SOME APPLICATIONS OF HOMOTOPY PERTURBATION METHOD

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CERTIFICATE

This is to certify that the project report entitled **"Some applications of homotopy perturbation method"** submitted by **Malika sahoo** to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of master of science in Mathematics and the review work is carried out by her under my supervision and guidance . It has fulfilled all the guidelines required for the submission of her research project paper for M.Sc. degree. In my opinion, the contents of this project submitted by her is worthy of consideration for M.Sc. degree and in my insight this work has not been submitted to any other institute or university for the award of any degree.

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ABSTRACT

In this thesis paper, I review the basic idea of Homotopy perturbation method (HPM), Modified Homotopy perturbation method (MHPM) and Homotopy perturbation transform method (HPTM). Then apply these on some higher order non-linear problems. Further, I tried to compare the results obtained from Modified homotopy perturbation method with HPM using the Sine-Gordon and fractional Klein-Gordon equation respectively. Homotopy perturbation transform method is the coupling of homotopy perturbation and Laplace transform method. Lastly, I applied the homotopy perturbation and homotopy perturbation transform method for solving linear and non-linear Schrödinger equations.

Keywords: Homotopy perturbation method, modified homotopy perturbation method, homotopy perturbation transform method, Sine-Gordon equation, Klein-Gordon equation, fractional Klein-Gordon equation, Linear and Nonlinear Schrödinger equations and He's Polynomials.

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CHAPTER-1

Introduction

Most scientific phenomena occur nonlinearly. We know that except a limited number, the majority of them don't have exact systematic arrangements. Therefore, these nonlinear equations are to be solved using other methods. In recent years, many researchers have paid attention to study the solutions of linear and nonlinear differential equations and also fractional order differential equation by using various analytical methods. Recently, some non-linear analytical techniques for solving non-linear problems have been dominated by the perturbation method. Perturbation method is one of the most well-known methods to solve nonlinear equations studied by a large number of researchers such as Bellman [2], Cole [3] and O'Malley [16]. Actually, these scientists had paid more attention to the mathematical aspects of the subject which included a loss of physical verification. This loss in the physical verification of the subject was recovered by Nayfeh [15] and Van Dyke [20]. But, like other non-linear analytical methods, perturbation methods have their own particular limitations.

Firstly, almost all perturbation methods are based on an assumption that a small parameters must exist in the equation. This is so called small parameter assumption greatly restricts utilizations of perturbation techniques. As well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. A suitable choice of small parameters leads to ideal results. However, an unsustainable choice of small parameters results is in bad effects. Thirdly, even if there exist suitable parameters, the approximate solutions are obtained by the perturbation methods valid in most cases, only for the small values of the parameters. So it is necessary to develop a kind of new non-linear analytical method which does not require small parameters at all.

Since there are some impediments with the common perturbation method, furthermore basis of the common perturbation method was upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different methods have recently introduced some ways to eliminate the small parameter, such as artificial parameter method introduced by Liu [12], the homotopy perturbation method by Ganji [4,5] and the variational iteration method by He [6,7,8].

Homotopy is an important part of differential topology. Homotopy techniques are generally connected to discover all bases of non-linear algebraic equations. The homotopy technique, or the continuous mapping procedure, embeds a parameter the embedding parameter p that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of a direct framework, when it is one; the equation is the same as the first one. So the embedded parameter $p[0, 1]$ can be considered as a small parameter. The coupling method of the homotopy techniques is called the homotopy perturbation method.

The Homotopy perturbation method (HPM) was presented by Ji-Huan He [He, 1999] of Shanghai University in 1998 which is the coupling method of the homotopy techniques and the perturbation technique. On the other hand, Homotopy perturbation transform method (HPTM) is combined form of the Laplace transform method with the homotopy perturbation method introduced by Y. khan and Q. Wu. The above methods find the solution without any discretisation or restrictive assumptions and avoid the round-off errors. The HPM is a special case of the Homotopy analysis method(HAM)[Liao S.,1992] developed by Liao Shijun in 1992.The HAM uses a so-called convergence control parameter to guarantee the convergence of approximations series over a given interval of physical parameters.

