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Study on Elliptic Partial Differential Equations

Final Year, M.Sc. Thesis

Department of Mathematics,
National Institute of Technology, Rourkela
2014-2015

Certificate

This is to certify that the dissertation entitled ***Study of Elliptic Partial Differential Equations*** is a bona-fide record of independent research work done by *Himanshu Singh* (Roll No. *410MA5076*) under my supervision and submitted to *National Institute of Technolgy, Rourkela* in partial fulfillment for the award of the degree of MASTER OF SCIENCE in *Mathematics*.

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Declaration

I *Himanshu Singh*, a bonafide student of ***Integrated M.Sc.*** in ***Mathematics*** of Department of Mathematics, National Institute of Technology, Rourkela would like to declare that the dissertation entitled ***Study of Elliptic Partial Differential Equations*** submitted by me in partial fulfillment of the requirements for the award of the degree of *Master of Science* in Mathematics is my original and authentic work accomplished by me.

PLACE: Rourkela
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Himanshu Singh

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[1]http://www.nitrkl.ac.in/IntraWeb/Department/UploadFaculty/Resume/1141128.pdf
[2]http://www.nitrkl.ac.in/Academic/1Department/Faculty.aspx?hrtuery765hd=MTI

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Abstract

The thesis entitled with *Study on Elliptic Partial Differential Equations* is a serious and a keen study towards the ambit of most influential and practical scenario of Mathematics i.e. Partial Differential Equations. Before in-sighting towards PDE, this thesis includes two chapters dedicated to partial differential equations. Chapter 1 discusses the classification of partial differential equations providing its detailed classification and definitions (see 1.1.1) and also, insight classification of partial differential equations is also included in B. Chapter 1 is an introductory step towards chapter 2 which provides the basic knowledge of symbols and notations that are immensely used in chapter 2. Chapter 1 keeps its concerns with Quasi-Linear Partial Differential Equations and Semi-Linear Partial Differential Equations.

Chapter 2 has a serious agenda of this thesis. Moving further in chapter 2, the governing definition of elliptic partial differential equations is provided (see 2.1.1) which is all the way important in the course of thesis. The section 2.1 provides the outline of all those aspects which have to analysed in section 2.2 which is, thus very important. Chapter 2 begins with the *Reisz- Representation Theorem.* This theorem establishes an important connection between a *Hilbert space* and its (continuous) dual space. Moving on, this thesis presents the *Bilinear Forms* in Elliptic Partial Differential Equations. The second last topic is *Poincr̆e Inequality* which helps in estimating the norm of a function in terms of a norm of its derivative. The *Lax-Milgram Theorem* incorporates as more general form of *Reisz-Representation Theorem*, as it applies to bilinear forms that are not necessarily symmetric.

Part I

Introduction

Chapter 1

Introduction to Partial Differential Equations

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Albert Einstein (1879–1955)

Before making any introductory step towards the ambit of Partial Differential Equation (PDEs), it must be subsumed that PDEs allows one to flex mathematical muscles which empowers one to overcome and provide the solution of many problems that arises in many fields of research principally in physics and engineering.

1.1 Partial Differential Equation

The key defining property of a PDE is that there is more than one independent variable $x, y \ldots$. There is a dependent variable that is an unknown function of these variables $u(x, y \ldots)$. We will often denote its derivatives by subscripts; thus $\partial u/\partial x = u_x$, and so on. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0$$

This is the most general PDE in two independent variables of *first order*. The order of an equation is the highest derivative that appears. The most general *second order* PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

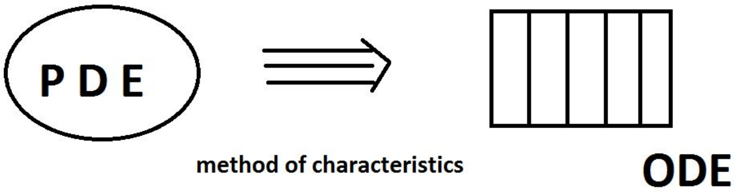
In the context of more mathematical profound grounds, one encounters partial differential equation as an equation involving an unknown function of two or more independent variables and certain of its partial derivatives with respect to those variables.
Say Ω as an open subset of $\mathbb{R}^n$, with $n \geq 2$, and for $k \geq 1$ a fixed integer, an expression of the form

$$E(x, u(x), Du(x), \ldots, D^k u(x)) = 0 \qquad in \ \ \Omega \tag{1.1}$$

is called a kth order partial differential equation, where $E : \mathbb{R}^n \times \mathbb{R} \times \ldots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function and $u : \Omega \to \mathbb{R}$ is an unknown function.
The partial differential equation can be classified as follows: (for insight classification of partial differential equations see B equations (B.10), (B.11) and (B.12) and (B.13) with definition B.2.2.)

Definition 1.1.1. The partial differential equation (1.1) can be studied by classifying them as follows

Figure 1.1: ***Diagramatic representation of MoC***



1. *linear* if it has the form of
$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u(x) = f(x);$$
where a_α (with $|\alpha| \leq k$, α a multindex) and f are given functions. This linear pde is *homogeneous* if $f = 0$.

