Project on

STUDY ON THE PROBLEM OF ESTIMATION OF PARAMETERS OF GENERALIZED EXPONENTIAL DISTRIBUTION

Submitted By
Sonu Munda
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(Roll No:-413MA2064)

Under the supervision of
Prof. Suchandan Kayal

Department of Mathematics
National Institute of Technology Rourkela
Rourkela-769008

Declaration

I hereby declare that the thesis entitled "Study on the problem of estimation of parameters of generalized exponential distribution" submitted for the M.Sc. degree is a revise work carried out by me and the thesis has not formed elsewhere for the award of any degree, fellowship or diploma.

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Sonu Munda
Roll no.: 413MA2064

<u>Certificate</u>

This is to certify that the dissertation entitled “Study on the problem of estimation of parameters of Generalized Exponential distribution” is a bonafide record of independent reaserch work done by Sonu Munda, Roll no. 413MA2064 under the guidance of Dr. Suchandan Kayal and submitted to National Institute of Technology, Rourkela in partial fulfilment of the award of the degree of Master of Science in Mathematics.

Dr. Suchandan Kayal

Asistant Professor

Department of Mathematics

National Institute of Technology, Rourkela

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SONU MUNDA

ROLL NO.- 413MA2064

DEPT. OF MATHEMATICS

NIT ROURKELA

Abstract

Mudholkar and Srivastava, Freimer (1995) proposed three-parameter exponentiated Weibull distribution. Two-parameter exponentiated exponential or generalized exponential (GE) distribution is a particular member of the exponentiated Weibull distribution. we study the problem of estimation of unknown parameters of the GE distribution and describe the some estimation techniques which are very useful to estimate the unknown parameters of the GE distribution. We consider the methods of maximum likelihood estimator, moments estimator, percentiles estimator, least square estimator, weighted least square estimator.

Contents

Chapter 1

1. Introduction

Exponential distribution plays a central role in analyses of lifetime or survival data. This distribution attracts attention from several researchers because of its convenient statistical theory, its important 'lack of memory' property and its constant hazard rates. In few circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as gamma, Weibull and lognormal distributions are in common use. Adding parameters to a well established family of distributions is a time honoured device for obtaining more flexible new families of distributions.

Gupta and Kundu (1999) proposed the generalized exponential (GE) distribution as an alternative to the well known Weibull or gamma distribution. It is observed that the proposed two parameter GE distribution has several desirable properties. In many situations it fits better than the Weibull or gamma distribution. Extensive work has been done since its introduction to establish several properties of the GE distribution. This distribution is used quite effectively in analysing many lifetime data, particularly in place of two parameter gamma and two parameter Weibull distributions.

The properties of the two-parameter GE distribution are quite close to the corresponding properties of the two-parameter gamma distribution. Both gamma and GE distributions have concave densities. The mean of both distributions diverges to ∞ as the shape parameter goes to ∞. The two parameter GE distribution can also have increasing and decreasing hazard rates depending on the shape parameter. If the shape parameter is greater than '1', then for both gamma and GE distributions, hazard rate increases from 0 to λ and if the shape parameter is less than '1', then they decrease from ∞ to λ. The tail behaviour of these two distributions is also quite similar.

The GE distribution also has some nice physical interpretations. Consider a parallel system, consisting of n components, i.e., the system works, only when at least one of the n-components works. If the lifetime distributions of the components are independently, identically distributed exponential, then the lifetime distribution of the system becomes $F(x \mid n, \lambda)$, where $F(x \mid n, \lambda)$ is given by (1.2), with $\alpha = n$ and $\lambda > 0$.

The probability density function of the two parameter GE distribution is given by

$$f(x \mid \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha - 1} e^{-\lambda x}; \qquad \alpha, \lambda, x > 0. \tag{1.1}$$

Also, the cumulative distribution function (cdf), survival function and hazard function of this distribution are given by

$$F(x \mid \alpha, \lambda) = (1 - e^{-\lambda x})^{\alpha}; \qquad \alpha, \lambda, x > 0 \tag{1.2}$$

$$s(x \mid \alpha, \lambda) = 1 - (1 - e^{-\lambda x})^{\alpha}; \qquad \alpha, \lambda, x > 0 \tag{1.3}$$

and

$$h(x \mid \alpha, \lambda) = \frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1} e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}; \qquad \alpha, \lambda, x > 0 \tag{1.4}$$

respectively.

