Numerical Solution of Static and Dynamic Problems of Imprecisely Defined Structural Systems

A THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE

OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

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NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

ROURKELA 769 008, ODISHA, INDIA

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DECLARATION

I hereby declare that the work which is being presented in the thesis entitled "**Numerical Solution of Static and Dynamic Problems of Imprecisely Defined Structural Systems**" for the award of the degree of Doctor of Philosophy in Mathematics, submitted in the Department of Mathematics, National Institute of Technology Rourkela, Rourkela 769008, Odisha, India, is an authentic record of my own work carried out under the supervision of Prof. (Dr.) S. Chakraverty.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

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CERTIFICATE

This to certify that the thesis entitled "**Numerical Solution of Static and Dynamic Problems of Imprecisely Defined Structural Systems**" is being submitted by **Mr. Diptiranjan Behera** for the award of the degree of Doctor of Philosophy in Mathematics at National Institute of Technology Rourkela, Rourkela-769008, Odisha, India is a record of bonafide research work carried out by him under my supervision and guidance. Mr. Diptiranjan Behera has worked for four years on the above problem in the Department of Mathematics, National Institute of Technology Rourkela and this has reached the standard for fulfilling the requirements and the regulation relating to the degree. The contents of this thesis, in full or part, have not been submitted to any other university or institution for the award of any degree or diploma.

Dr. S. CHAKRAVERTY

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Dedicated To

My Parents

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Diptiranjan Behera

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ABSTRACT

Static and dynamic problems with deterministic structural parameters are well studied. In this regard, good number of investigations have been done by many authors. Usually, structural analysis depends upon the system parameters such as mass, geometry, material properties, external loads and boundary conditions which are defined exactly or considered as deterministic. But, rather than the deterministic or exact values we may have only the vague, imprecise and incomplete informations about the variables and parameters being a result of errors in measurements, observations, experiments, applying different operating conditions or it may be due to maintenance induced errors, etc. which are uncertain in nature. Hence, it is an important issue to model these types of uncertainties. Basically these may be modelled through a probabilistic, interval or fuzzy approach. Unfortunately, probabilistic methods may not be able to deliver reliable results at the required precision without sufficient experimental data. It may be due to the probability density functions involved in it. As such, in recent decades, interval analysis and fuzzy theory are becoming powerful tools. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices.

In general, structural problems for uncertain static analysis with interval or fuzzy parameters simplify to interval or fuzzy system of linear equations whereas interval or fuzzy eigenvalue problem may be obtained for the dynamic analysis. Accordingly, this thesis develops new methods for finding the solution of fuzzy and interval system of linear equations and eigenvalue problems. Various methods based on fuzzy centre, radius, addition, subtraction, linear programming approach and double parametric form of fuzzy numbers have been proposed for the solution of system of linear equations with fuzzy parameters. An algorithm based on fuzzy centre has been proposed for solving the generalized fuzzy eigenvalue problem. Moreover, a fuzzy based iterative scheme with Taylor series expansion has been developed for the identification of structural parameters from uncertain dynamic data. Also, dynamic responses of fractionally damped discrete and continuous structural systems with crisp and fuzzy initial conditions have been obtained using homotopy perturbation method based on the proposed double parametric form of fuzzy numbers.

Numerical examples and application problems are solved to demonstrate the efficiency and capabilities of the developed methods. In this regard, imprecisely defined structures such as bar, beam, truss, simplified bridge, rectangular sheet with fuzzy/interval material and geometric properties along with uncertain external forces have been considered for the static analysis. Fuzzy and interval finite element method have been applied to obtain the uncertain static responses. Structural problems viz. multistorey shear building, spring mass mechanical system and stepped beam structures with uncertain structural parameters have been considered for dynamic analysis. In the identification problem, column stiffnesses of a multistorey frame structure have been identified using uncertain dynamic data based on the proposed algorithm. In order to get the dynamic responses, a single degree of freedom fractionally damped spring-mass mechanical system and fractionally damped viscoelastic continuous beam with crisp and fuzzy initial conditions are also investigated. Obtained results are compared in special cases for the validation of proposed methods.

Keywords: Fuzzy set, fuzzy number, fuzzy centre, fuzzy radius, $\alpha-$cut, double parametric form of fuzzy numbers, fuzzy and fully fuzzy system of linear equations, fuzzy eigenvalue problem, fuzzy and interval finite element method, bar, beam, truss, simplified bridge, spring-mass system, shear building, multistory frame, Taylor series, fractional derivative, fuzzy initial condition, Homotopy Perturbation Method (HPM).

Chapter 1

Introduction

Chapter 1

Introduction

Design and analysis of structures play a vital role in the field of structural engineering. Most of the structures fail due to the poor design. Normally a design process involves the system parameters such as mass, geometry, material properties, external loads and boundary conditions which are defined exactly or considered as deterministic. But rather than the deterministic or exact value, we may have only the vague, imprecise and incomplete information about the variables and parameters being a result of errors in measurement, observation, experiment, applying different operating condition or it may be due to maintenance induced error, etc. which are uncertain in nature. Moreover, variations in the structural response arises (Muhanna and Mullen 1999; Rama Rao 2004; Zhang 2005) due to the uncertainties involved in material and geometric properties, service loads or boundary conditions. Hence, it is an important issue to model these engineering systems with uncertainties. As such, computationally efficient methods need to be developed accordingly. There are various ways to classify these uncertainties, but mainly in engineering practice these can be categorized (Zhang 2005) as “aleatory” and “epistemic” uncertainty.

Aleatory uncertainty mainly deals with stochastic or statistical uncertainty which occurs due to the natural randomness in the process. It is generally expressed by a probability density or frequency distribution function. For the estimation of the distribution, it requires sufficient information about the variables and parameters involved in it. But, epistemic uncertainty refers to the uncertainty when we have lack of knowledge or incomplete information about the variables and parameters is present. In general, probabilistic approaches are extensively used to model aleatory uncertainty, but to represent epistemic uncertainty using probabilistic methods is often a subject of debate (Elishakoff 1995; Ben-Haim 1994; Ferson and Ginzburg 1996; Ferson 1996; Ferson et al. 2003). Therefore, various researchers investigated non-probabilistic approaches such as fuzzy set theory (Zadeh 1965), interval analysis (Moore 1966), convex model (Ben-Haim and Elishakoff 1990), Dempster-Shafer evidence theory (Dempster 1967; Shafer 1976), imprecise probabilities (Walley 1991) and so on to define epistemic uncertainty. Among these, fuzzy and interval theory have been used in this research for the uncertainty analysis of structures. Accordingly, in the following sections, the probabilistic concept, interval approach and fuzzy set theory are introduced and discussed with respect to the structural mechanics under uncertainty.

1.1. Probabilistic Concept

Probability theory is concerned with the analysis of (natural) random phenomena. The main objects of probability theory are described by random variables and stochastic process. Random variables are the variables subject to changes due to the randomness involved in it. Stochastic process is the collection of all random values, which are often used to represent the evolution of random variables or systems over time. For uncertainty analysis, Monte Carlo simulation method, first order and second order reliability methods (FORM and SORM) and response surface methods are frequently used. In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions. However, if the structural parameters and the external load are modeled as random variables with known probability density functions, the response of the structure can be predicted using the theory of probability and stochastic processes which have been studied by several authors such as Elishakoff (1983). Elishakoff and Colombi (1993) have also combined probabilistic and convex models to study uncertainty.

The probabilistic concept is well established for the extension of the deterministic finite element method towards uncertain assessment. This has led to a number of probabilistic and stochastic finite element procedures (Kiureghian and Ke 1988; Besterfield et al. 1990; Haldar and Mohadevan 2000; Antonio and Hoffbauer 2010). In addition, Vanmarcke and Grigoriu (1983) developed stochastic finite element method for simple beam problems. Modal approaches have been applied (Van den Nieuwenhof and Coyette 2003) for the stochastic finite element analysis of structures with material and geometric uncertainties. Unfortunately, probabilistic methods may not be able to deliver reliable results at the required precision without sufficient experimental data (Ben-Haim 1994; Elishakoff 2000). It may be due to the probability density functions involved in it. As such, interval and fuzzy theories are becoming powerful tools in recent decades for many real life applications. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices.

1.2. Interval Approach

Interval analysis was first introduced by Moore (1966) and subsequently various aspects of interval analysis along with applications are further explained by Moore (1979). Thereafter, several excellent books have also been written by various authors related to interval analysis, e.g. Alefeld and Herzberger 1983; Hansen 1992b; Neumaier 1990; Moore 2009 and it has been applied in a variety of science and engineering problems. In general, structural problems are static or dynamic in nature. As such, under interval uncertainty, static problems turn out to be an interval system of linear equations and the dynamic problems to be an interval eigenvalue problem. Accordingly, in the following paragraphs, interval system of linear equations and interval eigenvalue problems are first discussed.

1.2.1. Interval system of linear equations

System of linear equations with interval parameters can be defined as interval system of linear equations. In the system, elements of the coefficient matrix, right hand side vector and unknown vector are considered as interval number. In this regard, (Rohn 1989; Neumaier 1990; Rump 1992; Hansen 1992a) investigated various methodologies for the solution of interval system of linear equations. Rohn (1989) applied iterative methods in the solution process. An excellent book has been written by Neumaier (1990) in this regard. By modifying the algorithm proposed by Neumaier (1990), Rump (1992) has developed an iterative technique with the necessary and sufficient conditions for stopping criteria. Interval Newton's method has been applied by Hansen (1992a) for the solution of such systems. Rohn and Kreinovich (1995) proved that it is NP-hard to compute the exact component wise bounds on solutions of all the interval linear systems. Some topological and graph theoretical properties have been incorporated by Jansson (1997) for the solution set of linear algebraic systems with interval coefficients. There, the author found the exact bound of the solution. Aberth (1997) used a linear programming method to obtain the solution of linear interval equations. Skalna (2003) investigated the solution of linear equations of structural mechanics with interval parameters. Polyak and Nazin (2004) introduced a solution methodology to find "the best" interval solution of an interval algebraic system of linear equations. Skalna (2006) presented a new method to find the tight enclosure of the solution bound of system of a linear equations depending linearly on interval parameters. Neumaier and Pownuk (2007) have studied the solution of linear systems with large uncertainties with an application of truss structure. A technique for the non-negative solution of interval linear systems have been developed by Shary (2011) constructing the maximal inner estimation of the solution set. An algorithm for computing the hull of the solution set of interval linear equations has been developed by Rohn (2011). Myskova (2005, 2012) studied the solution of interval equation using max-plus algebra. A new approach based on the concept of inclusion principle for obtaining the algebraic solution of interval linear systems has been presented by Allahviranloo and Ghanbari (2012a). Recently, weak and strong solvability of interval linear systems of equations and inequalities have been studied by Hladik (2013b). Kolev (2014) resolved the interval hull solution of linear interval parameter system using an iterative scheme.

1.2.2. Interval eigenvalue problems

Interval eigenvalue problems have a great importance for studying the uncertainty quantification of real life problems. There exist various well known methods to handle deterministic eigenvalue problems. But solution methods of interval eigenvalue problems are scarce. As such, related problems are reviewed below.

An efficient method to solve the standard interval eigenvalue problem has been studied by Deif (1991). In their approach, the authors have used nonlinear programming and eigenvalue inequalities with the assumption that the signs of the components of eigenvectors remain invariant. Hertz (1992) investigated the stability analysis of dynamic symmetric interval systems in the field of control theory, as it depends on the bounds of extreme eigenvalue. Rohn and Deif (1992) proposed a method for obtaining the real eigenvalue of an interval matrix using the sign of central eigenvectors. Real eigenvalues of singular interval matrices have been studied by Rohn (1993). Qiu et al. (2001b) presented an approximate method based on interval perturbation theory for the standard interval eigenvalue problem of real non-symmetric interval matrices. Modares et al. (2006) dealt with the interval eigenvalue problem for the frequency analysis of a structure with uncertain structural parameter. Outer estimation of interval solution of the eigenvalue problem using affine interval approximation has been addressed by Kolev (2006). Leng and He (2007) have proposed a method to solve the generalized interval eigenvalue value problem by using the perturbation theory.

Recently, interval eigenvalue problems have also been addressed by various other authors. Leng et al. (2008) proposed an algorithm using interval centre and radius for computing the exact real eigenvalue bounds of standard interval eigenvalue problems. Next, Leng and He (2010) extended the approach of Leng et al. (2008) to generalize interval eigenvalue problems. Hladik et al. (2010) computed the outer approximation of the eigenvalue sets of general and symmetric interval matrices. Also Hladik et al. (2011b) presented an inner approximation algorithm to estimate the exact bounds of interval eigenvalue. Moreover, they have (Hladik et al. 2011a) also proposed a filtering method for the approximation of real eigenvalue set of an interval matrix. Matcovschi et al. (2012) analyzed a procedure based on global optimization to evaluate the right bounds of the eigenvalue ranges of interval matrices. Eigenvalue bounds for both real and complex interval matrices are examined by Hladik (2013a). Very recently, based on some known sufficient conditions for the regularity of interval matrices, Leng (2014) developed an algorithm to solve both standard and generalized real interval eigenvalue problems.

1.2.3. Uncertainty analysis of structures through interval approach

In this section, literature related to structural analysis under interval uncertainty are reviewed to have a better insight. If only incomplete information is available, it is possible to establish the minimum and maximum favorable response of the structures using interval analysis (Ben-Haim and Elishakoff 1990; Ganzerli and Pantelides 2000).

Firstly, literatures related to static analysis of structures with interval parameters are reviewed. As such, static analysis of structures under interval uncertainty has been studied by Rao and Berke (1997). In their approach, Gaussian elimination and combinatorial approach is used to obtain the solution of interval linear system of equations. Qiu and Elishakoff (1998) studied the anti-optimization analysis of structures through interval. They reported that the numerical results obtained by subinterval perturbation method yields tighter bounds than interval perturbation method. Kulpa et al. (1998) presented the application of interval methods in (qualitative) mechanical systems viz. truss and frame structures with parameter uncertainties. Chen and Yang (2000) developed a new interval finite element method to solve the uncertain problems of beam structures where the beam characteristics are assumed as interval parameters. Shu-xiang and Zhen-zhou (2001) studied the static linear interval finite element method and proposed a solution procedure using interval arithmetic for solving interval system of linear equations. They applied the method for the uncertainty analysis of a six bar truss structure. Mc Williams (2001) reported the anti-optimization technique for structures using interval analysis. Muhanna and Mullen (2001) and Qiu (2003) used interval finite element method to obtain interval static response of structures considering the parameters as interval. Skalna (2003) presented solution methods for solving interval system of linear equations by improving the approach of Rump and Neumaier and applied those methods for the uncertain static response of truss structures. Next, Muhanna et al. (2005) reported interval static responses where they have used an element by element technique in the solution procedure. Qiu et al. (2006) developed two new techniques for the estimation of interval static displacement of structures. In their study, vertex solution theorem with Cramer's rule has been used to find the upper and lower bounds of the solution set of linear interval equations. A three-stepped beam and a 10-bar truss have been taken into consideration to illustrate the computational aspects of their proposed methods. Neumaier and Pownuk (2007) studied the solution procedure for interval linear systems with large uncertainties with the applications to truss structures. Sub interval perturbed finite element method and anti-slide stability analysis methods were studied by Guo-jian and Jing-bo (2007) and the formula for computing the bounds of stability factor has been given. Muhanna et al. (2007) presented an interval approach for the treatment of uncertain parameter for linear static structural mechanics problems where uncertain parameters are introduced in the form of unknown but bounded quantities (intervals). They applied interval finite element method to analyze the system response under uncertain stiffness and loading. A new method called the interval factor method in the finite element analysis of truss structures with interval parameters has been proposed by Gao (2007). Very recently, Wang and Qiu (2013) proposed a method to obtain the exact solution of interval system of linear equations by converting the interval system into some deterministic systems. They have considered a six-bar truss example to validate the developed method. Modified interval and sub interval perturbation method for uncertain static analysis of structures with interval parameters has been presented by Xia and Yu (2014). This study avoids the unpredictable effect of neglecting the higher-order terms of Neumann series applied in the traditional interval perturbation method.

Next, literature related to dynamic analysis of structures under interval uncertainty are cited and discussed. In this regard, Qiu et al. (1994) applied matrix perturbation theory for the vibration analysis of a multi storey frame structure with interval parameters. A method has been proposed by Chen et al. (1995) for computing the upper and lower bounds of frequencies of structures with interval parameters. Rayleigh quotient and max-min theorem of eigenvalues have been used by them in their approach. Interval analysis for vibrating systems has been discussed by Dimarogonas (1995) and Qiu et al. (1995a). The Rayleigh quotient iteration method has also been applied by Qiu et al. (1995b) for the vibration analysis. Based on the interval finite element method, Yang et al. (2001) presented a new method to determine the bounds of complex eigenvalues of a damping structure with interval uncertainty. Interval finite element method has been used by Chen et al. (2003) for interval eigenvalue analysis for structures. Moens and Vandepitte (2004) studied an interval finite element approach for the calculation of envelope frequency response functions. Chen and Wu (2004a, 2004b) presented an interval optimization method for the dynamic response of structures with interval parameters. They derived a method by combining the interval extension of a function and the perturbation theory for finding interval dynamic response of a truss and a frame structure. Qiu and Wang (2005a) proposed solution theorems for the standard eigenvalue problem of structures with interval parameters. Vertex method has been used by Qiu et al. (2005) for finding the eigenvalue bounds of structures. They have compared the results obtained by the eigenvalue inclusion principle and the interval perturbation method. Qiu and Wang (2005b) presented vertex and interval perturbation methods for the generalized complex eigenvalue problem with bounded uncertainties of damped structure. A numerical example of a seven degree of freedom spring-damping-mass system has been considered by them. Modares et al. (2006) analysed the uncertain frequency of a structural system with bounded uncertainty. An interval (set theoretic) approach has been used by them for the uncertainty quantification. Modal analysis of structures with uncertain-but-bounded parameters via interval analysis has been investigated by Sim et al. (2007). Gao (2007) computed natural frequency and mode shape of structures for both random and interval parameter using random and interval factor method. They have considered truss structure for the analysis. Eigenvalue and frequency response function analysis of structures with uncertain parameters using interval finite element method have been studied by Gersem et al. (2007). Random Factor Method (RFM) and Interval Factor Method (IFM) have been used by Wei (2007) for finding the natural frequency and mode shape of truss structures with uncertain parameters. They have compared the structural natural frequency and mode shape solutions between RFM and IFM for truss structures. Frequency response function of uncertain structure with interval parameters has been studied by Moens et al. (2007) using interval finite element method, which is based on a hybrid interval and modal superposition principle. A spring–mass system with damping has been taken into consideration for the analysis. To compute the interval eigenvalue bounds of structures, Leng and He (2007) used perturbation theory. Recently, Modares and Mullen (2014) investigated the dynamic spectrum analysis of structures viz. spring mass system and truss with interval uncertainty.

1.3. Fuzzy Set Theory

Fuzzy set theoretical concept was first developed by Zadeh (1965) and is further widely used for the uncertain analysis of various science and engineering problems. Moreover, several excellent books (Dubois and Prade 1980; Kaufmann and Gupta 1985; Ross 2004; Hanss 2005; Zimmermann 2001; Chakraverty 2014) have also been written related to this. These books presented an extensive review and various aspects of fuzzy theory along with applications. However, for uncertain static and dynamic analysis of structures with fuzzy parameters and external loads, the corresponding problem converts to fuzzy algebraic linear systems or eigenvalue problems in general. As such, in the following paragraphs, literatures related to fuzzy linear systems, eigenvalue problem and their applications to structures are discussed. It is a gigantic task to include all the literatures available, so only important and related references are cited below.

