# Review work on splints of classical root system and related studies on untwisted affine Kac-Moody algebra

*Thesis submitted in partial fulfillment of the requirements for the degree of*

**Master of Science**

by

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*Under the guidance of*

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# Certificate

This is to certify that the thesis entitled **"Review work on splints of classical root system and related studies on untwisted affine Kac-Moody algebra"**, which is being submitted by **Sweta Sinha**, **Roll Number 413MA2073** in the Department of Mathematics, *National Institute of Technology, Rourkela*, in partial fulfilment for the award of the degree of **Master of Science**, is a record of bonafide review work carried out by her in the Department of Mathematics under the guidance of mine. She has worked as a project student in this Institute for one year. In my opinion the work has reached the standard, fulfilling the requirements of the regulations related to the Master of Science degree.

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# Abstract

The thesis addresses and makes a review work on a new term splint introduced by David A. Richter and classifies the splints of the classical root systems. Further, a related studies on affine Kac-Moody algebras and discuss the roots of untwisted affine Kac-Moody algebras which will be helpful in determining the splints of Kac-Moody algebras and classified the splints of type $B_n^1$ and $C_n^1$.

# Contents

# Chapter 1

# Introduction and Motivation

The term Lie algebras were discovered by Sophus Lie (1842-1899) while he was attempting to classify certain smooth subgroups of general linear groups. The groups he considered are now called Lie groups. He found that by taking the tangent space at the identity element of such a group, one obtained a Lie algebra. However, Lie algebras can also be studied independently in its own interest. Also E. Cartan and W. Killing investigated about the finite dimensional simple Lie algebras over the complex field (1890-1990). Later on H. Weyl also contributed in basic ideas on the structure and representation theory. The finite dimensional simple Lie algebras has found many application in mathematics and mathematical physics and it is considered as one of the classical branches of mathematics.

Kac-Moody algebras were introduced by V.G Kac and R.V Moody in 1967. The theory of Kac-Moody algebras extended finite dimensional Lie algebras to infinite dimensional. It has also attracted much attention in both mathematics and physics. Moreover, it turned out to be many applications in area of mathematics, including number theory, topology, combinatorics, differential equations and invariant theory. The representation theory of affine Kac-Moody algebras has its application in mathematical physics like string theory, statistical physics and conformal field theory.

In view of these applications the theory of Lie algebra of finite and affine types motivated us to study a new term ”splint” which was introduced by D. A Richter (2008) where the classification of splints for simple Lie algebras is obtained. The motivation comes from representation theory of semisimple Lie algebras. Many of the concepts that arise in the theory of affine Lie algebras play a very similar role and actually were developed in the theory of simple Lie algebra.

I also got motivated to study the theory of Lie algebras by inspiring course of lecture given by Prof. Kishor Chandra Pati at National Institute of Technology Rourkela.

The content of this thesis is summarised as follows. The basic definition of algebra, Lie algebra, Weyl group, Cartan subalgebra, and Cartan matrix are given, Representation theory of $sl_2$ is been discuss which is the foundation for introduction of the term "Splints", and revision of work by D. A Ritcher has been done on splints of classical root system. In Chapter 3 I have studied the few basics concept about genralised Cartan matrices and its classification, and discussed about roots of Kac-Moody algebras of affine type.

# Chapter 2

# Preliminary

In this chapter some definitions and basic results are discussed which are very much essential for the development of the entire project work. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required. The main objective is to establish notation, while more detailed information on these structures appears in the references.

## 2.1 Some Basic Definitions

**Definition 2.1 (Algebra)** *An algebra* $\mathfrak{a}$ *is a vector space endowed with a bilinear operation,*

$$\lozenge : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$$

**Definition 2.2 (Lie Algebra)** *A* **Liealgebra** $\mathfrak{g}$ *is an algebra such that* $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *obeys :*

1. $[x, x] = 0$ *for all* $x \in \mathfrak{g}$ *(antisymmetry)*
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in \mathfrak{g}$ *(Jacobi identity)*

**Definition 2.3 (Simple Lie algebra)** *A simple Lie algebra is a Lie algebra which contains no proper ideals and which is not abelian.*