Here α is the shape parameter and $1/\lambda$ is the scale parameter. The GE distribution with these parameters is denoted by GE (α, λ). When the shape parameter α equals to 1, then it coincides with one parameter exponential distribution. Also, the mean and variance of this distribution are given by

$E(X) = \frac{\alpha}{\lambda}$ *and* $\mathrm{var}(X) = \frac{\alpha}{\lambda^2}$, respectively.

Chapter 2

2. Literature Review and Summary

The three parameter gamma and Weibull distributions are the most popular distributions for analysing life data. Both distributions have been studied in the literature and both have applications in the various fields of science and technology. In this direction we refer to Masuyama & Kuroiwa (1952), van Kinken (1961), Alexander (1962) and Jackson (1969). The three parameters in both distributions represent location, scale and shape, and because of them both distributions have quite a bit of flexibility for analysing skewed data. Mudholkar, Srivastava and Freimer (1995) proposed a three parameter (one scale and two shapes) distribution, the exponentiated Weibull distribution (see also Mudholkar and Srivastava, 1993). Both papers analyse certain datasets and show that the exponentiated Weibull, which has three parameters and has a better fit than the two parameter (taking location parameter to zero) Weibull or one parameter exponential distribution which are special cases of the exponentiated Weibull distribution. Gupta and Kundu (1997) considered a special case (exponentiated exponential) of the exponentiated Weibull model assuming the location parameter to be zero and compared its performances with the two parameter gamma family and the two parameter Weibull family, mainly though data analysis and computer simulations. For some other exponentiated families (exponentiated Pareto or exponentiated gamma) one may see Gupta et al. (1998) who mainly discussed the different hazard rate properties under different situations. Gupta and Kundu (1999) compared the maximum likelihood estimators (MLE) with the other estimators such as the method of moments estimators (MME), estimators based on percentiles (PCE), least square estimators (LSE), weighted least square estimators (WLSE), and the estimators based on the linear combination of order statistics with respect to their biases and mean squared errors (MSE) using extensive simulation techniques. Also the three parameter generalized exponential distribution has been studied by Gupta and Kundu (1999). Raqab and Ahsanullah (2001) and Raqab (2002) studied the properties of order and record statistics from the two parameter generalized exponential distribution and their inferences, respectively. Kundu, Gupta and Manglick (2005) presented the discriminating between the log-normal and the GE distributions. Gupta and Kundu (2007) studied the two parameter GE distribution of several properties, different estimation procedures and their properties, estimation of the stress-strength parameter, closeness of this distribution to some of the well-known distribution functions. Also Kundu and Gupta (2007) studied that a very convenient way to generate gamma random variables using GE distribution, when the shape parameter lies between 0 and 1. The new method is compared with the most popular Ahrens & Dieter method and the method proposed by Best. Gupta and Kundu (2008) proposed that the Bayes estimators of the unknown parameters under the assumptions of gamma priors on both the shape and scale parameters.

In this thesis, we study the problem of estimation of unknown parameters of the GE distribution. In chapter 3, we describe about some estimation techniques which are very useful to estimate the unknown parameters of the GE distribution. We consider the methods of maximum likelihood estimator, moments estimator, percentiles estimator, least square estimator, weighted least square estimator. Finally, we include a section containing concluding remarks and future work.

Chapter 3

3. Methodology

In this chapter we discuss about some methods of estimation of unknown parameters of the GE distribution.

3.1. Maximum Likelihood Estimator

Here, we obtain the maximum likelihood estimator of the unknown parameters of the GE distribution with pdf given in (1.1).

