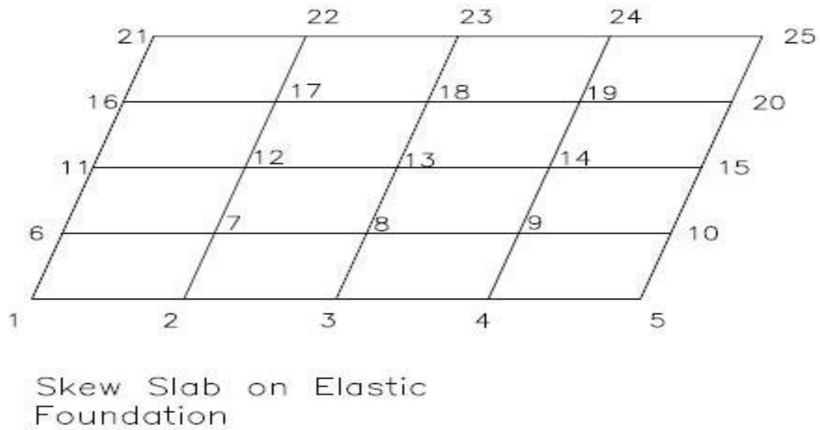

Figure 10: Nodes of Skew Slab on Elastic Foundation

Solution:

Displacements at all the nodes is given in the table

Nodes	Vetical Displacement W(in 10^{-2}cm)	Orthogonal Rotation θ_x	Orthogonal Rotation θ_y
1	2.72	00	-0.03
2	2.73	00	-0.02
3	2.74	00	00
4	2.73	00	0.02
5	2.72	00	0.03
6	2.72	00	-0.03
7	2.73	00	-0.02
8	2.74	00	00
9	2.73	00	0.02
10	2.72	00	0.03
11	2.72	00	-0.03
12	2.73	00	-0.02
13	2.74	00	00
14	2.73	00	0.02
15	2.72	00	0.03
16	2.72	00	-0.03
17	2.73	00	-0.02
18	2.74	00	00
19	2.73	00	0.02
20	2.72	00	0.03
21	2.72	00	-0.03
22	2.73	00	-0.02
23	2.74	00	00
24	2.73	00	0.02
25	2.72	00	0.03

Figure 11

	Maximum Displacement	Maximum Stress
Theoretical	$3.15X10^{-2}$cm	143.41KPa
Present	$2.74X10^{-2}$cm	124.74KPa

Table 4: Comparing results from theory

Absolute error:

$$\varepsilon_2 = \frac{3.15X10^{-2} - 2.74X10^{-2}}{3.15X10^{-2}} X100$$

$$\varepsilon_2 = 13\%$$

Absolute Mean Error

For problem 1:

$$\varepsilon_1 = \frac{4.9X10^{-3} - 4.63X10^{-3}}{4.9X10^{-3}} \text{X100}$$

ε_1 = 5.51%

For problem 2:

$$\varepsilon_2 = \frac{3.15X10^{-2} - 2.74X10^{-2}}{3.15X10^{-2}} \text{X100}$$

ε_2 = 13%

Absolute Mean Error (AME) $\varepsilon = \frac{\varepsilon_1 + \varepsilon_2}{2}$

$$\varepsilon = \frac{5.51 + 13}{2}$$

(AME) $\varepsilon = 9.25\%$

Conclusion and Future Scope

The Matlab program written was used to find values of displacement and stresses of the slab on elastic foundation. The program has a good performance and a reasonable prediction accuracy while using Winkler's Model. The reliability of the program was evaluated by computing absolute mean error between exact and predicted values. We were able to obtain an Absolute Mean Error (AME) of 9.25% which represents a good degree of accuracy.

The results suggest that FEM with the Winkler model can perform good predictions with least error and finally finite element method could be an important tool for slab analysis on elastic foundation.

Future studies on this project can incorporate using of other soil-structure interaction models like elastic continuum model to perform static analysis of slabs on elastic foundation using finite element as tool.

