Existence and Uniqueness Results for Nonlinear Boundary Value Problems of

Elliptic P.D.Es

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Submitted by

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DECLARATION

I hereby declare that the thesis entitled "Existence and Uniqueness Results for Nonlinear Boundary Value Problems of Elliptic P.D.Es" which is being submitted by me to National Institute of Technology for the award of degree of Master of Science is original and authentic work conducted by me in the Department of Mathematics, National Institute of Technology Rourkela, under the supervision of Prof. Debajyoti Choudhuri, Department of Mathematics, National Institute of Technology, Rourkela. No part or full form of this thesis work has been submitted elsewhere for a similar or any other degree.

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CERTIFICATE

This is to certify that the project report entitled "Existence and Uniqueness Results for Nonlinear Boundary Value Problems of Elliptic P.D.Es" submitted by **Sajan Kumar** for the partial fulfillment of M.Sc. degree in Mathematics, National Institute of Technology Rourkela, Odisha, is a bona fide record of review work carried out by him under my supervision and guidance. The content of this report, in full or in parts, has not been submitted to any other institute or university for the award of any degree or diploma.

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ABSTRACT

In my research project I review some elementary application of fixed point principles to prove existence and uniqueness of results for solutions of boundary value problems of ordinary and partial differential equations. The approach is based on the L^p space theory of certain linear differential operators subjected to certain boundary constraints.

CONTENTS

INTRODUCTION

In my research project I present a review of an elementary approach to existence and uniqueness theory of nonlinear boundary value problems. The approach is based on the L^P theory of linear differential operators subject to boundary conditions. The first chapter of the project is devoted to nonlinear perturbations of the Laplacian and boundary value problems for system of ordinary differential equations. The concept of Dirichlet condition has been discussed. Some notation has used.

In the second chapter the concept of Banach space and Hilbert space has been discussed.

Some theorem based on Banach space and Hilbert space has been proved. Riesz Representation Theorem has been proved in this chapter. The concept of bounded linear functional has been discussed.

CHAPTER 1: Introduction and review of the problem

1.Semi linear elliptic problems

Some Definitions:

Dirichlet Condition:

If the value of the unknown variable of a Partial differential equation is specified on the boundary of the domain then such a boundary condition is named as the Dirichlet boundary condition.

As an example consider,

$$\nabla^2 x + x = 0,$$

where ∇^2 denotes the Laplacian. The Dirichlet condition is given by $x|_{\partial\Omega} = \alpha$ for every $x \in \partial\Omega$.

L^p Space:

For a real number $p \geq 1$, L^p-norm of x is defined as

$$\|\boldsymbol{x}\|_{\boldsymbol{p}} = (|\boldsymbol{x_1}|^{\boldsymbol{p}} + |\boldsymbol{x_2}|^{\boldsymbol{p}} + \ldots + |\boldsymbol{x_n}|^{\boldsymbol{p}})^{\frac{1}{\boldsymbol{p}}}$$

Caratheodory conditions- let $G \subset R^n$ be an open set $J = [a, b] \subset R, a < b$.we can say that $f: J \times G \to R^m$satisfies the Caratheodory conditions on $J \times G$ written as $f \in Car(J \times G)$,if

1) $f(.,x): J \to R^m$is measurable for every $x \in G$.

2) $(t,.): G \to R^m$is continuous for almost every $t \in J$.

3) for each compact set $K \subset G$ the function

$h_K(t) = \sup\{\|f(t,x)\|: x \in K\}$ is Lebesgue integrable on J, where $\|.\|$ is the norm in R^m.

Let Ω be a domain in $\Re^n$ and

$$f: \Omega \times \Re \times \Re^n \to \boldsymbol{\Re}$$

be a mapping satisfying Caratheodory conditions.

We consider the Dirichlet problem

$$\Delta u = f(x, u, \nabla u), \qquad x \in \Omega$$

$$u|_{\partial\Omega}=0, \qquad (1.1)$$

where $\partial\Omega$ is the boundary of Ω.

The following notations have been used hereafter:

1. $|\cdot|$ stands for the absolute value in $\Re$ and Euclidean norm in $\Re^n$
2. $||\cdot||_p$ stands for the norm in $L^p(\Omega)$.