1.3.1. Fuzzy linear systems

As mentioned earlier, the system of linear equations has great applications in real life problems. It is simple and straight forward when the variables are defined as deterministic or crisp. But in actual case, the parameters may be uncertain or a vague estimation about the variables is known. Therefore, to overcome the uncertainty and vagueness, one may use the fuzzy numbers in place of the crisp numbers. Thus, the crisp system of linear equations becomes a Fuzzy System of Linear Equations (FSLE) or Fully Fuzzy System of Linear Equations (FFSLE). There is a difference between a fuzzy linear system and fully fuzzy linear system. The coefficient matrix is treated as crisp in the fuzzy linear system, but in the fully fuzzy linear system, the parameters as well as variables are considered to be fuzzy numbers.

Fuzzy system of linear equations was studied by Friedman et al. (1998). An embedding approach has been used by them in the solution process. They have converted

$n \times n$ fuzzy linear system to $2n \times 2n$ crisp system of linear equations to obtain the final solution. A method for solving fuzzy linear systems using fuzzy centre has been developed by Abbasbandy and Alavi (2005). Asady et al. (2005) considered $m \times n$ fuzzy general linear systems where they have assumed $m \leq n$. Conjugate gradient method has been considered by Abbasbandy et al. (2005) for the solution of fuzzy symmetric positive definite system of linear equations. Steepest descent method for the solution of fuzzy system of linear equations has been applied by Abbasbandy and Jafarian (2006). Nehi et al. (2006) solved a fuzzy linear system by solving its canonical form. In addition to this, Wang and Zheng (2006a) and Zheng and Wang (2006) presented various methods for the solution of FSLE. Wang and Zheng (2006a) studied an inconsistent fuzzy linear system whereas Zheng and Wang (2006) investigated the solution of $m \times n$ fuzzy general linear systems using an embedding approach. They have used the matrix inversion in the methodology. Allahviranloo and Kermani (2006) incorporated pseudo inverse properties in the solution process. Abbasbandy et al. (2008) investigated the existence of a minimal solution of general dual fuzzy linear systems. Necessary and sufficient conditions for the existence of minimal solution are also given by them. Horcik (2008) has applied interval theory for the solution of a system of linear equations with fuzzy numbers. Garg and Singh (2008) introduced Gaussian fuzzy number in the solution of fuzzy system of equations. Sun and Guo (2009) proposed a solution methodology along with its necessary and sufficient condition for FSLE. In addition to these, various other methods for the solution of FSLE have also been presented by different authors (Wang et al. 2009; Li et al. 2010; Senthilkumar and Rajendran 2011a). Ghanbari and Amiri (2010) used ranking functions and ABS algorithms for the solutions of LR fuzzy linear systems. Allahviranloo and Salahshour (2011) studied a simple and practical method to solve FSLE using 1-cut of fuzzy numbers. Also, they have presented the maximal and minimal solution of the system. Ezzati (2011) proposed an embedding approach for solving fuzzy linear systems including existence and uniqueness. Allahviranloo and Ghanbari (2012b) also studied the algebraic solution of fuzzy linear systems using interval theory. Amirfakhrian (2012) used fuzzy distance approach for the solution. Very recently, Chakraverty and Behera (2013b) investigated the solution using fuzzy centre and radius. Also, fuzzy addition and subtraction concepts have been incorporated by Behera and Chakraverty (2013f) for the solution of FSLE.

Numerous numerical and semi analytical methods have also been investigated by different authors for handling such problems. In view of this, Adomian decomposition method for FSLE has been taken into consideration by Allahviranloo (2005a, 2005b). Successive over relaxation methods for fuzzy linear systems has also been used by Wang and Zheng (2006b). Block Jacobi two stage methods with Gauss Seidel inner iterations has been implemented by Allahviranloo et al. (2006) for the solution of fuzzy systems. Dehghan and Hashemi (2006a) applied iterative methods and Abbasbandy et al. (2006) used LU decomposition method for solving fuzzy system of linear equations. Wang and Zheng (2007) used block iterative methods for fuzzy linear systems. Splitting iterative methods for fuzzy system of linear equations have been applied by Yin and Wang (2009). Homotopy analysis method has been applied by Jafari et al. (2009) for solving fuzzy system of linear equations. Tian et al. (2010) pursued perturbation analysis of fuzzy linear systems. Guo and Gong (2010) considered block Gaussian elimination method for the solution of fuzzy matrix equations. Fuzzy least square method has been presented by Gong and Guo (2011). Miao (2011) applied block homotopy perturbation method for solving fuzzy linear systems. Modified Adomian decomposition method for the solution of fuzzy polynomial equations has been studied by Otadi and Mosleh (2011b).

Next, fuzzy complex number was first proposed by Buckley (1989). Qiu et al. (2000, 2001a) discussed the sequence and series of fuzzy complex numbers and their convergence. Solution of fuzzy complex system of linear equations was described by Rahgooy et al. (2009) and applied to a circuit analysis problem. Jahantigh et al. (2010) developed a numerical procedure for solving complex fuzzy linear systems. Behera and Chakraverty (2012) proposed a new and simple centre and radius based method for solving fuzzy real and complex system of linear equations. Also, Behera and chakraverty (2013b, 2014a) proposed fuzzy arithmetic based methods for fuzzy complex system.

On the other hand, FFSLE is becoming an important upcoming area of research due to the vast applications in engineering and science problems. Accordingly, Buckley and Qu (1991) solved a fully fuzzy system of linear equations. In this respect, several computational and iterative techniques may be found in Dehghan et al. (2006, 2007). Decomposition procedure has been implemented by Dehghan and Hashemi (2006b). In Vroman et al. (2007a, 2007b), the authors have used parametric functions to obtain the solution. Abbasbandy et al. (2008) presented the minimal solution of general dual fuzzy linear systems. Homomorphic solution of FFSLE has been developed by Allahviranloo et al. (2008). They have obtained the fuzzy solution by converting the original system into three crisp systems of linear equations. Kumar et al. (2011) considered a generalised fully fuzzy linear system with unrestricted fuzzy coefficient matrix. Iterative Jacobi and Gauss-Seidel methods have been considered for the numerical approximation of the solution by Liu (2010). Senthilkumar and Rajendran (2011b) used Cholesky method for finding the non-negative solution of a symmetric fully fuzzy linear system. Allahviranloo et al. (2011) found the maximal and minimal symmetric solutions of fully fuzzy linear systems using 1-cut approach. Ezzati et al. (2012) studied the positive solution of a generalised fully fuzzy linear system and proved that a positive system always has a solution. Numerical solutions of fully fuzzy linear systems in dual form have also been investigated by Salahshour and Nejad (2013). Moloudzadeh et al. (2013) described a new method for solving an arbitrary fully fuzzy linear system with triangular fuzzy numbers. They have imposed 0-cut and 1-cut techniques in their methodology. Otadi and Mosleh (2011a) and Mosleh (2013) evaluated the numerical solution of dual fully fuzzy linear systems and FFSLE by fuzzy neural network technique respectively.

Moreover, Buckley (1992) explained the use of fuzzy equations in economics and finance problems too. Numerical solution of fuzzy system of linear equations has been applied by Rao and Chen (1998) in the field of structural mechanics. Muzzioli and Reynaerts (2007) applied fuzzy linear systems in financial applications. For uncertain static response of structures, Skalna et al. (2008) also described the solution of fuzzy system of equations.

1.3.2. Fuzzy eigenvalue problem

A little effort has been made to find the solution of fuzzy eigenvalue problems. As such, Chiao (1998) used Zadeh's extension principle for finding the solution of fuzzy generalized eigenvalue problems. Salahshour et al. (2012b) computed eigenvalues and eigenvectors of a standard fuzzy eigenvalue problem. They have obtained the eigenvalue by determining the maximal and minimal solution of the corresponding system. Allahviranloo and Hooshangian (2013) have developed a new method for solving standard fuzzy eigenvalue problems. They have used the core solution of the considered system in their methodology to obtain the solution bounds.

1.3.3. Uncertainty analysis of structures through fuzzy theory

In this regard, nonlinear membership function for fuzzy optimization of mechanical and structural systems is discussed in Dhingra et al. (1992). An excellent review paper discussing the concepts and developments of structural analysis with fuzzy uncertainty has been studied by Moller et al. (2000). Fuzzy behavior of mechanical systems with uncertain boundary conditions has been investigated by Chekri et al. (2000). Transformation method has been applied for the analysis of structural systems with uncertain parameters by Hanss (2002). In this regard, an important book has been written by Hanss (2005) in which applications of fuzzy arithmetic has been used in a variety of engineering problems. Massa et al. (2006) presented an efficient method for the static design of imprecise structures with fuzzy data. Rao et al. (2010) investigated the transient response of structures with fuzzy structural parameters. Fuzzy arithmetical approach has been used for modeling and analysis of uncertain systems of automotive crash and landslide failure by Hanss and Turrin (2010). In both applications, epistemic uncertainties are considered which arise from lack of knowledge. Recently, Farkas et al. (2012) presented an optimization study of a vehicle bumper subsystem with fuzzy parameters. Various generalized models of uncertainty have been applied to finite element methods too for solving the structural problems with fuzzy parameters. As such few papers that are related to Fuzzy Finite Element Method (FFEM) are discussed next.

First, few important literatures related to fuzzy static analyses of structures are described. Accordingly, fuzzy finite element approach has been applied to study static response of structural systems with imprecisely defined parameters by Rao and Sawyer (1995). Fuzzy finite element method has also been developed by Muhanna and Mullen (1999) for mechanics problems using alpha cut technique. Mullen and Muhanna (1999) introduced a new treatment of load uncertainties in structural problems based on fuzzy set theory. Hanss and Willner (2000) used fuzzy arithmetic approach for the static solution of a bar with fuzzy parameters. Rao and Reddy (2007) considered a cable-stayed bridge with multiple uncertainties to obtain fuzzy static responses of structures using fuzzy finite element approach. Fuzzy finite element method has been applied by Skalna et al. (2008) for the uncertain static problems of structural mechanics such as cantilever truss structure and two bay two floor frame. Verhaeghe et al. (2010) presented interval based computation with fuzzy finite element analysis to explain the static analysis of structures.

Balu and Rao (2011a, 2011b) investigated static analysis of structural problems with fuzzy parameters. They have used High Dimensional Model Representation (HDMR) along with finite element method for the analysis. Balu and Rao (2012) also investigated both static and dynamic responses of structures using FFEM with HDMR. Recently, Behera and Chakraverty (2013a, 2013c, 2013d) obtained uncertain static responses of imprecisely defined structures such as bar, beam, truss and simplified bridge using fuzzy finite element method. Fuzzy arithmetic based computations have been used by them for the solution.

Next, literatures related to uncertain dynamic analysis of structures with fuzzy parameters are reviewed. As such, Chen and Rao (1997) proposed a fuzzy finite element approach for the vibration analysis of imprecisely defined systems. Akpan et al. (2001a) derived fuzzy finite element method for the dynamic analysis of smart structures. Both fuzzy static and dynamic analysis of structures have also been explained by Akpan et al. (2001b) using fuzzy finite element approach. Vertex method and VAST software has been used for fuzzy finite element analysis. Donders et al. (2005) proposed a Short Transformation Method (STM) for the uncertainty assessment of dynamic response of structures. A clamped plate and a car front cradle with uncertain design parameters are demonstrated by them. Interval and fuzzy finite element method for the eigenvalue and frequency response function analysis of structures with uncertain parameters have been studied by Gersem et al. (2007). Fuzzy eigenvalues and eigenvectors of a finite element model with fuzzy parameters have been determined by Massa et al. (2008). Giannini and Hanss (2008) applied the transformation method to characterize the dynamic behavior of the structure with fuzzy parameters. They have considered a beam problem to show the performance of the proposed method. An optimization algorithm has been developed by Munck et al. (2008) for obtaining response surface by calculating fuzzy envelope and response functions. Morales et al. (2012) used fuzzy finite element method for active vibration control of uncertain structures. This work provides a tool for studying the influence of uncertainty propagation on both stability and performance of a vibration control system. Xia and Friswell (2014) proposed a method, based on the fundamental perturbation principle and vertex theory to solve fuzzy eigenvalue problem. They have verified the method by considering a simple cantilever beam.

In recent years, fractional order differential equations have also been used to model physical and engineering problems. Since, it is too difficult to obtain the exact solution of fractional differential equations, one may need a reliable and efficient numerical technique for the solution of fractional differential equations. Many important works have been reported regarding fractional calculus in the last few decades. Related to this field, several excellent books have also been written by different authors representing the scope and various aspects of fractional calculus such as in Miller and Ross (1993), Oldham and Spanier (1974), Kiryakov (1993) and Podlubny (1999). These books give an extensive review on fractional derivative and fractional differential equations which may help the reader in understating the basic concepts of fractional calculus. In this regard, many authors have developed various methods to solve fractional ordinary and partial differential equations and integral equations of physical systems. However, few related papers are cited here. Suarez and Shokooh (1997) used an eigenvector expansion method for the solution of a mechanical spring-mass system containing fractional derivatives. The same type of problem has also been studied by Yuan and Agrawal (2002) when the damping factor is defined as fractional. Very recently, Behera and Chakraverty (2013e), and Chakraverty and Behera (2013a) applied homotopy perturbation method to solve a fractionally damped beam and spring-mass system respectively. In these studies, variables and parameters are considered as deterministic in nature.

On the other hand, very few work has been carried out when uncertainty has been taken into consideration for the structural system when damping factor is defined as fractional. Both fractional and uncertainty play an important role in the structural modeling and design. Hence, an attempt has been made to combine both for a better analysis. Some recent useful contributions on the theory of Fuzzy Fractional Differential Equations (FFDEs) may be seen in (Agarwal et al. 2010; Jeong 2010; Arshad and Lupulescu 2011; Mohammed et al. 2011; Allahviranloo et al. 2012; Salahshour et al. 2012a**;** Ahmadian et al. 2013; Mazandarani and Kamyad 2013; Ahmad et al. 2013; Arshad 2013; Takaci et al. 2014). The concept of FFDEs was first introduced by Agrawal et al. (2010). Mohammed et al. (2011) applied differential transform method for solving fuzzy fractional initial value problems. Allahviranloo et al. (2012) studied the explicit solution of fractional differential equations with uncertainty. Salahshour et al. (2012a) developed Riemann-Liouville differentiability using Hukuhara difference called Riemann-Liouville H-differentiability and solved FFDEs by Laplace transform. A Jacobi operational matrix has been studied by Ahmadian et al. (2013) for solving a fuzzy linear fractional differential equation. Mazandarani and Kamyad (2013) applied modified fractional Euler method for solving fuzzy fractional initial value problem. Ahmad et al.

(2013) studied the solution of FFDEs using Zadeh's extension principle. The exact and approximate solutions of FFDEs are obtained by Takaci et al. (2014) under the Caputo Hukuhara differentiability. In addition to these, Ghaemi et al. (2013) solved a fuzzy fractional kinetic equation of acid hydrolysis reaction. Salah et al. (2013) applied homotopy analysis transform method for the solution of fuzzy fractional heat equation. Recently, Behera and Chakraverty (2014b) investigated the uncertain semi analytical solution of a fuzzy fractionally damped mechanical spring mass system.

1.4. Gaps

Static and dynamic problems with deterministic structural parameters are well studied. In this regard, good number of research papers have been written by different authors. As mentioned earlier, the involved parameters or variables may have some type of uncertainty due to errors in the measurement, observation, experiment, applying different operating condition or it may be maintenance induced error. Recntly, these uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices.

In general, uncertain static problems with interval or fuzzy parameters simplify to interval or fuzzy system of equations whereas dynamic problems convert to interval or fuzzy eigenvalue problems. In this respect, few authors have developed different methods for the solution of these type of problems. But sometimes, the existing methods are not computationally efficient and general to deliver the solution. These methods may have various disadvantages like number of iterations, triangularisation, finding more number of determinants, etc. Review of literature reveals that very little effort has also been made for dynamical problem of parameter identification from uncertain dynamic data. Moreover, structural systems governed by crisp fractional differential equations have been studied by many authors. But very limited studies have been done for fuzzy fractional differential equations. For solving a fuzzy fractionally damped system, we may get a coupled system of crisp fractional differential equations using the existing fuzzy approach. This may be time consuming to solve. Hence, efficient methods have to be developed to obtain the solution of such systems with reduced computational cost.

As such, there are many gaps in the above focused problems. It is known that interval and fuzzy computations are themselves very complex to handle. As for a simple example, the subtraction of two equal intervals (fuzzy numbers) does not give a zero interval (fuzzy number). Also, intersection of a fuzzy set with its complement is not null. Similarly, union of a fuzzy set with its complement is not the universal set. Having these in mind, one has to develop efficient algorithms or methods very carefully to handle these problems.

1.5. Aims and Objectives

Recently, effort has been made by various researchers to solve these type of problems but a lot of important information is still missing in the existing literature. Further, some of the known methods are computationally expensive. The purpose of the present work is to fill these gaps. In view of the above, our aim in this research is to develop new methods for solving fuzzy/interval algebraic system of equations and eigenvalue problems. Proposed methods are validated by studying uncertain static and dynamic analysis of structural systems. Parameter identification of multi storey frame structures from uncertain dynamic data are also taken into consideration. Then, solution of fractionally damped discrete and continuous system with crisp parameters are analysed. Lastly, dynamic responses of the above considered fractionally damped systems with fuzzy initial conditions are investigated. As such, the broad objectives related to the present research may be summarized as below:

- new methods to solve fuzzy complex system of linear equations;
- new methods to solve fuzzy real system of linear equations;
- non-negative solution of fully fuzzy system of linear equations using proposed double parametric form of fuzzy numbers;
- generalised solution of fully fuzzy system of linear equations using linear programming problem approach;
- new algorithm for solving fuzzy generalized eigenvalue problem;
- validation of the proposed methods through uncertain static and dynamic problems of structures;
- identification of system parameters of multistorey frame structure from uncertain dynamic data;
- solution of crisp fractionally damped discrete and continuous system;
- uncertain dynamic responses of above fractionally damped discrete and continuous systems with uncertain initial conditions.

1.6. Organisation of the Thesis

Present work, deals with the uncertainty analysis of structures through fuzzy and interval approach. Accordingly, this thesis consists of ten chapters which investigate new methods for fuzzy algebraic system of equations, eigenvalue problems, fuzzy fractional differential equations and system identification along with application problems. In view of this, the contents of the ten chapters of the thesis are organized in the following manner.

Chapter 1 (that in the present chapter) addresses the background of interval and fuzzy set theory along with an overview of its application in the field of structural mechanics. Related literatures are systematically reviewed here. Gaps as well as aim and objectives of the present study are also included here.

In Chapter 2 we recall definitions and details about interval and fuzzy computations relevant to the present investigation.

Chapter 3 proposes various methods for solving uncertain algebraic system of linear equations. As such, fuzzy complex and real system of linear equations, nonnegative solution of fully fuzzy system of linear equations and generalized fully fuzzy system of linear equations are taken into consideration. Fuzzy addition, subtraction, centre, radius, width and a newly developed double parametric form of fuzzy numbers have been used in the solution process. Linear programming problem has also been applied for the solution of fully fuzzy linear system.

Next, in Chapter 4, static analysis of imprecisely defined structures has been investigated using the proposed methodologies discussed in Chapter 3. Fuzzy and interval finite element methods have also been applied here to obtain the uncertain static responses of structures. In this chapter we have validated the proposed methods by analysing the static problems of structures such as bar, beam, truss, simplified bridge, thin plate and rectangular sheet with fuzzy/interval material and geometric properties along with uncertain external forces.

Chapter 5 presents algorithm for solving uncertain generalized eigenvalue problem to obtain the uncertain vibration characteristics of an imprecisely defined structure. The structural parameters viz. the material and geometrical properties are considered as uncertain and have been taken as fuzzy numbers in term of triangular and trapezoidal convex normalized fuzzy sets. In the proposed algorithm, fuzzy computations are handled through fuzzy arithmetic with the alpha cut form of fuzzy numbers. For verification, a lumped mass structural system viz. multistorey shear building, spring mass mechanical system and stepped beam structures have been analysed. Computed frequency parameters and corresponding mode shapes are compared with the existing results in special cases. Also, this chapter investigates the identification procedure of the column stiffness of multistorey frame structures by using the prior known uncertain parameters and dynamic data. Uncertainties are modelled through triangular convex normalized fuzzy sets. Bounds of the identified uncertain stiffness are obtained by using a proposed fuzzy based iteration algorithm associated with the Taylor series expansion. Example problems are solved to demonstrate the reliability and efficiency of the identification process.