2. *semilinear* if it is able to satisfy
$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u(x) + E_0(x, u(x), Du(x), \ldots, D^{k-1}u(x)) = 0 \tag{1.2}$$

3. *quasilinear* if possess
$$\sum_{|\alpha|\leq k} a_\alpha(x, u(x), Du(x), \ldots, D^{k-1}u(x))D^\alpha u(x) + E_0(x, u(x), Du(x), \ldots, D^{k-1}u(x)) = 0 \tag{1.3}$$

4. *fully non-linear* otherwise, i.e. if it depends non-linearly upon the highest order derivatives.

Partial differential equations arise naturally as models for many physical phenomena. The unknown function u then describes the state of a physical system (for example, the temperature distribution or the shape of a soap film realizing the least surface area amongst all surfaces spanned by a wire) and the given function E describes the physical laws according to which the state evolves or behaves (possibly also including interaction with external forces).

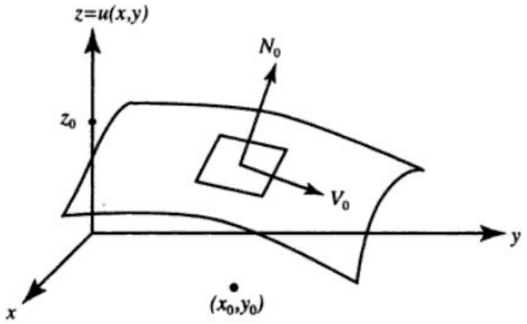
1.1.1 Discussion on Quasilinear and Semi-linear PDE

1. **Quasilinear Partial Differential Equation**
Eq. (1.3) can be transform in simpler terms as follows
$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1.4}$$
where a, b nd c are *continuous* in x ,y and u (function of two variables). By [1] we subsumed the fact that U is an *open subset* of $\mathbb{R}^n$ and $u : U \to \mathbb{R}^n$ is unknown.

§ Method of Characteristic (MoC)

Method of Characteristic abbreviated as MoC is a technique for solving partial differential equation applied over first order PDE, which reduces PDE to a family of ordinary differential equaitons along with the solution can be integrated from some initial provided data.
1.1 describes diagrammatically in a simpler way of inference of MoC.
Letting $z = u(x, y)$ and assuming it to be the solution of eq. (1.4), then we must construe the representation depicted in the figure 1.2 As we see clearly in 1.2, the surface $z = u(x, y)$ has the normal $N_0 = <-u_x, -u_y, 1>$ at $(x_0, y_0, u(x_0, y_0))$ and the vector V_0 (say) defined as
$$V_0 = <a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0)>$$
and one can easily realize that V_0 is $\perp$ to N_0 as V_0 is the tangent plane to the graph of $z = u(x, y)$ (see 1.2).
Generalising $V = <a, b, c>$ on (x, y, z) defines a *vector field* in $\mathbb{R}^3$ surface which are tangent to a vector field in $\mathbb{R}^3$ are called as ***Integral Surfaces*** and same goes for ***Integral Curves***.

Figure 1.2: ***Vector field V tangent to the graph***



The Cauchy Problem Our aim is to find the integral surface containing a given curve $\Gamma \subset \mathbb{R}^3$ which leads to the following problem stipulated as

> *Can we find a solution u of the first order partial differential equation for a curve Γ in $\mathbb{R}^3$ whose graph contains Γ.*

Methodology to solve the Cauchy Problem Using the characteristics curves which are the integral curves of V i.e.

$$\chi = (x(t), y(t), z(t))$$

is characteristics if it is able to satisfy following system of ordinary differential equations generally called as *characteristic equations*

$$\begin{cases} \dfrac{dx}{dt} = a(x,y,z) & \text{(1.5a)} \\ \dfrac{dy}{dt} = b(x,y,z) & \text{(1.5b)} \\ \dfrac{dz}{dt} = c(x,y,z) & \text{(1.5c)} \end{cases}$$

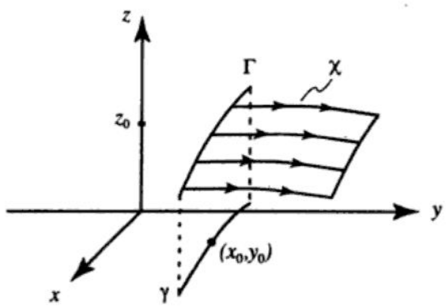
Now, (1.5) can be solved uniquely for $|t - t_0|$, provided the initial conditions as

$$x(t_0) = x_0; \; y(t_0) = y_0 \;\&\; z(t_0) = z_0$$

assuming a, b and c are all continuously differentiable in x, y and z.