1.1 Perturbation Theory

Perturbation theory comprises mathematical methods which are used to find the approximate solution to a problem which cannot be a solved accurately, by starting from the accurate solution of a related problem. Perturbation theory leads to an expression for the desired solution in terms of a formal power series in small parameter (ϵ)-known as perturbation series that quantifies the deviation from the exactly solvable problem and further terms describe the deviation in the solution.

Consider,

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

Here x_0 be the known solution to the exactly solvable initial problem and $x_1, x_2 \ldots$ are the higher order terms. For small ϵ these higher order terms are successively smaller. An approximate “perturbation solution” is obtained by truncating series, usually by keeping only the first two terms.

1.2 Regular Perturbation Theory

Very often, we cannot be solved a mathematical problem exactly, or if the exact solution is available it exhibits such an intricate dependency in the parameters that is hard to use as such. It may be the cases however, that a parameter can be identified, say ϵ such that the solution is available and reasonably simple for $\epsilon = 0$. Then one may wonder how this solution is altered for non zero but small parameter ϵ.

1.3 Singular perturbation Theory

It concerns the investigation of issues highlighting a parameter for which the solutions of the problem at a restricting value of the parameter are different in character from the limit of the solution of the general problem. For regular perturbation problems, the solution of the general problem converge to the solution of the limit problem as the parameter approaches the limit value.

1.4 Homotopy perturbation method

In the homotopy perturbation technique we will first propose a new perturbation technique coupled with the homotopy technique. In topology two continuous functions from one topological space to another is called "homo-topic". Formally a homotopy between two continuous functions from f and g from a topological space X to a topological space Y is defined to be a continuous function

$$H: X \times [0,1] \to Y$$

Such that

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x) \ \forall\, x \in X$$

The homotopy perturbation method does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter.

1.5 Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral operator of $\alpha \geq 0$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0, t > 0$$

In particular, $J^0 f(x) = f(x)$

For $\beta \geq 0$, $\gamma \geq -1$.The operator J^α has the following properties:

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

(iii) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

1.6 Caputo fractional derivative

The Caputo fractional derivative operator is given by

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) \quad = \quad \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{m}(t) dt$$

Where $m-1 < \alpha \leq m,\ m \in N,\ x > 0$. It has the flowing properties

(i) $D^{\alpha}\, [J^{\alpha} f(x)] = f(x)$

(ii) $J^{\alpha}\, [D^{\alpha} f(x)] \ = f(x) - \sum_{k=0}^{m-1} f^{k}(0)\, (\frac{x^{k}}{k!})$

1.7 Laplace's transform

The Laplace transforms of a function $f(t)$, denoted by $F(s)$, is defined by the equation

$$F(s) = L\{f(t), s\} = \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} e^{-t} e^{zt} dt = \frac{1}{1-z}$$

CHAPTER-2

Model of the Problems

2.1 Linear Schrodinger equation

Consider the linear Schrödinger equation from A. M. Wazwaz [1]

$$u_t + iu_{xx} = 0 \tag{2.1.1}$$

With the initial condition

$$u(x,0) = 1 + \cosh(2\text{x}) \tag{2.1.2}$$

Where $u(x,t)$ is a complex function and $i^2 = -1$

2.2 Nonlinear Schrodinger equation

Consider the non-linear Schrodinger equation from A. M. Wazwaz [1]

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{2.2.1}$$

With the initial condition

$$u(x,0) = e^{ix} \tag{2.2.2}$$

Where $|u|^2 = u\bar{u}$ and $\bar{u}$ is the conjugate of $u, u(x,t)$ is a complex function.