Chapter 6 investigates the numerical solution of a fractionally damped dynamic system. A single degree of freedom spring-mass mechanical system with fractional damping of order 1/2 is considered for the analysis. Homotopy Perturbation Method (HPM) is used to compute the dynamic responses of the system subjected to unit step and unit impulse loads. Obtained results are depicted in terms of plots. Comparisons are made with Podlunby (1999), Suarez and Shokooh (1997) and Yan and Agrawal (2002) to show the effectiveness and validation of the present analysis.

Chapter 7 introduces the numerical solution of a viscoelastic continuous beam whose damping behaviours are defined in terms of fractional derivatives of arbitrary order. HPM has been used to obtain the dynamic responses. Unit step and impulse function responses are considered for the analysis. Results are depicted in terms of plots and comparisons are made with the analytical solutions obtained by Zu-feng and Xiao-yan (2007).

Chapter 8 describes the numerical solution of imprecisely defined fractional order discrete system, subjected to unit impulse and step loads. A mechanical spring mass system having fractional damping of order $1/2$ with fuzzy initial condition has been taken into consideration. Fuzziness appeared in the initial conditions are modeled through different types of convex normalised fuzzy sets viz. triangular, trapezoidal and Gaussian fuzzy numbers. Homotopy perturbation method is used with fuzzy based approach to obtain the uncertain impulse response. Numerical examples are solved by symbolic computations and those are validated in crisp cases.

In Chapter 9, fuzzy fractionally damped beam has been studied using the double parametric form of fuzzy numbers subject to unit step and impulse loads. Triangular convex normalized fuzzy sets are used for the analysis. Using the alpha cut form, corresponding beam equation is first converted to an interval based equation. Next, it has been transformed to crisp form by applying double parametric form of fuzzy numbers. Finally, HPM is used for obtaining the fuzzy response. Various numerical examples are taken into consideration and the results are compared in special cases.

Based on the present work, Chapter 10 summarizes the main findings and conclusions of the study. Finally, suggestions for future work are incorporated.

Chapter 2

Preliminaries

Chapter 2

Preliminaries

This chapter presents the notations, definitions of interval, fuzzy numbers viz. triangular, trapezoidal and Gaussian, double parametric form of fuzzy numbers, Riemann–Liouville integral, Caputo derivative, theorem and lemma related to fuzzy/fuzzy fractional differential equations, homotopy perturbation method and fuzzy/interval arithmetic, which are relevant to the present investigation. Several excellent books related to this have also been written by different authors representing the scope and various aspects of interval and fuzzy set theory such as in (Jaulin et al. 2001; Zimmermann 2001; Ross 2004; Moore 2009; Chakraverty 2014). These books also give an extensive review on fuzzy set theory and its applications which may help the reader in understating the basic concepts of fuzzy set theory and its application.

Definition 2.1 Interval

An interval $\tilde{x}$ is denoted by $[\underline{x}, \overline{x}]$ on the set of real numbers R given by

$$\tilde{x} = [\underline{x}, \overline{x}] = \{x \in R : \underline{x} \le x \le \overline{x}\}. \tag{2.1}$$

Here we have only considered closed intervals throughout this thesis, although there exists various other types of intervals such as open and half open intervals. $\underline{x}$ and $\overline{x}$ are known as the left and right endpoints of the interval $\tilde{x}$ in the above expression (2.1) respectively.

Let us now consider two arbitrary intervals $\tilde{x} = [\underline{x}, \overline{x}]$ and $\tilde{y} = [\underline{y}, \overline{y}]$. These two intervals are said to be equal if they are in the same set. Mathematically, it only happens when corresponding end points are equal. Hence, one may write

$$\tilde{x} = \tilde{y} \text{ if and only if } \underline{x} = \underline{y} \text{ and } \overline{x} = \overline{y}. \tag{2.2}$$

For the above two arbitrary intervals $\tilde{x} = [\underline{x}, \overline{x}]$ and $\tilde{y} = [\underline{y}, \overline{y}]$, interval arithmetic operations such as addition (+), subtraction (-), multiplication ($\times$) and division (/) are defined as follows:

$$\tilde{x} + \tilde{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \tag{2.3}$$

$$\tilde{x} - \tilde{y} = [\underline{x} - \bar{y}, \bar{x} - \underline{y}], \tag{2.4}$$

$$\tilde{x} \times \tilde{y} = [\min S, \max S], \text{ where } S = \{\underline{x} \times \underline{y}, \underline{x} \times \bar{y}, \bar{x} \times \underline{y}, \bar{x} \times \bar{y}\}, \tag{2.5}$$

$$\text{and } \tilde{x} / \tilde{y} = [\underline{x}, \bar{x}] \times [\frac{1}{\bar{y}}, \frac{1}{\underline{y}}] \text{ if } 0 \notin \tilde{y}. \tag{2.6}$$

Now if k is a real number and $\tilde{x} = [\underline{x}, \bar{x}]$ is an interval, then the multiplication of them are given by

$$k\tilde{x} = \begin{cases} [k\bar{x}, k\underline{x}], k < 0, \\ [k\underline{x}, k\bar{x}], k \geq 0. \end{cases} \tag{2.7}$$

Definition 2.2 Fuzzy Set

A fuzzy set $\tilde{U}$ on the real line R is defined as the set of ordered pairs such that

$$\tilde{U} = \{(x, \mu_{\tilde{U}}(x)) \mid x \in R, \mu_{\tilde{U}}(x) \in [0,1]\}, \tag{2.8}$$

where, $\mu_{\tilde{U}}(x)$ is called the membership function.

Definition 2.3 Fuzzy Number

A fuzzy number $\tilde{U}$ is a convex normalised fuzzy set $\tilde{U}$ of the real line R such that

$$\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall\, x \in R\} \tag{2.9}$$

where, $\mu_{\tilde{U}}$ is called the membership function and it is piecewise continuous.

There exists a variety of fuzzy numbers. But in this study we have used only the triangular, trapezoidal and Gaussian fuzzy numbers. So, we define these three fuzzy numbers below.

Definition 2.4 Triangular Fuzzy Number (TFN)

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such that

- There exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of U), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set.
- $\mu_{\tilde{U}}(x)$ is piecewise continuous.

Let us consider an arbitrary triangular fuzzy number $\tilde{U}=(a,b,c)$ as shown in Fig. 2.1. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be defined as follows

$$\mu_{\tilde{U}}(x)=\begin{cases} 0, & x\le a \\ \dfrac{x-a}{b-a}, & a\le x\le b \\ \dfrac{c-x}{c-b}, & b\le x\le c \\ 0, & x\ge c. \end{cases} \tag{2.10}$$

The triangular fuzzy number $\tilde{U}=(a,b,c)$ can be represented with an ordered pair of functions through $\alpha-$cut approach viz.

$$[\underline{u}(\alpha),\overline{u}(\alpha)]=[(b-a)\alpha+a,-(c-b)\alpha+c] \text{ where, } \alpha\in[0,1].$$

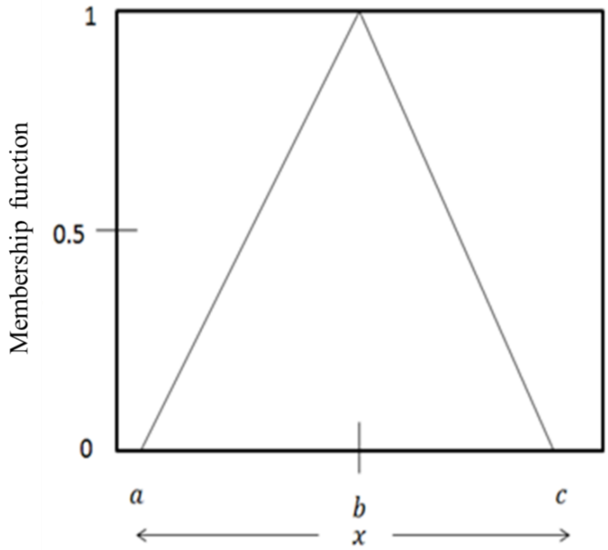

Fig. 2.1 Triangular fuzzy number

Definition 2.5 Trapezoidal Fuzzy Number (TrFN)

We have now considered an arbitrary trapezoidal fuzzy number $\tilde{U}=(a,b,c,d)$ as given in Fig. 2.2. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be interpreted as follows

$$\mu_{\tilde{U}}(x)=\begin{cases} 0, & x\le a \\ \dfrac{x-a}{b-a}, & a\le x\le b \\ 1, & b\le x\le c \\ \dfrac{d-x}{d-c}, & c\le x\le d \\ 0, & x\ge d. \end{cases} \tag{2.11}$$

The trapezoidal fuzzy number $\tilde{U} = (a, b, c, d)$ can be represented with an ordered pair of functions through $\alpha -$cut approach that is

$$[\underline{u}(\alpha), \overline{u}(\alpha)] = [(b-a)\alpha + a, -(d-c)\alpha + d] \text{ where, } \alpha \in [0,1].$$

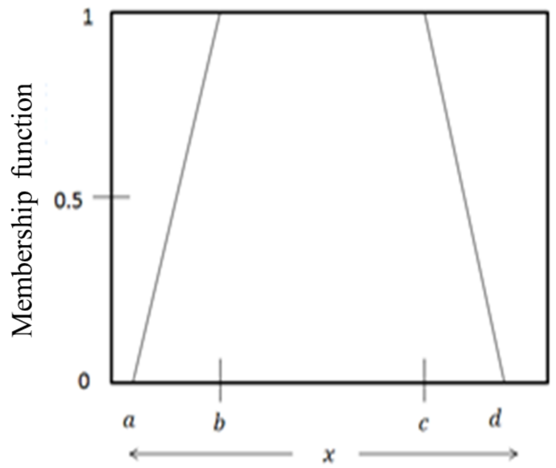

Fig. 2.2 Trapezoidal fuzzy number

Definition 2.6 Gaussian Fuzzy Number (GFN)

Let us define an arbitrary asymmetrical Gaussian fuzzy number, $\tilde{U} = (r, \sigma_l, \sigma_r)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} \exp[-(x-r)^2 / 2\sigma_l^2] & \text{for } x \le r \\ \exp[-(x-r)^2 / 2\sigma_r^2] & \text{for } x \ge r \end{cases} \quad \forall\ x \in R \tag{2.12}$$

where, the modal value is denoted as r and σ_l, σ_r denote the left-hand and right-hand spreads (fuzziness) corresponding to the Gaussian distribution.

For symmetric Gaussian fuzzy number, the left-hand and right-hand spreads are equal i.e. $\sigma_l = \sigma_r = \sigma$. So the symmetric Gaussian fuzzy number may be written as $\tilde{U} = (r, \sigma, \sigma)$ and corresponding membership function may be defined as $\mu_{\tilde{U}}(x) = \exp\{-\beta(x-r)^2\}$ $\forall\, x \in R$ where, $\beta = 1/2\sigma^2$. The symmetric Gaussian fuzzy number in parametric form as shown in Fig. 2.3 can be represented as

$$\tilde{U} = [\underline{u}(\alpha), \overline{u}(\alpha)] = \left[r - \sqrt{-\frac{(\log_e \alpha)}{\beta}}, r + \sqrt{-\frac{(\log_e \alpha)}{\beta}} \right] \text{ where, } \alpha \in [0, 1].$$

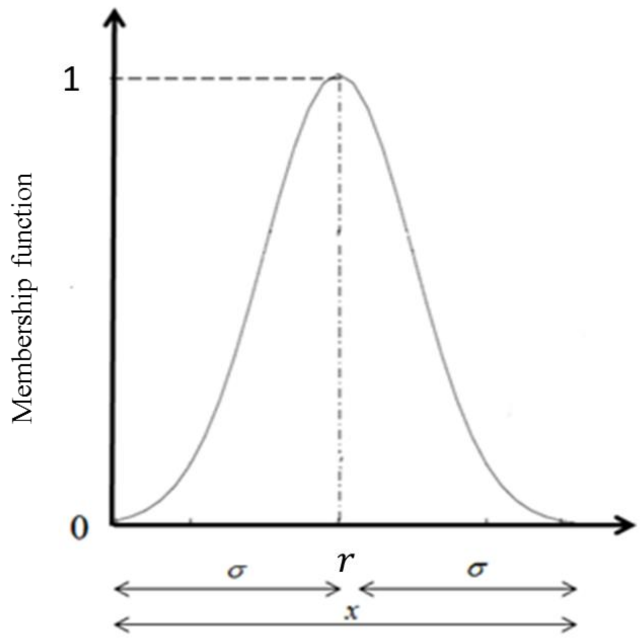

Fig. 2.3 Gaussian fuzzy number

For all the above type of fuzzy numbers, the left and right bounds (lower or upper bounds) of the fuzzy numbers satisfy the following requirements

- $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over [0, 1].
- $\overline{u}(\alpha)$ is a bounded right continuous non-increasing function over [0, 1].
- $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1.$

Definition 2.7 Double Parametric form of Fuzzy Number

Using the $\alpha -$cut approach as discussed in (Definition 2.4 to 2.6) for all the fuzzy numbers, we have $\tilde{U} = [\underline{u}(\alpha), \overline{u}(\alpha)]$. Now, one may write this as a crisp number with double parametric form as $\tilde{U}(\alpha, \beta) = \beta(\overline{u}(\alpha) - \underline{u}(\alpha)) + \underline{u}(\alpha)$ where α and $\beta \in [0,1]$. And to obtain the lower and upper bounds of the solution in single parametric form, we may put $\beta = 0$ and 1 respectively. This may be represented as $\tilde{U}(\alpha, 0) = \underline{u}(\alpha)$ and $\tilde{U}(\alpha, 1) = \overline{u}(\alpha)$.

Definition 2.8 Fuzzy Centre

Fuzzy centre of an arbitrary fuzzy number $\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$ is defined as $\tilde{x}^c = \dfrac{\underline{x}(\alpha) + \overline{x}(\alpha)}{2}$ for all $0 \le \alpha \le 1$.

Definition 2.9 Fuzzy Radius

Fuzzy radius of an arbitrary fuzzy number $\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$ is defined as $\Delta\tilde{x} = \dfrac{\overline{x}(\alpha) - \underline{x}(\alpha)}{2}$ for all $0 \le \alpha \le 1$.

For any two arbitrary fuzzy numbers $\tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)]$, $\tilde{y} = [\underline{y}(\alpha), \overline{y}(\alpha)]$ and scalar k, the fuzzy arithmetic is similar to the interval arithmetic as defined above.

Definition 2.10 Positive Fuzzy Number

A fuzzy number $\tilde{U}$ is said to be positive, denoted by $\tilde{U} > 0$ if its membership function $\mu_{\tilde{U}}(x)$ satisfies $\mu_{\tilde{U}}(x) = 0, \forall x \le 0$.

Definition 2.11 Non-negative Fuzzy Number

A fuzzy number $\tilde{U}$ is said to be non-negative, denoted by $\tilde{U} \ge 0$ if its membership function $\mu_{\tilde{U}}(x)$ satisfies $\mu_{\tilde{U}}(x) = 0, \forall x < 0$.

Definition 2.12 Fuzzy Complex Number

An arbitrary fuzzy complex number may be represented as $\tilde{X} = \tilde{p} + i\tilde{q}$, where $\tilde{p} = [\underline{p}(\alpha), \overline{p}(\alpha)]$ and $\tilde{q} = [\underline{q}(\alpha), \overline{q}(\alpha)]$ are two real fuzzy numbers, for all $0 \le \alpha \le 1$.

As such, the above can be written as

$$\tilde{X} = [\underline{p}(\alpha), \overline{p}(\alpha)] + i\,[\underline{q}(\alpha), \overline{q}(\alpha)] = [\underline{p}(\alpha) + i\underline{q}(\alpha), \overline{p}(\alpha) + i\,\overline{q}(\alpha)].$$

Now writing $\tilde{X} = [\underline{X}, \overline{X}]$ it gives $\underline{X} = \underline{p}(\alpha) + i\underline{q}(\alpha)$ and $\overline{X} = \overline{p}(\alpha) + i\,\overline{q}(\alpha)$.

Definition 2.13 Fuzzy Matrix

A matrix $[\tilde{A}] = (\tilde{a}_{kj})$ is called a fuzzy matrix, if each element of $[\tilde{A}]$ is a fuzzy number.

A fuzzy matrix $[\tilde{A}]$ will be non-negative, denoted by $[\tilde{A}] \ge 0$, if each element of $[\tilde{A}]$ will be a non-negative fuzzy number.

Definition 2.14 Fuzzy System of Linear Equations

The $n \times n$ fuzzy system of linear equations can be represented as

$$a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n = \tilde{b}_1$$
$$a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n = \tilde{b}_2$$
$$\vdots \tag{2.13}$$
$$a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n = \tilde{b}_n.$$

In matrix notation, the above system may be written as $[A]\{\tilde{X}\} = \{\tilde{b}\}$, where the coefficient matrix $[A] = (a_{kj})$, $1 \le k \le n$, $j \le n$ is a crisp real $n \times n$ matrix, $\{\tilde{b}\} = \{\tilde{b}_k\}$, $1 \le k$ is a column vector of fuzzy number and $\{\tilde{X}\} = \{\tilde{x}_j\}$ is the vector of fuzzy unknown.

Definition 2.15 Fully Fuzzy System of Linear Equations

The $n \times n$ fully fuzzy system of linear equations can be represented as

$$\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1$$
$$\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2$$
$$\vdots \tag{2.14}$$
$$\tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n.$$

In matrix notation, the above system may be written as $[\tilde{A}]\{\tilde{X}\} = \{\tilde{b}\}$, where the coefficient matrix $[\tilde{A}] = (\tilde{a}_{kj})$, $1 \le k \le n$, $j \le n$ is a fuzzy $n \times n$ matrix, $\{\tilde{b}\} = \{\tilde{b}_k\}$, $1 \le k$ is a column vector of fuzzy number and $\{\tilde{X}\} = \{\tilde{x}_j\}$ is the vector of fuzzy unknown.

Definition 2.16 Caputo Derivative

The fractional derivative of $f(t)$ in the Caputo sense is defined as below

$$D^{\lambda} f(t) = J^{m-\lambda} D^m f(t)$$
$$= \begin{cases} \dfrac{1}{\Gamma(m-\lambda)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\lambda+1-m}}, & m-1 < \lambda < m, \quad m \in N \\ \dfrac{d^m}{dt^m} f(t), & \lambda = m, \quad m \in N. \end{cases} \tag{2.15}$$

Some basic properties of the fractional operator are

(i) $J^{\lambda} J^{\gamma} f(t) = J^{\lambda+\gamma} f(t), \quad \lambda, \gamma \ge 0$

(ii) $J^{\lambda}(t^{\gamma}) = \left\{ \dfrac{\Gamma(\gamma+1) t^{\alpha+\gamma}}{\Gamma(\lambda+\gamma+1)}, \lambda > 0,\ \gamma > \text{-}1, t > 0 \right.$.

- **Homotopy Perturbation Method (HPM)**

To illustrate the basic ideas of this method (He 1999; He 2000), we consider the following nonlinear differential equation of the form.