Note 1.1.1. $z = u(x,y)$ is a smooth surface S which is a union of characteristic curve, then at each point of (x_0, y_0, z_0) the tangent plane contains the vector $V(x_0, y_0, z_0)$ hence S is supposed to be *integral surface* (see 1.3)

Figure 1.3: ***Smooth Surface***



realising that $x_0 = f(s)$, $y_0 = g(s)$ and $z_0 = h(s)$ as initial conditions which suggests analytically that, the construction of an integral surface Γ as to be contained (see [3]).

2. **Semi-linear Equations**
Cauchy problem for semi-linear equations in two variable is given as

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u \tag{1.6}$$

and Γ is parametrised as $(f(s), g(s), h(s)$ and characteristic equations (as in (1.5)) with initial conditions formulated as

$$\begin{cases} x(s,0) = f(s) & (1.7a) \\ y(s,0) = g(s) & (1.7b) \\ z(s,0) = h(s) & (1.7c) \end{cases}$$

and we have projected characteristic curve χ.

Example 1.1.1. *Consider semi-linear problem $u_x + 2u_y = u^2$ with $u(x,0) = h(x)$ assuming Γ lies in xz plane.*

Solution. Parametrising Γ as $(s, 0, h(s))$ and also, we see that,

$$\begin{cases} \dfrac{dx}{dt} = 1 & (1.8a) \\ \dfrac{dy}{dt} = 2 & (1.8b) \\ \dfrac{dz}{dt} = z^2 & (1.8c) \end{cases}$$

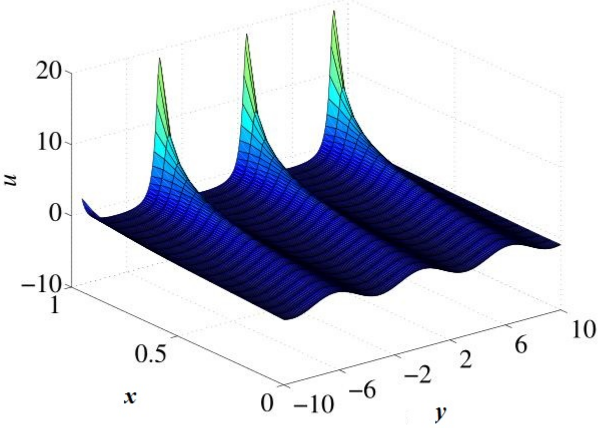
which makes us to realise that

$$x = t + c_1(s)$$
$$y = 2t + c_2(s)$$
$$z = \frac{-1}{t + c_3(s)}$$

So, the solution of the given problem is supposed to be $z = \frac{h(x-\frac{y}{2})}{1-\frac{y}{2}h(x-\frac{y}{2})}$ ♠

Note 1.1.2. If $h(x) = \cos x$, then the solution to the above problem can be evaluated analytically as follows

Figure 1.4: ***Example 1.1.1 with*** $h(x) = \cos x$



Part II

Study on Elliptic Partial Differential Equations

Chapter 2

Elliptic Partial Differential Equations

"There has to be a mathematical explaination for how bad that tie is".

Russell Crowe as
John Forbes Nash Jr. in
A Beautiful Mind (2001)

2.1 Definition

If the boundary problem is posed as follows

$$\begin{cases} Lu = f & \text{in } U \quad (2.1a) \\ u = 0 & \text{on } \partial U \quad (2.1b) \end{cases}$$

where U is open bounded subset of $\mathbb{R}^n$ and $u : \bar{U} \to \mathbb{R}$ with $f : U \to \mathbb{R}$, where L is second order partial differential operator having following form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{2.2}$$

for given coefficients a^{ij}, b^i and c and for symmetry condition $a^{ij} = a^{ji}$ (for more details see B.2 of B).

Definition 2.1.1. We say the partial differential operator L is uniformly elliptic $\exists$ a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta \left|\xi\right|^2 \tag{2.3}$$

for $a.e.$ $x \in U$ and $\forall$ $\xi \in \mathbb{R}^n$.

Ellipticity thus means that for each point $x \in U$, the symmetric matrix $\mathbf{A}(x) = (a^{ij}(x))$ is positive definite, with smallest eigenvalue greater than or equal to θ.

2.1.1 Adjoints and Weak Solutions

In particular, weak solutions may be defined for linear equations using integration by parts and a 'test functions': this leads to the notion of the "adjoint"of a linear operator.

Definition 2.1.2. A function f defined on Ω is called as ***test function*** if $f \in C^\infty(\Omega)$ and there is a compact set $K \subset \Omega$ such that the support of f lies in K. The set of all test function is represented as $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

Consider

$$\begin{aligned}
\langle Lu, v\rangle &= \int_U (\sum a^{ij}(x) u_{x_i x_j}) v \\
&= \sum \int_U a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} v \\
&= -\sum \int_U a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \\
&= \sum \int_U a^{ij} u \frac{\partial^2 v}{\partial x_j \partial x_i} \\
&= \int_U u \sum a^{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} \\
&= \langle u, L^* v\rangle
\end{aligned}$$

thus,

$$\langle Lu, v\rangle = \langle u, L^* v\rangle \tag{2.4}$$

Here, we have repeated the integration by parts process and also, we can repeat integration by parts with any combination of derivatives (see [3]), $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ to obtain

$$\int_U (D^\alpha u) v \, dx = (-1)^m \int_U u D^\alpha v \, dx \qquad (m = |\alpha|) \tag{2.5}$$