2.3 Sine-Gordon equation

Consider the sine-Gordon equation suggested by M.J. Ablowitz et al. in [14]

$$u_{tt} - u_{xx} + \sin(u) = 0 \tag{2.3.1}$$

With the initial conditions

$$u(x,0) = 0 \quad , u_t(x,0) = 4\text{sech}(x) \tag{2.3.2}$$

2.4 Fractional-order Klein-Gordon equation

Consider the fractional-order cubically nonlinear Klein-Gordon problem

$$D_t{}^{\alpha}u - D_x{}^{\beta} + u^3 = f(x,t),\ x \geq 0,\ t > 0,\ \alpha,\ \beta\ \epsilon\ (1,2] \tag{2.4.1}$$

With the initial conditions

$$u(x,0) = 0 \ , \ u_t(x,0) = 0 \tag{2.4.2}$$

$$f(x,t) = \Gamma(\alpha+1)\, x^{\beta} - \Gamma(\beta+1)\, t^{\alpha} + x^{3\beta}\, t^{3\alpha}$$

2.5 One-dimensional linear inhomogeneous Fractional-order Klein-Gordon equation

Consider the fractional order Klein-Gordon equation

$$D_t{}^{\alpha}\, u - u_{xx} + u = 6x^3 \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + (x^3 - 6x)t^3 \tag{2.5.1}$$

$$t > 0,\ \ x \in R, 1 < \alpha \le 2$$

With the initial conditions:

$$u(x,0)=0 \ \ , \ u_t(x,0) = 0 \tag{2.5.2}$$

CHAPTER-3

Analysis and Interpretation

3.1 Homotopy Perturbation Method

3.1.1 Analysis of HPM

To illustrate the basic concept of this method, we consider the following differential equation:

$$A(u) - f(r) = 0\,, \quad r \in \Omega \qquad (3.1.1.1)$$

With boundary conditions:

$$B\left(u\,,\ \frac{\partial u}{\partial n}\right) = 0 \quad , \quad r \in \Gamma \qquad (3.1.1.2)$$

Where A is a general differential operator, B is boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, Where L is linear and N is nonlinear. Then equation (3.1.1.1) can be written as follows:

$$L(u) + N(u) - f(r) = 0 \quad , \ r \in \Omega \qquad (3.1.1.3)$$

By the Homotopy technique, we construct a homotopy structure:

$$H(v,p) = (1-p)\ [(L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \qquad (3.1.1.4)$$

$$v(r,p) :\ \Omega \times [0,1] \rightarrow \mathrm{R} \qquad (3.1.1.5)$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary conditions. Now the solution of (3.1.1.4) can be written as a power series of p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \qquad (3.1.1.6)$$

and the best approximation solution is:

$$u = \lim_{p\rightarrow 1} v = v_0 + v_1 + v_2 + \cdots \qquad (3.1.1.7)$$

Now for fractional differential equation:

$$D_t{}^{\alpha} u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t) \quad , \ t > 0 \qquad (3.1.1.8)$$

Where L is a linear operator, N is a nonlinear operator, f is a known analytic function and D^{α}, is the Caputo fractional derivative of order α , where $m-1<\alpha<m$ subject to the initial conditions:

$$u^{k}(x,0)=h_{k}(x), \qquad k=0,1,2,3,\ldots,m-1 \tag{3.1.1.9}$$

Homotopy structure is:

$$D_t{}^{\alpha}u - L(u,u_x,u_{xx}) - f(x,t) = p[\,N(u,u_x,u_{xx})\,] \tag{3.1.1.10}$$

Or $$D_t{}^{\alpha}u - f(x,t) = p\,[\,L(u,u_x,u_{xx}) + N(u,u_x,u_{xx})\,] \tag{3.1.1.11}$$

Theorem 1:

Let f satisfy he Lipschitz condition and then the problem (3.1.1.8) has unique solution $u(x,t)$ whenever $0<\gamma<1$.

Theorem 2:

Let $\mathrm{u_n(x,t)}$ and $\mathrm{u(x,t)}$ be defined in Banach space $(\mathrm{C}[0,\mathrm{T}]\,,\ \|\ .\ \|)$. Then the series solution $\sum_{n=1}^{\infty}\mathrm{u_n(x,t)}$ defined by (3.1.1.7) converges to the solution of (3.1.1.8), if $0<\gamma<1$.