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.16}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad t \in \Gamma \tag{2.17}$$

where, A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Therefore, Eq. (2.16) may be written as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{2.18}$$

By the homotopy technique, we construct a homotopy $U(r, p) : \Omega \times [0,1] \to R$, which satisfies:

$$H(U, p) = (1-p)[L(U) - L(v_0)] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{2.19}$$

or

$$H(U, p) = L(U) - L(v_0) + pL(v_0) + p[N(U) - f(r)] = 0, \tag{2.20}$$

where, $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, v_0 is an initial approximation of Eq. (2.16). Hence, it is obvious that

$$H(U,0) = L(U) - L(v_0) = 0, \tag{2.21}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{2.22}$$

and the changing process of p from 0 to 1, is just that of $U(r, p)$ from $v_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(U) - L(v_0)$, $A(U) - f(r)$ are called homotopic. Applying the perturbation technique due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of Eq. (2.19) or (2.20) can be represented as a power series in p as follows,

$$U = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \tag{2.23}$$

when $p \to 1$, Eq. (2.19) or (2.20) corresponds to Eqs. (2.16) and (2.23) and becomes the approximate solution of Eq. (2.19), i.e.,

$$u = \lim_{p \to 1} U = u_0 + u_1 + u_2 + u_3 + \cdots \tag{2.24}$$

The convergence of the series (2.24) has been proved in He (1999, 2000).

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Chapter 3

Uncertain Algebraic System of Linear Equations

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Chapter 3

Uncertain Algebraic System of Linear Equations

Design and analysis of various science and engineering problems require the solution of linear system of equations. For example, the finite element formulation of equilibrium and steady state problems lead to an algebraic system of linear equations. For the sake of simplicity, variables and parameters of these systems are defined exactly or considered as deterministic in the modelling. But in actual practice, the parameters and variables may be uncertain or vague because those are found in general by some experiment or experience. As mentioned in the previous chapters, the uncertainty may be handled by interval and fuzzy numbers.

In general, uncertain system of linear equations with fuzzy number can be categorized as:

- Fuzzy System of Linear Equations (FSLE)
- Fully Fuzzy System of Linear Equations (FFSLE).

There is a difference between FSLE and FFSLE. The coefficient matrix is treated as deterministic in the fuzzy system of linear equations, but in the fully fuzzy system of linear equations all the parameters and variables are considered to be fuzzy numbers. Literature review given in Chapter 1 reveals that there are many shortcomings in the existing methods. As such, it is an important issue to develop mathematical models and numerical techniques that would appropriately treat the general fuzzy or fully fuzzy linear systems. This chapter aims to propose new methods for fuzzy and fully fuzzy system of linear equations in the following sections.

3.1. Fuzzy System of Linear Equations (FSLE)

Depending upon the nature of fuzzy number viz. real or complex fuzzy number, FSLE can be defined as

- Fuzzy complex system of linear equations
- Fuzzy real system of linear equations.

Here, we have considered both fuzzy complex and fuzzy real system of linear equations. Two methods have been proposed with related theorems for fuzzy complex system of linear equations and five methods for fuzzy real system of linear equations.

3.1.1. Fuzzy complex system of linear equations

The $n \times n$ fuzzy complex system of linear equations may be considered as

$$\begin{aligned} c_{11}\tilde{z}_1 + c_{12}\tilde{z}_2 + \cdots + c_{1n}\tilde{z}_n &= \tilde{w}_1 \\ c_{21}\tilde{z}_1 + c_{22}\tilde{z}_2 + \cdots + c_{2n}\tilde{z}_n &= \tilde{w}_2 \\ &\vdots \\ c_{n1}\tilde{z}_1 + c_{n2}\tilde{z}_2 + \cdots + c_{nn}\tilde{z}_n &= \tilde{w}_n. \end{aligned} \tag{3.1}$$

In matrix notation, we may then write the above as $[C]\{\tilde{Z}\} = \{\tilde{W}\}$, where the coefficient matrix $[C] = (c_{kj})$, $1 \le k \le n$, $j \le n$ is a complex $n \times n$ matrix, $\{\tilde{W}\} = \{\tilde{w}_k\}$, $1 \le k$ is a column vector of fuzzy complex number and $\{\tilde{Z}\} = \{\tilde{z}_j\}$ is the vector of fuzzy complex unknown.

System (3.1) can be represented as

$$\sum_{j=1}^{n} c_{kj}\tilde{z}_j = \tilde{w}_k, \qquad \text{for } k = 1, 2, \cdots, n. \tag{3.2}$$

The complex coefficient matrix, fuzzy complex unknown and the right hand fuzzy complex number vector may be written respectively as

$c_{kj} = a_{kj} + ib_{kj}$,

$\tilde{Z} = \tilde{z}_j = \tilde{p}_j + i\tilde{q}_j = [\underline{p}_j(\alpha), \overline{p}_j(\alpha)] + i\,[\underline{q}_j(\alpha), \overline{q}_j(\alpha)]$

and $\tilde{W} = \tilde{w}_k = \tilde{u}_k + i\tilde{v}_k = [\underline{u}_k(\alpha), \overline{u}_k(\alpha)] + i\,[\underline{v}_k(\alpha), \overline{v}_k(\alpha)]$.

Writing $\tilde{Z} = [\underline{Z}, \overline{Z}] = [\underline{z}_j(\alpha), \overline{z}_j(\alpha)]$ and $\tilde{W} = [\underline{W}, \overline{W}] = [\underline{w}_k(\alpha), \overline{w}_k(\alpha)]$ we have

$$\begin{aligned} \underline{Z} &= \underline{z}_j(\alpha) = \underline{p}_j(\alpha) + i\underline{q}_j(\alpha), \\ \overline{Z} &= \overline{z}_j(\alpha) = \overline{p}_j(\alpha) + i\overline{q}_j(\alpha), \end{aligned} \tag{3.3a}$$

and

$$\begin{aligned} \underline{W} &= \underline{w}_k(\alpha) = \underline{u}_k(\alpha) + i\underline{v}_k(\alpha), \\ \overline{W} &= \overline{w}_k(\alpha) = \overline{u}_k(\alpha) + i\overline{v}_k(\alpha). \end{aligned} \tag{3.3b}$$

Next, the following equation is obtained by substituting $\tilde{Z}$ and $\tilde{W}$ in Eq. (3.2)

$$\sum_{j=1}^{n}(a_{kj}+ib_{kj})[\underline{z}_j(\alpha),\bar{z}_j(\alpha)]=[\underline{w}_k(\alpha),\bar{w}_k(\alpha)],\quad \text{for } k=1,2,\cdots,n. \tag{3.4}$$

Substituting the expressions of (3.3a) and (3.3b) in Eq. (3.4) one may get

$$\sum_{j=1}^{n}(a_{kj}+ib_{kj})[\underline{p}_j(\alpha)+i\underline{q}_j(\alpha),\bar{p}_j(\alpha)+i\bar{q}_j(\alpha)]=[\underline{u}_k(\alpha)+i\underline{v}_k(\alpha),\bar{u}_k(\alpha)+i\bar{v}_k(\alpha)].$$

The above equation can now be written as

$$\sum_{j=1}^{n}a_{kj}[\underline{p}_j(\alpha)+i\underline{q}_j(\alpha),\bar{p}_j(\alpha)+i\bar{q}_j(\alpha)]+i\sum_{j=1}^{n}b_{kj}[\underline{p}_j(\alpha)+i\underline{q}_j(\alpha),\bar{p}_j(\alpha)+i\bar{q}_j(\alpha)]$$
$$=[\underline{u}_k(\alpha)+i\underline{v}_k(\alpha),\bar{u}_k(\alpha)+i\bar{v}_k(\alpha)].$$

Here, a_{kj} and b_{kj} both may be positive and/or negative. To handle the positive and negative values of a_{kj} and b_{kj}, the above equation is written as below (Eqs. (3.5) and (3.6))

$$\begin{aligned}&\left\{\sum_{a_{kj}\ge 0}a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}<0}a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}\\&+i\left\{\sum_{b_{kj}\ge 0}b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}<0}b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}=\underline{u}_k(\alpha)+i\underline{v}_k(\alpha)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\left\{\sum_{a_{kj}<0}a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}\ge 0}a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}\\&+i\left\{\sum_{b_{kj}<0}b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}\ge 0}b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}=\bar{u}_k(\alpha)+i\bar{v}_k(\alpha),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\underline{u}_k(\alpha)&=\sum_{a_{kj}\ge 0}a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}<0}a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right),\\ \underline{v}_k(\alpha)&=\sum_{b_{kj}\ge 0}b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}<0}b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\bar{u}_k(\alpha) &= \sum_{a_{kj}<0} a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}\geq 0} a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right),\\ \bar{v}_k(\alpha) &= \sum_{b_{kj}<0} b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}\geq 0} b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right).\end{aligned} \tag{3.8}$$

- **(Method 1) Fuzzy complex centre based method**

A new method has been proposed using the fuzzy complex centre. Firstly, the fuzzy centre solution is obtained and then the lower bound is written in terms of fuzzy centre and upper bound. From this we find the upper bound of the fuzzy solution. Similarly, lower bound can be obtained. Few theorems related to the investigation are stated and proved.

Our aim is now to solve Eq. (3.2) in terms of fuzzy complex centre. One may write Eq. (3.2) using centre as

$$\sum_{j=1}^{n}(a_{kj}+ib_{kj})\tilde{z}_j^{\,c}=\tilde{w}_k^{\,c},\qquad \text{for } k=1,2,\cdots,n. \tag{3.9}$$

Here, $\tilde{z}_j^{\,c}=\tilde{p}_j^{\,c}+i\tilde{q}_j^{\,c}$ and $\tilde{w}_k^{\,c}=\tilde{u}_k^{\,c}+i\tilde{v}_k^{\,c}$, where

$$\tilde{p}_j^{\,c}=\frac{\underline{p}_j(\alpha)+\bar{p}_j(\alpha)}{2},\;\; \tilde{q}_j^{\,c}=\frac{\underline{q}_j(\alpha)+\bar{q}_j(\alpha)}{2},$$

$$\tilde{u}_k^{\,c}=\frac{\underline{u}_k(\alpha)+\bar{u}_k(\alpha)}{2}\;\;\text{and}\;\;\tilde{v}_k^{\,c}=\frac{\underline{v}_k(\alpha)+\bar{v}_k(\alpha)}{2}.$$

Eq. (3.9) reduces to the following equation using fuzzy centre expressions

$$\left(\sum_{j=1}^{n}a_{kj}\tilde{p}_j^{\,c}-\sum_{j=1}^{n}b_{kj}\tilde{q}_j^{\,c}\right)+i\left(\sum_{j=1}^{n}a_{kj}\tilde{q}_j^{\,c}+\sum_{j=1}^{n}b_{kj}\tilde{p}_j^{\,c}\right)=\tilde{u}_k^{\,c}+i\tilde{v}_k^{\,c}.$$

Equating the real and imaginary part of the above equation we have

$$\sum_{j=1}^{n}a_{kj}\tilde{p}_j^{\,c}-\sum_{j=1}^{n}b_{kj}\tilde{q}_j^{\,c}=\tilde{u}_k^{\,c}$$

and

$$\sum_{j=1}^{n} a_{kj}\tilde{q}_j^{\,c} + \sum_{j=1}^{n} b_{kj}\tilde{p}_j^{\,c} = \tilde{v}_k^{\,c}.$$

The above two equations are now in crisp (deterministic) form so one may solve it easily to obtain $\tilde{p}_j^{\,c}$ and $\tilde{q}_j^{\,c}$. Correspondingly, the centre solution of complex fuzzy system of linear equations may be written as $\tilde{z}_j^{\,c} = \tilde{p}_j^{\,c} + i\tilde{q}_j^{\,c}$.

Few theorems are now stated and proved below related to the proposed method.

Theorem 3.1 The monotonic increasing solution vector $\{\underline{z}_j\}$ can be obtained by replacing $\{\bar{z}_j\}$ in terms of $\tilde{z}_j^{\,c}$ and $\{\underline{z}_j\}$ in Eq. (3.4).

Proof. As per the definition of complex fuzzy centre, $\tilde{z}_j^{\,c} = \dfrac{\underline{z}_j + \bar{z}_j}{2}$. From this, $\bar{z}_j$ can be written in terms of $\tilde{z}_j^{\,c}$ and $\underline{z}_j$ as $\bar{z}_j = 2\tilde{z}_j^{\,c} - \underline{z}_j$.

Substituting $\bar{z}_j$ in Eq. (3.4) we have

$$\sum_{j=1}^{n} (a_{kj} + ib_{kj})[\underline{z}_j,\, 2\tilde{z}_j^{\,c} - \underline{z}_j] = [\underline{w}_k, \overline{w}_k], \quad \text{for } k = 1,2,\cdots,n. \tag{3.10}$$

Eq. (3.10) is now represented equivalently for the expressions of $\underline{z}_j$, $\tilde{z}_j^{\,c}$, $\underline{w}_k$ and $\overline{w}_k$ as

$$\begin{aligned}
&\sum_{j=1}^{n} a_{kj}[\underline{p}_j(\alpha) + i\underline{q}_j(\alpha),\, 2(\tilde{p}_j^{\,c} + i\tilde{q}_j^{\,c}) - (\underline{p}_j(\alpha) + i\underline{q}_j(\alpha))] \\
&+ i\sum_{j=1}^{n} b_{kj}[\underline{p}_j(\alpha) + i\underline{q}_j(\alpha),\, 2(\tilde{p}_j^{\,c} + i\tilde{q}_j^{\,c}) - (\underline{p}_j(\alpha) + i\underline{q}_j(\alpha))] \\
&= [\underline{u}_k(\alpha) + i\underline{v}_k(\alpha), \bar{u}_k(\alpha) + i\bar{v}_k(\alpha)].
\end{aligned} \tag{3.11}$$

Next, Eq. (3.11) is expressed into the following two crisp complex systems (Eqs. 3.12 and 3.13) by equating the left and right bounds respectively

$$\sum_{a_{kj}\geq 0} a_{kj}(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)) + \sum_{a_{kj}<0} a_{kj}\left\{2(\tilde{p}_j^{\,c}+i\tilde{q}_j^{\,c})-(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))\right\}$$
$$+ i\left\{\sum_{b_{kj}\geq 0} b_{kj}(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)) + \sum_{b_{kj}<0} b_{kj}\left\{2(\tilde{p}_j^{\,c}+i\tilde{q}_j^{\,c})-(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))\right\}\right\}$$
$$= \underline{u}_k(\alpha)+i\underline{v}_k(\alpha) \tag{3.12}$$

and

$$\sum_{a_{kj}\geq 0} a_{kj}\left\{2(\tilde{p}_j^{\,c}+i\tilde{q}_j^{\,c})-(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))\right\} + \sum_{a_{kj}<0} a_{kj}(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))$$
$$+ i\left\{\sum_{b_{kj}\geq 0} b_{kj}\left\{2(\tilde{p}_j^{\,c}+i\tilde{q}_j^{\,c})-(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))\right\} + \sum_{b_{kj}<0} b_{kj}(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha))\right\}$$
$$= \overline{u}_k(\alpha)+i\overline{v}_k(\alpha) \tag{3.13}$$

Now, any one of the above two systems viz. Eq. (3.12) or (3.13) may be solved to obtain $\underline{p}_j(\alpha)$ and $\underline{q}_j(\alpha)$. Which give the solution $\underline{z}_j$ as $\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)$. This proves the Theorem 3.1. □

Note: Solution of any one of the above two systems (Eq. 3.12 or 3.13) gives the same result, which is proved in the following theorem.

Theorem 3.2 For a fuzzy complex linear system defined by Eq. (3.1), if the upper bound of the fuzzy complex variable $\tilde{z}_j$ that is $\{\overline{z}_j\}$ is replaced by lower bound with fuzzy complex centre, where the system (3.1) is first solved by total fuzzy centre (that is a crisp complex system), the set of equations thus obtained in terms of left and right bound of the fuzzy complex interval equations are exactly same.

Proof. In Theorem 3.1, it is seen that the fuzzy complex system (3.11) can be written as crisp complex systems (3.12) and (3.13) equivalently for left and right bound respectively. Now, it has to be shown that the two systems (3.12) and (3.13) are the same or in the other words, the solution given by the respective systems are equal.

As the values of a_{kj}, b_{kj}, $\tilde{p}_j^{\,c}$, $\tilde{q}_j^{\,c}$, $\bar{u}_k(\alpha)$ and $\bar{v}_k(\alpha)$ are known, so rearranging Eq. (3.12) and separating the real and imaginary parts of the system gives the following systems

$$\begin{aligned}&\sum_{a_{kj}\geq 0} a_{kj}\,\underline{p}_j(\alpha)-\sum_{a_{kj}<0} a_{kj}\,\underline{p}_j(\alpha)-\sum_{b_{kj}\geq 0} b_{kj}\,\underline{q}_j(\alpha)+\sum_{b_{kj}<0} b_{kj}\,\underline{q}_j(\alpha)\\&=\underline{u}_k(\alpha)-\left(\sum_{a_{kj}<0} 2a_{kj}\,\tilde{p}_j^{\,c}-\sum_{b_{kj}<0} 2b_{kj}\,\tilde{q}_j^{\,c}\right)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\sum_{a_{kj}\geq 0} a_{kj}\,\underline{q}_j(\alpha)-\sum_{a_{kj}<0} a_{kj}\,\underline{q}_j(\alpha)+\sum_{b_{kj}\geq 0} b_{kj}\,\underline{p}_j(\alpha)-\sum_{b_{kj}<0} b_{kj}\,\underline{p}_j(\alpha)\\&=\underline{v}_k(\alpha)-\left(\sum_{a_{kj}<0} 2a_{kj}\,\tilde{q}_j^{\,c}+\sum_{b_{kj}<0} 2b_{kj}\,\tilde{p}_j^{\,c}\right)\end{aligned} \tag{3.15}$$

Similarly, expanding Eq. (3.13) as above by separating the real and imaginary parts we get Eqs. (3.16) and (3.17) as follows

$$\begin{aligned}&\sum_{a_{kj}\geq 0} a_{kj}\,\underline{p}_j(\alpha)-\sum_{a_{kj}<0} a_{kj}\,\underline{p}_j(\alpha)-\sum_{b_{kj}\geq 0} b_{kj}\,\underline{q}_j(\alpha)+\sum_{b_{kj}<0} b_{kj}\,\underline{q}_j(\alpha)\\&=\left(\sum_{a_{kj}\geq 0} 2a_{kj}\,\tilde{p}_j^{\,c}-\sum_{b_{kj}\geq 0} 2b_{kj}\,\tilde{q}_j^{\,c}\right)-\bar{u}_k(\alpha)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}&\sum_{a_{kj}\geq 0} a_{kj}\,\underline{q}_j(\alpha)-\sum_{a_{kj}<0} a_{kj}\,\underline{q}_j(\alpha)+\sum_{b_{kj}\geq 0} b_{kj}\,\underline{p}_j(\alpha)-\sum_{b_{kj}<0} b_{kj}\,\underline{p}_j(\alpha)\\&=\left(\sum_{a_{kj}\geq 0} 2a_{kj}\,\tilde{q}_j^{\,c}+\sum_{b_{kj}\geq 0} 2b_{kj}\,\tilde{p}_j^{\,c}\right)-\bar{v}_k(\alpha)\end{aligned} \tag{3.17}$$

For proving the theorem it is necessary to show that the Eqs. (3.14) and (3.15) are the same as the Eqs. (3.16) and (3.17) respectively. Comparing Eq. (3.14) with Eq. (3.16) and Eq. (3.15) with Eq. (3.17), it is observed that the left hand sides of the equations are equal. So, for proving the theorem, it is now sufficient to show that the right hand sides of the equations are equal i.e. to show

$$\underline{u}_k(\alpha)-\left(\sum_{a_{kj}<0}2a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}<0}2b_{kj}\tilde{q}_j^{\,c}\right)=\left(\sum_{a_{kj}\geq 0}2a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}\geq 0}2b_{kj}\tilde{q}_j^{\,c}\right)-\bar{u}_k(\alpha) \tag{3.18}$$

and

$$\underline{v}_k(\alpha)-\left(\sum_{a_{kj}<0}2a_{kj}\tilde{q}_j^{\,c}+\sum_{b_{kj}<0}2b_{kj}\tilde{p}_j^{\,c}\right)=\left(\sum_{a_{kj}\geq 0}2a_{kj}\tilde{q}_j^{\,c}+\sum_{b_{kj}\geq 0}2b_{kj}\tilde{p}_j^{\,c}\right)-\bar{v}_k(\alpha) \tag{3.19}$$

Let us first consider the left hand side of Eq. (3.18) as

$$\underline{u}_k(\alpha)-\left(\sum_{a_{kj}<0}2a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}<0}2b_{kj}\tilde{q}_j^{\,c}\right) \tag{3.20}$$

Now, $\underline{u}_k(\alpha)$ can be represented as $2\tilde{u}_k^{\,c}-\bar{u}_k(\alpha)$, where

$$\tilde{u}_k^{\,c}=\sum_{j=1}^{n}a_{kj}\tilde{p}_j^{\,c}-\sum_{j=1}^{n}b_{kj}\tilde{q}_j^{\,c}.$$

Expression of $\tilde{u}_k^{\,c}$ contains all real values of a_{kj} and b_{kj} so, $\tilde{u}_k^{\,c}$ may be expressed as

$$\sum_{a_{kj}\geq 0}a_{kj}\tilde{p}_j^{\,c}+\sum_{a_{kj}<0}a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}\geq 0}b_{kj}\tilde{q}_j^{\,c}-\sum_{b_{kj}<0}b_{kj}\tilde{q}_j^{\,c}.$$

Consequently, we may have

$$\underline{u}_k(\alpha)=2\left(\sum_{a_{kj}\geq 0}a_{kj}\tilde{p}_j^{\,c}+\sum_{a_{kj}<0}a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}\geq 0}b_{kj}\tilde{q}_j^{\,c}-\sum_{b_{kj}<0}b_{kj}\tilde{q}_j^{\,c}\right)-\bar{u}_k(\alpha).$$

Substituting the above value of $\underline{u}_k(\alpha)$ in Eq. (3.20) we have

$$\left\{2\left(\sum_{a_{kj}\geq 0}a_{kj}\tilde{p}_j^{\,c}+\sum_{a_{kj}<0}a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}\geq 0}b_{kj}\tilde{q}_j^{\,c}-\sum_{b_{kj}<0}b_{kj}\tilde{q}_j^{\,c}\right)-\bar{u}_k(\alpha)\right\}$$
$$-\left(\sum_{a_{kj}<0}2a_{kj}\tilde{p}_j^{\,c}-\sum_{b_{kj}<0}2b_{kj}\tilde{q}_j^{\,c}\right)$$

Simplifying the above equation it turns out to be

$$\left(\sum_{a_{kj} \geq 0} 2a_{kj} \tilde{p}_j{}^c - \sum_{b_{kj} \geq 0} 2b_{kj} \tilde{q}_j{}^c \right) - \bar{u}_k(\alpha). \tag{3.21}$$

We may see that Eq. (3.21) is same as that of Eq. (3.20). This implies that Eq. (3.18) holds good. Similarly, Eq. (3.19) can also be shown to hold.
Thus Theorem 3.2 is proved. □

Theorem 3.3 The monotonic decreasing solution vector $\{\bar{z}_j\}$ can be obtained by replacing $\{\underline{z}_j\}$ in terms of $\tilde{z}_j{}^c$ and $\{\bar{z}_j\}$ in Eq. (3.4).

