PARTICULARS OF NON-LINEAR OPTIMIZATION

A Project Report Submitted by

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A thesis presented for the degree of
Master of Science

Department of Mathematics
National Institute of Technology, Rourkela
India

May 2015

Certificate

This is to certify that the one year master's project report entitled with *PARTICULARS OF NON-LINEAR OPTIMIZATION* submitted by ***Devendar Mittal*** to the Department of Mathematics of National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirement for the degree of Master of Science in *Mathematics* is a bonafide record of review work carried out by him under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

May, 2015

Prof. Anil Kumar

Declaration

I *Devendar Mittal*, a bonafide student of Integrated M.Sc. in Mathematics of Department of Mathematics, National Institute of Technology, Rourkela would like to declare that the dissertation entitled with ***PARTICULARS OF NON-LINEAR OPTIMIZATION*** submitted by me in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics is an authentic work carried out by me.

PLACE: Rourkela
DATE : 9th May, 2015

Devendar Mittal

Acknowledgment

It is my pleasure to thank to the people, for whom this dissertation is possible. A special thank to my supervisor ***Prof. Anil Kumar***, for his keen guidance and encouragement towards the thesis and also the study of numerous topics which I came across during my coursework.

I would also like to thank the faculty members of *Department of Mathematics* for their co-operation. I thank the Director, ***Prof. Sunil Sarangi***, National Institute of Technology Rourkela, for providing the facilities to pursue my postgraduate degree in ***Integrated M.Sc.***

I thank all my classmates and friends for making my stay memorable at National Institute of Technology Rourkela. Finally, I also thank my parents for their incessant inspiration and motivation.

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Abstract

We are providing a concise introduction to some methods for solving non-linear optimization problems. In mathematics, non-linear programming (NLP) is the process of solving an optimization problem defined by a system of equalities and inequalities, collectively termed constraints, over a set of unknown real variables, along with an objective function to be maximized or minimized, where some of the constraints or the objective function are non-linear. It is the sub-field of mathematical optimization that deals with problems that are not linear. This dissertation conducts its study on the theory that are necessary for understanding and implementing the optimization and an investigation of the algorithms such as *Wolfe's Algorithm*, *Dinkelbach's Algorithm* and etc. are available for solving a special class of the non-linear programming problem, quadratic programming problem which is included in the course of study.

Optimization problems arise continuously in a wide range of fields such as *Power System Control* (see [2]) and thus create the need for effective methods of solving them. We discuss the fundamental theory necessary for the understanding of optimization problems, with particular programming problems and the algorithms that solve such problems.

Contents

Introduction

Chapter 1

Introduction to Non-Linear Programming

1.1 Definition of Non-Linear Programming

Definition 1.1.1. Let z be a real valued function of n variables define by

$$z = f(x_1, \ldots, x_n).$$

Suppose $d_1, d_2, \ldots, d_m$ be a set of constraints such that

$$g_i(x_1, x_2, \ldots, x_n)\{\leq, \geq, or =\}\ d_i$$

for all $i = 1, 2, \ldots, m$, where $g_i's$ are real value function of n variables $x_1, \ldots, x_n$. Finally, assuming $x_j \geq 0$, $j = 1, \ldots, n$. If either $f(x_1, \ldots, x_n)$ or some $g_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$ or both are non-linear, then a problem to find an *optimal* solution of z for n-type $(x_1, \ldots, x_n)$ variables and satisfies the constraints is called as a *general non-linear programming problem.*

1.2 Convex Sets and Functions

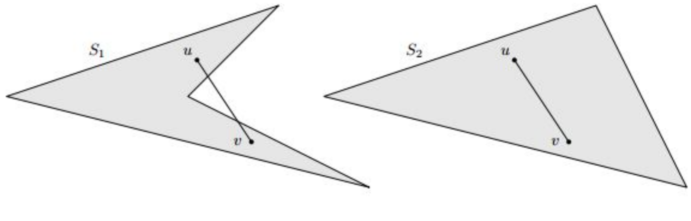
Convex Sets

Definition 1.2.1. A set $S \subset \mathbb{R}^n$ is called a *convex set* if

$$\mathbf{x_1}, \mathbf{x_2} \in S \Rightarrow \lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \in S\ \forall\ 0 \leq \lambda \leq 1 \tag{1.1}$$

Example 1.2.1. *The geometrical interpretation of* (1.1) *is depicted in figure 1.1. We see that S_2 is a convex set while the same not happens with S_1.*

Figure 1.1: ***Example of Convex Set***



Theorem 1.2.1. *The convex set in $\mathbb{R}^n$ satisfies the following properties:*

1. *The intersection of any collection of convex set is a convex set.*

2. *If C is a convex set and $\lambda \in \mathbb{R}$, then λC is a convex set.*

3. *If C and D are convex sets, then $C + D$ is a convex set.*

Convex Functions

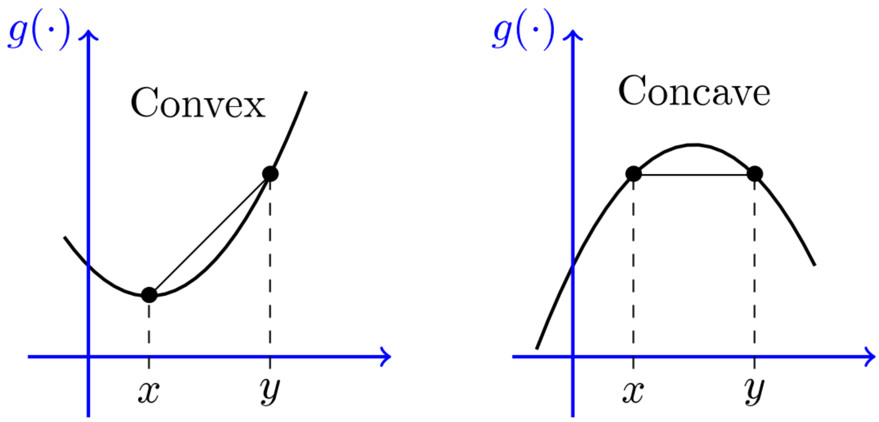
A function f defined on $T \subset \mathbb{R}^n$ is said to be convex at a point $x_0 \in T$ if $x_1 \in T$, $0 \leq \lambda \leq 1$ and $\lambda x_0 + (1-\lambda)x_1 \in T$ implies the following

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1) \tag{1.2}$$

A function f is said to be convex on T if it is convex on every point of T.