Appendix

Matlab Programming

Matlab Program for a rectangular slab with uniformly distributed load

```
clear all
meshX=4;   %mesh in X direction
meshY=4;   %mesh in Y direcion
prompt = 'Provide no. of nodes per element';
nnode = input(prompt);
prompt = 'Provide Modulus of elasticity';
E = input(prompt);
prompt = 'Provide Poissons Ratio';
nu = input(prompt);
prompt = 'Provide Thickness';
t = input(prompt);
ndofn=3;
nodes=(meshX+1)*(meshY+1);%total no of nodes
tdofs=nodes*ndofn;
K=zeros(tdofs);
loadMat=zeros(tdofs,1);
nelem=meshX*meshY;  %total no of elements
ielem=1;
prompt = 'Provide Pressure Load';
q = input(prompt);
xycord=zeros(nodes,2);%xy coordinates of all nodes initialized
cnode=1;  %node count
ndy1=-1;%node1 eta coordinate
%xy coordinates of all nodes stored
for i=1:meshY+1
   ndx1=-1;%node1 zye coordinate
   for j=1:meshX+1
```

```
        xycord(cnode,1)=ndx1;
        xycord(cnode,2)=ndy1;
        cnode=cnode+1;
        ndx1=ndx1+(2/meshX);
    end
    ndy1=ndy1+(2/meshY);
end
%assemblage of stiffness matrix and load matrix
cnt=0;
for ielemY=1:meshY
    for ielemX=1:meshX
        gbdof=[];
        node=ielem+cnt;
        ndcon=[node node+1 node+meshX+2 node+meshX+1];
     for inode=1:nnode
    xx(inode)=xycord(ndcon(inode),1);
    yy(inode)=xycord(ndcon(inode),2);
     end
    s1=xx(1);
    s2=xx(2);
    s3=xx(3);
    s4=xx(4);
    n1=yy(1);
    n2=yy(2);
    n3=yy(3);
    n4=yy(4);
     for inode=1:nnode
         for idofn=1:ndofn
             gbdof=[gbdof (ndcon(inode)-1)*ndofn+idofn];
         end
```

```
    end
   %find stiffness matrix and load matrix for typical element
syms s n
length=0.805;%length of plate in X dir
bredth=0.805;% width of plate in Y dir
a=length/2;
b=bredth/2;
w=[1 s n s^2 s*n n^2 s^3 s^2*n s*n^2 n^3 s^3*n s*n^3];
dwds=diff(w,s);
dwdn=diff(w,n);
A1=[w;dwdn;-dwds];
A=[subs(A1,{s,n},{s1,n1});subs(A1,{s,n},{s2,n2});subs(A1,{s,n},{s3,n3});subs(A1,{s,n},{s
4,n4})];
D=((E*t^3)/(12*(1-nu^2)))*[1 nu 0;nu 1 0;0 0 (1-nu)*0.5];
N=w*inv(A);
dNds=diff(N,s);
dNds2=diff(N,s,2);
dNdn2=diff(N,n,2);
dNdsn=diff(dNds,n);
B=-[(1/a^2)*dNds2;(1/b^2)*dNdn2;(2/(a*b))*dNdsn];
j=a*b;
k=j*int(int((transpose(B)*D)*B,s,s1,s2),n,n1,n4);
f=j*(int(int((transpose(N).*q),s,s1,s2),n,n1,n4));
   K(gbdof,gbdof)=K(gbdof,gbdof)+k;
   loadMat(gbdof)=loadMat(gbdof)+f;
   ielem=ielem+1;
   end
   cnt=cnt+1;
end
%equivalent stiffness matrix
al=zeros(5,5);
```

```
alp=zeros(75,75);
xx=meshX+1;
yy=meshY+1;
for i=1:xx
    for j=1:yy
        al(i,j)=1;
    end
end
for i=1:xx
    al(i,1)=0.5;
    al(i,meshY+1)=0.5;
end
for i=1:yy
    al(1,i)=0.5;
    al(meshX+1,i)=0.5;
end
al(1,1)=0.25;
al(meshX+1,1)=0.25;
al(1,meshY+1)=0.25;
al(meshX+1,meshY+1)=0.25;
kk=1;
for j=1:meshY+1
   for i=1:meshX+1
     alp(kk*3,kk*3)=al(i,j);
     alp(kk*3-1,kk*3-1)=al(i,j);
     alp(kk*3-2,kk*3-2)=al(i,j);
   kk=kk+1;
   end
end
DD=(E*t^3)/(12*(1-nu^2));
```

```
prompt = 'Provide mod';
mod = input(prompt);
dg=(DD/(15*a*b));
qq=dg*K + a*b*mod*alp;
f=inv(qq);
disp=f*loadMat
%Calculating Stresses
z=((meshX+1)*(meshY+1)+1)/2;
X=zeros(tdofs,1);
ss=zeros(3,1);
for i=1:4

    if (i==1)
    a=z;
    elseif i==2
       a=z+1;
    elseif 1==3
       a=z+n+1;
    elseif i==4
       a=z+n+2;
    end
       for j=1:4
          X(j*3-2,1) = disp(a*3-2,1);
          X(j*3-1,1) = disp(a*3-1,1);
          X(j*3,1)   = disp(a*3,1);
       end
    end
    ss = D*B*X;
    s1 = ss(1);
    for i=1:4
```

```
if i==1
a=z-n-1;
elseif i==2
   a=z-n;
elseif 1==3
   a=z+1;
elseif i==4
   a=z;
end
   for j=1:4
       X(j*3-2,1) = disp(a*3-2,1);
       X(j*3-1,1) = disp(a*3-1,1);
       X(j*3,1)   = disp(a*3,1);
   end
end
ss = D*B*X;
s2 = ss(1);
for i=1:4

if i==1
a=z-n-2;
elseif i==2
   a=z-n-1;
elseif 1==3
   a=z;
elseif i==4
   a=z-1;
end
   for j=1:4
```

```
        X(j*3-2,1) = disp(a*3-2,1);
        X(j*3-1,1) = disp(a*3-1,1);
        X(j*3,1)   = disp(a*3,1);
    end
end
ss = D*B*X;
s3 = ss(1);
for i=1:4

if i==1
a=z-1;
elseif i==2
    a=z;
elseif i==3
    a=z+n;
elseif i==4
    a=z+n+1;
end
    for j=1:4
        X(j*3-2,1) = disp(a*3-2,1);
        X(j*3-1,1) = disp(a*3-1,1);
        X(j*3,1)   = disp(a*3,1);
    end
end
ss = D*B*X;
s4 = ss(1);
stress = (s1+s2+s3+s4)/4
```