Proof. The proof is straight forward as Theorem 3.1. □

Theorem 3.4 For a fuzzy complex linear system defined by Eq. (3.1), if the lower bound of the fuzzy complex variable $\tilde{z}_j$ (that is $\{\underline{z}_j\}$) is replaced by upper bound with fuzzy complex centre, where the system (3.1) is first solved by total fuzzy centre (that is a crisp complex system), the set of equations thus obtained in terms of left and right bounds of the fuzzy complex interval equations are exactly same.

Proof. The proof is similar to Theorem 3.2. □

- **(Method 2) Addition and subtraction of fuzzy complex numbers based method**

In this section, a new and simple solution method is proposed here for solving general fuzzy complex system of linear equations. In this method, the general system is initially solved by adding and subtracting the left and right bounds of the fuzzy complex unknown and right hand side fuzzy complex vector respectively. Then, obtained solutions are used to get the final solution of the original system. Two theorems are stated and proved below related to Method 2.

Theorem 3.5 If $[C]\{\tilde{Z}\} = \{\tilde{W}\}$, then $\{\underline{Z} + \overline{Z}\}$ is the solution vector of the system $[C]\{\underline{Z} + \overline{Z}\} = \{\underline{W} + \overline{W}\}$.

Proof. Let us consider the expression in the left hand side of the system $[C]\{\underline{Z}+\overline{Z}\}=\{\underline{W}+\overline{W}\}$ as

$$\sum_{j=1}^{n}(a_{kj}+ib_{kj})\{\underline{z}_j(\alpha)+\overline{z}_j(\alpha)\}, \quad \text{for } k=1,2,\cdots,n.$$

This is equivalent to the following

$$\sum_{j=1}^{n}(a_{kj}+ib_{kj})\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)+\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha)\}, \quad \text{for } k=1,2,\cdots,n.$$

The above expression can be written as

$$\sum_{j=1}^{n}a_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)+\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha)\}$$
$$+i\sum_{j=1}^{n}b_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)+\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha)\}.$$

Explicitly, now we have the following,

$$(M+N)++i\,(S+T)$$

where

$$M=\sum_{a_{kj}\ge 0}a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}<0}a_{kj}\,\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha),$$

$$N=\sum_{a_{kj}<0}a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}\ge 0}a_{kj}\,\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha),$$

$$S=\sum_{b_{kj}\ge 0}b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}<0}b_{kj}\,\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha) \text{ and}$$

$$T=\sum_{b_{kj}<0}b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}\ge 0}b_{kj}\,\overline{p}_j(\alpha)+i\,\overline{q}_j(\alpha).$$

Using the values of $\underline{u}_k(\alpha)$, $\overline{u}_k(\alpha)$, $\underline{v}_k(\alpha)$ and $\overline{v}_k(\alpha)$ from Eqs. (3.7) and (3.8) and rearranging, we get,

$$\{\underline{u}_k(\alpha)+i\underline{v}_k(\alpha)\}+\{\overline{u}_k(\alpha)+i\overline{v}_k(\alpha)\}.$$

Following Eq. (3.3b) it gives

$$\{\underline{w}_k(\alpha)+\overline{w}_k(\alpha)\}=\{\underline{W}+\overline{W}\}.$$

Thus we have

$$[C]\{\underline{Z}+\overline{Z}\}=\{\underline{W}+\overline{W}\}$$

and it is concluded that $\{\underline{Z}+\overline{Z}\}$ is the solution vector of the given system. □

Theorem 3.6 If $[C]\{\widetilde{Z}\}=\{\widetilde{W}\}$, then $\{\underline{Z}-\overline{Z}\}$ is the solution vector of the system $[C^*]\{\underline{Z}-\overline{Z}\}=\{\underline{W}-\overline{W}\}$. Here $[C^*]=(c^*{}_{kj})$, where, $c^*{}_{kj}=a^*{}_{kj}+ib^*{}_{kj}$ are the elements of $[C]$ by changing all the negative real and imaginary part of the elements (if those exist) into positive. Here, all $a^*{}_{kj}\geq 0$ and $b^*{}_{kj}\geq 0$.

Proof. Let us first consider the left hand side of the system $[C^*]\{\underline{Z}-\overline{Z}\}=\{\underline{W}-\overline{W}\}$ as

$$\sum_{j=1}^{n}(a^*{}_{kj}+ib^*{}_{kj})\{\underline{z}_j(\alpha)-\overline{z}_j(\alpha)\},\quad \text{for } k=1,2,\cdots,n.$$

Equivalent expression (by using Eq. (3.3a)) may be written as

$$\sum_{j=1}^{n}(a^*{}_{kj}+ib^*{}_{kj})\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)-\overline{p}_j(\alpha)-i\overline{q}_j(\alpha)\},\quad \text{for } k=1,2,\cdots,n.$$

The above equation may now be expressed as

$$\sum_{j=1}^{n}a^*{}_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)-\overline{p}_j(\alpha)-i\overline{q}_j(\alpha)\}$$
$$+i\sum_{j=1}^{n}b^*{}_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)-\overline{p}_j(\alpha)-i\overline{q}_j(\alpha)\}.$$

So we may write

$$\begin{aligned}[C^*]\{\underline{Z}-\overline{Z}\}=&\sum_{j=1}^{n}a^*{}_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)-\overline{p}_j(\alpha)-i\overline{q}_j(\alpha)\}\\&+i\sum_{j=1}^{n}b^*{}_{kj}\{\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)-\overline{p}_j(\alpha)-i\overline{q}_j(\alpha)\}.\end{aligned}\tag{3.22}$$

Next we consider the right hand side of the system $[C^*]\{\underline{Z}-\overline{Z}\}=\{\underline{W}-\overline{W}\}$ and the same may be represented as (using Eq. (3.3b)) $\{\underline{w}_k(\alpha)-\overline{w}_k(\alpha)\}$.

Substituting the expressions of $\underline{w}_k(\alpha)$ and $\overline{w}_k(\alpha)$ from Eq. (3.3b) in the above, one may get

$$\{\underline{u}_k(\alpha)+i\underline{v}_k(\alpha)-\bar{u}_k(\alpha)-i\bar{v}_k(\alpha)\}.$$

Using the values of $\underline{u}_k(\alpha), \bar{u}_k(\alpha), \underline{v}_k(\alpha)$ and $\bar{v}_k(\alpha)$ from Eqs. (3.7) and (3.8) in the above expression we get

$$\left\{\sum_{a_{kj}\ge 0} a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}<0} a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}$$

$$+i\left\{\sum_{b_{kj}\ge 0} b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}<0} b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}$$

$$-\left\{\sum_{a_{kj}<0} a_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}\ge 0} a_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}$$

$$-i\left\{\sum_{b_{kj}<0} b_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}\ge 0} b_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)\right\}.$$

By changing all negative coefficients into positive, the above expression becomes

$$(M^*-N^*)++i\,(S^*-T^*)$$

where

$$M^* = \sum_{a_{kj}\ge 0} a^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{a_{kj}\ge 0} a^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right),$$

$$N^* = \sum_{a_{kj}\ge 0} a^*{}_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)+\sum_{a_{kj}\ge 0} a^*{}_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right),$$

$$S^* = \sum_{b_{kj}\ge 0} b^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)+\sum_{b_{kj}\ge 0} b^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right) \text{ and}$$

$$T^* = \sum_{b_{kj}\ge 0} b^*{}_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right)+\sum_{b_{kj}\ge 0} b^*{}_{kj}\left(\bar{p}_j(\alpha)+i\,\bar{q}_j(\alpha)\right).$$

This is then expressed as

$$\left\{\sum_{j=1}^{n} a^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)-\sum_{j=1}^{n} a^*{}_{kj}\left(\bar{p}_j(\alpha)+i\bar{q}_j(\alpha)\right)\right\}$$

$$+i\left\{\sum_{j=1}^{n} b^*{}_{kj}\left(\underline{p}_j(\alpha)+i\underline{q}_j(\alpha)\right)-\sum_{j=1}^{n} b^*{}_{kj}\left(\bar{p}_j(\alpha)+i\bar{q}_j(\alpha)\right)\right\}.$$

It may now equivalently be written as

$$\sum_{j=1}^{n} a^{*}{}_{kj}\{\underline{p}_{j}(\alpha)+i\underline{q}_{j}(\alpha)-\overline{p}_{j}(\alpha)-i\overline{q}_{j}(\alpha)\}$$
$$+i\sum_{j=1}^{n} b^{*}{}_{kj}\{\underline{p}_{j}(\alpha)+i\underline{q}_{j}(\alpha)-\overline{p}_{j}(\alpha)-i\overline{q}_{j}(\alpha)\}.$$

Therefore, we have

$$\{\underline{W}-\overline{W}\}=\sum_{j=1}^{n} a^{*}{}_{kj}\{\underline{p}_{j}(\alpha)+i\underline{q}_{j}(\alpha)-\overline{p}_{j}(\alpha)-i\overline{q}_{j}(\alpha)\}$$
$$+i\sum_{j=1}^{n} b^{*}{}_{kj}\{\underline{p}_{j}(\alpha)+i\underline{q}_{j}(\alpha)-\overline{p}_{j}(\alpha)-i\overline{q}_{j}(\alpha)\} \tag{3.23}$$

and finally Eqs. (3.22) and (3.23) give

$$[C^{*}]\{\underline{Z}-\overline{Z}\}=\{\underline{W}-\overline{W}\}.$$

Thus $\{\underline{Z}-\overline{Z}\}$ is the solution vector of the given system. □

In the proposed method, Theorems 3.5 and 3.6 are used to obtain the solution vector $\{\underline{Z}+\overline{Z}\}$ and $\{\underline{Z}-\overline{Z}\}$ respectively. Then solving the solution vectors $\{\underline{Z}+\overline{Z}\}$ and $\{\underline{Z}-\overline{Z}\}$ one may obtain $\underline{Z}$ and $\overline{Z}$.

To verify the above proposed methods (Methods 1 and 2 for fuzzy complex system of linear equations) a mathematical example has been taken into consideration as below.

Example 3.1 Let us consider 2×2 complex fuzzy linear system as (Rahgooy et al. 2009)

$$(10-i7.5)\tilde{z}_1-(6-i5)\tilde{z}_2=[4+\alpha,6-\alpha]+i[-1+\alpha,1-\alpha]$$
$$-(6-i5)\tilde{z}_1+(16+i3)\tilde{z}_2=[-2+\alpha,-\alpha]+i[-3+\alpha,-1-\alpha].$$

Solution by Method 1:

For fuzzy complex centre, the above system can be represented as

$$(10-i7.5)\tilde{z}_1{}^{c}-(6-i5)\tilde{z}_2{}^{c}=5$$
$$-(6-i5)\tilde{z}_1{}^{c}+(16+i3)\tilde{z}_2{}^{c}=-1-2i.$$

Solving this system for fuzzy complex centre solution we have $\tilde{z}_1^{\,c} = 0.3452 + i0.108$ and $\tilde{z}_2^{\,c} = 0.0654 - i0.2072$.

Using the fuzzy complex centre, the left bounds of the solution vector can be obtained by Theorem 3.1 as

$$\underline{z}_1 = (0.3164 + 0.0378\alpha) + i(0.0708 + 0.0378\alpha)$$

$$\underline{z}_2 = (0.0347 + 0.0307\alpha) + i(-0.2380 + 0.0307\alpha).$$

Similarly Theorem 3.3 with fuzzy complex centre gives the upper bounds of the solution vector as

$$\bar{z}_1 = (0.3920 - 0.0378\alpha) + i(0.1464 - 0.0378\alpha)$$

$$\bar{z}_2 = (0.0961 - 0.0307\alpha) + i(-0.1765 - 0.0307\alpha).$$

Finally, the solution may be written as

$$\tilde{z}_1 = [0.3164 + 0.0378\alpha, 0.3920 - 0.0378\alpha] + i[0.0708 + 0.0378\alpha, 0.1464 - 0.0378\alpha]$$

and

$$\tilde{z}_2 = [0.0347 + 0.0307\alpha, 0.0961 - 0.0307\alpha] + i[-0.2380 + 0.0307\alpha, -0.1765 - 0.0307\alpha].$$

Solution by Method 2:

Applying Theorems 3.5 and 3.6, we have the solution vectors as

$$\begin{bmatrix} \underline{z}_1 + \bar{z}_1 \\ \underline{z}_2 + \bar{z}_2 \end{bmatrix} = \begin{bmatrix} 0.7084 + i0.2172 \\ 0.1308 - i0.4144 \end{bmatrix}$$

and

$$\begin{bmatrix} \underline{z}_1 - \bar{z}_1 \\ \underline{z}_2 - \bar{z}_2 \end{bmatrix} = \begin{bmatrix} (-0.0756 + 0.0756\alpha) + i(-0.0756 + 0.0756\alpha) \\ (-0.0614 + 0.0614\alpha) + i(-0.0614 + 0.0614\alpha) \end{bmatrix}$$

respectively.

Solving the corresponding vector elements we have

$$\underline{z}_1 = (0.3164 + 0.0378\alpha) + i(0.0708 + 0.0378\alpha)$$

$$\bar{z}_1 = (0.3920 - 0.0378\alpha) + i(0.1464 - 0.0378\alpha)$$

$$\underline{z}_2 = (0.0347 + 0.0307\alpha) + i(-0.2380 + 0.0307\alpha)$$

$$\bar{z}_2 = (0.0961 - 0.0307\alpha) + i(-0.1765 - 0.0307\alpha).$$

As such, the solution may finally be written as

$$\tilde{z}_1 = [0.3164 + 0.0378\alpha, 0.3920 - 0.0378\alpha] + i[0.0708 + 0.0378\alpha, 0.1464 - 0.0378\alpha]$$

and

$$\tilde{z}_2 = [0.0347 + 0.0307\alpha, 0.0961 - 0.0307\alpha] + i[-0.2380 + 0.0307\alpha, -0.1765 - 0.0307\alpha].$$

Obtained results are compared with Rahgooy et al. (2009) and found to be in good agreement.

3.1.2. Fuzzy real system of linear equations

The $n \times n$ fuzzy system of linear equations may be written as

$$\begin{aligned} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \cdots + a_{1n}\tilde{x}_n &= \tilde{b}_1 \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \cdots + a_{2n}\tilde{x}_n &= \tilde{b}_2 \\ &\vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \cdots + a_{nn}\tilde{x}_n &= \tilde{b}_n . \end{aligned} \tag{3.24}$$

In matrix notation, the above system may be written as $[A]\{\tilde{X}\} = \{\tilde{b}\}$, where the coefficient matrix $[A] = (a_{kj})$, $1 \le k \le n$, $j \le n$ is a crisp real $n \times n$ matrix, $\{\tilde{b}\} = \{\tilde{b}_k\}$, $1 \le k$ is a column vector of fuzzy number and $\{\tilde{X}\} = \{\tilde{x}_j\}$ is the vector of fuzzy unknown.

The above system $[A]\{\tilde{X}\} = \{\tilde{b}\}$ can be written as

$$\sum_{j=1}^{n} a_{kj}\tilde{x}_j = \tilde{b}_k, \quad \text{for } k = 1, 2, \cdots, n. \tag{3.25}$$

As per the $\alpha -$cut form, we may write the real fuzzy unknown and the right hand real fuzzy number vector as $\tilde{x}_j = \tilde{x}_j(\alpha) = [\underline{x}_j(\alpha), \overline{x}_j(\alpha)]$ and $\tilde{b}_k = \tilde{b}_k(\alpha) = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]$.

Substituting these expressions in Eq. (3.25), we have

$$\sum_{j=1}^{n} a_{kj}[\underline{x}_j(\alpha), \overline{x}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)], \quad \text{for } k = 1, 2, \cdots, n. \tag{3.26}$$

Eq. (3.26) can equivalently be written as the following two equations

$$\sum_{a_{kj} \ge 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj} < 0} a_{kj}\overline{x}_j(\alpha) = \underline{b}_k(\alpha) \tag{3.27}$$

and

$$\sum_{a_{kj}\geq 0} a_{kj}\,\overline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\,\underline{x}_j(\alpha) = \overline{b}_k(\alpha) \tag{3.28}$$

In the following section, new methods are proposed for solving the fuzzy system of linear equations as defined in Eq. (3.24).

- **(Method 1) Method based on fuzzy left and right spread of fuzzy number with core solution**

Here, a new method has been proposed to solve fuzzy system of linear equations. It is worth mentioning that, the method is only valid for triangular and Gaussian fuzzy number matrices. First, the system has been solved for the core solution and then it is used to obtain left and right spreads to have the final solution.

Now, 1 – cut (for $\alpha = 1$) of the system (3.26) can be written as

$$\sum_{j=1}^{n} a_{kj}\,[\underline{x}_j(1), \overline{x}_j(1)] = [\underline{b}_k(1), \overline{b}_k(1)]. \tag{3.29}$$

One may obtain $\tilde{x}_j(1)$ by solving Eq. (3.25). Here, $\tilde{x}_j(1)$ is known as the core of $\tilde{x}_j(\alpha)$. Eq. (3.29) in general (for triangular or Gaussian fuzzy number matrix) converted to a crisp system for the core solution. So we have $\underline{b}_k(1) = \overline{b}_k(1)$ and $\underline{x}_j(1) = \overline{x}_j(1) = \tilde{x}_j(1)$.