$\forall\ u \in C^m(U)$ and $v \in C_0^m(U)$. After confronting eqs. (2.4) and (2.5), we deduce that

$$L^* v = \sum_{\alpha \le m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) v) \tag{2.6}$$

referring L^* as the adjoint of L and is an m^{th} -order linear differential operator with continuous coefficients ($a_\alpha \in C^\alpha(U)$).
Now, if u satisfies $Lu = f$ in U, then

$$\int_U u L^* v \, dx = \int_U f v \, dx \tag{2.7}$$

holds for every $v \in C_0^m(U)$. But (2.7) no longer requires u to have continuous derivatives; in general, u (and f) need only be integrable function compact subsets of U (i.e. belong to $L^1_{loc}(U)$) and thus, this leads to define a function $u \in L^1_{loc}(U)$ to be a *weak solution* of $Lu = f$ if (2.7) holds for every $v \in C_0^m(U)$. Particularity, if $L = \partial/\partial x_k$ and u is a weak solution of $\partial u/\partial x_k = f$, then we say f is the *weak derivative* of u.

2.1.2 Distributions

We now define the space of distributions. Here, we wish to have a generalized notion of a 'function'.

Definition 2.1.3. A **distribution** or **generalized function** is a linear mapping $\phi \mapsto (f, \phi)$ from $\mathcal{D}(\Omega)$ to $\mathbb{R}$ which is continuous in the following sense: If $\phi_n \to \phi$, then $(f, \phi_n) \to (f, \phi)$. The set of all distributions is called as $\mathcal{D}'(\Omega)$.

Example 2.1.1. *A current flowing along a curve $\mathcal{C} \subset \mathbb{R}^3$ is an example of a vector-valued distribution. If $\boldsymbol{j} : \mathcal{C} \to \mathbb{R}^3$ is integrable, then for $\boldsymbol{\phi} \in \mathbb{R}^3$ we define*

$$(\boldsymbol{j}, \boldsymbol{\phi}) = \int_{\mathcal{C}} \boldsymbol{j}(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x})$$

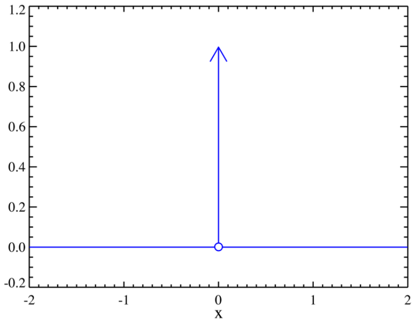
where $d\sigma(\boldsymbol{x})$ indicates integration with respect to arc length on $\mathcal{C}$.

The most important example for *distributions* is *Dirac delta function.* We assume that Ω contains the origin, and we define

$$(\delta, \phi) = \phi(\mathbf{0}) \tag{2.8}$$

It must be noted that the continuity of the functional follow from the fact that convergence of a sequence implies point wise convergence.
Figure 2.1 is the schematic representation of the Dirac delta function by a line surmounted by an arrow. The height of the arrow is usually used to specify the value of any multiplicative constant, which will give the area under the function.

Figure 2.1: ***Dirac delta function***



2.1.3 Notion of Weak Derivative

We denote by $L^1_{loc}(\mathbb{R})$ the space of locally integrable functions $f : \mathbb{R} \to \mathbb{R}$. These are the Lebesgue measurable functions which are integrable over every bounded interval.

Definition 2.1.4. The support of a function ϕ, denoted by $supp(\phi)$, is the closure of the set $x : \phi(x) \neq 0$ where ϕ does not vanish.

$$supp(\phi) = \overline{\{x \in \Omega \mid \phi(x) \neq 0\}} \tag{2.9}$$

Equation (2.9) can also be construed as follows (for details; follow [9],[10])

$$supp(\phi) = \Omega \backslash \{y \in \Omega \mid \exists\ neighbourhood\ y \in \mathcal{U} : \phi_{\mathcal{U}} = 0\} \tag{2.10}$$

Example 2.1.2. *Consider*

$$\phi(x) = \begin{cases} 1 - x^2 & if\ \ |x| < 1 \\ 0 & if\ \ |x| \geq 1 \end{cases} \tag{2.11}$$

then the $supp(\phi)$ *is* $[-1, 1]$ *and because* $\phi \neq 0$ *in* $(-1, 1)$, *also the closure of* $(-1, 1)$ *is* $[-1, 1]$.