Let X_1, X_2, X_3, ….., X_n be the random sample of size n drawn from the distribution with pdf given in (1.1). Then the likelihood function of $X=(X_1, X_2, X_3, \ldots, X_n)$ is given by

$$L(x\,|\,\alpha,\lambda)=\prod_{i=1}^{n} f(x_i\,|\,\alpha,\lambda)$$

$$=(\alpha\lambda)^n\sum_{i=1}^{n}(1-e^{-\lambda x_i})^{\alpha-1}.e^{-\lambda x_i};\alpha,\lambda,x_i>0. \qquad (3.1.1)$$

Taking logarithm both sides of (3.1.1) we get,

$$l=\log L(x\,|\,\alpha,\lambda)=n\log\alpha+n\log\lambda+(\alpha-1)\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})-\lambda\sum_{i=1}^{n}x_i. \qquad (3.1.2)$$

Differentiating partially (3.1.2) with respect to α and λ successively, we get

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})\ , \qquad (3.1.3)$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}-\sum_{i=1}^{n}x_i\ . \qquad (3.1.4)$$

We have the likelihood equations as

$$\frac{\partial l}{\partial\alpha}=0 \quad and \quad \frac{\partial l}{\partial\lambda}=0\ .$$

Therefore from (3.1.3) and (3.1.4), we get

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})=0\ . \qquad (3.1.5)$$

and

$$\Rightarrow \frac{n}{\lambda}-\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}=0 \ . \tag{3.1.6}$$

From (3.1.5),

$$\frac{n}{\alpha}=-\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})$$

$$\Rightarrow \alpha=-\frac{n}{-\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})} \ . \tag{3.1.7}$$

From (3.1.6) and (3.1.7), we conclude that the analytical solution in α and λ does not exist. Hence to get the MLEs of α and λ, one need to apply numerical technique. Here, we apply Newton-Raphson method. Using (3.1.7) in (3.1.6), we get

$$\phi(\lambda)=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i-\left(\frac{n}{\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})}+1\right)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})}=0 \ . \tag{3.1.8}$$

$$\phi'(\lambda)=-\frac{n}{\lambda^2}-\left(\frac{n}{\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})}+1\right)\left[\sum_{i=1}^{n}\frac{(1-e^{-\lambda x_i})x_i e^{-\lambda x_i}(-x_i)-x_i e^{-\lambda x_i}(-e^{-\lambda x_i}.-x_i)}{(1-e^{-\lambda x_i})^2}\right]-\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}}\frac{-n}{1-e^{-\lambda x_i}}(-e^{-\lambda x_i}.-x_i)$$

$$=-\frac{n}{\lambda^2}-\left(\frac{n}{\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})}+1\right)\left[\sum_{i=1}^{n}\frac{-x_i^{2}(1-e^{-\lambda x_i})e^{-\lambda x_i}.-x_i^{2}e^{-2\lambda x_i}}{(1-e^{-\lambda x_i})^2}\right]+\sum_{i=1}^{n}\frac{n x_i^{2}e^{-2\lambda x_i}}{(1-e^{-\lambda x_i})^2}$$

$$\Rightarrow \phi'(\lambda)=-\frac{n}{\lambda^2}-\left(\frac{n}{\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})}+1\right)\sum_{i=1}^{n}\left(\frac{(1-e^{-\lambda x_i})+e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2}\right)x_i^{2}e^{-\lambda x_i}+\sum_{i=1}^{n}\frac{n x_i^{2}e^{-2\lambda x_i}}{(1-e^{-\lambda x_i})^2}$$

$$\Rightarrow \phi'(\lambda)=-\frac{n}{\lambda^2}-\left(\frac{n}{\sum_{i=1}^{n}\log(1-e^{-\lambda x_i})}+1\right)\sum_{i=1}^{n}\frac{x_i^{2}e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2}+\sum_{i=1}^{n}\frac{n x_i^{2}e^{-2\lambda x_i}}{(1-e^{-\lambda x_i})^2} \ . \tag{3.1.9}$$

Now by Newton-Raphson method,

$$\lambda_{k+1}=\lambda_k-\frac{f(\lambda_k)}{f'(\lambda_k)}, k=0,1,2,............$$

Therefore,

$$\lambda_{k+1} = \lambda_k - \frac{\dfrac{n}{\lambda_k} - \sum_{i=1}^{n} x_i - \left(\dfrac{n}{\sum_{i=1}^{n} \log(1-e^{-\lambda_k x_i})} + 1 \right) \sum_{i=1}^{n} \dfrac{x_i e^{-\lambda_k x_i}}{(1-e^{-\lambda_k x_i})}}{-\dfrac{n}{\lambda_k{}^2} + \sum_{i=1}^{n} \dfrac{n x_i{}^2 e^{-2\lambda_k x_i}}{(1-e^{-\lambda_k x_i})^2} + \left(\dfrac{n}{\sum_{i=1}^{n} \log(1-e^{-\lambda_k x_i})} + 1 \right) \sum_{i=1}^{n} \dfrac{x_i{}^2 e^{-\lambda_k x_i}}{(1-e^{-\lambda_k x_i})^2}} . \tag{3.1.10}$$

3.2 Method of Moments Estimator

Here, we obtain the method of moments estimation of the unknown parameters of the GE distribution with pdf given in (1.1).