Solution vector $\tilde{x}_j(\alpha)$ can be represented for triangular or Gaussian fuzzy number as

$$[\tilde{x}_j(1) - \delta(\alpha), \tilde{x}_j(1) + \gamma(\alpha)]$$

where $\delta(\alpha)$ and $\gamma(\alpha)$ are left and right spreads of the solution vector for $\alpha \in [0,1]$, which are to be determined.

As such, Eq. (3.24) can be expressed for triangular or Gaussian fuzzy number as

$$\sum_{j=1}^{n} a_{kj}\,[\tilde{x}_j(1) - \delta(\alpha), \tilde{x}_j(1) + \gamma(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]. \tag{3.30}$$

Hence, Eq. (3.30) may equivalently be written as the following two equations

$$\sum_{a_{kj}\geq 0} a_{kj}\left\{\tilde{x}_j(1) - \delta(\alpha)\right\} + \sum_{a_{kj}<0} a_{kj}\left\{\tilde{x}_j(1) + \gamma(\alpha)\right\} = \underline{b}_k(\alpha) \tag{3.31}$$

and

$$\sum_{a_{kj}\ge 0} a_{kj}\left\{\tilde{x}_j(1)+\gamma(\alpha)\right\}+\sum_{a_{kj}<0} a_{kj}\left\{\tilde{x}_j(1)-\delta(\alpha)\right\}=\bar{b}_k(\alpha) \tag{3.32}$$

Finally, Eqs. (3.31) and (3.32) are solved to find the left and right spread viz. $\delta(\alpha)$ and $\gamma(\alpha)$ of the solution vector. Hence, for triangular or Gaussian fuzzy number system, the solution vector can be written as $[\tilde{x}_j(1)-\delta(\alpha), \tilde{x}_j(1)+\gamma(\alpha)]$.

This method is not in general for all fuzzy numbers. Therefore, to overcome the limitations of the above method, the following methods are developed.

- **(Method 2) Fuzzy centre and radius based method**

First, the system is solved in terms of fuzzy centre then this solution is used to get the radius of the final solution. Related theorems are stated and proven below.

Theorem 3.7 Let $\tilde{X}$ be a fuzzy solution of fuzzy system of linear equations (viz. Eq. (3.24)) where $[A]$ is the crisp real $n\times n$ nonsingular matrix and $\{\tilde{b}\}$ is a fuzzy number vector. Then $\{X^c\}$ is the solution vector of the crisp system

$$[A]\{X^c\}=\{b^c\}$$

where $\{X^c\}=\{(\underline{x}_j(\alpha)+\overline{x}_j(\alpha))/2\}$ and $\{b^c\}=\{(\underline{b}_k(\alpha)+\overline{b}_k(\alpha))/2\}$.

Proof. Let us now first consider the left hand side of the system $[A]\{X^c\}=\{b^c\}$.

Hence, one may write $[A]\{X^c\}$ as

$$\sum_{j=1}^{n} a_{kj}\{(\underline{x}_j(\alpha)+\overline{x}_j(\alpha))/2\},\ \text{for } k=1,2,\cdots,n.$$

This can be written as

$$\sum_{a_{kj}\ge 0} a_{kj}\{(\underline{x}_j(\alpha)+\overline{x}_j(\alpha))/2\}+\sum_{a_{kj}<0} a_{kj}\{(\underline{x}_j(\alpha)+\overline{x}_j(\alpha))/2\}.$$

It is equivalent to

$$\sum_{a_{kj}\ge 0} a_{kj}\frac{\underline{x}_j(\alpha)}{2}+\sum_{a_{kj}\ge 0} a_{kj}\frac{\overline{x}_j(\alpha)}{2}+\sum_{a_{kj}<0} a_{kj}\frac{\underline{x}_j(\alpha)}{2}+\sum_{a_{kj}<0} a_{kj}\frac{\overline{x}_j(\alpha)}{2}. \tag{3.33}$$

By combining first with fourth term and second with third term in the above equation and using Eqs. (3.27) and (3.28) we get

$$\{(\underline{b}_k(\alpha)+\overline{b}_k(\alpha))/2\}=\{b^c\}.$$

Thus, we have $[A]\{X^c\}=\{b^c\}$. which proves that $\{X^c\}$ is the solution vector of the system $[A]\{X^c\}=\{b^c\}$. □

Unknown fuzzy number vector $\{\tilde{x}_j\}$ and the right hand side fuzzy number vector $\{\tilde{b}_k\}$ of the system (3.25), using fuzzy centre and radius can be represented respectively as

$$\{\tilde{x}_j\}=\{[\underline{x}_j(\alpha),\overline{x}_j(\alpha)]\}=\{[x_j^c-\Delta x_j,\,x_j^c+\Delta x_j]\}$$

and

$$\{\tilde{b}_k\}=\{[\underline{b}_k(\alpha),\overline{b}_k(\alpha)]\}=\{[b_k^c-\Delta b_k,\,b_k^c+\Delta b_k]\}.$$

Substituting the expression for left and right bound of $\{\tilde{x}_j\}$ in Eqs. (3.27) and (3.28) we have

$$\sum_{a_{kj}\geq 0} a_{kj}(x_j^c-\Delta x_j)+\sum_{a_{kj}<0} a_{kj}(x_j^c+\Delta x_j)=\underline{b}_k(\alpha) \tag{3.34}$$

and

$$\sum_{a_{kj}\geq 0} a_{kj}(x_j^c+\Delta x_j)+\sum_{a_{kj}<0} a_{kj}(x_j^c-\Delta x_j)=\overline{b}_k(\alpha). \tag{3.35}$$

Let us now substitute the value of $\{x_j^c\}$ of Theorem 3.7 in Eqs. (3.34) and (3.35) and solve any one of the above system (viz. Eq. (3.34) or (3.35)) to obtain Δx_j. This gives the solution $\tilde{x}_j=[x_j^c-\Delta x_j,\,x_j^c+\Delta x_j]$. It is important now to show that the solution vector $\{\Delta x_j\}$ obtained by Eq. (3.34) or (3.35) is same. This is illustrated in the following theorem.

Theorem 3.8 Crisp linear systems (viz. Eqs. 3.34 and 3.35) give exactly same radius when the left and right bounds of the fuzzy variables are replaced by centre solution of Eq. (3.25).

Proof. Let us first consider Eq. (3.34)

$$\sum_{a_{kj}\geq 0} a_{kj}(x_j^c - \Delta x_j) + \sum_{a_{kj}<0} a_{kj}(x_j^c + \Delta x_j) = \underline{b}_k(\alpha).$$

The above expression may equivalently be written as

$$\sum_{a_{kj}\geq 0} a_{kj}x_j^c + \sum_{a_{kj}<0} a_{kj}x_j^c - \sum_{a_{kj}\geq 0} a_{kj}\Delta x_j + \sum_{a_{kj}<0} a_{kj}\Delta x_j = b_k^c - \Delta b_k. \tag{3.36}$$

The central fuzzy system $[A]\{X^c\} = \{b^c\}$ is written as

$$\sum_{j=1}^{n} a_{kj}x_j^c = b_k^c.$$

It may now be equivalently expressed as

$$\sum_{a_{kj}\geq 0} a_{kj}x_j^c + \sum_{a_{kj}<0} a_{kj}x_j^c = b_k^c.$$

Substituting this in Eq. (3.32) one may have

$$-\sum_{a_{kj}\geq 0} a_{kj}\Delta x_j + \sum_{a_{kj}<0} a_{kj}\Delta x_j = -\Delta b_k.$$

This can be represented as

$$\sum_{a_{kj}\geq 0} a_{kj}\Delta x_j - \sum_{a_{kj}<0} a_{kj}\Delta x_j = \Delta b_k. \tag{3.37}$$

From this, one may conclude that Eq. (3.34) is equivalent to Eq. (3.37). Similarly, it can also be proved that Eq. (3.35) is also equivalent to Eq. (3.37). Hence, it may be concluded that Eqs. (3.34) and (3.35) are exactly same. Thus Theorem 3.8 is proved. □

- **(Method 3) Fuzzy addition and subtraction based method**

In this method, the coefficient matrix has been considered as real crisp, whereas the unknown variable vector and right hand side vector are considered as fuzzy. The general system is initially solved by adding and subtracting the left and right bounds of the vectors respectively. Conditions for consistent and computational complexity analysis are also presented.

For this case, let us now define

$$\tilde{X} = [\underline{X}, \overline{X}] = [\underline{x}_j(\alpha), \overline{x}_j(\alpha)] \text{ and } \tilde{b} = [\underline{b}, \overline{b}] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)].$$

In this regard, related theorems are now stated and proved below.

Theorem 3.9 If $[A]\{\tilde{X}\}=\{\tilde{b}\}$, then $\{\underline{X}+\overline{X}\}$ is the solution vector of the system $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$.

Proof. Let us consider the right hand side of the system $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$ as

$$[A]\{\underline{X}+\overline{X}\}.$$

One may write $[A]\{\underline{X}+\overline{X}\}$ as

$$\sum_{j=1}^{n} a_{kj}[\underline{x}_j(\alpha)+\overline{x}_j(\alpha)] \text{ for } k=1,2,\cdots,n.$$

The above can be represented as

$$\sum_{a_{kj}\geq 0} a_{kj}\{\underline{x}_j(\alpha)+\overline{x}_j(\alpha)\}+\sum_{a_{kj}<0} a_{kj}\{\underline{x}_j(\alpha)+\overline{x}_j(\alpha)\}.$$

This is equivalent to

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)+\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)+\sum_{a_{kj}<0} a_{kj}\underline{x}_j(\alpha)+\sum_{a_{kj}<0} a_{kj}\overline{x}_j(\alpha).$$

Using Eqs. (3. 27) and (3.28), the above expression can be written as

$$[\underline{b}_k(\alpha)+\overline{b}_k(\alpha)]=[\underline{b}+\overline{b}].$$

Accordingly, one may conclude $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$. This proves $\{\underline{X}+\overline{X}\}$ is a solution vector of the system $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$. □

Theorem 3.10 If $[A]\{\tilde{X}\}=\{\tilde{b}\}$, then $\{\underline{X}-\overline{X}\}$ is the solution vector of the system $[A^*]\{\underline{X}-\overline{X}\}=\{\underline{b}-\overline{b}\}$, where $[A^*]=(a^*_{kj})$. Here, $a^*_{kj}\geq 0$ are the elements of $[A]$ but changing all negative elements (if those exist) into positive.

Proof. Let us consider the expression $[A^*]\{\underline{X}-\overline{X}\}$, which may be written as

$$\sum_{j=1}^{n} a^*_{kj}[\underline{x}_j(\alpha)-\overline{x}_j(\alpha)] \text{ for } k=1,2,\cdots,n.$$

This can be expressed as $\sum_{j=1}^{n} a^*_{kj}\underline{x}_j(\alpha)-\sum_{j=1}^{n} a^*_{kj}\overline{x}_j(\alpha)$.

One may get then

$$[A^*]\{\underline{X}-\overline{X}\}=\sum_{j=1}^{n} a_{kj}^* \underline{x}_j(\alpha)-\sum_{j=1}^{n} a_{kj}^* \overline{x}_j(\alpha). \tag{3.38}$$

By subtracting Eq. (3.28) from Eq. (3.27), it gives

$$\{\underline{b}_k(\alpha)-\overline{b}_k(\alpha)\}=\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)+\sum_{a_{kj}<0} a_{kj}\overline{x}_j(\alpha)-\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)-\sum_{a_{kj}<0} a_{kj}\underline{x}_j(\alpha)$$

$$=\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)-\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)-\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)+\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)$$

$$=\left(\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)+\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha)\right)-\left(\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)+\sum_{a_{kj}\geq 0} a_{kj}\overline{x}_j(\alpha)\right)$$

Here, all $a_{kj}\geq 0$, so one may write the above expression as

$$\{\underline{b}_k(\alpha)-\overline{b}_k(\alpha)\}=\sum_{j=1}^{n} a_{kj}^* \underline{x}_j(\alpha)-\sum_{j=1}^{n} a_{kj}^* \overline{x}_j(\alpha). \tag{3.39}$$

From Eqs. (3.38) and (3.39) we have

$$[A^*]\{\underline{X}-\overline{X}\}=\{\underline{b}_k(\alpha)-\overline{b}_k(\alpha)\}=\{\underline{b}-\overline{b}\}.$$

Hence, it can be concluded that $\{\underline{X}-\overline{X}\}$ is the solution vector of the discussed system as mentioned. Thus the theorem is proved. □

It may be noted that, Theorems 1 and 2 may be utilised to obtain the solution vector $\{\underline{X}+\overline{X}\}$ and $\{\underline{X}-\overline{X}\}$ respectively. Then solving the solution vectors $\{\underline{X}+\overline{X}\}$ and $\{\underline{X}-\overline{X}\}$ one, may obtain $\{\underline{X}\}$ and $\{\overline{X}\}$.

Here, it is interesting to note that if a fuzzy system of linear equations has no solution or inconsistent, then the converted crisp systems also have no solution or inconsistent and vice versa. One related theorem is stated and proved below.

Theorem 3.11 Fuzzy system of linear equations $[A]\{\tilde{X}\}=\{\tilde{b}\}$ is inconsistent iff $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$ and $[A^*]\{\underline{X}-\overline{X}\}=\{\underline{b}-\overline{b}\}$ are inconsistent.

Proof. First we will prove the necessary part of the theorem. Let us consider the system $[A]\{\tilde{X}\} = \{\tilde{b}\}$ be consistent. Using fuzzy centre, $[A]\{\tilde{X}\} = \{\tilde{b}\}$ can be represented as

$$[A]\{X^c\} = \{b^c\}, \tag{3.40}$$

where $X^c = \dfrac{\underline{X} + \overline{X}}{2}$ and $b^c = \dfrac{\underline{b} + \overline{b}}{2}$.

One may conclude that Eq. (3.40) is also inconsistent because $\{X^c\}$ is the middle point of $\{\tilde{X}\}$. From Eq. (3.40), we may conclude that $[A]\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$ is inconsistent.

Next, the system $[A]\{\tilde{X}\} = \{\tilde{b}\}$ can be represented by radius of the fuzzy solution as

$$[A^*]\{\Delta X\} = \{\Delta b\}, \text{ where } \{\Delta X\} = \left\{\frac{\overline{X} - \underline{X}}{2}\right\} \text{ and } \{\Delta b\} = \left\{\frac{\overline{b} - \underline{b}}{2}\right\}.$$

So $[A^*]\left\{\dfrac{\overline{X} - \underline{X}}{2}\right\} = \left\{\dfrac{\overline{b} - \underline{b}}{2}\right\}$ is inconsistent, because $\{\Delta X\}$ is the radius of the solution. From this, we conclude that $[A^*]\{\underline{X} - \overline{X}\} = \{\underline{b} - \overline{b}\}$ is inconsistent. Hence, necessary part holds good.

Conversely, we will proceed for the proof of sufficient part of the theorem. Let us now consider $[A]\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$ and $[A^*]\{\underline{X} - \overline{X}\} = \{\underline{b} - \overline{b}\}$ are inconsistent. Now, we have to prove $[A]\{\tilde{X}\} = \{\tilde{b}\}$ is inconsistent.

$[A]\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$ can be equivalently written as $[A]\left\{\dfrac{\underline{X} + \overline{X}}{2}\right\} = \left\{\dfrac{\underline{b} + \overline{b}}{2}\right\}$ and accordingly we have $[A]\{X^c\} = \{b^c\}$. Hence, the centre solution of $[A]\{\tilde{X}\} = \{\tilde{b}\}$ is inconsistent as this is the equivalent from of $[A]\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$. According to Method 2 of Section 3.2.1, solution vector $[\underline{X}, \overline{X}]$ can be represented as $[X^c - \Delta X, X^c + \Delta X]$. As the centre solution is inconsistent, hence $[A]\{\tilde{X}\} = \{\tilde{b}\}$ is inconsistent because one may not proceed for $[X^c - \Delta X, X^c + \Delta X]$. This proves the sufficient part. □

Similarly, one may have the condition for consistency of the fuzzy linear system of equations. The related theorem may be proposed as below.

Theorem 3.12 The fuzzy system of linear equations $[A]\{\tilde{X}\}=\{\tilde{b}\}$ is consistent iff $[A]\{\underline{X}+\overline{X}\}=\{\underline{b}+\overline{b}\}$ and $[A^*]\{\underline{X}-\overline{X}\}=\{\underline{b}-\overline{b}\}$ are consistent.

Proof. The proof is straight forward as Theorem 3.11. □

Computational complexities are investigated for this method. Similar interpretations may be made for other methods too. For computational complexity, the present method is compared with standard method of Friedman et al. (1998). First study related to fuzzy system of linear equations has been done by Friedman et al. (1998). In Friedman et al. (1998), they used an embedding approach where $n \times n$ fuzzy system of linear equations is converted to $2n \times 2n$ crisp system of linear equations. Then solving the $2n \times 2n$ crisp system of linear equations, they found the final solution of fuzzy system of linear equations. For large systems, one may use Gaussian elimination method (or any other) for the solution. If it is Gaussian elimination, then the total number of operations needed to solve $2n \times 2n$ crisp system of linear equations is equivalent to the aggregation of operation count for forward and backward substitution. According to Kreyszig (2004), the total number of operations involved in the Gaussian elimination procedure is equivalent to $\frac{16}{3}n^3+4n^2$.

But in the present procedure, first the system is solved by Theorems 3.9 and 3.10. That means, first we have to solve two $n \times n$ crisp systems of linear equations. So for this, the number of operations involved to find the numerical solution using Gaussian elimination procedure is $\frac{4}{3}n^3+2n^2$. After getting $\{\underline{X}+\overline{X}\}$ and $\{\underline{X}-\overline{X}\}$ from Theorems 3.9 and 3.10, again we have to solve n times 2×2 system of linear equations to find the corresponding fuzzy solution. So here, the number of operations involved is equal to $\frac{20}{3}n$. Hence for the present method, the total number of operations involved is $\left(\frac{4}{3}n^3+2n^2\right)+\frac{20}{3}n$.

To have a feeling of the number of operations, let us consider a 100×100 fuzzy system of linear equations. The operation count for both the methods are computed by following the procedure as above and are cited in Fig. 3.1. One may notice from Fig. 3.1 that for large systems the number of operations count for the present method is less than the method of Friedman et al. (1998). So we may conclude that the present method may be computationally efficient.

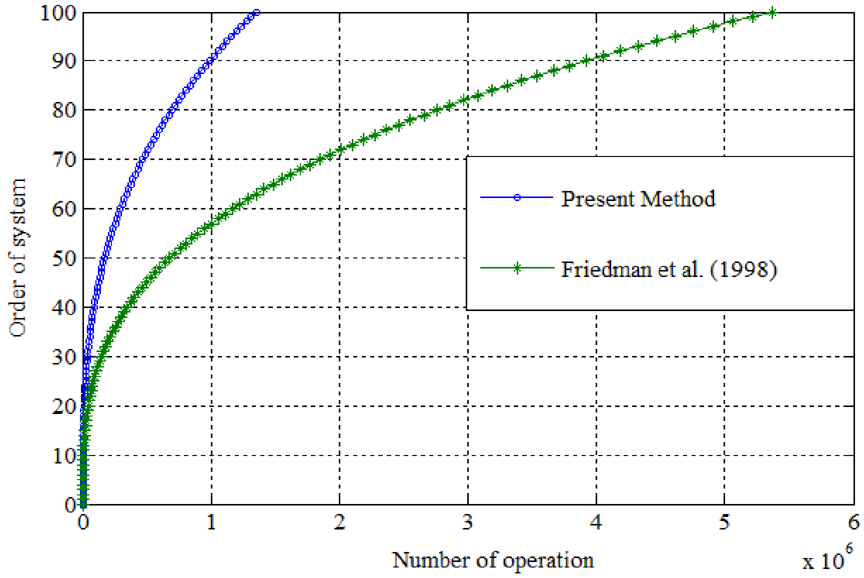


Fig. 3.1 Comparison plot of operations count between the present method (Method 3) and Friedman et al. (1998)

Remark 3.1 In the following method (Method 4), centre and radius of fuzzy numbers are computed separately to obtain the final solution. This method is similar to the above method. Related theorem is stated and proved below.