Matlab Program for a rectangular skew slab with concentrated load

```
clear all
meshX=4;   %mesh in X dir
meshY=4;   %mesh in Y dir
prompt = 'Provide no. of nodes per element';
nnode = input(prompt);
prompt = 'Provide Modulus of elasticity';
E = input(prompt);
prompt = 'Provide Poissons Ratio';
nu = input(prompt);
prompt = 'Provide Thickness';
t = input(prompt);
ndofn=3;
nodes=(meshX+1)*(meshY+1);%total no of nodes
tdofs=nodes*ndofn;
K=zeros(tdofs);
loadMat=zeros(tdofs,1);
nelem=meshX*meshY;  %total no of elements
ielem=1;
prompt = 'Provide Pressure Load';
q = input(prompt);
xycord=zeros(nodes,2);%xy coordinates of all nodes initialized
cnode=1;  %node count
ndy1=-1;%node1 eta coordinate
%xy coordinates of all nodes stored
for i=1:meshY+1
   ndx1=-1;%node1 zye coordinate
   for j=1:meshX+1
```

```
        xycord(cnode,1)=ndx1;
        xycord(cnode,2)=ndy1;
        cnode=cnode+1;
        ndx1=ndx1+(2/meshX);
    end
    ndy1=ndy1+(2/meshY);
end
%assemblage of stiffness matrix and load matrix
cnt=0;
for ielemY=1:meshY
    for ielemX=1:meshX
        gbdof=[];
        node=ielem+cnt;
        ndcon=[node node+1 node+meshX+2 node+meshX+1];
     for inode=1:nnode
    xx(inode)=xycord(ndcon(inode),1);
    yy(inode)=xycord(ndcon(inode),2);
     end
    s1=xx(1);
    s2=xx(2);
    s3=xx(3);
    s4=xx(4);
    n1=yy(1);
    n2=yy(2);
    n3=yy(3);
    n4=yy(4);
     for inode=1:nnode
         for idofn=1:ndofn
             gbdof=[gbdof (ndcon(inode)-1)*ndofn+idofn];
         end
```

```
    end
   %find stiffness matrix and load matrix for typical element
syms s n
length=10;%length of plate in X dir
bredth=10;% width of plate in Y dir
a=length/2;
b=bredth/2;
w=[1 s n s^2 s*n n^2 s^3 s^2*n s*n^2 n^3 s^3*n s*n^3];
dwds=diff(w,s);
dwdn=diff(w,n);
A1=[w;dwdn;-dwds];
A=[subs(A1,{s,n},{s1,n1});subs(A1,{s,n},{s2,n2});subs(A1,{s,n},{s3,n3});subs(A1,{s,n},{s
4,n4})];
D=((E*t^3)/(12*(1-nu^2)))*[1 nu 0;nu 1 0;0 0 (1-nu)*0.5];
N=w*inv(A);
dNds=diff(N,s);
dNds2=diff(N,s,2);
dNdn2=diff(N,n,2);
dNdsn=diff(dNds,n);
B=-[(1/a^2)*dNds2;(1/b^2)*dNdn2;(2/(a*b))*dNdsn];
j=a*b;
y=71.565;%thetha
double dd=0
dd=1/(tan(y));
double ee=0;
ee=1/(sin(y));
H=[1,0,0;dd^2,ee^2,dd*ee;2*dd,0,ee];
