Some Studies On Control Theory Involving Schrodinger Group

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Himani Garg

Under The Supervision of

Prof.K.C.Pati

National Institute of Technology
Rourkela, India

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Place: NIT Rourkela

Date: May 2015

(Himani Garg)

Roll No-413MA2060

Certificate

This is to certify that the thesis entitled **"Some studies On Control Theory involving Schrodinger Group"**, which is being submitted by **Himani Garg** in the Department of Mathematics, National Institute of Technology, Rourkela, in partial fulfillment for the award of the degree of **Master of Science**, is a record of bonafide review work carried out by her in the Department of Mathematics under my guidance. She has worked as a project student in this Institute for one year. In my opinion the work has reached the standard, fulfilling the requirements of the regulations related to the Master of Science degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

Place: NIT Rourkela
Date: May 2015

Dr. K.C.PATI
Professor
Department of Mathematics
NIT Rourkela-769008
India

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Introduction

1 Some Terms Associated With Control Theory

The theme of this chapter is to introduce some basic concepts and results related to Control theory which will be required latter. The first two sections deals with control theory on a vector space .The particular section contains some definitions and results for control systems on a vector space.Where as the last section provides an preface of geometric control theory. A description of control theory on a manifold is given.The chapter ends with some well-known results of control theory on Lie groups.A general exposition is given and some results without proofs are presented. Further details can be found in the cited references.

1.1 Control System

Let **V** be an n-dimensional vector space , called the state space,and let $(x\mathbf{V}$ be a state vector. A control system σ on **V** is defined by

$$\dot{y} = f(y, u(t)), x(t_0) = x_0 \tag{1}$$

where the control functions u belongs to a class **U**of admissible controls with values in a subset of R^m and f is continuously differentiable.It is provided that sufficiently smooth control function $(u\epsilon\mathbf{U})$, is a solution of the system termed as a trajectory, is determined. Such type of solution can be explained using the transition function ϕ. Specifically $\phi(t, t_0, x_0, u)$ denotes the state that results at time t if the system was in state x_0 at time t_0 and the control u was applied.

Definition the state z can be **reached from** the state x if and only if there is a trajectory of σ whose initial state is x and whose final state is z,that is , if there exist $u\mathbf{U}$ such that $\phi(t_f, 0, x, u) = z$. One can also say that x can be **controlled** to z. The controllable set at t_1 is the set of initial states that can be controlled to the origin in time t_1 using an admissible control , that is,

$$C(t_1) = x_0 : \phi(t_1, 0, x_0, u) = 0 for some u \epsilon U \quad (2)$$

The **controllable set C** is the set of states that can be controlled to the origin in any finite time i.e,

$$C = u_{t_1 >= 0} C(t_1)$$

.

The system σ is called **controllable** at x if z can be controlled to x for all $z \epsilon V$.Therefore , σ is controllable at the origin if and only if **C=V**.
If all initial states can be controlled to x for all $x \epsilon V$, then the system σ is **controllable**.

1.2 Linear Control Systems

A linear control system σ is defined as

$$\dot{x} = Ax + Bu$$

where A(n,n)and B(n,m) are scalar matrices and the dimension of the state space is n and the control $u \epsilon U$, where **U** is the class of integrable functions of t. The solution of the system 2 starting at x_0 has the form

$$x(t) =$$

we can define the exponential of a matrix by the using the definition of infinite series

$$exp(A) = \frac{A^k}{k}$$

It follows that $x_0 \epsilon C(t_1)$ if and only if there is an admissible control $u \epsilon U$ such that

$$x_0 = -exp(-A$$

The following lemma shows some controllability equivalences for a linear system.

Lemma 1. If σ is a linear control system , then
(i) State space(x) is controllable to another State space (z) iff the origin 0 is controllable to $z - exp(At)x$.
(ii) The Control system σ is controllable iff the origin is controllable to y for all $y \epsilon R^m$.

Proof . **(i)** Note that x can be controlled to z that implies there exist an admissible control $u \epsilon U$ such that

$$z = exp(At)(x+$$

$$z - exp(At)x = exp(At)(exp(-A)$$

the origin can be controlled to $z - exp(At)x$ by definition.

Proposition C(t_1)and C are both symmetric and convex.

Example Consider the linear system given by the following state equations

$$\dot{x_1} = x_1 + u, \dot{x_2} = x_2 + u$$

where $u \epsilon U_b$ and the matrices A and B are given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3)$$

so we have that x=(x_1,x_2) belongs to C(t_1) if

$$x_1 = -\int_0^{t_1} exp(-\tau)u_1 d\tau = x_2$$

since $|u| \leq 1$ then $|x_1| \leq 1 - exp(-t_1)$.
Therefore, C(t_1) is the closed diagonal segment C(t_1) =x_1=x_2:$|x_1| \leq 1 - exp(-t_1)$ and C is the open diagonal segment C=x_1=x_2:—$x_1 \leq |1$.