- **(Method 4) Another method based on fuzzy centre and radius**

In this section, first we have to compute the fuzzy centre solution $\{X^c\}$ using the proposed Theorem 3.7. Next, using the following theorem, one may obtain the radius $\{\Delta X\}$ of fuzzy solution directly.

Theorem 3.13 If $[A]\{\tilde{X}\}=\{\tilde{b}\}$ then $\{\Delta X\}$ is the solution vector of the system

$$[A]\{\Delta X\}=\{\Delta b\}$$

where $\Delta X=\dfrac{\overline{x}_j(\alpha)-\underline{x}_j(\alpha)}{2}$, $\Delta b=\dfrac{\overline{b}_k(\alpha)-\underline{b}_k(\alpha)}{2}$ and $[A^*]=(a_{kj}^*)$. Here $a_{kj}^*\geq 0$ are the elements of $[A]$ but changing all negative elements (if those exist) into positive.

Proof. The proof of the theorem is equivalent to Theorem 3.10. □

Hence using the centre and radius of the fuzzy solution, we may write the final solution as

$$[x_j^c - \Delta x_j, x_j^c + \Delta x_j].$$

Remark 3.2 Next, a new methodology (Method 5) is presented using the concept of fuzzy addition. Hence, related theorems are stated and proved accordingly for the completeness of the present study as follows.

- **(Method 5) Fuzzy addition based method**

Using Theorem 3.9, one can find the solution vector $\{\underline{X} + \overline{X}\}$ of the system $[A]\{\underline{X} + \overline{X}\} = \{\underline{b} + \overline{b}\}$. Let us consider the solution vector $\{\underline{X} + \overline{X}\}$ as $\{P\} = \{P_j(\alpha)\}$. So, this can be written as $\{\underline{x}_j(\alpha) + \overline{x}_j(\alpha)\} = \{P_j(\alpha)\}$. The lower and upper bounds of the solution vector may be obtained as

$$\{\underline{x}_j(\alpha)\} = \{P_j(\alpha) - \overline{x}_j(\alpha)\} \text{ and } \{\overline{x}_j(\alpha)\} = \{P_j(\alpha) - \underline{x}_j(\alpha)\}$$

respectively.

Theorem 3.14 The monotonic increasing solution vector $\{\underline{x}_j(\alpha)\}$ can be obtained by replacing $\{\overline{x}_j(\alpha)\}$ in terms of $\{\underline{x}_j(\alpha)\}$ and $\{P_j(\alpha)\}$ in Eq. (3.26).

Proof. We get two crisp systems (3.27) and (3.28) from Eq. (3.26). Substituting $\{\overline{x}_j(\alpha)\} = \{P_j(\alpha) - \underline{x}_j(\alpha)\}$ in any one of these equations and solving we may find $\{\underline{x}_j(\alpha)\}$. So this proves the theorem. □

Theorem 3.15 Crisp linear systems (viz. Eqs. 3.27 and 3.28) give exactly the same $\{\underline{x}_j(\alpha)\}$ when the upper bound of the fuzzy variable $\{\overline{x}_j(\alpha)\}$ is replaced by $\{P_j(\alpha) - \underline{x}_j(\alpha)\}$ in Eq. (3.26).

Proof. Let us first consider Eq. (3.27),

$$\sum_{a_{kj} \geq 0} a_{kj}\,\underline{x}_j(\alpha) + \sum_{a_{kj} < 0} a_{kj}\,\overline{x}_j(\alpha) = \underline{b}_k(\alpha).$$

Substituting $\{\bar{x}_j(\alpha)\} = \{P_j(\alpha) - \underline{x}_j(\alpha)\}$ in the above equation we have

$$\sum_{a_{kj} \geq 0} a_{kj} \underline{x}_j(\alpha) + \sum_{a_{kj} < 0} a_{kj} \{P_j(\alpha) - \underline{x}_j(\alpha)\} = \underline{b}_k(\alpha).$$

The above expression may equivalently be written as

$$\sum_{a_{kj} \geq 0} a_{kj} \underline{x}_j(\alpha) + \sum_{a_{kj} < 0} a_{kj} \{P_j(\alpha)\} - \sum_{a_{kj} < 0} a_{kj} \{\underline{x}_j(\alpha)\} = \underline{b}_k(\alpha). \qquad (3.41)$$

Which may be expressed as

$$\sum_{a_{kj} \geq 0} a_{kj} \underline{x}_j(\alpha) + \sum_{a_{kj} < 0} a_{kj} \{P_j(\alpha)\} = \underline{b}_k(\alpha). \qquad (3.42)$$

But, the fuzzy system $[A]\{\underline{X} + \bar{X}\} = \{\underline{b} + \bar{b}\}$ gives

$$\sum_{j=1}^{n} a_{kj} \{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}.$$

This can be expressed as

$$\sum_{a_{kj} \geq 0} a_{kj} \{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} + \sum_{a_{kj} < 0} a_{kj} \{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}.$$

Introducing $P_j(\alpha)$ we get

$$\sum_{a_{kj} \geq 0} a_{kj} \{P_j(\alpha)\} + \sum_{a_{kj} < 0} a_{kj} \{P_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}.$$

This is similar to

$$\sum_{a_{kj} < 0} a_{kj} \{P_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\} - \sum_{a_{kj} \geq 0} a_{kj} \{P_j(\alpha)\}.$$

Substituting the above in Eq. (3.42) we have

$$\sum_{a_{kj} \geq 0} a_{kj} \underline{x}_j(\alpha) = \sum_{a_{kj} \geq 0} a_{kj} \{P_j(\alpha)\} - \bar{b}_k(\alpha). \qquad (3.43)$$

One may see that Eq. (3.27) is equivalent to Eq. (3.43). Similarly, one may prove that Eq. (3.28) is equivalent to Eq. (3.43). Hence, it may be concluded that Eqs. (3.27) and (3.28) are exactly same. Thus Theorem 3.15 is proved. □

Theorem 3.16 The monotonic decreasing solution vector $\{\bar{x}_j(\alpha)\}$ can be obtained by replacing $\{\underline{x}_j(\alpha)\}$ in terms of $\{\bar{x}_j(\alpha)\}$ and $\{P_j(\alpha)\}$ in Eq. (3.26).

Proof. The proof is straight forward as Theorem 3.14. □

Theorem 3.17 Crisp linear systems (viz. Eqs. 3.27 and 3.28) give exactly same $\{\bar{x}_j(\alpha)\}$ when the lower bound of the fuzzy variable $\{\underline{x}_j(\alpha)\}$ is replaced by $\{P_j(\alpha)\}-\{\bar{x}_j(\alpha)\}$.

Proof. The proof is straight forward as Theorem 3.15. □

To validate the proposed methods (Methods 1 to 5) for fuzzy real system of linear equations, a numerical problem has been solved in Example 3.2.

Example 3.2 Let us consider a 3×3 real Gaussian fuzzy system as (Garg and Singh 2008)

$$4\tilde{x}_1+2\tilde{x}_2-\tilde{x}_3=[-20-\sqrt{-50\log_e\alpha},-20+\sqrt{-50\log_e\alpha}]$$
$$2\tilde{x}_1+7\tilde{x}_2+6\tilde{x}_3=[16-\sqrt{-100\log_e\alpha},16+\sqrt{-100\log_e\alpha}]$$
$$-\tilde{x}_1+6\tilde{x}_2+10\tilde{x}_3=[44-\sqrt{-100\log_e\alpha},44+\sqrt{-100\log_e\alpha}].$$

Using the proposed Methods 1 to 5 for real fuzzy system of linear equations we obtain the solution as

$$\tilde{x}_1=\left[-4-\frac{170}{113}\sqrt{-2\log_e\alpha}+\frac{90}{113}\sqrt{-\log_e\alpha},\ -4+\frac{170}{113}\sqrt{-2\log_e\alpha}-\frac{90}{113}\sqrt{-\log_e\alpha}\right],$$
$$\tilde{x}_2=\left[\frac{70}{113}\sqrt{-2\log_e\alpha}-\frac{170}{113}\sqrt{-\log_e\alpha},\ -\frac{70}{113}\sqrt{-2\log_e\alpha}+\frac{170}{113}\sqrt{-\log_e\alpha}\right]\text{ and}$$
$$\tilde{x}_3=\left[4-\frac{25}{113}\sqrt{-2\log_e\alpha}-\frac{20}{113}\sqrt{-\log_e\alpha},4+\frac{25}{113}\sqrt{-2\log_e\alpha}+\frac{20}{113}\sqrt{-\log_e\alpha}\right].$$

Obtained results are compared with Garg and Singh (2008) and found to be in good agreement. The results are also depicted in Table 3.1. It may be noted that the results obtained by the proposed methods exactly satisfy the fuzzy system. Whereas results obtained by Garg and Singh (2008) are a little bit far form the exact solution.

Table 3.1 Solution of Example 3.2

Solution bound	Present method (s)	Garg and Singh (2008)
$\underline{x}_1(\alpha)$	$-4-\frac{170}{113}\sqrt{-2\log_e\alpha}+\frac{90}{113}\sqrt{-\log_e\alpha}$	$-4-\frac{3}{10}\sqrt{-50\log_e\alpha}+\frac{7}{88}\sqrt{-100\log_e\alpha}$
$\overline{x}_1(\alpha)$	$-4+\frac{170}{113}\sqrt{-2\log_e\alpha}-\frac{90}{113}\sqrt{-\log_e\alpha}$	$-4+\frac{3}{10}\sqrt{-50\log_e\alpha}-\frac{7}{88}\sqrt{-100\log_e\alpha}$
$\underline{x}_2(\alpha)$	$\frac{70}{113}\sqrt{-2\log_e\alpha}-\frac{170}{113}\sqrt{-\log_e\alpha}$	$\frac{1}{8}\sqrt{-50\log_e\alpha}-\frac{3}{20}\sqrt{-100\log_e\alpha}$
$\overline{x}_2(\alpha)$	$-\frac{70}{113}\sqrt{-2\log_e\alpha}+\frac{170}{113}\sqrt{-\log_e\alpha}$	$-\frac{1}{8}\sqrt{-50\log_e\alpha}+\frac{14}{93}\sqrt{-100\log_e\alpha}$
$\underline{x}_3(\alpha)$	$4-\frac{25}{113}\sqrt{-2\log_e\alpha}-\frac{20}{113}\sqrt{-\log_e\alpha}$	$4-\frac{2}{45}\sqrt{-50\log_e\alpha}-\frac{1}{56}\sqrt{-100\log_e\alpha}$
$\overline{x}_3(\alpha)$	$4+\frac{25}{113}\sqrt{-2\log_e\alpha}+\frac{20}{113}\sqrt{-\log_e\alpha}$	$4+\frac{3}{68}\sqrt{-50\log_e\alpha}+\frac{1}{56}\sqrt{-100\log_e\alpha}$

One may note that all the proposed methods give exactly same results but the way these handle the problem is different with different computational efforts.

3.2. Fully Fuzzy System of Linear Equations (FFSLE)

Depending on the nature of the fuzzy number viz. real or complex fuzzy number, FFSLE can be defined as

- Fully fuzzy complex system of linear equations
- Fully fuzzy real system of linear equations.

But here, we have discussed only the fully fuzzy real system of linear equations in the following sections.

3.2.1. Fully fuzzy real system of linear equations

The $n \times n$ fully fuzzy real system of linear equations may be written as

$$\begin{aligned}
\tilde{a}_{11}\tilde{x}_1+\tilde{a}_{12}\tilde{x}_2+\cdots+\tilde{a}_{1n}\tilde{x}_n&=\tilde{b}_1,\\
\tilde{a}_{21}\tilde{x}_1+\tilde{a}_{22}\tilde{x}_2+\cdots+\tilde{a}_{2n}\tilde{x}_n&=\tilde{b}_2,\\
&\vdots\\
\tilde{a}_{n1}\tilde{x}_1+\tilde{a}_{n2}\tilde{x}_2+\cdots+\tilde{a}_{nn}\tilde{x}_n&=\tilde{b}_n.
\end{aligned} \tag{3.44}$$

In matrix notation, the above system may be presented as $[\tilde{A}]\{\tilde{X}\}=\{\tilde{b}\}$, where the coefficient matrix $[\tilde{A}]=(\tilde{a}_{kj}), 1\le k,\ j\le n$ is a fuzzy $n\times n$ matrix, $\{\tilde{b}\}=\{\tilde{b}_k\}, 1\le k$ is a column vector of fuzzy numbers and $\{\tilde{X}\}=\{\tilde{x}_j\}$ is the vector of fuzzy unknowns.

3.2.1.1. Non-negative solution of fully fuzzy system of linear equations with non-negative fuzzy coefficient matrix

Here two new methods are presented. The fuzzy system has been converted to a crisp system of linear equations by using single and double parametric form of fuzzy numbers to obtain the non-negative solution. Double parametric form of fuzzy numbers is defined and applied for the first time in this thesis. Using single parametric form, the $n\times n$ fully fuzzy system of linear equations has been converted to a $2n\times 2n$ crisp system of linear equations. On the other hand, double parametric form of fuzzy numbers converts the $n\times n$ fully fuzzy system of linear equations to a crisp system of same order.
In this section, we have to obtain a non-negative solution of a fully fuzzy linear system viz. $[\tilde{A}]\{\tilde{X}\}=\{\tilde{b}\}$, where we have assumed $\tilde{A}, \tilde{X}$ and $\tilde{b}\ge 0$.

- **Limitations of the existing (known) methods**

Here we have pointed out some short comings of the existing methods to solve the considered fully fuzzy system of linear equations.

1. Das and Chakraverty (2012) studied the solution of $n\times n$ fully fuzzy system of linear equations by converting it into a $2n\times 2n$ crisp system of linear equations. The matrices involved in the corresponding system are considered as positive.
2. Cholesky decomposition was adopted by Senthilkumar and Rajendran (2011b) for the solution of a symmetric fully fuzzy system of linear equations. Here, positive matrices are considered and the elements are assumed as triangular fuzzy number. In this method, symmetric coefficient matrix has been decomposed into two matrices and then the solution was obtained in three steps.
3. Fully fuzzy linear systems has also been solved by linear programming approach, Gauss elimination method, Cramer's rule, etc. (Dehghan et al. 2006 and 2007). These

computational methods have various disadvantages like number of iterations, triangularisation and finding value of large number of determinants, etc.

To overcome these drawbacks, we have introduced two new methods for solving fully fuzzy linear systems based on single and double parametric form of fuzzy numbers.

- **(Method 1) Single parametric form based method**

The above system, $[\tilde{A}]\{\tilde{X}\} = \{\tilde{b}\}$ can be represented as

$$\sum_{j=1}^{n} \tilde{a}_{kj}\tilde{x}_j = \tilde{b}_k \quad \text{for } k = 1,2,\cdots,n\,. \tag{3.45}$$

Using the parametric form of fuzzy number we may write the elements of the fuzzy coefficient matrix, real fuzzy unknown and the right hand real fuzzy number vector as $\tilde{a}_{kj} = [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)]$, $\tilde{x}_j = [\underline{x}_j(\alpha), \overline{x}_j(\alpha)]$ and $\tilde{b}_k = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]$ respectively.

Substituting the above expressions in Eq. (3.45), one may obtain

$$\sum_{j=1}^{n} [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)] \quad \text{for } k = 1,2,\cdots,n \tag{3.46}$$

By applying standard rule of fuzzy arithmetic, Eq. (3.46) can equivalently be expressed as the following two crisp equations

$$\sum_{j=1}^{n} \underline{a}_{kj}(\alpha)\underline{x}_j(\alpha) = \underline{b}_k(\alpha) \tag{3.47}$$

and

$$\sum_{j=1}^{n} \overline{a}_{kj}(\alpha)\overline{x}_j(\alpha) = \overline{b}_k(\alpha)\,. \tag{3.48}$$

One may write explicitly the combined form of Eqs. (3.47) and (3.48) as

$$\begin{pmatrix} S & O \\ O & D \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{3.49}$$

where,

$$S=\begin{pmatrix}\underline{a}_{11}(\alpha) & \underline{a}_{12}(\alpha) & \cdots & \underline{a}_{1n}(\alpha)\\ \underline{a}_{21}(\alpha) & \underline{a}_{22}(\alpha) & \cdots & \underline{a}_{2n}(\alpha)\\ \vdots & \vdots & \cdots & \vdots\\ \underline{a}_{n1}(\alpha) & \underline{a}_{n2}(\alpha) & \cdots & \underline{a}_{nn}(\alpha)\end{pmatrix},$$

$$D=\begin{pmatrix}\overline{a}_{11}(\alpha) & \overline{a}_{12}(\alpha) & \cdots & \overline{a}_{1n}(\alpha)\\ \overline{a}_{21}(\alpha) & \overline{a}_{22}(\alpha) & \cdots & \overline{a}_{2n}(\alpha)\\ \vdots & \vdots & \cdots & \vdots\\ \overline{a}_{n1}(\alpha) & \overline{a}_{n2}(\alpha) & \cdots & \overline{a}_{nn}(\alpha)\end{pmatrix},$$

$$y=\begin{pmatrix}\underline{x}_1(\alpha)\\ \underline{x}_2(\alpha)\\ \vdots\\ \underline{x}_n(\alpha)\end{pmatrix}, z=\begin{pmatrix}\overline{x}_1(\alpha)\\ \overline{x}_2(\alpha)\\ \vdots\\ \overline{x}_n(\alpha)\end{pmatrix},\ p=\begin{pmatrix}\underline{b}_1(\alpha)\\ \underline{b}_2(\alpha)\\ \vdots\\ \underline{b}_n(\alpha)\end{pmatrix},\ q=\begin{pmatrix}\overline{b}_1(\alpha)\\ \overline{b}_2(\alpha)\\ \vdots\\ \overline{b}_n(\alpha)\end{pmatrix}$$

and O represents $n\times n$ zero matrix.

Now one may solve either Eqs. (3.47) and (3.48) separately or Eq. (3.49) directly to obtain the lower and upper bounds of the solution vector.

One may note that the procedure converts the fuzzy system to crisp system for the solution. Here we have to solve either two $n\times n$ crisp system separately or a single $2n\times 2n$ system. Hence to reduce the computational cost, a new approach is proposed in the following section based on double parametric form of fuzzy numbers.