Observation 2.1.1. *We have following properties of supp as follows:*

Let f *and* $g \in C(\Omega)$, *then*

1. *if* $f = 0 \Leftrightarrow supp(f) = \emptyset$
2. $supp(f)$ *is closed in* Ω
3. $supp(f \cdot g) \subseteq supp(f) \cap supp(g)$
4. $supp(f)$ *is the compliment of the largest open subset of* Ω *where* f *does not vanish.*

By $C^\infty_c(\mathbb{R})$ we denote the space of continuous functions with compact support, having continuous derivatives of every order. Every locally integrable function $f \in L^1_{loc}(\mathbb{R})$ determines a linear functional $\Lambda_f : C^\infty_c(\mathbb{R} \mapsto \mathbb{R})$ namely

$$\Lambda_f(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\, dx \tag{2.12}$$

well defined $\forall\ \phi \in C^\infty_c(\mathbb{R})$ as ϕ vanishes outside a compact set.

Moving further assuming that f is continuously differentiable, in turn f' determines a linear functional on $C_c^\infty\mathbb{R}$ as follows

$$\Lambda_{f'}(\phi) = \int_{\mathbb{R}} f'(x)\phi(x)\,dx = \int_{\mathbb{R}} f(x)\phi'(x)\,dx \tag{2.13}$$

Definition 2.1.5. Given an integer $k \geq 1$ the *distributional derivative* of order k, of $f \in L^1_{loc}$ is the linear functional

$$\Lambda_{D^k f}(\phi) = (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x)\,dx$$

If there exists a locally integrable function g such that $\Lambda_{D^k f}(\phi) = \Lambda_g$ as

$$\int_{\mathbb{R}} g(x)\phi(x)\,dx = (-1)^k \int_{\mathbb{R}} f(x) D^k \phi(x)\,dx \quad \forall\ \phi\ \in\ C_c^\infty(\mathbb{R})$$

then we say that g is the *weak derivative* of order k of f.

Example 2.1.3. *Consider the function*

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \tag{2.14}$$

. Equation (2.14) *subsumes the distributional derivative which is as follows*

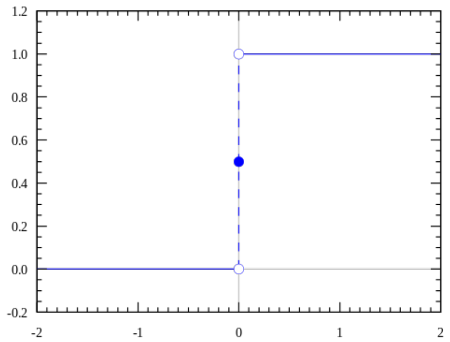
$$\Lambda(\phi) = \int_0^\infty x \cdot \phi'(x)\,dx = \int_0^\infty \phi(x)\,dx = -\int_{\mathbb{R}} H(x)\phi(x)\,dx$$

where

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \tag{2.15}$$

The Heaviside function H in eq (2.15) *is the weak derivative of f.*

Figure 2.2: ***Plot of** H **in*** (2.15)



§ ***Discussion on*** $H(x)$

As eq (2.7) suggests that (with the advantage of formal integration by parts), we deduce that $H'(x)$ should satisfy

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x)v(x)\,dx &= -\int_{-\infty}^{\infty} H(x)v'(x)\,dx \\ &= -\int_0^\infty v'(x)\,dx = v(0) - v(\infty) = v(0)\end{aligned}$$

for every $v \in C_0^1(\mathbb{R})$ and because $H'(x) = 0$ for $x \neq 0$, this suggests that $H'(0) = \infty$ in such a manner $\int H'(x)v(x)\,dx = v(0)$ and thus

$$\int_{-\infty}^{\infty} H''(x)v(x)\,dx = \int_{-\infty}^{\infty} H(x)v''(x)\,dx = \int_{0}^{\infty} v''(x)\,dx = -v'(0)$$

$$\int_{-\infty}^{\infty} H'''(x)v(x)\,dx = \int_{-\infty}^{\infty} H(x)v'''(x)\,dx = -\int_{0}^{\infty} v'''(x)\,dx = v''(0)$$

provided v is sufficiently smooth and has compact support $v \in c_0^\infty(\mathbb{R})$. This is called as *distributional derivatives* and the function in $C_0^\infty(\mathbb{R})$ are called as *test functions*.
In the context of delta distribution (see eqs (2.5), (2.7) & (2.8)) $\delta(x)$, we have

$$\int_{\mathbb{R}^n} \delta(x)v(x)\,dx = v(0) \tag{2.16}$$

and we also construe that $H'(x) = \delta(x)$ if $D^\alpha\delta(x)$ is something satisfying

$$D^\alpha(x)\delta(x)v(x)\,dx = (-1)^{|\alpha|}\int \delta(x)^\alpha v(x)\,dx = (-1)^{|\alpha|}D^\alpha v(0)$$

Note 2.1.1. The vector space $C_0^\infty(\Omega)$ of test functions is often denoted as $\mathcal{D}(\Omega)$ (see 2.1.3) & the space of distributions is denoted as its *dual space* $\mathcal{D}'(\Omega)$. If $v \in \mathcal{D}(\Omega)$ and $F \in \mathcal{D}'(\Omega)$, we also denote the action of F on v by $\langle F, v\rangle$ or even $\int_\Omega F(x)v(x)\,dx$ which is called as a *distributional integral*.