In general , it would be impossible to control both components simultaneously with identical controls.To control both components using the same control , the initial deviation of the component must be equal.

To get controllability there are two necessary conditions on the controllable set, namely it must have full dimension and be bounded.

1.3 The Reachable Set

In this section the properties of the reachable set for the linear systems are presented. These properties will be compared to results of linear systems on Lie Groups.

Define the reachable set R(x_0,t_1 as the set of points that can be reached from the initial state x_0 in time t_1. As we know there is a reciprocal relationship between reachable sets and the controllable sets , namely if $z \in R(x_0, t_1)$ then $x_0 \in C(z, t_1)$.
if the system σ is linear then $x_1 \in R(x_0, t_1)$ implies

$$x_1 = exp(At_1)(x_0 + \int_0^{t_1} exp(-A\tau)Bu(\tau)d\tau)$$

for some $u \in U$. The following results are straightforward.

Proposition. If the system σ is linear tehn $R(x_0, t) = exp(At)x_0 + R(0, t)$. The property that any point in R^n can be reached from the origin is called **reachability**.

2 Some Terms Associated With Lie Group and their Lie algebra

In simple geometry Lie groups are smooth manifolds.They can be studied using differential calculus ,They are the most general admirable space than that of topological spaces .

Definition: A well defined set G is termed as a Lie group if it satisfies the following conditions:
1)If Set G is a smooth manifold.

2)Must satisfies the Group properties.

3) The all Group operations must be smooth.

2.1 Lie Algebra

Let **K** be a field. A Lie algebra over **K** is an vector space L, along with a bi-linear map, known as Lie bracket given by:

$$L \times L \to L, (x, y \to [x, y])$$

this lie bracket satisfies the following properties:

$$[y, y] = 0 \tag{4}$$

$$[x, [y, z]] = [y, [z, x]] + [z, [x, y]] \tag{5}$$

This $[., .]$ is also known as commutator of Lie algebra.

2.2 Relation between Lie group and its Lie algebra

In the general theory it has been shown that the structure(vector space) of the Lie algebra of a Lie group is isomorphic to the tangent space at the identity element of the Lie group. Consider in **GL(n,C)** a subset of operators depends onto a real parameter t and satisfies $A(0) = 1$, where 1 is the identity operator on the given vector space V.Now we can consider the tangent vector at $t = 0$ with the aid of Taylor's expansion of A(t) upto first order term.

$$A(t) = A(0) + N(t) + O(t^2)$$

where N determines the derivative of A(t) at t=0.

Hence these linear operators N obtained by this way are the elements of Lie algebra of $GL(n, C)$. With the help of the basis elements $k_1, k_2, ..., k_n$ these operators can be represented by $n \times n$ matrices say n_{ij} .On consideration of all possible smooth curves through the unit element of the group ,we will be having a vector space of the Lie algebra whose dimension will be n^2 consisting of nn matrices. Now in order to obtain the Lie bracket of the elements i.e. for $M, N \in N(n, C)$, assume the group commutator:

$$C(t) = A(t)B(t)A^{-1}(t)B^{-1}(t)$$

Here $(0) = M$ and $(0) = N$ represents the tangent vectors of C(t).After a very simple calculation we will be having the following result:

$$[M, N] = MN - NM$$

It can be easily seen they both can be easily achieved by using exponential mapping which is given as:

$$exp : M \in gl(n, C) \rightarrow expM \in GL(n, C)$$

Hence we are provided with the technique to obtain the Lie algebra corresponding to Lie group.

3 Control Theory Using Lie Groups

3.1 Pontryagin's Principle

Pontryagin's Maximum Principle applies to a particular type of problem called a Bolzano Problem. Most optimization problems can be put into the form of a problem involves a number of state variables which can change over time where time t runs from 0 to T. Let us suppose the state variables are $Y_1(t), Y_2(t), Y_3(t)...Y_n(t)$ The target is to minimize

$$V(T) = c_1Y_1(T) + c_2Y_2(T) + ... + c_nY_n(T)$$

where we are provided with the initial points $Y_1(0), Y_2(0), ..., Y_n(0)$, here the coefficients are provided with certain conditions and T is some definite finite time. Suppose we are provided with steering functions for controlling the changes in the state variables I.e.

$$\frac{dY_1}{dt} = f_1(Y_1, Y_2, ..., Y_n, u_1, u_2..., u_n) \tag{6}$$

$$\frac{dY_2}{dt} = f_2(Y_1, Y_2, ..., Y_n, u_1, u_2..., u_n) \tag{7}$$

$$. \tag{8}$$

$$. \tag{9}$$

$$.\frac{dY_n}{dt} = f_n(Y_1, Y_2, ..., Y_n, u_1, u_2..., u_n) \tag{10}$$

where the variables $u_1, u_2, ..., u_n$ are functions of time and are called the control variables. Our aim is to determine the control variables at each point in order to steer the state variables from their starting states. i.e.

$$Y_1(0), Y_2(0), ..., Y_n(0)$$

, to some point

$$Y_1(T), Y_2(T), ..., Y_n(T)$$

such that

$$V(T) = c_1Y_1(T) + c_2Y_2(T) + ... + c_nY_n(T)$$

is minimized.