For GE distribution, we have

$$E(X) = \frac{\alpha}{\lambda} .$$

We denote

$$\mu_1 = \frac{\alpha}{\lambda} . \tag{3.2.1}$$

Also, $E(X^2) = \dfrac{\alpha}{\lambda^2}$.

Denote

$$\mu_2 = \frac{\alpha}{\lambda^2} . \tag{3.2.2}$$

From (3.2.1), we get

$$\alpha = \mu_1 \lambda .$$

Putting the above value in (3.2.2), we get

$$\mu_2 = \frac{\mu_1 \lambda}{\lambda^2} = \frac{\mu_1}{\lambda}$$

$$\Rightarrow \hat{\lambda} = \frac{\mu_1}{\mu_2} . \tag{3.2.3}$$

Again putting (3.2.3) in (3.2.1), we get

$$\hat{\alpha} = \mu_1 . \lambda$$

$$= \mu_1 . \frac{\mu_1}{\mu_2}$$

$$= \frac{\mu_1^{\ 2}}{\mu_2} \ . \qquad (3.2.4)$$

Therefore from (3.2.3) and (3.2.4), the method of moment estimators are given by

$$\hat{\lambda} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

and

$$\hat{\alpha} = \frac{(\frac{1}{n}\sum_{i=1}^{n} x_i)^2}{\frac{1}{n}\sum_{i=1}^{n} x_i^2} \ ,$$

where $\mu_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$ *and*

$$\mu_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \ .$$

3.3 Percentiles Estimator

Here, we obtain the percentiles estimators of the unknown parameters of the GE distribution with pdf given in (1.1).

Taking logarithm both sides of (1.2), we get

$$\log F = \alpha \log(1 - e^{-\lambda x})$$

$$\Rightarrow \log(1 - e^{-\lambda x}) = \frac{1}{\alpha}\log F = \log F^{1/\alpha}$$

$$\Rightarrow 1 - e^{-\lambda x} = e^{\log F^{1/\alpha}} = F^{\frac{1}{\alpha}}$$

$$\Rightarrow e^{-\lambda x} = 1 - F^{\frac{1}{\alpha}} \ . \qquad (3.3.1)$$

Again taking logarithm both sides of (3.3.1)

$$\log(e^{-\lambda x}) = \log(1 - F^{1/\alpha})$$

$$\Rightarrow -\lambda x \log e = \log(1-F^{1/\alpha})$$

$$\Rightarrow -\lambda x = \log(1-F^{1/\alpha})$$

$$\Rightarrow x = -\frac{1}{\lambda}\log(1-F^{1/\alpha}) \ .$$

Or,

$$x(F) = -\frac{1}{\lambda}\log(1-F^{1/\alpha}) \ . \qquad (3.3.2)$$

If P_i denotes some estimate of $F(x_i \mid \alpha,\lambda)$ then the estimate of α and λ can be obtained by minimizing

$$R = \sum_{i=1}^{n}\left[x_i - x(F)\right]^2$$

$$= \sum_{i=1}^{n}\left[x_i + \lambda^{-1}\log(1-P_i^{1/\alpha})\right]^2$$

$$= \sum_{i=1}^{n}\left[x_i^2 + \left\{\lambda^{-1}\log(1-P_i^{1/\alpha})\right\}^2 + 2x_i\lambda^{-1}\log(1-P_i^{1/\alpha})\right] \ . \qquad (3.3.3)$$

Now, differentiating (3.3.3) with respect to λ, we get

$$\frac{dR}{d\lambda} = \sum_{i=1}^{n}2\left\{\lambda^{-1}\log(1-P_i^{1/\alpha})\right\}\log(1-P_i^{1/\alpha})\log\lambda + \sum_{i=1}^{n}2x_i\log(1-P_i^{1/\alpha})\log\lambda \ .$$