Example 1.2.2. *The example of convex function can be understood by the follwoing figure 1.2:*

Figure 1.2: ***Example of Convex Functions***



Theorem 1.2.2. *Let f be twice differentiable function on an open convex set $T \subset \mathbb{R}^n$, then*

1. *f is convex on T if the $H(x)$ (Hessian Matrix) of f is positive semi-definite for each $x \in T$.*

2. *f is strictly convex on T, if the $H(x)$ of f is positive definite for each $x \in T$. The converse is not true.*

§ *Definiteness of Matrix*

Say we have

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Then, A has three 'second-order principles'submatrices and three 'leading principals submatrices'. The relationship between the principal minors and definiteness of a matrix are

1. An $n \times n$ matrix A is positive definite iff all its n leading principal minors are strictly positive.

2. An $n \times n$ matrix A is negative definite iff all its n leading principal minors alternate in signs, with the sign of the k^{th} order leading principal minor equal to $(-1)^k$, i.e.

$$x_{11} < 0 \; ; \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} > 0 \; ; \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} < 0$$

Non-Linear Programming

Chapter 2

Unconstrained Optimization

2.1 Unconstrained Univariate Optimization

Univariate optimization defined as optimization of a scalar function of a single variable:

$$y = p(x) \tag{2.1}$$

The importance of these optimization methods are given as follows:

1. There are many instances in engineering when we want to find the optimum of functions like optimum reactor temperature, etc.
2. Almost all multivariable optimization methods in commercial use today contain a line search step in their algorithm,
3. They are easy to illustrate and many of the fundamental ideas are directly taken over to multivariable optimization.

2.1.1 Necessary and Sufficient Conditions for an Optimum

For a twice continuously differentiable function $p(x)$ (see (2.1)), the point x^*, is an optimum iff:

$$\frac{dp}{dx} = 0 \ at \ x = x^* \tag{2.2}$$

$$\frac{d^2p}{dx^2} < 0 \ at \ x = x^* \tag{2.3}$$

or

$$\frac{d^2p}{dx^2} > 0 \ at \ x = x^* \tag{2.4}$$

Equation (2.2) provides the extreme points x^*, hence as a result, equation (2.3) as maximum or equation (2.4) as minimum.

2.2 Unconstrained Multivariate Optimization

Non-Linear Optimization techniques concerns with optimizing a scalar function of two or more than two variables as described mathematically in eq. (2.5)

$$y = p(x_1, x_2, x_3, \ldots, x_n) \tag{2.5}$$

2.2.1 Procedure for obtaining extreme points

Step 1 : Like we have seen in univariate unconstrained optimization, we have to obtain the extreme points by partial differentiating the function:

$$\left.\begin{aligned} \frac{\partial p}{\partial x_1} &= 0 && \text{(2.6a)} \\ \frac{\partial p}{\partial x_2} &= 0 && \text{(2.6b)} \\ &\vdots && \text{(2.6c)} \\ \frac{\partial p}{\partial x_n} &= 0 && \text{(2.6d)} \end{aligned}\right\}$$

Step 2 : The eq. (2.6) results into the stationary points which is denoted by $\bar{x}$.

Step 3 : Deduce the second derivative by substituting the stationary points and configure the matrix of these values calling it as *Hessian matrix* at $\bar{x}$.

Say, for three variables x_1, x_2 and x_3, and the function defined as $p = p(x_1, x_2, x_3)$ the Hessian matrix is given as

$$H(\bar{x}) = \begin{bmatrix} p_{x_1x_1} & p_{x_1x_2} & p_{x_1x_3} \\ p_{x_2x_1} & p_{x_2x_2} & p_{x_2x_2} \\ p_{x_3x_1} & p_{x_3x_2} & p_{x_3x_3} \end{bmatrix}$$

Step 4 : Consider the leading minors of $H(\bar{x})$ which is given as follows:

$$p_{x_1x_1}; \begin{vmatrix} p_{x_1x_1} & p_{x_1x_2} \\ p_{x_2x_1} & p_{x_2x_2} \end{vmatrix}; \begin{vmatrix} p_{x_1x_1} & p_{x_1x_2} & p_{x_1x_3} \\ p_{x_2x_1} & p_{x_2x_2} & p_{x_2x_2} \\ p_{x_3x_1} & p_{x_3x_2} & p_{x_3x_3} \end{vmatrix}$$

Step 5 : (a) If the positive sign appears in the leading minors then, the point $\bar{x}$ is minimum.

(b) If the leading minors are of alternate sign then the stationary point is maximum.

Chapter 3

Constrained Optimization

Consider a situation where the non linear programming is comprising of some differentiable objective functions and equality sign constraints, then the optimization is achieved by implementing *Lagrangian multipliers*.

Definition 3.0.1. Say $f(x_1, x_2)$ and $g(x_1, x_2)$ are differentiable functions with respect to x_1 and x_2. Consider the problem of maximizing and minimizing

$$z = f(x_1, x_2)$$

subject to constraints

$$g(x_1, x_2) = c$$

and $x_1 > 0$, $x_2 > 0$ and c is a constant. Introducing a function:

$$h(x_1, x_2) = g(x_1, x_2) - c$$

To find a necessary condition for maximizing or minimizing value of z a new function is constructed as follows:

$$L(x_1, x_2; \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2) \tag{3.1}$$

The λ is an unknown number called as *Lagrangian multiplier* whereas L is called as *Lagrangian function.*

3.1 Generalised Lagrangian Method to n-Dimesnsional Case

Consider the general NLPP
Maximise(or minimise) Z=$f(x_1, x_2, x_3, \dots, x_n)$ subject to the constraints

$$g_i(x_1, x_2, \dots, x_n) = c_i \text{ \& } x_i \geq 0, i = 1, 2, \dots, m (m \leq n)$$