I reviewed a few results pertaining to the problem in (1.1).

Banach contraction principle

This principle state that if (X, d) is a complete metric space and

$T: X \to X$ is contraction map, i.e., $d(Tx, Ty) \leq \lambda d(x, y)$ for all

$x, y \in X$, where $\lambda \in (0,1)$ is a constant. Then T has a unique fixed point.

Poincare inequality

Let $1 \leq p \leq \infty$ and Ω be a bounded, connected and open subset of the 'n' dimensional Euclidean space $\Re^n$ with a Lipschitz Boundary (i.e. Ω is a Lipschitz domain).

Then there exist a constant c , depending only on Ω and p , such that for every function *u* in $W^{1,p(\Omega)}$, $\|u - u_\Omega\|L^P(\Omega) \leq C\|\nabla u\|L^P(\Omega)$

Where $u_\Omega = \frac{1}{|\Omega|} \int u(y)dy$ is the average value of *u* over Ω.

$|\Omega|$ is the Lebesgue measure of the domain Ω. For a smooth, bounded domain Ω, since the Rayleigh quotient (defined below) for Laplace operator in the space $W_0^{1,2}(\Omega)$ is minimized by the eigen function corresponding to the minimal eigen value $\lambda_1$of the laplacian, hence for any $u \in W_0^{1,2}(\Omega)$

$\|u\|_{L^2}^2 \leq \lambda_1^{-1}\|\nabla u\|_{L^2}^2$, where L is the laplacian operator.

Rayleigh quotient: For a given complex Hermitian matrix M and nonzero vector x, the Rayleigh quotient $R(M,x)$, defines as

$$R(M,x) = \frac{x^*Mx}{x^*x}.$$

Theorem1.1: Let f satisfy

$$|f(x,u,v) - f(x,\tilde{u},\tilde{v})| \leq a|u - \tilde{u}| + b|v - \tilde{v}|$$

$\forall\, u, \tilde{u} \in \Re,\ v, \tilde{v} \in \Re^n$, where a and b are non-negative constants such that $\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1$ and λ_1 is the principle Eigen value of $-\Delta$ subject to homogeneous Dirichilet boundary conditions on $\partial\Omega$. Then the problem in (1.1) admits a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

Proof: For $v \in L^2(\Omega)$, let us put

$$Av = f(\cdot, \Delta^{-1}v, \nabla\Delta^{-1}v)... \qquad (1.2)$$

The operator A is a mapping of $L^2(\Omega)$ to itself. We shall now show that A is contracting mapping.

To see this, let $v_1, v_2 \in L^2(\Omega)$. Then

$$\|Av_1 - Av_2\|_2 \leq a\|\Delta^{-1}v_1 - \Delta^{-1}v_2\|_2 + \mathrm{b}||\nabla\Delta^{-1}v_1 - \nabla\Delta^{-1}v_2||_2. \tag{1.3}$$

On the other hand it follows from the L^2 theory of Δ that

$$||\Delta^{-1}v_1 - \Delta^{-1}v_2||_2 \leq \frac{1}{\lambda_1}||v_1 - v_2||_2 \tag{1.4}$$

and from the Green's Identity we have

$$||\nabla\Delta^{-1}v_1 - \nabla\Delta^{-1}v_2||_2^2 \leq \left|v_1 - v_2\, ,\, \Delta^{-1}v_1 - \Delta^{-1}v_2\right| \leq \frac{1}{\lambda_1}\|v_1 - v_2\|_2^2 \tag{1.5}$$

Where $(\cdot,\cdot)$ is the $L^2 - inner\ product.$

Combining the equations (1.4) and (1.5) we obtain

$$\|Av_1 - Av_2\|_2 \leq \frac{a}{\lambda_1}\|v_1 - v_2\|_2 + \frac{b}{\sqrt{\lambda_1}}\|v_1 - v_2\|_2 \tag{1.6}$$

This shows that A is a contraction mapping and thus has a uniquely fixed point. On the other hand if $v \in L^2(\Omega)$ is a fixed point of A, then $u = \Delta^{-1}v$ is in $H_0^1(\Omega)\ and\ \Delta u \in L^2(\Omega)$ and u solves (1.1).