Pontryagin's Maximum Principle provides us a very clean,appreciable and systematic solution.

3.2 Hamiltonian function

To implement Pontryagin's method one defines a stable function which could help us in approaching the desired task.We are defining the Hamiltonian function which is given as follow :

$$H = \varphi_1 f_1 + \varphi_2 f_2 + ... + \varphi_n f_n$$
$$= \sum \varphi_i f_i$$

where these $\varphi_1, \varphi_2, ...\varphi_n$ are ad-joint variables, such that

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = -\frac{\partial H}{\partial Y_i}$$
$$= -\sum_i \varphi_j (\frac{\partial f_i}{\partial Y_i})$$

where $\varphi_i(T) = c_i$.

Hence the value of control variables are obtained at time **T** which maximizes **H** are termed to be optimal values.

3.3 Poisson structure

A Poisson structure on a smooth manifold **SM** is a Lie bracket $\{.,.\}$ on the algebra of smooth functions which subjects to the lebinitz rule , It is given by:

$$\{AB, C\} = A\{B, C\} + B\{A, C\}$$

where A,B,C are the smooth functions of the manifold .

Else in simplest form we can define this structure on a vector space of Smooth functions on **SM** as a Lie algebra given by:

$$Y_A = \{A, .\} \vdots \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$$

such that Y_A will act as a vector field for each smooth function A.

Definition

Let us consider **SM** be a smooth manifold and $\mathbb{C}^{\infty}(SM)$ represents the algebra of smooth functions defined on **SM**. Hence **Poisson structure** on **SM** is given by:

$$\{.,.\}:\mathbb{C}^{\infty}(SM)\times\mathbb{C}^{\infty}(SM)\rightarrow\mathbb{C}^{\infty}(SM)$$

satisfies the following three conditions:

$$\{A,B\}=-\{B,A\} \tag{11}$$

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0. \tag{12}$$

$$\{AB,C\}=A\{B,C\}+B\{A,C\} \tag{13}$$

4 Schrodinger Group

The Schrodinger group for one dimensional particle is given by :

$$i\hbar\frac{\partial\psi(y,t))}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(y,t)}{\partial y^2}$$

where the particle has to be described by the wave function which is given by $\psi(y',t')$. as it satisfies

$$i\hbar\frac{\partial\psi(y',t'))}{\partial t'} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(y',t')}{\partial y'^2}$$

It ha s already proven that the Schrodinger equation is invariant under conformal coordinate transformations

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}, y' = \frac{ay+ut+c}{\gamma t+\delta}, a^2 = \delta\alpha - \beta\gamma \neq 0$$

Hence as a result the it forms a group known as Schrodinger group, whose basis elements are given by:

$$A_1 = -i\hbar\frac{\partial}{\partial y} \tag{14}$$

$$A_2 = \frac{A_1^2}{2m} \tag{15}$$

$$A_3 = tA_1 - my \tag{16}$$

$$A_4 = tA_2 - \frac{1}{4}(yA_1 + A_1y) \tag{17}$$

$$A_5 = t^2A_2 - \frac{t}{2}(yA_1 + A_1y) + \frac{m}{2}y^2, A_6 = 0. \tag{18}$$

Now using the Exponential mapping we can obtain the Schrodinger Lie Algebra from the group. Also it can be very easily seen that the basis elements for this Schrodinger Group also satisfies all the properties of Lie Bracket. Hence they form a algebra. The commutation table for the elements of Schrodinger Lie algebra is given by:

$$\begin{bmatrix} [.,.] & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_1 & 0 & -2A_1 & A_2 & 0 & A_4 & 0 \\ A_2 & 2A_1 & 0 & -2A_3 & A_4 & -A_5 & 0 \\ A_3 & -A_2 & 2A_3 & 0 & A_5 & 0 & 0 \\ A_4 & 0 & -A_4 & -A_5 & 0 & A_6 & 0 \\ A_5 & -A_4 & A_5 & 0 & & 0 & 0 \\ A_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

5 Left-Invariant Control Problem on the Schrodinger Group

In this section we are considering the left invariant control affine system on Schrodinger Lie Group. Here we are showing that any left invariant control problem can be lifted to a Hamiltonian system on the dual of Schrodinger lie algebra.Also here we are deriving the reduced Hamiltonian equations associated with extremal curves obtained by describing Lie-Poisson structure on dual space of Schrodinger Lie algebra .