The constraints can be written as

$$h_i(x_1, x_2, \dots, x_n) = 0 \text{ for } i = 1, 2, \dots, m$$

where,

$$h_i(x_1, x_2, \dots, x_n) = g_i(x_1, x_2, \dots, x_n) - c_i \tag{3.2}$$

3.1.1 Necessary Condition for Maximum(Minimum)

To find the necessary condition for a maximum or minimum of $f(x)$, the *Lagrangian function* $L(x,\lambda)$ is formed by introducing m *Lagrangian multipliers* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. The function is given as

$$L(x,\lambda) = f(x) = \sum_{i=1}^{m} \lambda_i h_i(x). \tag{3.3}$$

As we are working with only differentiable function, the necessary conditions for a maximum(minimum) of $f(x)$ are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(x)}{\partial x_j} = 0 \qquad j = 1, 2, \ldots, n \tag{3.4}$$

$$\frac{\partial L}{\partial \lambda_i} = -h_i(x) = 0 \qquad i = 1, 2, \ldots, m \tag{3.5}$$

These $m + n$ necessary conditions also become sufficient for a maximum(minimum) of the objective function if the objective function is concave(convex) and the side constraints are equality ones.

3.1.2 Sufficient Conditions for Maximum/Minimum of Objective Function(with single Equality Constraint)

Let the *Lagrangian function* for a general NLPP involving n variables and one constraint be:

$$L(x,\lambda) = f(x) - \lambda h(x). \tag{3.6}$$

The necessary conditions for a stationary point to be a maximum or minimum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 (j = 1, 2, \ldots, n) \tag{3.7}$$

and,

$$\frac{\partial L}{\partial \lambda} = -h(x) = 0 \tag{3.8}$$

The sufficient conditions for a maximum or minimum require the evaluation at each stationary point of $n - 1$ principal minors of the determinant given below:

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If $\Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 > 0, \ldots,$ the sign pattern being alternate, stationary points is local maximum, otherwise, the point of local minimum is observed if $\Delta_i = 0,\ \forall\ i \geq 2$.

3.1.3 Sufficient Conditions for Maximum/Minimum of Objective Function (with more than one equality constraints)

In this method we introduce the m *Lagrangian multipliers* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, let the *Lagrangian* function for a general NLPP with more than one constraint be

$$L(x,\lambda) = f(x) = -\sum_{i=1}^{m} \lambda_j h_j(x) (m < n). \tag{3.9}$$

The necessary condition for the stationary points are:

$$\frac{\partial L}{\partial x_i} = 0 \text{ and } \frac{\partial L}{\partial \lambda_j} = 0 \ (i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m)$$

The sufficiency conditions for the *Lagrange Multiplier Method* of stationary point of $f(x)$ to be maxima or minima is defined as: Let

$$V = \left(\frac{\partial^2 L(x, \lambda)}{\partial x_i \partial x_j} \right)_{n \times n} \tag{3.10}$$

be the matrix of the second order partial derivatives of $L(x, \lambda)$ with respect to decision variables

$$\text{and } U = [h_{ij}(x)]_{m \times n} \tag{3.11}$$

where

$$h_{ij}(x) = \frac{\partial h_i(x)}{\partial x_j}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

Now, define the *Boardered Hessian Matrix* i.e.

$$H^B = \begin{bmatrix} O & | & U \\ \ldots & | & \ldots \\ U^T & | & V \end{bmatrix}_{(m+n) \times (m+n)}$$

where O is a $m \times n$ null matrix, then the sufficient conditions for maximum and minimum stationary points are given as follows:

Let (x_0, λ_0) for the function $L(x; \lambda)$ be its stationary point.

1. x_0 is a maximum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B form an alternating sign pattern starting with $(-1)^{m+n}$
2. x_0 is a minimum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B have the sign of $(-1)^m$.

Example 3.1.1. *Solve the NLPP:*
Minimize $Z = 4x_1^2 + 2_x^2 + x_3^3 - 4x_1x_2$ *subject to the constraints* $x_1 + x_2 + x_3 = 15$, $2x_1 - x_2 + 2x_3 = 20$.

Solution. Say,

$$f(x) = 4x_1^2 + 2_x^2 + x_3^3 - 4x_1x_2$$
$$h_1(x) = x_1 + x_2 + x_3 - 15$$
$$h_2(x) = 2x_1 - x_2 + 2x_3 - 20$$

Constructing the *Lagrangian* function; which is given as follow:

$$\begin{aligned} L(x, \lambda) &= f(x) - \lambda_1 h_1(x) - \lambda_2 h_2(x) \\ &= 4x_1^2 + 2_x^2 + x_3^3 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20) \end{aligned}$$

Necessary conditions yielding stationary points are as follows:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \\ \frac{\partial L}{\partial x_2} &= 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial x_3} &= 2x_3 - \lambda_1 - 2\lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda_1} &= -[x_1 + x_2 + x_3 - 15] = 0 \\ \frac{\partial L}{\partial \lambda_2} &= -[2x_1 - x_2 + 2x_3 - 20] = 0 \end{aligned}$$

Solution to the problem is

$$x_0 = (x_1, x_2, x_3) = (33/9, 10/3, 8)$$

and

$$\lambda_0 = (\lambda_1, \lambda_2) = (40/9, 52/9)$$

Clearly, one sees that $n = 3$ and $m = 2$, which means that $n - m = 1$ and $(2m + 1) = 5$. Thus, one may clearly inspects that $H_0^B = 72 > 0$ i.e. the sign of H_0^B is $(-1)^2$. So, concluding that x_0 is a minimum point. □

3.2 Constrained Optimization with Inequality Constraints

3.2.1 Kuhn-Tucker Condition

Consider the general NLPP:

$$\text{Optimize } Z = p(x_1, x_2, \ldots, x_n), \text{ subject to the constraints}$$

$$g_i(x_1, x_2, \ldots, x_n) \leq c_i \text{ and } x_1, x_2, \ldots, x_n \geq 0 \text{ and } i = 1, 2, \ldots, m (m < n)$$

Convert the inequality into the equality by introducing some slack or surplus quantity and thus apply lastly, *Lagrangian Multiplier Method.* The Lagrangian Multiplier Method works best if the NLPP follows strictly the constraints inequality.