k=j*int(int((transpose(B)*(transpose(H))*D)*H*B,s,s1,s2),n,n1,n4);
f=j*(int(int((transpose(N).*q),s,s1,s2),n,n1,n4));
   K(gbdof,gbdof)=K(gbdof,gbdof)+k;
   loadMat(gbdof)=loadMat(gbdof)+f;
```

```
    ielem=ielem+1;
    end
    cnt=cnt+1;
end
%equivalent stiffness matrix
al=zeros(5,5);
alp=zeros(75,75);
xx=meshX+1;
yy=meshY+1;
for i=1:xx
    for j=1:yy
        al(i,j)=1;
    end
end
for i=1:xx
    al(i,1)=0.5;
    al(i,meshY+1)=0.5;
end
for i=1:yy
    al(1,i)=0.5;
    al(meshX+1,i)=0.5;
end
al(1,1)=0.25;
al(meshX+1,1)=0.25;
al(1,meshY+1)=0.25;
al(meshX+1,meshY+1)=0.25;
kk=1;
for j=1:meshY+1
   for i=1:meshX+1
     alp(kk*3,kk*3)=al(i,j);
```

```
    alp(kk*3-1,kk*3-1)=al(i,j);
    alp(kk*3-2,kk*3-2)=al(i,j);
  kk=kk+1;
  end
end
D=(E*t^3)/(12*(1-nu^2));
prompt = 'Provide mod';
mod = input(prompt);
dg=(D/(15*a*b));
qq=dg*K + a*b*mod*alp;
f=inv(qq);
disp=f*loadMat
%Calculating Stresses
z=((meshX+1)*(meshY+1)+1)/2;
X=zeros(tdofs,1);
ss=zeros(3,1);
for i=1:4
        if (i==1)
        a=z;
        elseif i==2
            a=z+1;
        elseif 1==3
            a=z+n+1;
        elseif i==4
            a=z+n+2;
        end
            for j=1:4
                X(j*3-2,1) = disp(a*3-2,1);
                X(j*3-1,1) = disp(a*3-1,1);
                X(j*3,1)   = disp(a*3,1);
```

```
    end
end
ss = D*B*X;
s1 = ss(1);
for i=1:4
if i==1
a=z-n-1;
elseif i==2
    a=z-n;
elseif 1==3
    a=z+1;
elseif i==4
    a=z;
end
    for j=1:4
        X(j*3-2,1) = disp(a*3-2,1);
        X(j*3-1,1) = disp(a*3-1,1);
        X(j*3,1)   = disp(a*3,1);
    end
end
ss = D*B*X;
s2 = ss(1);
for i=1:4
if i==1
a=z-n-2;
elseif i==2
    a=z-n-1;
elseif 1==3
    a=z;
elseif i==4
```

```
    a=z-1;
end
    for j=1:4
        X(j*3-2,1) = disp(a*3-2,1);
        X(j*3-1,1) = disp(a*3-1,1);
        X(j*3,1)   = disp(a*3,1);
    end
end
ss = D*B*X;
s3 = ss(1);
for i=1:4
if i==1
a=z-1;
elseif i==2
    a=z;
elseif i==3
    a=z+n;
elseif i==4
    a=z+n+1;
end
    for j=1:4
        X(j*3-2,1) = disp(a*3-2,1);
        X(j*3-1,1) = disp(a*3-1,1);
        X(j*3,1)   = disp(a*3,1);
    end
end
ss = D*B*X;
s4 = ss(1);
stress = (s1+s2+s3+s4)/4
```

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