Theorem The constants of structure of the Schrodinger Lie algebra are given by: where A_1,A_2,A_3,A_4,A_5,A_6 is the canonical basis of Schrodinger Lie algebra, i.e.:
Then we have successively:

$$[A_1, A_1] = c_{11}^k A_k$$

$$c_{11}^1 = c_{11}^2 = c_{11}^3 = c_{11}^4 = c_{11}^5 = c_{11}^6 = 0$$

$$[A_1, A_2] = c_{12}^k A_k$$

$$c_{12}^1 = c_{12}^2 = c_{12}^3 = c_{12}^4 = c_{12}^5 = c_{12}^6 = -2$$

$$[A_1, A_3] = c_{13}^k A_k$$

$$c_{13}^1 = c_{13}^2 = c_{13}^3 = c_{13}^4 = c_{13}^5 = c_{13}^6 = 1$$

$$[A_1, A_4] = c_{14}^k A_k$$

$$c_{14}^1 = c_{14}^2 = c_{14}^3 = c_{14}^4 = c_{14}^5 = c_{14}^6 = 0$$

$$[A_1, A_5] = c_{15}^k A_k$$

$$c_{15}^1 = c_{15}^2 = c_{15}^3 = c_{15}^4 = c_{15}^5 = c_{15}^6 = 1$$

$$[A_1, A_6] = c_{16}^k A_k$$

$$c_{16}^1 = c_{16}^2 = c_{16}^3 = c_{16}^4 = c_{16}^5 = c_{16}^6 = 0$$

$$[A_2, A_2] = c_{22}^k A_k$$

$$c_{22}^1 = c_{22}^2 = c_{22}^3 = c_{22}^4 = c_{22}^5 = c_{22}^6 = 0$$

$$[A_2, A_3] = c_{23}^k A_k$$
$$c_{23}^1 = c_{23}^2 = c_{23}^3 = c_{23}^4 = c_{23}^5 = c_{23}^6 = -2$$
$$[A_2, A_4] = c_{24}^k A_k$$
$$c_{24}^1 = c_{24}^2 = c_{24}^3 = c_{24}^4 = c_{24}^5 = c_{24}^6 = 1$$
$$[A_2, A_5] = c_{25}^k A_k$$
$$c_{25}^1 = c_{25}^2 = c_{25}^3 = c_{25}^4 = c_{25}^5 = c_{25}^6 = -1$$
$$[A_2, A_6] = c_{26}^k A_k$$
$$c_{26}^1 = c_{26}^2 = c_{26}^3 = c_{26}^4 = c_{26}^5 = c_{26}^6 = 0$$
$$[A_3, A_3] = c_{33}^k A_k$$
$$c_{33}^1 = c_{33}^2 = c_{33}^3 = c_{33}^4 = c_{33}^5 = c_{33}^6 = 0$$
$$[A_3, A_4] = c_{34}^k A_k$$
$$c_{34}^1 = c_{34}^2 = c_{34}^3 = c_{34}^4 = c_{34}^5 = c_{34}^6 = 1$$
$$[A_3, A_5] = c_{35}^k A_k$$
$$c_{35}^1 = c_{35}^2 = c_{35}^3 = c_{35}^4 = c_{35}^5 = c_{35}^6 = 0$$
$$[A_3, A_6] = c_{36}^k A_k$$
$$c_{36}^1 = c_{36}^2 = c_{36}^3 = c_{36}^4 = c_{36}^5 = c_{36}^6 = 0$$
$$[A_4, A_4] = c_{44}^k A_k$$
$$c_{44}^1 = c_{44}^2 = c_{44}^3 = c_{44}^4 = c_{44}^5 = c_{44}^6 = 0$$
$$[A_4, A_5] = c_{45}^k A_k$$
$$c_{45}^1 = c_{45}^2 = c_{45}^3 = c_{45}^4 = c_{45}^5 = c_{45}^6 = 1$$
$$[A_4, A_6] = c_{46}^k A_k$$
$$c_{46}^1 = c_{46}^2 = c_{46}^3 = c_{46}^4 = c_{46}^5 = c_{46}^6 = 0$$
$$[A_5, A_5] = c_{55}^k A_k$$
$$c_{55}^1 = c_{55}^2 = c_{55}^3 = c_{55}^4 = c_{55}^5 = c_{55}^6 = 0$$
$$[A_5, A_6] = c_{56}^k A_k$$
$$c_{56}^1 = c_{56}^2 = c_{56}^3 = c_{56}^4 = c_{56}^5 = c_{56}^6 = 0$$

$$[A_6, A_6] = c_{66}^k A_k$$
$$c_{66}^1 = c_{66}^2 = c_{66}^3 = c_{66}^4 = c_{66}^5 = c_{66}^6 = 0$$

As a consequence we obtain:

Theorem The minus Lie-Poisson structure on dual of Schrodinger Lie algebra is given by the matrix:

$$\pi_- = \begin{pmatrix} 0 & 2A_1 & -A_2 & 0 & -A_6 & 0 \\ -2A_1 & 0 & 2A_3 & -A_4 & A_5 & 0 \\ A_2 & -2A_3 & 0 & -A_5 & 0 & 0 \\ 0 & A_4 & A_5 & 0 & -A_6 & 0 \\ A_4 & -A_5 & 0 & A_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

Remark 1 It is easy to see that the function C given by

$$C(P_1, P_2, P_3, P_4, P_5, P_6) = P_1 P_2 P_3 P_4 P_5 P_6$$

is a Casimir of our configuration $(h(3), \pi_-)$ i.e.

$$(\triangle C)^t . \pi_- = 0$$

Theorem There exist the following two types of controllable drift-free left invariant systems on Schrodinger Lie Group, namely

$$\dot{X} = X.(A_1 u_1 + A_2 u_2 + A_3 u_3 + A_4 u_4 + A_5 u_5 + A_6 u_6)$$

and

$$\dot{X} = X.(A_1 u_1 + A_2 u_2 + A_3 u_3 + A_4 u_4 + A_5 u_5)$$

Proof

The proof is a consequence of the Table 1 and Chow's theorem.

6 An optimal control problem on Schrodinger Lie Group

Let

$$J(u_1,u_2,u_3,u_4,u_5,u_6) = 1/2(\int_0^{T_f}(c_1u_1^2+c_2u_2^2+c_3u_3^2+c_4u_4^2+c_5u_5^2+c_6u_6^2)dt$$

$(c_1,c_2,c_3,c_4,c_5,c_6>0)$ be the cost function. Then the problem which we intend to solve is the following: find u_1,u_2,u_3,u_4,u_5,u_6that minimize J and steer the above system from X = 0 at t = 0 to $X = X_f$ at $t = t_f$. We have the following results:

Theorem The optimal controls of the above problem for our system are given by

$$u_1 = P_1/c_1$$
$$u_2 = P_2/c_2$$
$$u_3 = P_3/c_3$$
$$u_4 = P_4/c_4$$
$$u_5 = P_5/c_5$$
$$u_6 = P_6/c_6$$

where P_i are solutions of the system:

$$\dot{P}_1 = \frac{2P_1P_2}{c_2} - \frac{P_2P_3}{c_3} - \frac{P_5^2}{c_5}$$
$$\dot{P}_2 = \frac{-2P_1^2}{c_1} - \frac{2P_3^2}{c_3} - \frac{P_4^2}{c_4} + \frac{P_5^2}{c_5}$$
$$\dot{P}_3 = \frac{P_1P_2}{c_1} - \frac{2P_3P_2}{c_2} - \frac{P_4P_5}{c_4}$$

$$\dot{P}_4 = \frac{P_4 P_2}{c_2} + \frac{P_3 P_5}{c_3} - \frac{P_6 P_5}{c_5}$$
$$\dot{P}_4 = \frac{P_4 P_1}{c_1} + \frac{P_2 P_5}{c_2} - \frac{P_6 P_4}{c_4}$$
$$\dot{P}_6 = 0$$

Proof.
Let us take the extended Hamiltonian H given by:

$$H = P_1 u_1 + P_2 u_2 + P_3 u_3 + \frac{1}{2}(c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2 + c_4 u_4^2 + c_5 u_5^2 + c_6 u_6^2)$$

Then using the maximum principle, we have the conditions:

$$\frac{\partial H}{\partial u_1} = 0$$
$$\frac{\partial H}{\partial u_2} = 0$$
$$\frac{\partial H}{\partial u_3} = 0$$
$$\frac{\partial H}{\partial u_4} = 0$$
$$\frac{\partial H}{\partial u_5} = 0$$
$$\frac{\partial H}{\partial u_6} = 0$$

which lead us to:

$$P_1 = c_1 u_1$$
$$P_2 = c_2 u_2$$
$$P_3 = c_3 u_3$$
$$P_4 = c_4 u_4$$
$$P_5 = c_5 u_5$$
$$P_6 = c_6 u_6$$

and so the reduced Hamiltonian (or the optimal Hamiltonian) is given by:

$$H = \frac{1}{2}(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} + \frac{P_3^2}{c_3} + \frac{P_4^2}{c_4} + \frac{P_5^2}{c_5} + \frac{P_6^2}{c_6})$$

It follows that the reduced Hamilton equations have the following expressions:

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \\ \dot{P}_5 \\ \dot{P}_6 \end{bmatrix} = \begin{bmatrix} 0 & 2A_1 & -A_2 & 0 & -A_6 & 0 \\ -2A_1 & 0 & 2A_3 & -A_4 & A_5 & 0 \\ A_2 & -2A_3 & 0 & -A_5 & 0 & 0 \\ 0 & A_4 & A_5 & 0 & -A_6 & 0 \\ A_4 & -A_5 & 0 & A_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{P_1}{c_1} \\ \frac{P_2}{c_2} \\ \frac{P_3}{c_3} \\ \frac{P_4}{c_4} \\ \frac{P_5}{c_5} \\ \frac{P_6}{c_6} \end{bmatrix}$$

as required. It is easy to see that the reduced Hamilton's equations can be put in the equivalent form:

$$\dot{P}_1 = \frac{2P_1P_2}{c_2} - \frac{P_2P_3}{c_3} - \frac{P_5^2}{c_5}$$

$$\dot{P}_2 = \frac{-2P_1^2}{c_1} + \frac{2P_3^2}{c_3} - \frac{2P_4^2}{c_4} + \frac{2P_5^2}{c_5}$$

$$\dot{P}_3 = \frac{P_1P_2}{c_1} - \frac{2P_2P_3}{c_2} - \frac{P_5P_4}{c_4}$$

$$\dot{P}_4 = \frac{P_4P_2}{c_2} + \frac{P_5P_3}{c_3} - \frac{kP_5}{c_5}$$

$$\dot{P}_5 = \frac{P_1P_4}{c_1} - \frac{P_2P_5}{c_2} - \frac{kP_4}{c_4}$$

$$P_6 = k$$

Theorem 5. The controls $u_1, u_2, ...u_n$ are given by sinusoidal s, More exactly

$$u_1 = \frac{l_1}{c_1} \cos \sqrt{\frac{c_1}{c_2}} \left(\frac{-p_2 \dot{p}_1 + p_{12}}{p_1^2 + p_2^2} \right) t + C_1$$

$$u_1 = \frac{l_2}{c_2} \sin \sqrt{\frac{c_3}{c_4}} \left(\frac{-p_4 \dot{p_3} + p_{34}}{p_3^2 + p_4^2} \right) t + C_2$$

$$u_1 = \frac{l_3}{c_3} \cos \sqrt{\frac{c_5}{c_6}} \left(\frac{-p_6 \dot{p_5} + p_{56}}{p_5^2 + p_6^2} \right) t + C_3$$

Proof Let us assume that :

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} = l_1^2$$

$$\frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} = l_2^2$$

$$\frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} = l_3^2$$

On Substituting these values in Reduced Hamiltonian system of equation , the equation becomes :

$$\frac{p_1^2}{c_1} + \frac{p_2^2}{c_2} + \frac{p_3^2}{c_3} + \frac{p_4^2}{c_4} + \frac{p_5^2}{c_5} + \frac{p_6^2}{c_6} = l^2$$

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2 = l^2$$

For the convenience of the proof let us assume

$$p_1 = l_1 \sqrt{c_1} \cos \Theta_1$$

$$p_2 = l_1 \sqrt{c_2} \sin \Theta_1$$

$$p_3 = l_2 \sqrt{c_3} \cos \Theta_2$$

$$p_4 = l_2 \sqrt{c_4} \sin \Theta_2$$

$$p_5 = l_3 \sqrt{c_5} \cos \Theta_3$$

$$p_1 = l_3 \sqrt{c_6} \sin \Theta_3$$

such that we have:

$$u_1 = \frac{p_1}{c_1} = \frac{l_1 \cos \Theta_1}{\sqrt{c_1}}$$

$$u_2 = \frac{p_2}{c_2} = \frac{l_1 \sin \Theta_1}{\sqrt{c_2}}$$

$$u_3 = \frac{p_3}{c_3} = \frac{l_2 \cos \Theta_2}{\sqrt{c_3}}$$

$$u_4 = \frac{p_4}{c_4} = \frac{l_2 \sin \Theta_2}{\sqrt{c_4}}$$

$$u_5 = \frac{p_5}{c_5} = \frac{l_3 \cos \Theta_3}{\sqrt{c_5}}$$

$$u_6 = \frac{p_6}{c_6} = \frac{l_3 \sin \Theta_3}{\sqrt{c_6}}$$

now on simplifying we have :

$$\Theta_1 = \sqrt{\frac{c_1}{c_2}} \arctan(\frac{p_2}{p_1})$$

$$\Theta_2 = \sqrt{\frac{c_3}{c_4}} \arctan(\frac{p_4}{p_3})$$

$$\Theta_3 = \sqrt{\frac{c_5}{c_6} \arctan(\frac{p_6}{p_5})}$$

take derivative of θ ,as a result we have :

$$\dot{\Theta}_1 = \sqrt{\frac{c_1}{c_2}}(\frac{p_{12} -_1 p_2}{p_1^2 + p_2^2})$$

$$\dot{\Theta}_2 = \sqrt{\frac{c_3}{c_4}}(\frac{p_{34} -_3 p_4}{p_3^2 + p_4^2})$$

$$\dot{\Theta}_3 = \sqrt{\frac{c_5}{c_6}}(\frac{p_{56} -_5 p_6}{p_5^2 + p_6^2})$$