So, to convert the constraints inequality into equality one we use new non-negative variable S called as *slack variable.* We use only non-negative slack variable to avoid an additional constraints $S \geq 0$.

Introducing m slack variables in m inequality constraints, $S = (S_1, S_2, \ldots, S_m)$, then the problem can be stated as:

$$\text{Optimize } Z = p(x_1, x_2, \ldots, x_n), \text{ subject to the constraints}$$

$$h_i(x_1, x_2, \ldots, x_n) + s_i^2 = 0$$

where,

$$h_i(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, x_n) - c_i \leq 0 \text{ and } x_1, x_2, \ldots, x_n \geq 0 \text{ and } i = 1, 2, \ldots, m$$

Then the Lagrangian function for the above NLPP with m constraints is

$$L(x, s, \lambda) = p(x) - \sum_{i=1}^{m} \lambda_i [h_i(x) + s_i^2] \tag{3.12}$$

where, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ are *Lagrangian Multiplier Vector.* The necessary condition for $p(x)$ to be a maximum are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial p}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \text{ for } j = 1, 2, \ldots, n \tag{3.13}$$

$$\frac{\partial L}{\partial \lambda_i} = h_i + s_i^2 = 0 \text{ for } i = 1, 2, \ldots, m \tag{3.14}$$

$$\frac{\partial L}{\partial s_i} = -2 s_i \lambda_i = 0 \text{ for } i = 1, 2, \ldots, m \tag{3.15}$$

where

$$L = L(x, S, \lambda), \ p = p(x) \text{ and } h_i = h_i(x)$$

Thus the *Kuhn-Tucker conditions* for a maximum is restated as:

$$p_j = \sum_{i=1}^{m} \lambda_i h_{ij}$$
$$\lambda_i h_i = 0$$
$$h_i \geq 0$$
$$\lambda_i \geq 0$$

For the NLPP of maximising $p(x_1, x_2, \ldots, x_n)$ subject to the inequality constraints $h_i(x) \leq 0$ $(i = 1, 2, \ldots, m)$, the *Kuhn-Tucker* conditions are also the *sufficient* conditions for a maximum if $p(x)$ is concave and all $h_i(x)$ a convex function.
The K-T conditions for a minimization NLPP can be obtained in the similar manner.

Example 3.2.1. *Use the Kuhn-Tucker conditions to solve the NLPP:*

$$Max.Z = 8x_1 + 10x_2 - x_1^2 - 2x_2^2$$

subject to $\quad 3x_1 + 2x_2 \leq 6, x_1 \geq 0, x_2 \geq 0$

Solution. Here,

$$f(x) = 8x_1 + 10x_2 - x_1^2 - 2x_2^2$$
$$g(x) = 3x_1 + 2x_2, \; c = 6$$
$$h(x) = g(x) - c = 3x_1 + 2x_2 - 6$$

The K-T conditions are

$$\frac{\partial f(x)}{\partial x_1} - \lambda \frac{\partial h(x)}{\partial x_1} = 0, \frac{\partial f(x)}{\partial x_2} - \lambda \frac{\partial h(x)}{\partial x_2} = 0$$

$\lambda h(x) = 0$, $h(x) \leq 0$, $\lambda \geq 0$, where λ is the Lagrangian multiplier. That is

$$8 - 2x_1 = 3\lambda \tag{3.16}$$
$$10 - 2x_2 = 2\lambda \tag{3.17}$$
$$\lambda[3x_1 + 2x_2 - 6] = 0 \tag{3.18}$$
$$3x_1 + 2x_2 - 6 = 0 \tag{3.19}$$
$$\lambda \geq 0 \tag{3.20}$$

From eq. (3.18) either $\lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$. For $\lambda = 0$ does not provide any optimal solution, but (3.19) provides an optimal solution at stationary point $x_0 = (x_1, x_2) = (4/13, 33/13)$. The K-T conditions are sufficient conditions for providing maximum. Hence, by x_0 the maximum value of Z is 21.3. □

3.2.2 Graphical Method for solving NLPP

The Linear Programming Problem (LPP) provides the optimal solution at one of the extreme points of the convex region which is generated by the constraints and the objective function of the problem. But in NLPP, it is not necessary to determine the solution at a corner or edge of the feasible region, as the following example depicts:

Example 3.2.2. *Graphically solve*

$$maximize\ z = 3x_1 + 5x_2$$

subject to

$$x_1 \leq 4$$
$$9x_1^2 + 5x_2^2 \leq 216$$

provided that $x_1, \; x_2 \geq 0$

Solution. Say,

$$x_1 = 4 \tag{3.21}$$

$$\frac{x_1^2}{\frac{216}{9}} + \frac{x_2^2}{\frac{216}{5}} = 1 \tag{3.22}$$

Eq. (3.22) is an ellipse
On differentiating the objective function by considering it $3x_1 + 5x_2 = \kappa$[1] with respect to x_1

$$\frac{dx_2}{dx_1} = -\frac{3}{5} \tag{3.23}$$

But the differentiation of (3.22) with respect to x_1 gives

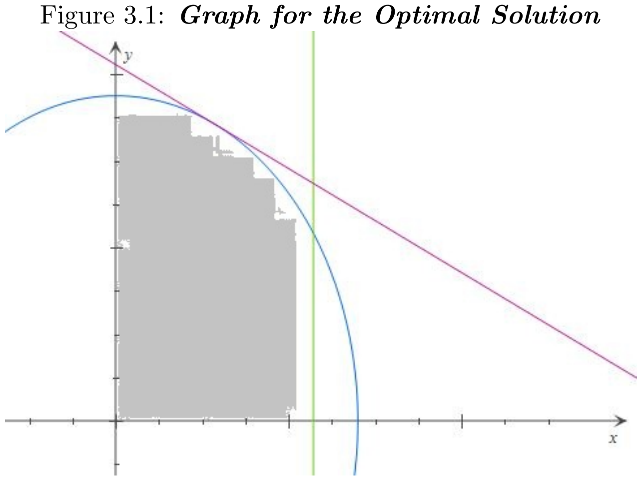
$$\frac{dx_2}{dx_1} = -\frac{18x_1}{10x_2} \tag{3.24}$$