Example

In the given theorem let us take the value of $N = 1$, $\Omega = (0, \mathrm{T})$, $\lambda_1 = \frac{\pi^2}{T^2}$ and the equation $\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1$ will become $\frac{aT^2}{\pi^2} + \frac{bT}{\pi} < 1$.

The same condition was also obtained by Mawhin [8] and using the spaces of continuous functions with weighted norm by Albrecht [2].

2. System of Ordinary Differential Equations

Consider the nonlinear boundary value problem

$$u'' + Ku' + f(x, u, u') = 0 \qquad 0 < x < T$$

$$u = 0 \qquad x \in \{0, T\}$$

Where

$$f: [0, T] \times H \times H \rightarrow H$$

And H is a Banach space with norm $| \cdot |$

Nemytskii operator: Given a function f satisfying the Caratheodory condition and a function $u: \Omega \rightarrow R^m$, define a new function $F(u): \Omega \rightarrow R$ by $F(u)(x) = f(x, u(x))$.

The function F is called a Nemytskii operator.

Theorem 1.2: Assume that the mapping f is continuous and satisfies the Lipschitz condition

$$|f(x,u,v) - f(x,\tilde{u},\tilde{v})| \le a|u-\tilde{u}| + b|v-\tilde{v}|$$

$$\forall u,\tilde{u} \in H\ ,\ v,\tilde{v} \in H\ ,\ x \in (0,T)$$

Where

$$a\phi(K) + b\psi(K) < 1$$

and ψ,ϕ are given by

$$\phi(K) = \frac{1}{|K|(1-e^{-|K|T})}\int_0^T \left(1-e^{-|K|s}\right)\left(1-e^{-|K|(T-s)}\right)ds$$

$$\psi(K) = \frac{2}{|K|}\frac{\left(1-e^{-\frac{|K|T}{2}}\right)^2}{1-e^{-|K|T}},$$

then the above problem has unique solution.

Proof: We observe that u is a solution of above problem iff

$v = u'$ solves

$$v' + kv + f\left(x, \int_0^x v, v\right) = 0 \qquad 0 < x < T$$

$$\int_0^T v = 0 \qquad (1.7)$$

We proceed to establish the existence of a unique solution of the above equations.

We first assume that $K > 0$. Let M be a subspace of $L^1((0,T);H)$ consisting of those with $\int_0^T v = 0$. Define $A: M \to M$ by

$$Av(x) = \frac{Ke^{-Kx}}{1 - e^{-KT}} \int_0^T e^{-Kx} \left(\int_0^x e^{ks} Nv(s) \right)$$

$$-e^{-kx} \int_0^x e^{ks} Nv(s)$$

Where N is Nemytskii operator

$$NV(x) = f\left(x, \int_0^x v, v\right)$$

Then v is a solution of the (1.7) if and only if v is a fixed point of A.

For $u, v \in N$, let $w = u - v$ and $p(x) = |Nu(x) - Nv(x)|$ then $p(x) \leq a\left|\int_0^x w\right| + +b|w(x)|$

$$Av(x) = \frac{e^{-Kx}}{1 - e^{-KT}} \left\{ \int_0^x (1 - e^{Ks})\, Nv(s) + \int_x^T \left(1 - e^{-K(T-s)}\right) Nv(s) \right\}$$

From which follows

$$\int_0^x |Au - Av| \leq 2 \int_0^T \frac{(1 - e^{Ks})\left(1 - e^{-K(T-s)}\right)}{K(1 - e^{KT})} p(s)$$

On the other hand since $\int_0^T w = 0$ it follows that $\left|\int_0^x w\right| \leq \frac{1}{2}\int_0^T |w| \quad x \in [0,T]$ further the function $(1-e^{Ks})\left(1-e^{-K(T-s)}\right)$

attains its maximum at $T/2$. Using these facts we obtain that

$$||Au - Av||_1 \leq \left(a\phi(K) + b\psi(K)\right)u||e - v_1||$$

and hence A is a contraction mapping on M and therefore it has a unique fixed point in M.

CHAPTER 2: Elementary Banach and Hilbert space theory

<u>Banach Space</u>: A normed space X is a vector space with norm defined on it. A *Banach* space is a complete normed linear space. (Complete in the metric defined by the norm)

Theorem 2.1(Completeness): If Y is a Banach space, then $B(X,Y)$ is a Banach space.