Theorem 3:

The maximum absolute truncation error of the series solution (3.1.1.7) of the problem (3.1.1.8) is estimated to be

$$|\,\mathrm{u(x,t)} - \textstyle\sum_{\mathrm{i=0}}^{\mathrm{m}}\mathrm{u_i(x,t)}\,| \le \frac{\gamma^{m+1}}{1-\gamma}\ \|\ \mathrm{u}_0(t)\ \|\,.$$

3.1.2 Implementation of the method

(A) Linear Schrödinger equation

For solving equation (2.1.1) we construct the following homotopy:

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} + i\,\frac{\partial^2 v}{\partial x^2}\right] = 0 \tag{3.1.2.1}$$

Substituting Eq. (3.1.1.6) into Eq. (3.1.2.1) and equating the same powers of p we have:

$$p^0:\ \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad , \qquad v_0(x,0) = 1+\cosh(2\mathrm{x})$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + i\,\frac{\partial^2 v_0}{\partial x^2} = 0 \quad , \quad v_1(x,0) = 0$$

$$p^2 : \frac{\partial v_2}{\partial t} + i\,\frac{\partial^2 v_1}{\partial x^2} = 0 \qquad , \qquad v_2(x,0) = 0$$

and so on. Consider $u_0(x,t) = 1 + \cosh 2x$ as a first approximation for the solution that satisfies the initial condition. Solving above equations by simple integral we obtained v_0, v_1, v_2, v_3 and so on. Hence the solution of Eq. (2.1.1) when $p \to 1$ will be as follows:

$$u(x,t) = (1 + \cosh(2x))(1 - 4it - 8t^2 + \tfrac{32}{3} it^3 + \cdots)$$

$$= 1 + e^{-4it}\cosh(2x)$$

This is an exact solution.

(B) Non-Linear Schrödinger equation

For solving equation (2.2.1) we construct the following homotopy:

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} - i\left(\frac{\partial^2 v}{\partial x^2} + 2v^2\bar{v}\right)\right] = 0 \qquad (3.1.2.2)$$

Substituting Eq. (3.1.1.6) into Eq. (3.1.2.2) and equating the same powers of p we have:

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad , \quad v_0(x,0) = e^{ix}$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - i\left(\frac{\partial^2 v_0}{\partial x^2} + 2{v_0}^2\overline{v_0}\right) = 0 \qquad , \quad v_1(x,0) = 0$$

$$p^2 : \frac{\partial v_2}{\partial t} - i\left(\frac{\partial^2 v_1}{\partial x^2} + 2\,({v_0}^2\overline{v_1} + 2\,v_1 v_0\overline{v_0})\right) = 0\,, \quad v_2(x,0) = 0$$

and so on. Consider $u_0(x,t) = e^{ix}$ as a first approximation for the solution that satisfies the initial condition. Solving above equations by simple integral we obtained v_0, v_1, v_2, v_3 and so on.

Hence the solution of Eq. (2.2.1) when $p \to 1$ will be as follows:

$$u(x,t) = e^{ix} + ite^{ix} + \frac{e^{ix^2}}{2!}e^{ix} + \ldots$$

$$= e^{i(x+t)}$$

This is an exact solution.

(C) Sine-Gordon equation

For solving equation (2.3.1) we construct the following homotopy:

$$u_{tt} - y_{0tt} + p y_{0tt} + p(-u_{xx} + \sin(pu)) = 0 \tag{3.1.2.3}$$

Substituting Eq. (3.1.1.6) into Eq. (3.1.2.3) and equating same powers of p we have:

$$p^0 : \ u_{0tt} - y_{0tt} = 0 \quad , \quad y_0 = 4\,\mathrm{sech}(x)\,t$$

$$p^1 : \ u_{1tt} - u_{0xx} + y_{0tt} = 0 \quad , \quad u_1(x, t_0) = 0 \quad , \ u_{1t}(x, t_0) = 0$$

$$p^2 : \ u_{2tt} - u_{1xx} + u_0 = 0 \quad , \quad u_2(x, t_0) = 0 \ , \ u_{2t}(x, t_0) = 0$$

$$p^3 : \ u_{3tt} - u_{2xx} + u_1 = 0 \quad , \quad u_3(x, t_0) = 0 \ , \ u_{3t}(x, t_0) = 0$$

$$p^4 : \ u_{4tt} - u_{3xx} + u_2 - \frac{{u_0}^3}{3!} = 0 \quad , \quad u_4(x, t_0) = 0 \ , \ u_{4t}(x, t_0) = 0$$

and so. Solving these equations by simple integral we obtained u_0, u_1, u_2, u_3, u_4 and so on.