2.1.4 Convolutions and Fundamental Solutions

Firstly, defining the *convolutions* for two functions f and g supposing that they lie in $\mathbb{R}^n$ as follows

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \tag{2.17}$$

considering the situation when the integral converges. Say for instance, $f,\ g \in L^1_{loc}(\mathbb{R}^n)$, and at least one exhibiting one compact support, then (2.17) converges and $f \star g$ is well defined.[1] Assuming that $v \in C_0^\infty(\mathbb{R}^n)$, considering $f \star g$ as a distribution, then the following computations explains about the value of linear functional $f \star g$ on v;

$$\begin{aligned}\langle f \star g, v\rangle &\equiv \int_{\mathbb{R}^n} f \star g(x)v(x)\,dx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)v(x)\,dydx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)g(y)v(z+y)\,dydz,\end{aligned} \tag{2.18}$$

Therefore, defining the *convolutions of distributions* F and G:

$$\langle F \star G, \rangle \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(z)G(y)v(z+y)\,dydz \tag{2.19}$$

which is well defined provided that F or G has compact support.

Lemma 2.1.1. *If $f \in C(\mathbb{R}^n)$ and $g \in L^1_{loc}(R)^n$, one of which has compact support, then $f \star g \in C(\mathbb{R}^n)$.*

Corollary 2.1.1. *If $f \in C^k(\mathbb{R}^n)$ and $g \in L^1_{loc}(R)^n$, one of which has compact support, then $f \star g \in C^k(\mathbb{R}^n)$.*

[1] And, one can show with the help of change of variables $z = x - y$, that

$$f \star g = g \star f,$$

thus commutative property is not annihilated in the scenario of convolution.

Discussing about the distributional derivatives of a convolution as follows:

$$\begin{aligned}\langle D^\alpha(F\star G), v\rangle &= (-1)^{|\alpha|}\langle F\star G, D^\alpha v\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(z)G(y)D_z^\alpha v(z+y)\,dydz\\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(z)G(y)D_y^\alpha v(z+y)\,dydz\end{aligned}$$

which in turn implies

$$D^\alpha(F\star G) = (D^\alpha F)\star G = F\star D^\alpha G \tag{2.20}$$

The convolution used in solving non-homogeneous equation;

$$Lu = \sum_{|\alpha|=m} = a_\alpha(x)D^\alpha u = f^2, \tag{2.21}$$

where a_α are constants. Solution $u = F$ being the *fundamental solution* of L, and

$$\langle F, L'v\rangle = \int_{\mathbb{R}^n} F(x)L'v(x)\,dx = v(0)$$

for all $v \in C_0^\infty(\mathbb{R}^n)$. Therefore, f is a distribution with compact support and form the convolution with F

$$u(x) = F\star f(x) = \int_{\mathbb{R}^n} F(x-y)f(y)\,dy$$

Then u is a distribution solution of eq (2.21) because $Lu = \sum_\alpha a_\alpha D^\alpha(F\star f) = \sum_\alpha a_\alpha D^\alpha F\star f = \delta\star f = f$

2.2 Analysis of Elliptic Partial Differential Equations

2.2.1 Reisz-Representation Theorem

Theorem 2.2.1. *The dual space of a Hilbert space is an isometric*[3] *to the Hilbert space itself. In particular,* $\forall\ x\ \in H$ *the linear functional on* H *is defined by*

$$l_x(y) = (x, y) \tag{2.22}$$

is bounded with norm $\|l_x\|_{H^*} = \|x\|_H$. *Moreover,* $\forall\ l\ \in\ H\ \exists\ !\ x\ \in\ H$, *such that*

$$l(y) = (x, y)\quad \forall\ y \in\ H \tag{2.23}$$

and furthermore, $\|x\|_H = \|l\|_{H^*}$

In other words, every bounded linear functional ϕ on H can be represented uniquely in the form of $\phi(u) = (u, v)$ with a suitable element v of H.

2.2.2 Bilinear Forms

Definition 2.2.1. 1. The bilinear form $B[\ ,\ ]$ associated with divergence form elliptic operator L defined by (2.2) is

$$B[u,v] = \int_\Omega \sum_{i,j=1}^n a^{i,j}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\,dx \tag{2.24}$$

$\forall\ u,\ v \in H_0^1(\Omega)$.

2. Also, we say that $u \in H^1(\Omega)$ is a weak solution of the boundary problem (2.1) if

$$B[u,v] = (f,v)\ \forall\ v\ \in\ H_0^1(\Omega) \tag{2.25}$$

where $(\ ,\ )$ denotes the *inner product* in $L^2(\Omega)$.

[2]Particlularly, replace f by the delta distribution δ, refer (2.8) here.