**Definition 2.4 (Semisimple Lie algebra)** *A semisimple Lie algebra is an algebra which is direct sum of simple Lie algebras.*

**Definition 2.5 (Root System)** *The roots form a finite set of non-zero elements of a real inner product space $E$ and have the following properties :*

1. *The roots span $E$.*
2. *If $\alpha$ is a root, then $-\alpha$ is a root and the only multiples of $\alpha$ that are roots are $\alpha$ and $-\alpha$.*
3. *If $\alpha$ is a root, let $w_\alpha$ denote the linear transformation of $E$ given by*

$$w_\alpha.\beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \tag{2.1}$$

   *then for all roots $\alpha$ and $\beta$, $w_\alpha.\beta$ is also a root.*

4. *If $\alpha$ and $\beta$ are roots, then the quantity $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ is an integer.*

*Any collection of vectors is a finite dimensional real inner product space having these properties is called root system.*

**Definition 2.6 (Direct sum of root system)** *If $\mathfrak{g}$ is complex semisimple, then its root system is a disjoint union of various copies of these, each corresponding to a summand of $\mathfrak{g}$ into simple components.*

*Suppose $\Delta_1$ and $\Delta_2$ are root systems. The direct sum of the root systems is then*

$$\Delta_1 \oplus \Delta_2 = \{(\alpha, 0) : \alpha \in \Delta_1\} \bigcup \{(0, \alpha) : \alpha \in \Delta_2\}$$

The direct sum operation has the following features:

1. If $\Delta = \Delta_1 + \Delta_2$ with $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$, then $\alpha_1 + \alpha_2 \notin \Delta$.
2. If there exist roots $\alpha_1, \alpha_2 \in \Delta$ such that $\alpha_1 + \alpha_2$ is also a root, then the root system cannot be written as a direct sum $\Delta = \Delta_1 + \Delta_2$ where $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$.
3. If one may write $\Delta = \Delta_1 \bigcup \Delta_2$ as a disjoint union where for all $\alpha_1 \in \Delta_1$ and all $\alpha_2 \in \Delta_2$, the vector $\alpha_1 + \alpha_2$ is not a root, then $\Delta$ is the direct sum of $\Delta_1$ and $\Delta_2$.

**Definition 2.7 (Cartan Matrix)** *Let $\mathfrak{G}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{H}$ and simple root system $\Delta^0$ = ($\alpha_1$, ..., $\alpha_r$ ). The Cartan matrix $A = (A_{ij})$ of the simple Lie algebras $\mathfrak{G}$ in the $r \times r$ matrix defined by*

$$A_{ij} = 2\frac{\alpha_i.\alpha_j}{\alpha_i.\alpha_i}$$

**Definition 2.8 (Cartan-Weyl basis)** *Let $\mathfrak{G}$ be a simple complex Lie algebra of dimension n. The Cartan Weyl basis of $\mathfrak{G}$ will be constituted br r generators $H_i$ and the n - r generators $E_\alpha$ satisfying the commutation relations :*

1. $[H_i, H_j] = 0 \qquad [H_i, E_\alpha] = \alpha^i E_\alpha$
2. $[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha^i H_i$
3. $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$

*The r- dimensional vector $\alpha$ = ($\alpha^1$, ..., $\alpha^r$) of $\mathbb{R}^r$ is said to be associated to the root generator $E_\alpha$. The coefficients $N_{\alpha\beta}$ satisfy for any pair of roots $\alpha$ and $\beta$*

$$N_{\alpha\beta}^2 = \frac{1}{2} k(k^{'} + 1)\alpha^2$$

*where k and $k^{'}$ are integers such that $\alpha + k\beta$ and $\alpha - k^{'}\beta$ are roots.*

**Definition 2.9 (Cartan subalgebra)** *The set $\mathfrak{H}$ generated by the generators ($H_1$, ..., $H_r$) is called cartan subalgebra of $\mathfrak{G}$*

## 2.2 Representation Theory

For finite-dimensional modules of a simple Lie algebra $\mathfrak{g}$, one can always find a basis such that the Cartan subalgebra acts diagonally. This is best understood by representation theory of $A_1 \cong sl_2$.