$$\begin{aligned} u = 4t\mathrm{sech}(x) - \frac{4t^3}{3}\mathrm{sech}^3(x) + \frac{4t^5}{5}\mathrm{sech}^5(x) - \frac{t^7\mathrm{sech}^7(x)}{90720}(51840 - 55296\cosh^2(x) \\ + 8784\cosh^4(x) + 114\cosh^6(x)) + \frac{t^9\mathrm{sech}^9(x)}{90720}(40320 - 60480\cosh^2(x) \\ + 23184\cosh^4(x) - 1640\cosh^6(x) + \cosh^8(x)) \end{aligned}$$

This is an approximate solution.

(D) Fractional-order Klein-Gordon equation

According to the homotopy (3.1.1.10), we obtain the following set of linear partial differential equations of fractional order

$$p^0 : \ {D_t}^{\alpha} u_0 = 0 \quad , \quad u_0(x,0) = 0 \quad , u_{0t}(x,0) = 0$$

$$p^1 : \ {D_t}^{\alpha} u_1 = {D_x}^{\beta} u_0 + f(x,t) \, , u_1(x,0) = 0 \ , \ u_{1t}(x,0) = 0$$

$$p^2 : \ {D_t}^{\alpha} u_2 = {D_x}^{\beta} u_1 \quad , \ u_2(x,0) = 0 \ , \ u_{2t}(x,0) = 0 \tag{3.1.2.4}$$

$$p^3 : \ {D_t}^{\alpha} u_3 = {D_x}^{\beta} u_2 - {u_0}^3 \quad , \ u_3(x,0) = 0 \ , \ u_{3t}(x,0) = 0$$

and so on.

Case-1: ($\alpha \epsilon$ (1,2] and $\beta = 2$) solving (3.1.2.4), we get

$$u_0 = 0$$

$$u_1 = t^\alpha x^2 - 2t^{2\alpha}\ \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + t^{4\alpha} x^6\ \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}$$

$$u_2 = 30\ t^{5\alpha} x^4\ \frac{\Gamma(1+3\alpha)}{\Gamma(1+5\alpha)} + 2t^{2\alpha}\ \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$$

and so on ,the solution is

$$u = u_0 + u_1 + u_2 + \cdots$$

Case-2: ($\alpha = 2$ and $\beta\ \epsilon$ (1,2]) solving (3.1.2.4), we get

$$u_0 = 0$$

$$u_1 = t^2 x^\beta + t^8 x^{3\beta} \frac{1}{56} - \frac{1}{12}\Gamma(1+\beta)t^4$$

$$u_2 = t^{10} x^{2\beta} \frac{\Gamma(1+3\beta)}{5040\Gamma(1+2\beta)} + \frac{1}{12}\Gamma(1+\beta)t^4$$

and so on, the solution is

$$u = u_0 + u_1 + u_2 + \cdots$$

Case-3: (both α and ϵ (1,2]) solving (3.1.2.4), we get

$$u_0 = 0$$

$$u_1 = t^\alpha x^\beta + t^{4\alpha} x^{3\beta}\ \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} - t^{2\alpha}\ \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+2\alpha)}$$

$$u_2 = t^{5\alpha} x^{2\beta}\ \frac{\Gamma(1+3\alpha)\,\Gamma(1+3\beta)}{\Gamma(1+5\alpha)\,\Gamma(1+2\beta)} + t^{2\alpha} \frac{\Gamma(1+\alpha)\,\Gamma(1+\beta)}{\Gamma(1+2\alpha)}$$

and so on the solution is

$$u = u_0 + u_1 + u_2 + \cdots$$

3.2 Modified Homotopy Perturbation Method

3.2.1 Analysis of MHPM

To illustrate the basic concept of this modification, we consider the following differential equation:

$$L(u) + N(u) - f(r) = 0\,,\ r \in \Omega \tag{3.2.1.1}$$

With the initial conditions:

$$u_j(x,0) = g_j(x), \quad j = 0,1,2,\ldots \tag{3.2.1.2}$$

Where L is a linear operator, N is a nonlinear operator, f is a known analytic function. From homotopy technique, we construct the following homotopy:

$$\frac{\partial u}{\partial t} + L(u) - f(r) = p\left[\frac{\partial u}{\partial t} - N(u)\right]\ ,\ p\epsilon[0,1] \tag{3.2.1.3}$$

or $$\frac{\partial u}{\partial t} - f(r) = p\left[\frac{\partial u}{\partial t} - L(u) - N(u)\right]\ ,\ p\epsilon[0,1] \tag{3.2.1.4}$$

The homotopy parameter p always changes from zero to unity. In case of $p = 0$ the equation (3.2.1.3) becomes the linearized equation:

$$\frac{\partial u}{\partial t} + L(u) = f(r)$$

and equation (3.2.1.4) becomes the linearized equation:

$$\frac{\partial u}{\partial t} = f(r)$$

In case of $p = 1$, equation (3.2.1.3) or (3.2.1.4) turns out to be the original differential equation. Where $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary conditions. We can assume that the solution of eq. (3.2.1.3) can be written as power series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{3.2.1.5}$$

and the best approximation solution is:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{3.2.1.6}$$

Now for fractional differential equation:

$$D_t{}^{\alpha} u(x,t) = L(u, u_x, u_{xx}) + N(u, u_x, u_{xx}) + f(x,t)\quad ,\ t > 0 \tag{3.2.1.7}$$

Where L is a linear operator, N is a nonlinear operator, f is a known analytic function and D^{α} is the Caputo fractional derivative of order α, where $m-1<\alpha<m$, with the initial conditions:

$$u^{k}(x,0)=h_{k}(x), \qquad k=0,1,2,3,\dots,m-1 \tag{3.2.1.8}$$

Homotopy structure is:

$$D_t{}^{\alpha}u-\ L(u,u_x,u_{xx})-f_0(x,t)=p[N(u,u_x,u_{xx})+f_1(x,t)] \tag{3.2.1.9}$$

$$\text{Or}\quad D_t{}^{\alpha}u-\ f_0(x,t)=p[L(u,u_x,u_{xx})+N(u,u_x,u_{xx})+f_1(x,t)] \tag{3.2.1.10}$$

3.2.2 Implementation of the method

(A) Sine-Gordon equation

According to modified homotopy perturbation method, we suppose that the solution (2.3.1) can be representing in power of p as (3.2.1.5). Then by equating the same powers of p, and the Taylor series expansion of $\sin(u)$ we get

$$p^0:\ u_{0tt}=0 \quad,\quad u_0(x,0)\ =0\ ,\quad u_{0t}(x,0)\ =4\,\text{sech}(x)$$

$$p^1:\ u_{1tt}-u_{0xx}+u_0=0\ ,\quad u_1(x,0)=0\ ,\quad u_{1t}(x,0)=0$$

$$p^2:\ u_{2tt}-u_{1xx}+u_1=0\ ,\quad u_2(x,0)=0\ ,\quad u_{2t}(x,0)=0$$

$$p^3:\ u_{3tt}-u_{2xx}+u_2=0\ ,\quad u_3(x,0)=0\ ,\quad u_{3t}(x,0)=0$$

and so on.

Solving these equations by simple integral yields u_0, u_1, u_2, u_3, u_4 and so on.

$$u=4t\,\text{sech}(x)\ -\ \frac{4t^3}{3}\text{sech}^3(x)\ +\ \frac{4t^5}{15}(2\ -\cosh(x))\,\text{sech}^5(x)+\frac{8t^5\text{sech}^3(x)}{15}-\frac{t^7}{315}(96-80\cosh(2x)+4\cosh(4x))$$

This gives the approximate solution.