It clearly implies :

$$\theta_1 = \sqrt{\frac{c_1}{c_2}} \left(\frac{-p_2 \dot{p}_1 + p_{12}}{p_1^2 + p_2^2} \right) t + C_1$$

$$\theta_2 = \sqrt{\frac{c_3}{c_4}} \left(\frac{-p_4 \dot{p}_3 + p_{34}}{p_3^2 + p_4^2} \right) t + C_2$$

$$\theta_3 = \sqrt{\frac{c_5}{c_6}} \left(\frac{-p_6 \dot{p}_5 + p_{56}}{p_5^2 + p_6^2} \right) t + C_3$$

as a result we obtain the solutions given by :

$$u_1 = \frac{l_1}{c_1} \cos \sqrt{\frac{c_1}{c_2}} \left(\frac{-p_2 \dot{p}_1 + p_{12}}{p_1^2 + p_2^2} \right) t + C_1$$

$$u_1 = \frac{l_2}{c_2} \sin \sqrt{\frac{c_3}{c_4}} \left(\frac{-p_4 \dot{p}_3 + p_{34}}{p_3^2 + p_4^2} \right) t + C_2$$

$$u_1 = \frac{l_3}{c_3} \cos \sqrt{\frac{c_5}{c_6}} \left(\frac{-p_6 \dot{p}_5 + p_{56}}{p_5^2 + p_6^2} \right) t + C_3$$

7 Stability associated with Lie Groups

7.1 Stability associated with Schrodinger Lie Algebra

We investigate the stability nature of the dynamical system shown above , The equilibrium states are

1. $P_e^M 1 = (M, 0, 0, 0, 0, 0)$,
2. $P_e^M 2 = (0, M, 0, 0, 0, 0)$,
3. $P_e^M 3 = (0, 0, M, 0, 0, 0)$,
4. $P_e^M 4 = (0, 0, 0, M, 0, 0)$,
5. $P_e^M 5 = (0, 0, 0, 0, M, 0)$,
6. $P_e^M 6 = (0, 0, 0, 0, 0, M)$,
here, M $\in$ R$\backslash$0) and the origin (0, 0, 0,0,0,0).

Proposition
The equilibrium state $P_e^M 1 = (M, 0, 0, 0, 0, 0)$ has the following behaviour:
1. If k(constant) is positive, then state is non linearly stable.
2. if k(constant) is negative , the state is not stable. For all $c_1, c_@, c_3, c_4, c_5, c_6$ the equilibrium state is unstable:

Proof : For calculating the eigen values of the derived dynamical system we have to obtain the linearization matrix which is actually the Jacobian matrix of the dynamics of the system. That is, we have $\dot{P} = F(P)$, so matrix of the linearization is the Jacobian of F.

$$D(F(p)) = \begin{bmatrix} \frac{2P_2}{c_2} & \frac{2p_1}{c_2} - \frac{P_3}{c_3} & \frac{-P_2}{c_3} & \frac{-P_5}{c_5} & \frac{-P_4}{c_5} & 0 \\ \frac{-4P_1}{c_1} & 0 & \frac{4P_3}{c_3} & \frac{-2P_4}{c_4} & \frac{2P_5}{c_5} & 0 \\ \frac{P_2}{c_1} & \frac{P_1}{c_1} - \frac{2P_3}{c_3} & \frac{-2P_2}{c_2} & \frac{-P_5}{c_4} & \frac{-P_4}{c_4} 0 & \\ 0 & \frac{P_4}{c_2} & \frac{P_5}{c_3} & \frac{P_2}{c_2} & \frac{P_3}{c_3} - \frac{k}{c_3} & 0 \\ \frac{P_4}{c_1} & \frac{-P_5}{c_2} & 0 & \frac{P_1}{c_1} & \frac{-P_2}{c_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

Thus The matrix of the linearization of the system at $P_e^M 1$ is

$$\begin{bmatrix} 0 & \frac{2M}{c_2} & 0 & 0 & 0 & 0 \\ \frac{-4M}{c_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{M}{c_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-k}{c_3} & 0 \\ 0 & 0 & 0 & \frac{M}{c_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

with eigenvalues are $\lambda = 0$, $\lambda = \sqrt{\frac{kM}{c_1 c_3}}$,$\lambda_1 = \sqrt{\frac{-8M^2}{c_1 c_2}}$ Since $c_1, c_@, c_3, c_4, c_5, c_6$ are ≥ 0.
For $k = positivevalue$ as a result the $Real(\lambda)$ are $\geq = 0$
Hence the state is unstable by using the theory of stability.
For k= non positive value ,the stability is to studied using casimir energy function: Let H_ψ be the (energy-Casimir) function given by