### 2.2.1 Representation theory of $A_1 \cong sl_2$.

The Lie algebra sl(2 F) is an important canonical example of a simple Lie algebra. The irreducible representations of sl(2 F) form the foundation of the representation theory of finite dimensional Lie algebras.

**Definition 2.10 (Weights and maximal vectors)** *Let $V$ be arbitrary L-Module. Let $h$ is semisimple then $h$ acts diagonally on $V$. This yeilds a decomposition of $V$ as direct sum of eigenspaces $V_\lambda = \{v \in V | h.v = \lambda v\}$, $\lambda \in F$. When $\lambda$ is not an eigenvalue for endomorphism of $V$ which represents $h$ then the subspace $V_\lambda = 0$. Whenever $V_\lambda \neq 0$, we define $\lambda$ a* ***weight*** *of $h$ in $V$ and we call $V_\lambda$ a* ***weight space****.*

**Lemma 2.1** *If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

**Remark 2.1** *The lemma implies that $x$ and $y$ are represented by nilpotent endomorphism of $V$.*

*Since $dimV < \infty$ $\exists$ $\lambda \in F$ such that$V_\lambda \neq 0$ but $V_{\lambda+2} = 0$ ( so in particular $x.v = 0$ for any $v \in V_\lambda$) a finite dimensional representation, (note not necessarily irreducible)*

**Definition 2.11 (Maximal vector weight)** *In general, any nonzero vector $V_\lambda$ annihilated by $x$ will be called a maximal vector weight $\lambda$.*

### 2.2.2 Classification of irreducible modules

Now assume that V is an irreducible L-module. Choose a maximal vector $V_0 \in V_\lambda$ ; set $V_{-1} = 0$ and $V_i = (1/i!)y^i.v_0$ for $i \geq 0$.

**Lemma 2.2** 1. $h.v_i = (\lambda - 2i)v_i$

2. $y.v_i = (i+1)v_{i+1}$

3. $x.v_i = (\lambda - i + 1)v_{i-1}$

**Theorem 2.1** *Let V be an irreducible module for L = sl(2, F).*

1. *Relative to h, V is the direct sum of weight spaces $V_\mu$, $\mu$ = m,m-2, . . . ,-(m-2),-m, where m + 1 = dim V and dim $V_\mu$ = 1 for each $\mu$.*

2. *V has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weight of V ) is m.*

3. *The action of L on V is given explicitly by the above formulas, if the basis is chosen in the prescribed fashion. In particular, there exists at most one irreducible L-module (up to isomorphism) of each possible dimension m + 1,m $\geqslant 0$.*

**Corollary 2.1** *Let V be any (finite dimensional) L-module, L = sl(2, F). Then the eigenvalues of h on V are all integers, and each occurs along with its negative (an equal number of times). Moreover, in any decomposition of V into direct sum of irreducible submodules, the number of summands is precisely $dimV_0 + dimV_1$.*

To finish the discussion of representations of sl(2, F) we explore the different representations we can obtain. For dim V = 1 we have the trivial module, for dim V = 2 we have the natural representation, and for dim V = 3 the adjoint representation.

**Definition 2.12 (Weyl Group)** *Let $\mathfrak{G}$ be a simple Lie algebra of rank r with root system $\Delta$ and coroot system $\Delta^v$. For any root $\alpha \in \Delta$ there is a transformation $w_\alpha$ in the weight space, called Weyl reflection, such that if $\lambda$ is a weight:*

$$w_\alpha(\lambda) = \lambda - 2\frac{\alpha.\lambda}{\alpha.\alpha}\alpha$$

*The set of Weyl reflections with respect to all the roots of $\mathfrak{G}$ forms a finite group $W$ called the Weyl group of $\mathfrak{G}$*

## 2.3 Splints of classical root system

Splints play a role in determining branching rules of a module over a complex semisimple Lie algebra.

The represenation theory of finite dimensional Lie algebars motivates us to study the basic concept of branching rule. Here we discuss the basic definition of branching rule and concept of splints and classify the splints of classical root system.