(B) Fractional-order Klein-Gordon equation

For solving the equation (2.5.1), we construct the following homotopy :

$$D_t{}^{\alpha}u-6x^3\,\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}=p[(x^3-6x)t^3+u_{xx}-u] \tag{3.2.2.1}$$

Where $f_0 = 6x^3\left(\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right)$ and $f_1 = (x^3 - 6x)t^3$

Substituting Eq. (3.2.1.5) and the initial conditions Eq. (2.5.2) into Eq. (3.2.2.1) and equating the same power of p , we obtain the following equations:

$$p^0 : \; {D_t}^{\alpha} u_0 = 6x^3 \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} \quad , \; u_0(x,0) = 0, \quad (u_0)_t(x,0) = 0$$

$$p^1 : \; {D_t}^{\alpha} u_1 = (x^3 - 6x)t^3 + (u_0)_{xx} - u_0 \quad , u_1(x,0) = 0, \quad (u_1)_t(x,0) = 0$$

$$p^2 : \; {D_t}^{\alpha} u_2 = (u_1)_{xx} - u_1 \quad , \; u_2(x,0) = 0, \quad (u_2)_t(x,0) = 0$$

and so on. Solving these equations by simple integral we obtained u_0, u_1, u_2, u_3 and so on.

$$u(x,t) = x^3 t^3$$

This is the required exact solution.

3.3 Homotopy Perturbation Transform Method

3.3.1 Analysis of HPTM

To illustrate the basic concept of this method, we consider the following differential equation:

$$D\,u(x,t) + R\,u(x,t) + N\,u(x,t) = g(x,t) \qquad (3.3.1.1)$$

With the initial conditions:

$$u(x,0) = h(x)\,,\; u_t(x,0) = f(x)$$

Where $D = \frac{\partial^2}{\partial t^2}$ is the second order linear differential operator, R is the differential operator of less order than D, N is the general nonlinear differential operator and $g(x,t)$ is the source term. Taking the Laplace transform L on both sides of equation (3.3.1.1), we have

$$L\,[D\,u(x,t)] + L\,[R\,u(x,t)] + L\,[N\,u(x,t)] = L\,[\,g(x,t)\,]. \qquad (3.3.1.2)$$

Using the differentiation property of the Laplace transform, we obtained

$$L\,[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}\,L\,[\,g(x,t)\,] - \frac{1}{s^2}\,L\,[\,R\,u(x,t)\,] - \frac{1}{s^2}\,L\,[\,N\,u(x,t)] \quad (3.3.1.3)$$

Operating with the Laplace inverse on both sides of equation (3.3.1.3), we have

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}\, L\,[\,R\,u(x,t)\; + \; N\,u(x,t)\,]\right] \qquad (3.3.1.4)$$

Where $G(x,t)$ is the term arising from the source term and the prescribed initial conditions. Now apply the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n\, u_n\,(x,t\,) \qquad (3.3.1.5)$$

and the nonlinear term can be decomposed as

$$N\,u(x,t) = \sum_{n=0}^{\infty} p^n H_n\,(u) \qquad (3.3.1.6)$$

For some He's polynomial $H_n(u)$ that are given by

$$H_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!}\,\frac{\partial^n}{\partial p^n}\left[\,N\left(\,\sum_{i=0}^{\infty} p^i u_i\right)\,\right]_{p=0}, n = 0,1,2,3, \dots \qquad (3.3.1.7)$$

Substituting equations (3.3.1.5) and (3.3.1.6) in Eq. (3.3.1.4) we have

$$\sum_{n=0}^{\infty} p^n\, u_n\,(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2} L\,[\,R\;\sum_{n=0}^{\infty}\; p^n\, u_n(x,t) + \sum_{n=0}^{\infty}\; p^n\, H_n(u)]\right]\right) \qquad (3.3.1.8)$$

Which is the coupling of the Laplace transform and the HPM using He's polynomials Comparing the coefficient of like powers of p, the following approximation are obtained.

$$p^0 :\; u_0(x,t) = G(x,t)$$

$$p^1 :\; u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L[R\,u_0(x,t) + H_0(u)\,\right]$$

$$p^2 :\; u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L\,[R\,u_1(x,t) + H_1(u)\,\right]$$

$$p^3 :\; u_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L[R\,u_2(x,t) + H_2(u)\,\right]$$

and so on.