[3] distance-preserving map between metric spaces

In more formal manner of defining *Bilinear form*, one can easily define as follows; (for more details see [11])

Definition 2.2.2. H being a Hilbert space, then $B : H \times H \to \mathbb{R}$ such that $a(x, y)$ is a linear in each $x,\ y \in H$ i.e. $\forall\ u_1,\ u_2,\ w \in\ H$ and $c_1,\ c_2\ \in\ \mathbb{R}$, we have

$$B(c_1u_1 + c_2u_2, w) = c_1B(u_1, w) + c_2B(u_2, w)$$

$$B(w, c_1u_1 + c_2w_2) = c_1B(w, u_1) + c_2B(w, u_2)$$

2.2.3 The Poincaŕe Inequality

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean.

Theorem 2.2.2 (The Poincaŕe Inequality)**.** *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded in some direction. Then $\exists$ a constant C such that*

$$\int_\Omega u^2\,dx \leq \int_\Omega C\,|Du|^2\ dx\ \forall\ u \in\ H_0^1(\Omega) \tag{2.26}$$

2.2.4 The Lax-Milgram Theorem

Theorem 2.2.3. *Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which $\exists$ constants α and $\beta \geq 0$ such that

$$|B[u, v]| \leq \alpha\|u\|\|v\|\ u,\ v\ \in\ H \tag{2.27}$$

and

$$\beta\|u\|^2 \leq B[u, v]\ u \in\ H. \tag{2.28}$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then $\exists$ a ! element $u \in H$ such that

$$B[u, v] = \langle f, v\rangle\ \forall\ v \in\ H \tag{2.29}$$

Note 2.2.1. With the assurance of [12], it should be noted that the Lax-Milgram theorem is not a particular case of the Riesz theorem. It is actually more general, since it applies to bilinear forms that are not necessarily symmetric, and it implies the Riesz theorem when the bilinear form is just the scalar product. An example of bilinear form is $\int_\Omega \nabla u \cdot \nabla v\, dx$ on $H_0^1(\Omega)$.

Part III

Final Discussion

Chapter 3

Final Discussion

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country".

David Hilbert (1862-1943)

Chapter 1 deals with the introduction of partial differential equations on the grounds of mathematical terms which are sincerely adopted and implemented in chapter 2. The *Method of Characteristic* is a tool that is discussed in chapter 1 to reduce a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hypersurface[1]. The method of characteristics discovers curves (called *characteristic curves* or simply *characteristics*) along which the PDE becomes an ordinary differential equation (ODE). Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE. A detailed procedure is stipulated in example 1.1.1.

3.1 Observation on MoC

Observation 3.1.1 ([17]). *One can use the crossings of the characteristics to find shock waves for potential flow in a compressible fluid. Intuitively, we can think of each characteristic line implying a solution to u along itself. Thus, when two characteristics cross, the function becomes multi-valued resulting in a non-physical solution. Physically, this contradiction is removed by the formation of a shock wave, a tangential discontinuity or a weak discontinuity and can result in non-potential flow, violating the initial assumptions.*

Chapter 2 is the main concern of this course which focusses its interest on elliptic partial differential equations, mathematically defined by eq. (2.3). Proceeding further, we have *adjoint* of second order partial differential equation operator defined by eq. (2.2). Some serious topics also encounters on this platform which are *weak solutions*, *distributions*, *weak derivatives* and *convolutions* and *fundamental solutions*. The last section of chapter 2 has the prime motive of the thesis, which has to be finally discussed here.

3.2 Observation on Elliptic Partial Differential Equations

Observation 3.2.1 (see [1] and [4]). *The inequality that appears in eq.* (2.28) *of the* Lax-Milgram Theorem *is referred as* **coercive** *when the mapping B is able to satisfy if for some $\beta > 0$. The inequality* (2.28) *can be thought of as an* energy estimate*. The inequality says that the energy (the norm squared) can only blow up as fast as the bilinear form). The* Lax-Milgram Theorem *incorporates as more general form of* Reisz-Representation Theorem*, as it applies to bilinear forms that are not necessarily symmetric.*

[1]A generalization of the concept of an ordinary surface in three-dimensional space to the case of an -dimensional space. The dimension of a hypersurface is one less than that of its ambient space (for more details see [16].

Part IV

Appendix

Appendix A

Differences between Ordinary and Partial Differential Equation

A.1 Preliminaries

Let us recall that a *differential equation* is an equation for an unknown function of several independent variables (and of functions of these variables) that relates the value of the function and of its derivatives of different orders. An *ordinary differential equation* (ODE) is a differential equation in which the functions that appear in the equation depend on a single independent variable.

A.2 Differences

1. A general solution of an ODE involves arbitrary constants. Obtaining a general solution for PDEs is difficult and a general solution would involve arbitrary functions. Let us look at a simple example now. Consider the PDE $u_x = 0$. Any arbitrary function of y solves this PDE. This is the simplest possible linear equation of first order and it has an infinite dimensional space of solutions. Compare this situation with that of a linear first order ODE $\frac{d\mathbf{y}}{dt} = 0$ where $\mathbf{y} = (y_1, \ldots, y_p)$, whose solution space is $\mathbb{R}^p$ which is finite dimensional.