So, we arrive at the conclusion that

$$3x_1 = x_2$$

after confronting (3.23) and (3.24), thus $x_1 = \pm 2$, and $x_2 = 6$ when $x_1 = 2$. This leads to the optimal solution which is max $z = 36$

Figure 3.1: ***Graph for the Optimal Solution***



The *shaded* region is for feasible region, bounded by objective function $3x_1 + 5x_2 = \kappa$ and (3.21) and (3.22). □

[1] κ is a constant

Chapter 4

Quadratic Programming

As the name suggests that, here we have to optimize the quadratic objective function subject to the linear inequality constraints.Unlike LPPs, the optimal solution to a NLPP can be found anywhere on the boundary of the feasible region and even some interior point of it. For LPPs, we have very efficient algorithm to solve but no such algorithm exist for solving NLPP.

Definition 4.0.1. Let x^T and $C \in R^n$. Let Q be a symmetric $m \times n$ real matrix. Then, the problem of maximising

$$f(x) = Cx + \frac{1}{2}x^TQx \text{ subject to the constraints}$$

$$Ax \leq b^T \text{ and } x \geq 0$$

where, $b^T \in R^n$ and A is a $m \times n$ real matrix is called a general *Quadratic Programming Problem.*

The function x^TQx relates a quadratic form. The quadratic form x^TQx is said to be positive definite if $x^TQx > 0$ for $x \neq 0$ and positive semi-definite if $x^TQx \geq 0$ for all x such that there is one $x \neq 0$ satisfying $x^TQx = 0$, then it is convex in x over all of R^n and vice versa.
These result help in enumerating whether the QPP $f(x)$ is concave/convex and the implication of the same on the sufficiency of the Kuhn-Tucker conditions for constrained maxima/minima of $f(x)$.

4.1 Wolfe's Algorithm:

Consider a Quadratic Programming Problem in the form:

$$Maximize = f(X) = \sum_{j=1}^{n} c_jx_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} x_jd_{jk}x_k$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i = 1, 2, \dots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \dots, n$$

where,$d_{jk} = d_{kj}$ for all j and k. Also we assume that the Quadratic form $x_jd_{jk}x_k$ be negative semi-definite.

The *Wolfe's Algorithm* for solution of QPP is summarised as follows:

Step 1 : Convert the inequality constraints into equality constraints by introducing the slack variables r_i^2 in the i^{th} constraints $i = 1, 2, \dots, m$, and the slack variables q_j^2 in the j^{th} non-negativity constraint $j = 1, 2, \dots, n$.

Step 2 : Construct the Lagrangian function

$$L(X,q,r,\lambda,\mu) = f(X) - \sum_{i=1}^{m} \lambda_i [\sum_{j=1}^{n} a_{ij}x_j - b_i + r_i^2] - \sum_{j=1}^{n} \mu_j(-x_j + q_j^2)$$

where $X = (x_1, x_2, \ldots, x_n), q = (q_1^2, q_2^2, \ldots, q_n^2), r = (r_1^2, r_2^2, \ldots, r_m^2), \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$.
On first order partial differentiating of function $L(x,q,r,\lambda,\mu)$ with respect to the components of x, q, r, λ, μ and on equating the above derivatives to zero, which results Kuhn-Tucker condition from the resulting equation.

Step 3 : Introduce the positive artificial variables W_j, $j = 1, 2, \ldots, n$ in the K-T condition

$$c_j + \sum_{k=1}^{n} d_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0$$

for $j = 1, 2, \ldots, n$ and deduce the objective function to

$$Z = W_1 + W_2 + \ldots + W_n$$

Step 4 : Now, obtain the initial basic feasible solution to the Linear Programming problem:

Minimize $Z = W_1 + W_2 + \ldots + W_n$ subject to the constraints

$$\sum_{k=1}^{n} d_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_i j + \mu_j + W_j = -c_j \qquad (j = 1, 2, \ldots, n)$$

$$\sum_{j=1}^{n} a_{ij}x_j + r_i^2 = b_i \qquad (i = 1, 2, \ldots, m)$$

$$W_i, \lambda_i, \mu_j, x_j \geq 0 \qquad (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$$

Also satisfying the complementary slackness conditions:

$$\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} r_i^2 \lambda_i = 0$$

Step 5 : Now, we can apply two phase simplex method to obtain an optimum solution to the LPP of step 4, provided the solution satisfies the complementary slackness condition.

Step 6 : The optimum solution which we get from step 5 is an optimum solution to the given QPP.

Notes:

1. If the Quadratic programming problem is given in the minimization form,then convert it into maximization one by suitable modification in $f(X)$ and the '$\geq$' constraints.
2. The solution of the given system is obtained by using *Phase I* of the simplex method. The solution does not need the consideration of *Phase II*.We have to maintain the condition $\lambda_i q_i^2 = 0 = \mu_j x_j$ every time.
3. Here we will observe that the *Phase I* will terminate in the usual manner i.e. the sum of all artificial variables equal to *zero* only if the feasible solution exists.