$$H_\psi(P_1, P_2, P_3, P_4, P_5, P_6) = \frac{1}{2}\left((\frac{P_1^2}{c_1}) + (\frac{P_2^2}{c_2}) + (\frac{P_3^2}{c_3}) + (\frac{P_4^2}{c_4}) + (\frac{P_5^2}{c_5}) + (\frac{P_6^2}{c_6}) \right) + H_\psi(P_1$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. The first variation is
$\partial H_\psi(P_1, P_2, P_3, P_4, P_5, P_6) = (\frac{P_1 \partial_1}{c_1}) + (\frac{P_2 \partial_2}{c_2}) + (\frac{P_3 \partial_3}{c_3}) + (\frac{P_4 \partial_4}{c_4}) + (\frac{P_5 \partial_5}{c_5}) + (\frac{P_6 \partial_6}{c_6}) +$
$\partial H_\psi(P_1 P_2 P_3 P_4 P_5 P_6)\,(\partial_1 P_2 P_3 P_4 P_5 P_6 + P_1 \partial_2 P_3 P_{4 P_5 P_6 + P_1 P_2 \partial_3 + P_4 P_5 P_6 + P_1 P_2 P_3 \partial_4 P_5 P_6 + P_1 P_2 P_3 P_4}$
equals to zero if and only if

$$\dot{\psi}(0) = 0$$

The second variation at $P_e^M 1 : \partial^2 H_\psi = \frac{\partial_1^2}{c_1} + \frac{\partial_2^2}{c_2} + \frac{\partial_3^2}{c_3} + \frac{\partial_4^2}{c_4} + \frac{\partial_5^2}{c_5}$
is positive definite for $P_e^M 1$ Hence by energy casimir function , the given Equilibrium state is nonlinearly stable.

In a similar manner, we can prove the following result.

Proposition. The equilibrium state $P_e^M 2 = (0, M, 0, 0, 0, 0)$ has the following behaviour:
1. If $c_1 c_3 \leq c_2^2$, then state is not linearly stable.
2. if $c_1 c_3 \geq c_2^2$, the state is non linearly stable.

Proof.The matrix of the linearization is the Jacobian of F is given by :

$$D(F(p)) = \begin{bmatrix} \frac{2P_2}{c_2} & \frac{2p_1}{c_2} - \frac{P_3}{c_3} & \frac{-P_2}{c_3} & \frac{-P_5}{c_5} & \frac{-P_4}{c_5} & 0 \\ \frac{-4P_1}{c_1} & 0 & \frac{4P_3}{c_3} & \frac{-2P_4}{c_4} & \frac{2P_5}{c_5} & 0 \\ \frac{P_2}{c_1} & \frac{P_1}{c_1} - \frac{2P_3}{c_3} & \frac{-2P_2}{c_2} & \frac{-P_5}{c_4} & \frac{-P_4}{c_4} & 0 \\ 0 & \frac{P_4}{c_2} & \frac{P_5}{c_3} & \frac{P_2}{c_2} & \frac{P_3}{c_3} - \frac{k}{c_3} & 0 \\ \frac{P_4}{c_1} & \frac{-P_5}{c_2} & 0 & \frac{P_1}{c_1} & \frac{-P_2}{c_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (23)$$

Thus The matrix of the linearization of the system at $P_e^M 2$ is

$$\begin{bmatrix} \frac{2M}{c_2} & 0 & \frac{-M}{c_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{M}{c_1} & 0 & \frac{-2M}{c_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{M}{c_2} & -\frac{k}{c_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{-M}{c_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (24)$$

after the calculation of characteristic matrix the eigenvalues can be found which are given by : $\lambda = \frac{M}{c_2}\sqrt{\frac{c_2^2 - 4c_1c_3}{c_1c_3}}$ using previous results we can easily proof the provided results.

Proposition. The equilibrium state $P_e^M 3 = (0,0,M,0,0,0)$, $P_e^M 4 = (0,0,0,M,0,0)$ and $P_e^M 5 = (0,0,0,0,M,0)$ are non linearly stable.

8 Conclusion

In this project we tried to cover the possible aspects of mathematical control theory, especially Left invariant optimal control on the very important physical Group "Schrodinger Lie Group". Here we tried to restrict our-selves only in the mathematical aspects,as a result we concluded the stability factors of Schrodinger Lie Group at various equilibrium states. However this group is related to many physical phenomenon, which are interesting to physicist .We hope our study will be lead to a small step towards such type of investigation.

9 Gaps

There are many things yet to be concluded:

1) We think Elliptic Jacobi integration may be possible for the optimal control that are determined by us. In future we will surely find a suitable way to solve this type of integration in this left invariant dynamical system.

2) More Casimir functions can be calculated in future.As only one Casimir operator has been determined in this scope of paper.

3) Here the most general form of left invariant control systems is adopted for calculation of optimal controls, in future more realistic and suitable left invariant dynamical systems can be defined.

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