**Definition 2.13 (Branching Rule)** *Let $\mathfrak{G}$ be a semi simple Lie algebra and $\mathfrak{K}$ a subalgebra of $\mathfrak{G}$. An irreducible representation $R(\mathfrak{G})$ of $\mathfrak{G}$ is a representation, in general reducible, $R(\mathfrak{K})$ of $\mathfrak{K}$. So the following formula holds*

$$R(\mathfrak{G}) = \bigoplus_i m_i R_i(\mathfrak{K})$$

*where $R_i(\mathfrak{K})$ is an irreducible representation of $\mathfrak{K}$ and $m_i \in \mathbb{Z}_{>0}$ is the number of times the representation $R_i(\mathfrak{K})$ appears in $R(\mathfrak{G})$. The determination of the above decomposition for any $R_{\mathfrak{G}}$ gives the branching rules of $\mathfrak{G}$ with respect to $\mathfrak{K}$*

**Definition 2.14 (Embedding)** *Embedding $\phi$ of a root system $\Delta_1$ into root system $\Delta$ is a bijective map of roots of $\Delta_1$ to a proper subset of $\Delta$ that commutes with vector composition law in $\Delta_1$ and $\Delta$.*

**Definition 2.15 (Metric Embedding)** *Suppose $\iota : \Delta_0 \longrightarrow \Delta$ is an embedding, and suppose that the inner products associated to $\Delta_0$ and $\Delta$, respectively, are $\langle , \rangle_0$ and $\langle \rangle$. Call the embedding $\iota$ metric if there is a positive scalar $\lambda$ such that $\langle \alpha, \beta \rangle_0 = \lambda \langle \iota(\alpha), \iota(\beta) \rangle$ for all $\alpha$, $\beta \in \Delta_0$ and non metric otherwise.*

**Definition 2.16 (Splints of a root system)** *A root system $\Delta$ "splinters" as $(\Delta_1, \Delta_2)$ if there are two embedings $\iota_1 : \Delta_1 \longrightarrow \Delta$ and $\iota_2 : \Delta_2 \longrightarrow \Delta$ where*
*(a)$\Delta$ is the disjoint union of $\iota_1$ and $\iota_2$*
*(b)neither the rank of $\Delta_1$ nor the rank of $\Delta_2$ exceeds $\Delta$.*

*Equivalently, $(\Delta_1, \Delta_2)$ is a splint of $\Delta$*

**Definition 2.17 (Equivalency of splints)** *If $\Delta$ is a simple root system with weyl group $W$, then the splints $(\Delta_1, \Delta_2)$ and $\left(\Delta_1^{'}, \Delta_2^{'}\right)$ of $\Delta$ are equivalent if there exists $\sigma \in W$ such that*

$$\sigma. \left(\Delta_1 \bigcup (-\Delta_1), \Delta_1 \bigcup (-\Delta_1)\right) = \left(\Delta_1^{'} \bigcup \left(-\Delta_1^{'}\right), \Delta_2^{'} \bigcup \left(-\Delta_2^{'}\right)\right)$$

### 2.3.1 Classification of splints of classical root system

The purpose of this section is is to define what it means for a root system to splinter. The following are the list of classical Lie algebra:
$A_r$ (r $\geq 1$), $B_r$ (r $\geq 2$), $C_r$ (r $\geq 3$), $D_r$ (r $\geq 4$), $E_6$, $E_7$, $E_8$, $F_4$, $G_2$,

**Example 2.1** *1. Case $\Delta = A_r$*

*For r = 1 there is no splints.*

*For r = 2 $(A_1, 2A_1)$*

*For r = 3 $(3A_1, 3A_1)$ , $(3A_1, A_2)$ and $(A_1 + A_2, 2A_1)$*

*In general for $r \geq 4$ there are only two types of splints $(rA_1, A_{r-1})$ and $(A_1 + A_{r-1}, (r-1)A_1)$*

2. *Case* $\Delta = D_r$

   *There is only one value of* $r \geq 4$ *for which* $D_r$ *splinters is* $r = 4$*. In this case there is only one splint* $(2A_2, 2A_2)$

3. *Case* $\Delta = E_6$*,* $E_7$ *and* $E_8$

   *In this case there is no splints.*

4. *Case* $\Delta = G_2$

   *There are two splints* $(A_2, A_2)$ *and* $(2A_1, B_2)$

5. *Case* $\Delta = F_4$

   $(D_4, D_4)$ *is the required splints.*

6. *Case* $\Delta = B_r$

   *For any* $r \geq 2$*, the root* $B_r$ *has a splint* $(D_r, rA_1)$

7. *Case* $\Delta = C_r$

   *For* $r \geq 4$ *the required splint is* $(rA_1, D_r)$

Here the case by case analysis of classical Lie algebra and its root system lead to the splintering of root system.