Example 4.1.1. *Maximize* $Z = 2x_1 + 3x_2 + 2x_1^2$ *subject to*
$x_1 + 4x_2 \leq 4, x_1 + x_2 \leq 2, x_1, x_2 \geq 0$

Solution. The solution is given as follows:

Step 1: First, introduce the slack variables in the constraints and deduce in equality form.Then the problem can be restated as

$$MaxZ = 2x_1 + 3x_2 + 2x_1^2$$

subject to

$$x_1 + 4x_2 + s_1^2 = 4, x_1 + x_2 + s_2^2 = 2, -x_1 + s_3^2 = 0, -x_2 + s_4^2 = 0$$

Step 2: Construct the Lagrangian function:

$$\begin{aligned} L(x_1, x_2, s_1, s_2, s_3, s_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2x_1 + 3x_2 + 2x_1^2) &- \lambda_1(x_1 + 4x_2 + s_1^2 - 4) \\ &- \lambda_2(x_1 + x_2 + s_2^2 - 2) - \lambda_3(-x_1 + s_3^2) \\ &- \lambda_4(-x_2 + s_4^2) \end{aligned}$$

Step 3: As $-x_1^2$ is negative semi-definite quadratic form hence the maxima of L will be maxima of $Z = 2x_1 + 3x_2 + 2x_1^2$.To get the necessary and sufficient condition for maxima of L , we equate the first order partial derivative of L with respect to the decision variable with zero.

$$\frac{\partial L}{\partial x_i} = 0, \frac{\partial L}{\partial s_j} = 0, \frac{\partial L}{\partial \lambda_j} = 0 \text{ for } i = \{1,2\} \ \& \ j = \{1,2,3,4\}$$

On simplification the above

$$\left.\begin{aligned} 4x_+\lambda_1 + \lambda_2 - \lambda_3 = 2,\ 4\lambda_1 + \lambda_2 - \lambda_4 = 3 \quad (4.1a) \\ x_1 + 4x_2 + s_1^2 = 4, x_1 + x_2 + s_2^2 = 2 \quad (4.1b) \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda_1 s_1^2 + \lambda_2 s_2^2 + x_1\lambda_3 + x_2\lambda_4 = 0 \quad (4.2a) \\ x_1, x_2, s_1^2, s_2^2, \lambda_1, \lambda_2\lambda_3, \lambda_4 \geq 0 \quad (4.2b) \end{aligned}\right\}$$

To determine the optimal solution of the given problem, we introduce the artificial variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first two constraints of (4.1).

Step 4: Modified LPP is configured as follows

$$Maximize \ \ Z = -A_1 - A_2$$

subject to

$$4x_+\lambda_1 + \lambda_2 - \lambda_3 + A_1 = 2$$

$$4\lambda_1 + \lambda_2 - \lambda_4 + A_2 = 3$$

$$x_1 + 4x_2 + x_3 = 4$$

$$x_1 + x_2 + x_4 = 2$$

$$x_1, x_2, x_3, x_4, A_1, A_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

satisfying the complimentary slackness condition $\sum \lambda_i x_i = 0$, where we replaced s_1^2 by x_3 and s_2^2 by x_4. An initial basic feasible solution to the LPP is provided as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix}$$

Step 5 : Now the LPP will be solved by Two Phase Method

Figure 4.1: ***Initial Table of Phase 1***

		C_i	0	0	0	0	0	0	0	0	−1	−1
C_B	Y_B	X_B	$x_1 \downarrow$	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4	A_1	A_2
−1	$\overleftarrow{A_1}$	2	4	0	0	0	1	1	−1	0	1	0
−1	A_2	3	0	0	0	0	4	1	0	−1	0	1
0	x_3	4	1	4	1	0	0	0	0	0	0	0
0	x_4	2	1	1	0	1	0	0	0	0	0	0
$z_j - c_j$		−5	−4	0	0	0	5	2	−1	−1	0	0

From the table 4.1, an observation is made that x_1, λ_1 or λ_2 can enter the basis. But λ_1 and λ_2 will not enter the basis, as x_3 and x_4 are in the basis.

Figure 4.2: ***Final Iteration:***

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4
0	x_1	5/16	1	0	0	0	0	3/16	−1/4	1/16
0	λ_1	3/4	0	0	0	0	1	1/4	0	−1/4
0	x_2	59/64	0	1	1/4	0	0	−3/64	1/16	−1/64
0	x_4	49/64	0	0	−1/4	1	0	−9/64	3/16	−3/64
$z_j - c_j$		0	0	0	0	0	0	0	0	0

Hence, the optimum solution is

$$x_1 = 5/16, x_2 = 59/64 \text{ and}$$
$$\text{Maximum of } Z = 3.19$$

. □

4.2 Beale's Method

Beale's algorithm is a very famous approach to solve quadratic programming problem which was suggested by *Beale*. Here we do not implement Kuhn-Tucker conditions. In this algorithm, we divide the variables into basic variables and non-basic variables arbitrarily and we use the result of classical calculus to get the optimum solution of the given quadratic programming problem.

Let the QPP given in the form

$$Maximize\ p(X) = C^T X + \frac{1}{2} X^T Q X$$

subject to

$$AX\{\geq, \leq \text{ or } =\}\ B^T \text{ and}$$

$X \geq 0, X \in R^n, B = (b_1, b_2, \ldots, b_m), C$ is $n \times 1$ and Q is $n \times n$ symmetric matrix.

Algorithm:

Step 1: Deduce the objective function to maximization type and introduce the slack or surplus variables to make the inequality constraints into equality one.

Step 2: Now, select the m variables as basic variables and remaining $n-m$ variables as non-basic variables arbitrarily. Denote the basic variables and non-basic variables as X_B and $X_N B$ respectively. Now, divide the constraints into basic and non-basic variables i.e. convert each basic variables in terms of non-basic variables. Now the constraints equation can be expressed as

$$SX_B + TX_{NB} = b \Rightarrow X_B = S^{-1}b - S^{-1}TX_{NB}$$

where the matrix A is converted into two sub matrices S and T corresponding to X_B and X_{NB} respectively.

Step 3: Also express $p(x)$ in terms of only non-basic variables and examine the partial differentiation of the $p(x)$ w.r.t. non-basic variable X_{NB}. Thus we observe that as we increase the value of any non-basic variables the value of objective function $p(x)$ is improved. Now the constraints become

$$S^{-1}TX_{NB} \leq S^{-1}B \quad (\text{since } X_B \geq 0)$$

Hence, any component of X_{NB} can increase only until $\frac{\partial p}{\partial X_{NB}} = 0$ or one or more components of $X_B = 0$.

Step 4: Now, we have $m+1$ non-zero variables and $m+1$ constraints which is basic solution to the modified set of constraints.