3.3.2 Implementation of the Method

(A) Linear Schrödinger equation

Taking the Laplace transform on both sides of Eq. (2.1.1) subject to the initial condition (2.1.2), we have

$$L\,[\,u(x,t)] = \frac{1+\cosh(2x)}{s} - \frac{1}{s}\,i\,L\,[\,u_{xx}\,] \qquad (3.3.2.1)$$

The inverse of Laplace transforms

$$u(x,t) = 1 + \cosh(2x) - L^{-1}\left[\frac{1}{s} iL[u_{xx}]\right] \tag{3.3.2.2}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n = 1 + \cosh 2x - p\left(L^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right]\right) \tag{3.3.2.3}$$

Comparing the coefficients of same powers of p, we have

$$p^0 : u_0(x,t) = 1 + \cosh 2x$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} iL\left[(u_0)_{xx}\right]\right] = (-4it)\cosh(2x)$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} iL\left[(u_1)_{xx}\right]\right] = \frac{(-4it)^2\cosh(2x)}{2!}$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s} iL\left[(u_2)_{xx}\right]\right] = \frac{(-4it)^3\cosh(2x)}{3!}$$

and so on. Therefore the solution $u(x,t)$is given by

$$u(x,t) = 1 + \cosh(2x) + (-4it)\cosh(2x) + \frac{(-4it)^2\cosh(2x)}{2!} + \frac{(-4it)^3\cosh(2x)}{3!} + \cdots$$

$$= 1 + e^{-4it}\cosh(2x)$$

This is an exact solution.

(B) Non- Linear Schrödinger equation

Taking the Laplace transform on both sides of Eq. (2.2.1) subject to the initial condition Eq. (2.2.2), we have

$$L[u(x,t)] = \frac{e^{ix}}{s} + \frac{1}{s} iL[u_{xx} + 2u^2\bar{u}] \tag{3.3.2.4}$$

The inverse of Laplace transform implies that

$$u(x,t) = e^{ix} + L^{-1}\left[\frac{1}{s} iL[u_{xx} + 2u^2\bar{u}]\right]. \tag{3.3.2.5}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{ix} + p\left(L^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{3.3.2.6}$$

The first few components of He's polynomial are given by

$$H_0(u) = 2{u_0}^2\overline{u_0}$$

$$H_1(u) = 2({u_0}^2\overline{u_1} + 2u_1u_0\overline{u_0})$$

and so on. Comparing the coefficients of same powers of p, we have

$$p^0 : \ u_0(x,t) = e^{ix}$$

$$p^1 : \ u_1(x,t) = \ L^{-1}\left[\tfrac{1}{s} iL[(u_0)_{xx} + H_0(u)]\right] = (it)e^{ix}$$

$$p^2 : \ u_2(x,t) = \ L^{-1}\left[\frac{1}{s} iL[(u_1)_{xx} + H_1(u)]\right] = \frac{(it)^2 e^{ix}}{2!}$$

$$p^3 : \ u_3(x,t) = \ L^{-1}\left[\frac{1}{s} iL[(u_2)_{xx} + H_2(u)]\right] = \frac{(it)^3 e^{ix}}{3!}$$

and so on. Therefore the solution $u(x,t)$is given by

$$u(x,t) = \ e^{ix}[1 + it + \tfrac{(it)^2}{2!} + \tfrac{(it)^3}{3!} + \cdots]$$

$$= \ e^{i(x+t)}$$

This is an exact solution.

Conclusion

Homotopy perturbation method (HPM) has the advantage of dealing directly with the problem without transformations, linearization, discretization or any unrealistic assumption and usually a few iterations lead to an accurate approximation of the exact solution. It is clear that HPM provides fast convergence to exact solutions. The comparative study between homotopy perturbation and modified homoopy perturbation methods show that the MHPM is better than HPM because this method can obtain the exact solution only in one iteration. So this method is very effective, simple and very fast convergence as compared to HPM.In homotopy perturbation transform method (HPTM) the solution procedure is simple by using He's polynomials. It is capable of reducing the volume of the computational work as compared to the classical homotopy perturbation method. Also it does not require any arbitrary initial guess L and v_0. Finally ,I observed that the Homotopy perturbation transform method is full advantage of all other given methods.

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