Since, $\frac{dR}{d\lambda} = 0$

So, $\sum_{i=1}^{n}\lambda^{-1}\log(1-P_i^{1/\alpha})\log(1-P_i^{1/\alpha}) + \sum_{i=1}^{n}x_i\log(1-P_i^{1/\alpha}) = 0$

$$\Rightarrow \lambda^{-1} = -\frac{\sum_{i=1}^{n}x_i\log(1-P_i^{1/\alpha})}{\sum_{i=1}^{n}\left[\log(1-P_i^{1/\alpha})\right]^2}$$

Or, $\hat{\lambda} = -\frac{\sum_{i=1}^{n}\left[\log(1-P_i^{1/\alpha})\right]^2}{\sum_{i=1}^{n}x_i\log(1-P_i^{1/\alpha})} \ .$

Again, similarly differentiating (3.3.3) with respect to α, we get

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} \log(P_i)\log(1-e^{-x_i})}{\sum_{i=1}^{n}\left[\log(1-e^{-x_i})\right]^2} .$$

3.4 Least square estimator

Here, we obtain the least square estimators of the unknown parameters of the GE distribution with pdf given in (1.1).

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n from GE distribution with cdf given by (1.2). Here, $X_{(j)}$; j=1,2, … ,n denotes the orderd sample.

For a sample size n, we have the following equations

$$E(F(X_j)) = \frac{j}{n+1}$$

$$Var(F(X_j)) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

$$Cov(F(X_j), F(x_k)) = \frac{j(n-k+1)}{(n+1)^2(n+2)},$$

for j<k.

Using the expectations and variances, the least squares method are used in minimizing $\sum_{j=1}^{n} w_j \left(F(X_j) - \frac{j}{n+2} \right)^2$ with respect to the parameters, where w_j are some weights. When $w_j=1$ for all j, we can obtain the least square estimators by minimizing $\sum_{j=1}^{n} w_j \left(F(X_j) - \frac{j}{n+2} \right)^2$ with respect to the unknown parameters for the GE distribution. We can obtain the least square estimator of α and λ by minimizing

$$R = \sum_{j=1}^{n} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right)^2$$

with respect to α and λ.

$$R = \sum_{j=1}^{n} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right)^2 . \tag{3.4.1}$$

Differentiating (3.4.1) with respect to α, we get-

$$\frac{dR}{d\alpha} = 2\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)\frac{d}{d\alpha}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)$$

$$= 2\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)(1-e^{-\lambda x_j})^{\alpha}\log(1-e^{-\lambda x_j})$$

$$Now,\ \frac{dR}{d\alpha} = 0$$

$$\Rightarrow \sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)(1-e^{-\lambda x_j})^{\alpha}\log(1-e^{-\lambda x_j}) = 0. \qquad (3.4.2)$$

Here,

$$\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right) = 0$$

$$\Rightarrow \sum_{j=1}^{n}(1-e^{-\lambda x_j})^{\alpha} = 0$$

$$\Rightarrow \sum_{j=1}^{n}\log(1-e^{-\lambda x_j}) = 0$$

Let us take,

$$\sum_{j=1}^{n}\log(1-e^{-\lambda x_j}) = 0. \qquad (3.4.3)$$

$$\frac{dR}{d\lambda} = 2\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)\frac{d}{d\lambda}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right),$$

again, differentiating (3.4.1) with respect to λ, we get

$$\frac{dR}{d\lambda} = 2\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)\alpha(1-e^{-\lambda x_j})^{\alpha-1}\frac{d}{d\lambda}(1-e^{-\lambda x_j})$$

$$= 2\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1}\right)\alpha(1-e^{-\lambda x_j})^{\alpha-1}e^{-\lambda x_j}x_j$$

$$Now,\ \frac{dR}{d\lambda} = 0$$

$$\Rightarrow \sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha}-\frac{j}{n+1}\right)\alpha(1-e^{-\lambda x_j})^{\alpha-1}e^{-\lambda x_j}x_j=0 \ . \tag{3.4.4}$$

Let us take,

$$\sum_{j=1}^{n}\left((1-e^{-\lambda x_j})^{\alpha}-\frac{j}{n+1}\right)=0 \ . \tag{3.4.5}$$

Now solving (3.4.3) and (3.4.5).