Step 5: Go to step 3 and repeat the procedure until the optimal basic feasible solution is reached.

Example 4.2.1. *Solve the problem using Beale's Method:*

$$\text{Max } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to $x_1 + 2x_2 \leq 2$
and $x_1, x_2 \geq 0$

Solution. Solving the problem as follows:

Step 1:

$$Max\ Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 \tag{4.3}$$

subject to

$$x_1 + 2x_2 + x_3 = 2 \tag{4.4}$$

and $x_1, x_2, x_3 \geq 0$
taking $X_B = (x_1)$ and $X_{NB}^T = (x_2, x_3)$. So, we can write

$$x_1 = 2 - 2x_2 - x_3 \tag{4.5}$$

Step 2: Apply (4.5) in (4.3), we get

$$\max\ f(x_2, x_3) = 4(2 - 2x_2 - x_3) + 6x_2 - 2(2 - 2x_2 - x_3)^{-}2(2 - 2x_2 - x_3)x_2 - 2x_2^2$$

$$\frac{\partial f}{\partial x_2} = -2 + 8(2 - 2x_2 - x_3) + 4x_2 - 2(2 - x_3)$$

$$\frac{\partial f}{\partial x_3} = -4 + 4(2 - 2x_2 - x_3) + 2x_2$$

Now,

$$\left(\frac{\partial f}{\partial x_2}\right)_{(0,0)} = 10$$
$$\left(\frac{\partial f}{\partial x_3}\right)_{(0,0)} = 4.$$

Here '+ve'value of $\frac{\partial f}{\partial x_i}$ indicates that the objective function will increase if x_i increased. In the same manner, '-ve'value of $\frac{\partial f}{\partial x_i}$ represents the decrement of the objective function.
Therefore, in order to have better improvement in objective function we have to increase x_2.

Step 3: Increase in x_2 to a value greater than 1, x_1 results negative.
Since $x_1 = 2 - 2x_2 - x_3$

$$x_3 = 0; \ \frac{\partial f}{\partial x_2} = 0$$

That implies $x_2 = 5/6$.
$\min(1, 5/6) = 5/6$.
Now, the new basic variables is x_2.

Second Iteration:

Step 1: Now, $X_B = (x_2)$ and $X_{NB} = (x_1 \ x_3)^T$

$$x_2 = 1 - \frac{1}{2}(x_1 + x_3) \tag{4.6}$$

Step 2: Substitute 4.6 in 4.3

$$\max \ f(x_1, x_3) = 4x_1 + 6(1 - \frac{1}{2}(x_1 + x_3)) - 2x_1^2 - 2x_1(1 - \frac{1}{2}(x_1 + x_3)) - 2(1 - \frac{1}{2}(x_1 + x_3))^2$$

$$\frac{\partial f}{\partial x_1} = 1 - 3x_1,$$
$$\frac{\partial f}{\partial x_3} = -1 - x_3$$
$$\left(\frac{\partial f}{\partial x_1}\right)_{(0,0)} = 1$$
$$\left(\frac{\partial f}{\partial x_3}\right)_{(0,0)} = -1$$

This implies that x_1 can be introduced to increased objective function.

Step 3: Now, $x_2 = 1 - \frac{1}{2}(x_1 + x_3)$ and $x_3 = 0$.
Increase in x_1 to a value greater than 2, x_2 results negative.

$$\frac{\partial f}{\partial x_1} = 0$$

which implies that $x_1 = \frac{1}{3}$
$\min(2, 1/3) = 1/3$ Thus, $x_1 = 1/3$.
Hence $x_1 = 1/3$, $x_2 = 5/6$, $x_3 = 0$ Therefore, the solution of $Max f(x) = 25/6$.

□

Chapter 5

Fractional Programming

5.1 Linear Fractional Programming

Linear fractional programming is unique type of non-linear programming in which the objective function is the fraction of two linear function subject to linear constraints. Lots of practical application of fractional programming exists such as construction planning, hospital planning and economic planning etc.

A linear fractional programming is an optimization problem of the form:

$$\text{Minimize} \frac{c^T x + c_0}{d^T x + d_0} \tag{5.1}$$

subject to,

$$Ax = b. \tag{5.2}$$

$$x \geq 0.$$

It should be noticed that the objective function is the quotient of the two linear function.

5.1.1 Charnes & Cooper Algorithm

Charnes & Cooper developed very simple technique to optimize the linear fractional programming. The basic idea of the development is that convert the linear fractional programming into linear programming problem and then use simplex or any other method to solve that linear programming.

The algorithm to solve the linear fractional programming are as follows:

Step 1: Let

$$v = d^T x + d_0,\ y_0 = \frac{1}{v} > 0,\ y = y_0 x \tag{5.3}$$

Then,

$$\frac{c^T x + c_0}{d^T x + d_0} = y_0(c^T x + c_0) = c^T y + c_0 y_0$$

Step 2: Now the problem can be written as:
Minimize$(c^T y + c_0 y_0)$
subject to
$Ay \geq y_0 b$,
$d^T y + d_0 y_0 = 1$,
$y \geq 0, y_0 \geq 0$

Step 3: Now the above linear fractional programming can be solved by the *simplex method*. Also, the optimal solution are

Example 5.1.1. *Apply the Charnes & Cooper Algorithm to solve:*

$$Minimize\ z = \left(\frac{-6x_1 - 5x_2}{2x_1 + 7}\right)$$

subject to

$$x_1 + 2x_2 \leq 3,$$
$$3x_1 + 2x_2 \leq 6,$$
$$x_1 \geq 0, x_2 \geq 0$$

.