Proof: We consider the Cauchy sequence (T_n) in $B(X,Y)$ and show that (T_n) converges to an operator $T \in B(X,Y)$.

Since (T_n) is Cauchy for every $\epsilon > 0$ there is a N such that

$$\|T_n - T_m\| < \epsilon \tag{2.1}$$

For all $x \in X$ and $m, n > N$ we thus obtain

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\|x\| < \epsilon\|x\| \tag{2.2}$$

Now for any fixed x and given $\tilde{\epsilon}$ we may choose $\epsilon = \epsilon_x$ so that $\epsilon_x\|x\| \leq \tilde{\epsilon}$. Then from (2.2) we have $\|T_n x - T_m x\| < \tilde{\epsilon}$ and see that $(T_n x)$ is a Cauchy sequence in Y. Since Y is complete $(T_n x)$ converges, say, $T_n x \to y$. clearly the limit $y \ in \ Y$ depends up on the choice $x \in X$. This defines an operator $T: X \to Y$ where $y = Tx$. The operator T is linear since

$$lim\, T_n(\alpha x + \beta y) = \lim(\alpha T_n x + \beta T_n x) = \alpha \lim T_n x + \beta \lim T_n x$$

We prove that T is bounded and $T_n \to T$ that is $\|T_n - T\| \to 0$. Since (2.2) holds for every $m > N$ and $T_m x \to Tx$, we may let $m \to \infty$.Using the continuity of the norm, we then obtain from (2.2) for every $n > N$ and all $x \in X$.

$$\|T_n x - Tx\| = \left\|T_n x - \lim_{m\to\infty} T_m x\right\| = \lim_{m\to\infty} \|T_n x - T_m x\| \leq \epsilon\|x\|$$

(2.3)

This shows that $(T_n - T)$ with $n > N$ is a bounded linear operator. Since T_n is bounded, $T = T_n - (T_n - T)$ is bounded, that is, $T \in B(X, Y)$.Furthermore, if in (2.4) we take the supremum over all x of norm 1, we obtain

$$\|T_n - T\| \leq \epsilon \qquad (n > N)$$

Hence $\|T_n - T\| \to 0$.

Theorem 2.2 (Riesz Lemma): Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval (0,1) there is a $z \in Z$ such that

$$\|z\| = 1 \qquad \|z - y\| \geqq \theta \qquad \forall\, y \in Y$$

Proof: We consider any $v \in Z - Y$ and denote its distance from Y by a, that is $a = \inf_{y\in Y} ||v - y||$.

Clearly $a > 0$a and since $Y$is closed. We now take any $\theta \in (0,1)$. By the definition of an infimum there is a $y_0 \in Y$ such that

$$a \leqq \|v - y_0\| \leqq \frac{a}{\theta} \tag{2.4}$$

(Note that $\frac{a}{\theta} > a$ since $0 < \theta < 1$). Let

$$z = c(u - y_0) \quad where \quad c = \frac{1}{||v - y_0||}$$

Then $\|z\| = 1$ ans we show that $||z - y_0|| \geqq \theta$ for every $y \in Y$. We have

$$\begin{aligned} ||z - y_0|| &= \|c(v - y_0) - y\| \\ &= c\|v - y_0 - c^{-1}y\| \\ &= c||v - y_1|| \end{aligned}$$

Where $y_1 = y_0 + c^{-1}y$

The form of $y_1$shows that $y_1 \in Y$. Hence $\|v - y_1\| \geqq a$, by the definition of a. Writing c out and using (2.4) we obtain

$$\|z - y_0\| = c\|v - y_1\| \geqq ca = \frac{a}{\|v - y_0\|} \geqq \frac{a}{a/\theta} = \theta$$

Since $y \in Y$ was arbitrary, this completes the proof.

Bounded linear functional: A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $D(f)$ lies. Thus there exists a real number c such that for all $x \in D(f)$.

$$|f(x)| \leqq c||x||$$

Furthermore, the norm of f is

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{||x||}$$

Or

$$\|f\| = \sup_{\substack{x \in D(f) \\ ||x||=0}} |f(x)|$$

Combining the above two $|f(x)| \leqq \|f\|\|x\|$

Theorem 2.3(Orthogonal): Let M be a complete subspace Y and $x \in X$ fixed. Then $z = x - y$ is orthogonal to Y.