From (3.4.3),

$$\sum_{j=1}^{n}\log(1-e^{-\lambda x_j})=0$$

$$\Rightarrow \sum_{j=1}^{n}(1-e^{-\lambda x_j})=1$$

$$\Rightarrow n-\sum_{j=1}^{n}e^{-\lambda x_j}=1$$

$$\Rightarrow \sum_{j=1}^{n}e^{-\lambda x_j}=n-1$$

$$\Rightarrow \sum_{j=1}^{n}-\lambda x_j=\log(n-1)$$

$$\Rightarrow \hat{\lambda}=-\frac{\log(n-1)}{\sum_{j=1}^{n}x_j} \ . \tag{3.4.6}$$

Using the value of $\hat{\lambda}$ in (3.4.5)

$$\sum_{j=1}^{n}\left[\left\{1-\exp\left(\frac{x_j\log(n-1)}{\sum_{j=1}^{n}x_j}\right)\right\}^{\alpha}-\frac{j}{n+1}\right]=0$$

$$\Rightarrow \sum_{j=1}^{n}\left\{1-\exp\left(\frac{x_j \log(n-1)}{\sum_{j=1}^{n} x_j}\right)\right\}^{\alpha} = \sum_{j=1}^{n}\frac{j}{n+1}$$

$$\Rightarrow \alpha \log \sum_{j=1}^{n} \left\{1-\exp\left(\frac{x_j \log(n-1)}{\sum_{j=1}^{n} x_j}\right)\right\} = \log(\sum_{j=1}^{n}\frac{j}{n+1})$$

$$\Rightarrow \hat{\alpha} = \frac{\log(\sum_{j=1}^{n}\frac{j}{n+1})}{\log \sum_{j=1}^{n} \left\{1-\exp\left(\frac{x_j \log(n-1)}{\sum_{j=1}^{n} x_j}\right)\right\}} \; . \qquad (3.4.7)$$

From (3.4.6) and (3.4.7), the estimators are-

$$\hat{\lambda} = -\frac{\log(n-1)}{\sum_{j=1}^{n} x_j} \quad \textit{and}$$

$$\Rightarrow \hat{\alpha} = \frac{\log(\sum_{j=1}^{n}\frac{j}{n+1})}{\log \sum_{j=1}^{n} \left\{1-\exp\left(\frac{x_j \log(n-1)}{\sum_{j=1}^{n} x_j}\right)\right\}} \; .$$

3.5 Weighted Least Square Estimator

Here, we obtain the weighted least square estimators of the unknown parameter of the GE distribution with pdf given in (1.1).

The weighted least square estimators can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left(F(X_j) - \frac{j}{n+1} \right)^2$$

with respect to two unknown parameters with

$$w_j = \frac{1}{Var(F(X_j))} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

For the GE distribution, we can obtain the weighted least squares estimators of of α and λ by minimizing

$$R = \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right)^2 \tag{3.5.1}$$

with respect to α and λ .

Differentiating (3.5.1) with respect to α ,

$$\frac{dR}{d\alpha} = 2\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right) \frac{d}{d\alpha} \left\{ (1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right\}$$

$$= 2\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right) (1-e^{-\lambda x_j})^{\alpha} \log(1-e^{-\lambda x_j}) \ .$$

$$Now,\ \frac{dR}{d\alpha} = 0$$

$$\Rightarrow \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left((1-e^{-\lambda x_j})^{\alpha} - \frac{j}{n+1} \right) (1-e^{-\lambda x_j})^{\alpha} \log(1-e^{-\lambda x_j}) = 0 \ . \tag{3.5.2}$$

Chapter 4

4.1. Concluding Remarks and Future work

In this thesis, we discuss on several estimation techniques to estimate the unknown parameters of the GE distribution. Particularly, we focus on the methods of maximum likelihood estimator, moments estimator, percentiles estimator, least square estimator, weighted least square estimator. My future work plan is to find out the risk or MSE (mean square error) of the estimators, obtained in the previous section. In this purpose we will take square error and linux loss function. Square error loss function is a balanced loss function. Linux loss function is appropriate in the situation where over-estimation or under-estimation is dangerous than other. To compute the risk, we will use Monte-Carlo simulation.

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