Solution. Let $y = y_0 x$. Applying step 1 of the above algorithm then the problem becomes:

$$Minimize\ z = -6y_1 - 5y_2$$

subject to

$$y_1 + 2y_2 - 3y_0 \leq 0, \tag{5.4}$$
$$3y_1 + 2y_2 - 6y_0 \leq 0, \tag{5.5}$$
$$2y_1 + 7y_0 = 1, \tag{5.6}$$

$y_0 \geq 0, y_1 \geq 0, y_2 \geq 0$. After introducing the required slack variables in constraints (5.4) and (5.5) respectively, and the artificial variable $v_1 \geq 0$ in constraint (5.6), we minimize the infeasibility form $v = v_1$. The iterations of the two ways simplex method for an optimal solutions are given below:

Figure 5.1: ***Table 1***

BASIC VARIABLE	y_0	y_1	y_2	y_3	y_4	v_1	CONSTANTS
y_3	-3	1	2	1	0	0	0
y_4	-6	3	2	0	1	0	0
$\leftarrow v_1$	7	2	0	0	0	1	1
$-z$	0	-6	-5	0	0	0	0
$-v$	-7	-2	0	0	0	0	-1
	$\uparrow$						

where y_3 and y_4 are slack variables.

Figure 5.2: ***Table 2***

BASIC VARIABLE	y_0	y_1	y_2	y_3	y_4	CONSTANTS
$\leftarrow y_3$	0	0	40/33	1	$-13/33$	1/11
y_1	0	1	14/33	0	7/33	2/11
y_0	1	0	$-4/33$	0	$-2/33$	1/11
$-z$	0	0	$-27/11$	0	14/11	12/11
			$\uparrow$			

The final iteration yields $y_0^0 = \frac{1}{10} > 0$. Thus, the optimal solution is

$$x_1^0 = {y_1^0}/{y_0^0} = 3/2$$

$$x_2^0 = {y_2^0}/{y_0^0} = 3/4$$

Therefore, min $z = -51/40$. □

5.2 Non-Linear Fractional Programming

As the name of the concerned problem implies that the objective function is non-linear i.e. the ratio of non-linear and linear function or linear and non-linear function or non-linear and non-linear function. Here, we will use the concept of *quadratic programming* to solve the problem. To get the optimal solution of the problem, a algorithm is given by scientist Dinkelbach known as *Dinkelbach Algorithm.*

A non linear fractional program is an optimization problem of the form:

$$\text{Maximize } \frac{N(x)}{D(x)} \text{ for } x \in T(\subset R^n).$$

Let $F(q) = max\,[N(x) - qD(x)]$ be the optimal value of the non linear fractional problem where q is known real number.

Necessary and Sufficient Condition:
The necessary and sufficient conditions for

$$q_0 = \frac{N(x_0)}{D(x_0)} = \max \frac{N(x_0)}{D(x_0)}$$

is

$$F(q_0) = F(q_0, x_0) = \max\,[N(x) - q_0 D(x)] = 0$$

It is noticed here that x_0 is the optimal solution for the non-linear fractional programming. From the necessary and sufficient condition of the non-linear programming, as $F(q)$ is continuous, we can convert the non-linear programming as:

Find an $x_n \in T$ and $q_n = N(x_n)/D(x_n)$ such that for any $\delta > 0$,

$$F(q_n) - F(q_0) = F(q_n) < \delta.$$

5.2.1 Dinkelbach Algorithm:

The algorithm can be started with $q = 0$ or by any feasible point $x_1 \in T$ such that $q(x_1) = N(x_1)/D(x_1) \geq 0$

Step 1: Take $q_2 = 0$ or $q_2 = N(x_1)/D(x_1)$ and proceed to step 2 with $k = 2$.

Step 2: To find an $x_k \in T$ use a suitable convex programming methods that maximizes $[N(x) - q_k D(x)]$. Calculate

$$F(q_k) = N(x_k) - q_k D(x_k)$$

.

Step 3: If $F(q_k) < \delta$, terminate and we have

$$x_k = \begin{cases} x_0 & \text{if } F(q_k) = 0 \\ x_n & \text{if } F(q_k) > 0 \end{cases} \tag{5.7}$$

where x_0 is an optimal solution and x_n an appropriate optimal solution to non-linear programming problem. If $F(q_k) \geq \delta$, evaluate $q_{k+1} = N(x_k)/D(x_k)$ and go to step 2, replacing q_k by q_{k+1}.

Example 5.2.1. *Solve the following problem by Dinkelbach Algorithm:*
Maximize $z = (2x_1 + 2x_2 + 1)/(x_1^2 + x_2^2 + 3)$
subject to
$x_1 + x_2 \leq 3,$
$x_1, x_2 \geq 0$

Solution. Here, $N(x) = 2x_1 + 2x_2 + 1,\ D(x) = x_1^2 + x_2^2 + 3$, and
˜ $T = x : x_1 + x_2 \leq 3,\ x_1 \geq 0,\ x_2 \geq 0.$

Clearly, $D(x) > 0$, $N(x) isconcave$, and $D(x)$ is convex. Suppose $\delta = 0.01$. To start the algorithm, we let $q_2 = 0$. Now, we have to maximize $(2x_1 + 2x_2 + 1)$ for $x \in T$. An optimal solution to this linear programming by simplex method is

$x_2 = (3, 0)^T$

Since $F(q_2) = 7 > \delta$, we find $q_3 = N(x_2)/D(x_2) = 7/12$ and maximize

$[(2x_1 + 2x_2 + 1) - \frac{7}{12}(x_1^2 + x_2^2 + 3)]$ for $x \in T$

The optimal solution to this *quadratic program* is found to be $x_3 = (3/2, 3/2)^T$. Now, $F(q_3) = 21/8 > \delta$ and $q_4 = N(x_3)/D(x_3) = 14/15$. Hence, we maximize

$[(2x_1 + 2x_2 + 1) - \frac{14}{15}(x_1^2 + x_2^2 + 3)]$ for $x \in T$

The optimal solution to this quadratic program is

$x_4 = (15/14, 15/14)^T$

Again, since $F(q_4) = 12/35 > \delta$ and $q_5 = N(x_4)/D(x_4) = 518/519$, we maximize

$[(2x_1 + 2x_2 + 1) - \frac{518}{519}(x_1^2 + x_2^2 + 3)]$ for $x \in T$

The optimal solution to this program is

$x_5 = (519/518, 519/518)^T$.

Since $F(q_5) < \delta$, we terminate the algorithm, and an approxcimate optimal solution to the given program is

$x_5 = (519/518, 519/518)^T$.

□

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