# Chapter 3

# Affine Kac-Moody algebra

## 3.1 Generalised Cartan matrices and Kac-Moody algebras

In this chapter we come to the first object of our interest, the theory of affine Lie algebras. In these we discuss basic definitions of generalised Cartan matrix, Realisations and Kac-Moody algebras. Section 1.2 describes the root system and dynkin diagram of affine algebras. In section 1.3 we discuss the properties of highest weight modules of affine algebras. In section 1.4 we discuss about branching rules for embedding of affine algebras.

We start with a complex n $\times$ n matrix A = $(a_{ij})_{i,j=1}^{n}$ of rank $l$ and we will associate with it a complex Lie algebra $g(A)$.

**Definition 3.1 (Generalised Cartan Matrix)** *A matrix A is called generalised cartan matrix if it satifies the following conditions :*

1. $a_{ii} = 2$ *for i = 1,...,n;*
2. $a_{ij}$ *are non positive integers for* $i \neq j$*;*
3. $a_{ij} = 0 \Rightarrow a_{ji} = 0$*;*

**Definition 3.2 (Realization)** *A realization of A is a triple* $(h,\Pi,\ \Pi^v)$*, where* $h$ *is a complex vector space,* $\Pi =\ \alpha_1,\ ...,\ \alpha_n \subset h^*$ *and* $\Pi^v =\ \alpha_1^v,\ ...,\ \alpha_n^v \subset h$ *are indexed subsets in* $h^*$*,* $h$*, respectively, satisfying the following three conditions :*

1. *both sets* $\Pi$*,* $\Pi^v$ *are linearly independent;*
2. $\langle \alpha_i^v, \alpha_j \rangle = a_{ij} \quad (i,j = 1,...,n)$*;*
3. $n - l$ *= dim h - n*

**Definition 3.3 (Kac-Moody algebra)** *Let* $e_i$*,* $f_i$ *and* $h_i$ *denote the Chevalley generators. The KacMoody algebra is defined as the Lie algebra* $\mathfrak{g}$ *together with following relations :*

1. $[h_i, h_j]$ *= 0*
2. $[e_i, f_i]$ *=* $h_i$
3. $[e_i, f_j]$ *= 0*
4. $[h_i, e_j]$ *=* $a_{ij}e_j$
5. $[h_i, f_j]$ *= -*$a_{ij}f_j$
6. $(ade_i)^{1-a_{ij}}e_j$ *= 0 if* $i \neq j$
7. $(adf_i)^{1-a_{ij}}f_j$ *= 0 if* $i \neq j$

## 3.2 The classification of generalised Cartan matrices

**Definition 3.4** *A generalised Cartan matrices A has finite type if*

1. *det* $A \neq 0$*.*

2. *there exists u > 0 with Au > 0.*

3. *Au≥ 0 implies u>0 or u = 0.*

*The generalised Cartan matrices A has affine type if*

1. *corank A = 1.*

2. *there exists u > 0 with Au = 0.*

3. *Au≥ 0 implies Au = 0.*

*The generalised Cartan matrices A has indefinite type if*

1. *there exists u > 0 with Au < 0.*

2. *Au≥ 0 and u≥ 0 imply u = 0.*

## 3.3 Roots of Kac-Moody algebra

In case of affine Kac-Moody algebra there exists real root as well as imaginary roots. When A has affine type, then there exists u>0 with Au = 0. The vector u is determined to within a scalar multiple. Thus there is unique such u whose entries are positive integers with no common factor. Let u = ($a_1$, ..., $a_n$). Let $\delta$ = $a_1\alpha_1$ + ... + $a_n\alpha_n$. Then the imaginary roots of L(A) are the elements k$\delta$ for k$\in \mathbb{Z}$, k$\neq$ 0. Also $\alpha_i j$ = $e_i$ - $e_j$