Proof: If $z \perp Y$ were false, there would be a $y_1 \in Y$ such that

$$< z, y_1 >= \beta \neq 0$$

Cleary $y_1 \neq 0$ since otherwise $< z, y_1 > = 0$. Furthermore for any scalar α

$$\|z - \alpha y_1\|^2 =< z - \alpha y_1, z - \alpha y_1 >$$
$$=< z, z > -\bar{\alpha} < z, y_1 > -\alpha[< y_1, z > -\bar{\alpha} < y_1, y_1 >]$$
$$=< z, z > -\bar{\alpha}\beta - \alpha\left[\bar{\beta} - \bar{\alpha} < y_1, y_1 >\right]$$

The expression in the brackets [...] is zero if we choose

$$\bar{\alpha} = \frac{\bar{\beta}}{< y_1, y_1 >}$$

By minimizing vector theorem we have $\|z\| = \|x - y\| = \delta$ so that the equation yields

$$\|z - \alpha y_1\| = \|z\|^2 - \frac{\bar{\beta}}{< y_1, y_1 >} < \delta$$

But this is impossible because we have

$z - \alpha y_1 = x - y_2$ where $y_2 = \ y + \alpha y_1 \in Y$,

So that $\|z - \alpha y_1\| \geqq \delta$. Hence the assumption does not hold and the above theorem is proved.

Theorem 3.4 (Riesz Representation Theorem): let H_1, H_2 be Hilbert spaces and

$$h: H_1 \times H_2 \to K$$

a bounded sesquilinear form. Then h has a representation

$$h(x, y) =< Sx, y >$$

where $S: H_1 \to H_2$ is a bonded linear operator. S is uniquely determined by h and has norm $\|S\| = \|h\|$

Proof: We consider $\overline{h(x, y)}$. This is linear in y because of the bar. To make Reisz Theorem applicable we keep x fixed. Then that theorem yields a representation in which y is variable, say,

$$\overline{h(x, y)} =< y, z >$$

Hence

$$h(x,y) =< z,y >$$

Let $z \in H_2$ is unique but, of course, depends on our fixed $x \in H_1$. It follows that with the variable x defines an operator

$$S: H_1 \rightarrow H_2 \text{ given by } z = Sx$$

Substituting $z = Sx$ in the above we get one result of the theorem.

S is linear. In fact, its domain is the vector space H_1 and the sesquilinearity we obtain

$$\begin{aligned} < S(\alpha x_1 + \beta x_2), y > &= h(\alpha x_1 + \beta x_2, y) \\ &= \alpha h(x_1, y) + \beta h(x_2, y) \\ &= \alpha < Sx_1 , y > + \beta < Sx_2, y > \\ &=< \alpha Sx_1 + \beta Sx_2, y > \end{aligned}$$

For all y in H_1 $S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2$

S is bonded. Indeed leaving the trivial case $S = 0$ we have

$$\|h\| \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|< Sx, y >|}{\|x\| \|y\|} \geqq \sup_{\substack{x \neq 0 \\ Sx \neq 0}} \frac{|< Sx, Sx >|}{\|x\| \|Sx\|} = \sup_{x \neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|$$

The above proves $\|h\| > \|S\|$

By Cauchy Schwarz application we get

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{\|< Sx, y >\|}{\|x\|\|y\|} \leqq \sup_{x\neq 0} \frac{\|Sx\|\|y\|}{\|x\|\|y\|} = \|S\|$$

S is unique. In fact, assuming that there is a linear operator $T: H_1 \to H_2$ such that for all $x \in H_1$ and $y \in H_2$ we have

$$h(x, y) =< Sx, y \geq< Tx, y >,$$

We see that $Sx = Tx$ by the lemma of equality for all $x \in H_1$. Hence $S = T$ by definition. Combining all the above cases the theorem is proved.

CONCLUSION

A review of an existence and uniqueness result was made in this thesis for a class of elliptic partial differential equations. In the whole process a special class of Banach Space, namely the Sobolev space was studied. The thesis also showed an application of the basic results in analysis such as the Banach's contraction principle and the Poincare inequality.

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