2. In differential equations the unknown function has the interpretation of the state of a system when the equations describes the evolution of a physical system in time. For ODEs the independent variable is time and for PDEs one of the independent variables has the interpretation of time. Now the initial state (state of the system at time $t = 0$) for ODEs is prescribed as an element of $R^n(n$ is the length of the unknown vector $y)$; while for PDEs the initial state varies in a function space. Thus solving a PDE means finding the states of the system at different times and each of these states vary in an infinite dimensional space of function while solving ODE means finding the states of the system but are in a finite dimensional space.

3. Linear ODEs have global solutions. Linear PDEs posed on $\mathbb{R}^2$ do not necessarily have solutions defined on $\mathbb{R}^2$.

Appendix B

Insight Classification of PDEs

Some linear, second-order partial differential equations can be classified as parabolic, hyperbolic and elliptic. The classification provides a guide to appropriate initial and boundary conditions, and to smoothness of the solutions.

B.1 The Symbol of a Differential Equation

The notation of multi-indices is very convenient in avoiding excessively cumbersome notations in PDEs. A multi-index is a vector

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

whose components are non-negative integers. The notation $\alpha \geq \beta$ indicates that $\alpha_i \geq \beta_i \ \forall \ i$. For any multi-index α, we make the following definitions:

(B.1)
$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

Moreover, any vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \ \in \ \mathbb{R}^n$, we set

(B.2)
$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

The following notation goes for partial derivatives:

(B.3)
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

For example, if $a\alpha = (1, 2)$

$$D^\alpha u = \frac{\partial^3 u}{\partial x_1 \partial x_2^2}$$

Now, consider linear differential expression

(B.4)
$$L(\mathbf{x}, D)u = \sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) D^\alpha u$$

where $u : \mathbb{R}^n \to \mathbb{R}$. With this analytic operation on functions we associate an algebraic operation called the *symbol*.

Definition B.1.1. The symbol of (B.4) is given by

(B.5)
$$L(\mathbf{x}, \iota\xi) = \sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) (\iota\xi)^\alpha$$

and the **principal part** of the symbol is

(B.6)
$$L^p(\mathbf{x}, \iota\xi) = \sum_{|\alpha| = m} a_\alpha(\mathbf{x}) (\iota\xi)^\alpha$$

Example B.1.1. *Symbol of Laplace's operator* $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ *is* $-\xi_1^2 - \xi_2^2$, *heat operator* $\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}$ *is* $\iota\xi_1 + \xi_2^2$ *and that of wave operator* $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ *is* $-\xi_1^2 + \xi_2^2$. *For the Laplace and wave operator, the symbols are equal to their principal parts; the principal part for the heat operator is* ξ_2^2.

B.2 Scalar Equation of Second Order

Say

$$Lu = a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y) \tag{B.7}$$

The principal part of symbol of (B.7) is

$$L^p(x,y;\iota\xi,\iota\eta) = -a(x,y)\xi^2 - b(x,y)\xi\eta - c(x,y)\eta^2 \tag{B.8}$$

One should realize that (B.8) is quadratic form, and hence after deducing the matrix form of the same is as follows

$$L^p(x,y;\iota\xi,\iota\eta) = (\xi,\eta)\begin{pmatrix} -a(x,y) & -1/2b(x,y) \\ -1/2b(x,y) & -c(x,y) \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{B.9}$$

Recall that a quadratic form is called *definite* if the associated symmetric matrix is (positive or negative) definite, it is called indefinite if the matrix has eigenvalues of both signs, and it is called *degenerate* if the matrix is singular.

Definition B.2.1. The differential equation (B.7) is called *elliptic* if the quadratic form given by (B.8) is strictly definite, *hyperbolic* if it is indefinite and *parabolic* if it is degenerate.

Example B.2.1. *Laplace's equation is elliptic, the heat equation is parabolic and the wave equation is hyperbolic. For these three cases, the matrices associated with the principal part of the symbol are*

Elliptic

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.10}$$

Parabolic

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.11}$$

Hyperbolic

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{B.12}$$

Consider, now a second-order PDE in n space dimensions:

$$Lu = a_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(\mathbf{x})\frac{\partial u}{\partial x_i} + c(\mathbf{x})u = 0 \tag{B.13}$$

Because the matrix of second partials of u is symmetric, we may assume without loss of generality that $a_{ij} = a_{ji}$. The principal symbol of this second-order PDE is still a quadratic form in ξ; we can represent this quadratic form as $\xi^T A(\mathbf{x})\xi$, where A is the $n \times n$ matrix with components $-a_{ij}$.

Definition B.2.2. Equation (B.13) is called *elliptic* if all eigenvalues of A have the same sign, *parabolic* if A is singular and *hyperbolic* if all but one of the eigenvalues of A have the same sign and one has the opposite sign.

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