We have studied the roots of untwisted affine Kac-Moody algebras. Here $e_i's$ are basis vectors where $i, j \in \mathbb{Z}$ and It is given as follows :

1. $A_l^{(1)}$ ( $l \geq 1$) :
   $\Delta = \{e_i - e_j + n\delta, m\delta, 1 \leq i, j \leq l+1, i \neq j, 0 \neq m \in \mathbb{Z}, n \in \mathbb{Z}\}$

2. $B_l^{(1)}$ ( $l \geq 2$) :
   $\Delta = \{\pm e_{ij} \pm n\delta, \pm(e_i \pm e_j) + n\delta, m\delta, 1 \leq i, j \leq l, i \neq j, 0 \neq m \in \mathbb{Z}, n \in \mathbb{Z}\}$

3. $C_l^{(1)}$ ( $l \geq 2$) :
   $\Delta = \{\pm 2e_{ij} \pm n\delta, \pm(e_i \pm e_j) + n\delta, m\delta, 1 \leq i, j \leq l, i \neq j, 0 \neq m \in \mathbb{Z}, n \in \mathbb{Z}\}$

4. $D_l^{(1)}$ ( $l \geq 4$) :
   $\Delta = \{\pm(e_i \pm e_j) + n\delta, m\delta, 1 \leq i, j \leq l, i \neq j, 0 \neq m \in \mathbb{Z}, n \in \mathbb{Z}\}$

5. $E_6^{(1)}$ :
   $\Delta = \{\pm(e_i \pm e_j) + n\delta, \pm 1/2(e_8 - e_7 - e_6 + \Sigma(-1)^{v(i)}e_i) + n\delta(\Sigma v(i)e_i)i < j, n \in \mathbb{Z}\}$

6. $E_7^{(1)}$ :
   $\Delta = \{\pm(e_i \pm e_j) + n\delta, \pm(e_7 - e_8) + n\delta, \pm 1/2(e_8 - e_7 - e_6 + \Sigma(-1)^{v(i)}e_i) + n\delta(\Sigma v(i)e_i)i < j, n \in \mathbb{Z}\}$

7. $E_8^{(1)}$ :
   $\Delta = \{\pm(e_i \pm e_j) + n\delta, \Sigma(-1)^{v(i)}e_i) + n\delta(\Sigma v(i)e_i)i < j, n \in \mathbb{Z}\}$

8. $F_4^{(1)}$ :
   $\Delta = \{\pm(e_i \pm e_j) + n\delta, 1/2(\pm e_1 \pm e_2 \pm e_3 \pm e_4) + n\delta i < j, n \in \mathbb{Z}\}$

9. $G_2^{(1)}$ :
   $\Delta = \{\pm(\phi_i) + n\delta(1 \leq i \leq 3) \pm (\phi_i - \phi_j) + n\delta, m\delta(1 \leq i \leq j \leq 3), n, m \in \mathbb{Z}\}$
   Where $\phi_i = e_i$ - $1/3(e_1 + e_2 + e_3)$ i=1,2,3

After studying the roots of affine Kac-Moody algebras we can extend our work to splints of affine Kac-Moody algebras. Here is the few examples of splints of untwisted affine Kac-Moody algebra.

**Example 3.1** *1. Case* $\Delta = B_n^{(1)}$
*Splints of* $B_n^{(1)}$ *is* $(B_m^{(1)}, D_{n-m}^{(1)})$ *where* $n \geq 3$, $m \geq 1$

*2. Case* $\Delta = C_n^{(1)}$
*Splints of* $C_n^{(1)}$ *is* $(C_m^{(1)}, C_{n-m}^{(1)})$ *where* $n \geq 3$, $m \geq 1$

# Conclusions and Scope of Future Works

Due to the time constraint I could not complete the whole work and the result obtained, I could not generalised it. In future more analysis is to be done on representation theory of affine Kac-Moody algebra so that we can classify the splints of affine Kac-Moody algebra in more generalized term.

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