- **(Method 2) Double parametric form based method**

We have Eq. (3.46) in single parametric form as

$$\sum_{j=1}^{n}[\underline{a}_{kj}(\alpha),\overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\overline{x}_j(\alpha)]=[\underline{b}_k(\alpha),\overline{b}_k(\alpha)] \quad \text{for } k=1,2,\cdots,n.$$

Using the double parametric form of fuzzy numbers, the elements of the fuzzy coefficient matrix, fuzzy unknown vector and right hand side fuzzy number vector of the above system can be expressed respectively as

$$[\underline{a}_{kj}(\alpha),\overline{a}_{kj}(\alpha)]=\beta(\overline{a}_{kj}(\alpha)-\underline{a}_{kj}(\alpha))+\underline{a}_{kj}(\alpha),$$

$$[\underline{x}_j(\alpha),\overline{x}_j(\alpha)]=\beta(\overline{x}_j(\alpha)-\underline{x}_j(\alpha))+\underline{x}_j(\alpha)$$

$$\text{and } [\underline{b}_k(\alpha),\overline{b}_k(\alpha)]=\beta(\overline{b}_k(\alpha)-\underline{b}_k(\alpha))+\underline{b}_k(\alpha).$$

Substituting these expressions in the above system (Eq. (3.46)) we may have

$$\sum_{j=1}^{n}\{\beta(\overline{a}_{kj}(\alpha)-\underline{a}_{kj}(\alpha))+\underline{a}_{kj}(\alpha)\}\{\beta(\overline{x}_j(\alpha)-\underline{x}_j(\alpha))+\underline{x}_j(\alpha)\} \\ =\beta(\overline{b}_k(\alpha)-\underline{b}_k(\alpha))+\underline{b}_k(\alpha). \tag{3.50}$$

Let us define $\beta(\overline{x}_j(\alpha)-\underline{x}_j(\alpha))+\underline{x}_j(\alpha)=\tilde{x}_j(\alpha,\beta)$ and then we substitute this in Eq. (3.50) to get

$$\sum_{j=1}^{n}\{\beta(\overline{a}_{kj}(\alpha)-\underline{a}_{kj}(\alpha))+\underline{a}_{kj}(\alpha)\}\{\tilde{x}_j(\alpha,\beta)\}=\beta(\overline{b}_k(\alpha)-\underline{b}_k(\alpha))+\underline{b}_k(\alpha). \tag{3.51}$$

Eq. (3.51) is now symbolically solved to obtain $\tilde{x}_j(\alpha,\beta)$. After getting the expression of $\tilde{x}_j(\alpha,\beta)$, one may substitute $\beta=0$ and 1 to get the lower and upper bounds of the fuzzy solution vector respectively. Accordingly, this gives

$$\tilde{x}_j(\alpha,0)=\underline{x}_j(\alpha) \text{ and } \tilde{x}_j(\alpha,1)=\overline{x}_j(\alpha).$$

The order of the main system remains unaltered in this solution procedure. So the method is computationally efficient in comparison with other methods. Also, the method is straight forward and easy to handle because the fuzzy system turns into a crisp system using double parametric form of fuzzy numbers.

- **Existence of a suitable solution**

A theorem is stated and proved as below for the existence of a solution. Using the double parametric form of the fuzzy number, one may express the non-negative system (3.45) as

$$[\tilde{A}(\alpha,\beta)]\{\tilde{X}(\alpha,\beta)\}=\{\tilde{b}(\alpha,\beta)\}. \tag{3.52}$$

Theorem 3.18 Let $\tilde{A}(\alpha,\beta)\geq 0$, $\tilde{b}(\alpha,\beta)\geq 0$ and $\tilde{A}(\alpha,\beta)$ correspond to a permutation matrix. Then the non-negative fully fuzzy system of linear equations has a non-negative consistent fuzzy solution.

Proof. Hypotheses imply that $[\tilde{A}(\alpha,\beta)]^{-1}$ exists as non-negative matrix (DeMarr 1972). So we have $\{\tilde{X}(\alpha,\beta)\}=[\tilde{A}(\alpha,\beta)]^{-1}\{\tilde{b}(\alpha,\beta)\}\geq 0$. Hence one may conclude that $\{\tilde{X}(\alpha,\beta)\}$ is a non-negative solution of the required system. □

To illustrate the applicability and effectiveness of the proposed methods an example problem has been solved below.

Example 3.3 Let us consider a 2×2 fully fuzzy system of linear equations (Allahviranloo and Mikaeilvand 2011, Dehegan et al. 2006)

$$[4+\alpha, 6-\alpha]\tilde{x}_1 + [5+\alpha, 8-2\alpha]\tilde{x}_2 = [40+10\alpha, 67-17\alpha],$$
$$[6+\alpha, 7]\tilde{x}_1 + [4, 5-\alpha]\tilde{x}_2 = [43+5\alpha, 55-7\alpha].$$

- **Solution using single parametric form:** By Eqs. (3.47) and (3.48) this can be written as

$$(4+\alpha)\underline{x}_1(\alpha) + (5+\alpha)\underline{x}_2(\alpha) = 40+10\alpha,$$
$$(6+\alpha)\underline{x}_1(\alpha) + 4\underline{x}_2(\alpha) = 43+5\alpha,$$
$$(6-\alpha)\overline{x}_1(\alpha) + (8-2\alpha)\overline{x}_2(\alpha) = 67-17\alpha,$$
$$7\overline{x}_1(\alpha) + (5-\alpha)\overline{x}_2(\alpha) = 55-7\alpha.$$

In matrix notation the above system can be written as

$$\begin{pmatrix} 4+\alpha & 5+\alpha & 0 & 0 \\ 6+\alpha & 4 & 0 & 0 \\ 0 & 0 & 6-\alpha & 8-2\alpha \\ 0 & 0 & 7 & 5-\alpha \end{pmatrix} \begin{pmatrix} \underline{x}_1(\alpha) \\ \underline{x}_2(\alpha) \\ \overline{x}_1(\alpha) \\ \overline{x}_2(\alpha) \end{pmatrix} = \begin{pmatrix} 40+10\alpha \\ 43+5\alpha \\ 67-17\alpha \\ 55-7\alpha \end{pmatrix}.$$

Now solving the above system of linear equations one may have

$\underline{x}_1(\alpha) = \dfrac{5\alpha^2 + 28\alpha + 55}{\alpha^2 + 7\alpha + 14}$, $\underline{x}_2(\alpha) = \dfrac{5\alpha^2 + 37\alpha + 68}{\alpha^2 + 7\alpha + 14}$, $\overline{x}_1(\alpha) = \dfrac{3\alpha^2 + 14\alpha - 105}{\alpha^2 + 3\alpha - 26}$ and

$\overline{x}_2(\alpha) = \dfrac{7\alpha^2 + 22\alpha - 139}{\alpha^2 + 3\alpha - 26}$.

Then we get the elements of the fuzzy solution vector as

$$\tilde{x}_1 = [\underline{x}_1(\alpha), \overline{x}_1(\alpha)] = \left[\frac{5\alpha^2 + 28\alpha + 55}{\alpha^2 + 7\alpha + 14}, \frac{3\alpha^2 + 14\alpha - 105}{\alpha^2 + 3\alpha - 26}\right],$$

and

$$\tilde{x}_2 = [\underline{x}_2(\alpha), x_2(\alpha)] = \left[\frac{5\alpha^2 + 37\alpha + 68}{\alpha^2 + 7\alpha + 14}, \frac{7\alpha^2 + 22\alpha - 139}{\alpha^2 + 3\alpha - 26}\right].$$

- **Solution using double parametric form:** The original system can be represented by using the double parametric form of fuzzy numbers as

$$\{\beta((6-\alpha)-(4+\alpha))+(4+\alpha)\}\{\beta(\bar{x}_1(\alpha)-\underline{x}_1(\alpha))+\underline{x}_1(\alpha)\}$$
$$+\{\beta((8-2\alpha)-(5+\alpha))+(5+\alpha)\}\{\beta(\bar{x}_2(\alpha)-\underline{x}_2(\alpha))+\underline{x}_2(\alpha)\}$$
$$=\beta((67-17\alpha)-(40+10\alpha))+(40+10\alpha),$$

$$\{\beta(7-(6+\alpha))+(6+\alpha)\}\{\beta(\bar{x}_1(\alpha)-\underline{x}_1(\alpha))+\underline{x}_1(\alpha)\}$$
$$+\{\beta((5-\alpha)-4)+4\}\{\beta(\bar{x}_2(\alpha)-\underline{x}_2(\alpha))+\underline{x}_2(\alpha)\}$$
$$=\beta((55-7\alpha)-(43+5\alpha))+(43+5\alpha).$$

Let us consider $\beta(\bar{x}_j(\alpha)-\underline{x}_j(\alpha))+\underline{x}_j(\alpha)=\tilde{x}_j(\alpha,\beta)$ for $j=1,2$. So, substituting this value in the above system, it can be represented as

$$\{\beta((6-\alpha)-(4+\alpha))+(4+\alpha)\}\tilde{x}_1(\alpha,\beta)+\{\beta((8-2\alpha)-(5+\alpha))+(5+\alpha)\}\tilde{x}_2(\alpha,\beta)$$
$$=\beta((67-17\alpha)-(40+10\alpha))+(40+10\alpha),$$

$$\{\beta(7-(6+\alpha))+(6+\alpha)\}\tilde{x}_1(\alpha,\beta)+\{\beta((5-\alpha)-4)+4\}\tilde{x}_2(\alpha,\beta)$$
$$=\beta((55-7\alpha)-(43+5\alpha))+(43+5\alpha).$$

Solving the above, one may get

$$\tilde{x}_1(\alpha,\beta)=\frac{9\alpha^2\beta^2-17\alpha^2\beta+5\alpha^2-18\alpha\beta^2-24\alpha\beta+28\alpha+9\beta^2+41\beta+55}{\alpha^2\beta^2-3\alpha^2\beta+\alpha^2-2\alpha\beta^2-8\alpha\beta+7\alpha+\beta^2+11\beta+14},$$

$$\tilde{x}_2(\alpha,\beta)=\frac{3\alpha^2\beta^2-15\alpha^2\beta+5\alpha^2-6\alpha\beta^2-53\alpha\beta+37\alpha+3\beta^2+68\beta+68}{\alpha^2\beta^2-3\alpha^2\beta+\alpha^2-2\alpha\beta^2-8\alpha\beta+7\alpha+\beta^2+11\beta+14}.$$

Substituting $\beta=0$ and 1 in $\tilde{x}_1(\alpha,\beta)$, one may get the lower and upper bounds of the fuzzy solution respectively as

$$\tilde{x}_1(\alpha,0)=\underline{x}_1(\alpha)=\frac{5\alpha^2+28\alpha+55}{\alpha^2+7\alpha+14} \text{ and}$$

$$\tilde{x}_1(\alpha,1)=\bar{x}_1(\alpha)=\frac{3\alpha^2+14\alpha-105}{\alpha^2+3\alpha-26}.$$

Similarly, substituting $\beta=0$ and 1 in $\tilde{x}_2(\alpha,\beta)$ we have

$$\tilde{x}_2(\alpha,0)=\underline{x}_2(\alpha)=\frac{5\alpha^2+37\alpha+68}{\alpha^2+7\alpha+14} \text{ and}$$

$$\tilde{x}_2(\alpha,1)=\bar{x}_2(\alpha)=\frac{7\alpha^2+22\alpha-139}{\alpha^2+3\alpha-26}.$$

Allahviranloo and Mikaeilvand (2011) and Dehghan et al. (2006) have also solved this problem. Hence results obtained by the proposed methods are compared with results of

Allahviranloo and Mikaeilvand (2011) and Dehghan et al. (2006) and are shown in Table 3.2.

Table 3.2 Comparison between Allahviranloo and Mikaeilvand (2011), Dehghan et al. (2006) and present method(s)

Solution bounds	Allahviranloo and Mikaeilvand (2011)	Dehghan et al. (2006)	Present method(s)
$\underline{x}_1(\alpha)$	$\frac{5\alpha^2+28\alpha+55}{\alpha^2+7\alpha+14}$	$\frac{43}{11}+\frac{\alpha}{11}$	$\frac{5\alpha^2+28\alpha+55}{\alpha^2+7\alpha+14}$
$\overline{x}_1(\alpha)$	$\frac{3\alpha^2+14\alpha-105}{\alpha^2+3\alpha-26}$	4	$\frac{3\alpha^2+14\alpha-105}{\alpha^2+3\alpha-26}$
$\underline{x}_2(\alpha)$	$\frac{5\alpha^2+37\alpha+68}{\alpha^2+7\alpha+14}$	$\frac{54}{11}+\frac{\alpha}{11}$	$\frac{5\alpha^2+37\alpha+68}{\alpha^2+7\alpha+14}$
$\overline{x}_2(\alpha)$	$\frac{7\alpha^2+22\alpha-139}{\alpha^2+3\alpha-26}$	$\frac{21}{4}-\frac{\alpha}{4}$	$\frac{7\alpha^2+22\alpha-139}{\alpha^2+3\alpha-26}$

It is interesting to note that the method of Dehghan et al. (2006) gives the approximate solution, wherever proposed methods give exact solution. Also, one may notice that the results obtained by the present methods are exactly the same as Allahviranloo and Mikaeilvand (2011).

3.2.1.2. Non-negative solution of fully fuzzy system of linear equations with unrestricted fuzzy coefficient matrix

Here we have considered the fully fuzzy real system of linear equations as defined in Eq. (3.44). We assume that $\{\tilde{X}\} \geq 0$ and there is no restriction on the fuzzy coefficient matrix and right hand side fuzzy column vector.

From Eq. (3.44) we have

$$\sum_{j=1}^{n}[\underline{a}_{kj}(\alpha),\overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\overline{x}_j(\alpha)]=[\underline{b}_k(\alpha),\overline{b}_k(\alpha)].$$

Applying the standard fuzzy arithmetic we can write the above equation as

$$\sum_{\underline{a}_{kj}(\alpha)\geq 0}\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{\underline{a}_{kj}(\alpha)<0}\underline{a}_{kj}(\alpha)\overline{x}_j(\alpha)=\underline{b}_k(\alpha) \quad (3.53)$$

and

$$\sum_{\bar{a}_{kj}(\alpha)\geq 0} \bar{a}_{kj}(\alpha)\bar{x}_j(\alpha) + \sum_{\bar{a}_{kj}(\alpha)<0} \bar{a}_{kj}(\alpha)\underline{x}_j(\alpha) = \bar{b}_k(\alpha). \tag{3.54}$$

Eqs. (3.53) and (3.54) are then written in matrix form as

$$\begin{pmatrix} E & -E \\ -F & F \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} \tag{3.55}$$

where

$$E = \begin{pmatrix} \underline{a}_{11}(\alpha) & \underline{a}_{12}(\alpha) & \cdots & \underline{a}_{1n}(\alpha) \\ \underline{a}_{21}(\alpha) & \underline{a}_{22}(\alpha) & \cdots & \underline{a}_{2n}(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ \underline{a}_{n1}(\alpha) & \underline{a}_{n2}(\alpha) & \cdots & \underline{a}_{nn}(\alpha) \end{pmatrix},$$

$$F = \begin{pmatrix} \bar{a}_{11}(\alpha) & \bar{a}_{12}(\alpha) & \cdots & \bar{a}_{1n}(\alpha) \\ \bar{a}_{21}(\alpha) & \bar{a}_{22}(\alpha) & \cdots & \bar{a}_{2n}(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n1}(\alpha) & \bar{a}_{n2}(\alpha) & \cdots & \bar{a}_{nn}(\alpha) \end{pmatrix},$$

$$s = \begin{pmatrix} \underline{x}_1(\alpha) \\ \underline{x}_2(\alpha) \\ \vdots \\ \underline{x}_n(\alpha) \end{pmatrix},$$

$$t = \begin{pmatrix} \bar{x}_1(\alpha) \\ \bar{x}_2(\alpha) \\ \vdots \\ \bar{x}_n(\alpha) \end{pmatrix},$$

$$m = \begin{pmatrix} \underline{b}_1(\alpha) \\ \underline{b}_2(\alpha) \\ \vdots \\ \underline{b}_n(\alpha) \end{pmatrix}$$

and

$$n = \begin{pmatrix} \bar{b}_1(\alpha) \\ \bar{b}_2(\alpha) \\ \vdots \\ \bar{b}_n(\alpha) \end{pmatrix}.$$

Solving the above crisp system of linear equations, one may get the lower and upper bound of the fuzzy solution vector. The non-positive solution of the fully fuzzy system of linear equations may be obtained in the similar manner.

3.2.1.3. Generalised fully fuzzy system of linear equations with unrestricted fuzzy coefficient matrix

We consider the fully fuzzy system as $[\tilde{A}]\{\tilde{X}\}=\{\tilde{b}\}$, where the coefficient matrix $[\tilde{A}]=(\tilde{a}_{kj}), 1\le k\le n, j\le n$ is a fuzzy $n\times n$ matrix, $\{\tilde{b}\}=\{\tilde{b}_k\}, 1\le k$ is a column vector of fuzzy numbers and $\{\tilde{X}\}=\{\tilde{x}_j\}$ is the vector of fuzzy unknowns. We assume $0\notin\tilde{x}_j$. That means zero is not an inner point of the elements of unknown solution vector. As such, a new method is proposed based on a linear programming problem approach. First the sign of the solution vector is determined and then interval based fuzzy arithmetic is used in linear programming approach. To show the effectiveness of the proposed method, an example problem viz. Example 3.4 has been considered.

- **Limitations of the existing (known) methods**

In this section we have pointed out some short comings of the existing methods for solving fuzzy and fully fuzzy system of linear equations.

1. There exist different solution procedures (Behera and Chakraverty 2012, Abbasbandy and Jafarian 2006, Abbasbandy et al. 2005, Allahviranloo 2005a and 2005b, Sun and Guo 2009, Yin and Wang 2009, Friedman et al. 1998) for fuzzy system of linear equations where the coefficient matrices are considered as crisp real matrix. These methods are not applicable when the system is fully fuzzy.
2. Various methodologies (Das and Chakraverty 2012, Senthilkumar and Rajendran 2011, Dehgan et al. 2006 and 2007, Muzzioli and Reynaerts 2007, Otadi and Mosleh 2012, Allahviranloo and Mikaeilvand 2011) have been proposed to solve FFSLE where all the elements of fuzzy matrices are considered as non-negative. The existing methods are not suitable to solve when one may consider non-positive matrix elements as defined in Example 3.4.
3. Recently (Otadi and Mosleh 2012, Babbar et al. 2013) proposed solution technique for FFSLE. There is no restriction for the coefficient matrix. But the authors have found the non-negative solution of fuzzy system of equations. These methods are not applicable when the unknown solution vector consists of only non-positive elements or both non-negative and non-positive elements.

As such, we propose now a new method based on linear programming problem approach which may avoid the above limitations.

- **Proposed method**

Eq. (3.44) may be written as

$$\sum_{j=1}^{n} \tilde{a}_{kj}\tilde{x}_j = \tilde{b}_k, \text{ for } k = 1,2,\cdots,n. \tag{3.56}$$

Let us now define the solution set for the system (3.56) as follows

$$\Omega = \{x_j \mid \sum_{j=1}^{n} a_{kj}x_j = b_k \text{ where } a_{kj} \in \tilde{a}_{kj} \text{ and } b_k \in \tilde{b}_k\}. \tag{3.57}$$

In parametric form, we may write the fuzzy coefficient matrix, real fuzzy unknown and the right hand real fuzzy number vector as $\tilde{a}_{kj} = \tilde{a}_{kj}(\alpha) = [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)]$, $\tilde{x}_j = \tilde{x}_j(\alpha) = [\underline{x}_j(\alpha), \overline{x}_j(\alpha)]$ and $\tilde{b}_k = \tilde{b}_k(\alpha) = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]$ respectively. Substituting the above expressions in Eq. (3.56), one may have

$$\sum_{j=1}^{n} \tilde{a}_{kj}(\alpha)\tilde{x}_j(\alpha) = \tilde{b}_k(\alpha), \text{ for } k = 1,2,\cdots,n.. \tag{3.58}$$

or

$$\sum_{j=1}^{n} [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]. \tag{3.59}$$

From Eq. (3.59) one may predict the sign of the elements of solution vector by the following Theorem.

Theorem 3.19 If $\sum_{j=1}^{n} \tilde{a}_{kj}\tilde{x}_j = \tilde{b}_k$, for $1 \le k \le n$ where $0 \notin \tilde{x}_j$, then sign of the elements of the fuzzy solution vector can be predicted by solving $\sum_{j=1}^{n} \tilde{a}_{kj}{}^c x_j = \tilde{b}_k{}^c$ where

$$\tilde{a}_{kj}{}^c = \frac{\underline{a}_{kj}(\alpha) + \overline{a}_{kj}(\alpha)}{2}, \quad \tilde{b}_k{}^c = \frac{\underline{b}_k(\alpha) + \overline{b}_k(\alpha)}{2}.$$

Proof. The solution x_j can be obtained by solving the crisp system $\sum_{j=1}^{n} \tilde{a}_{kj}{}^{c} x_j = \tilde{b}_k{}^{c}$. From the definition of the solution set of Eq. (3.56), one may easily conclude that x_j ($1 \le j \le n$) are the inner points of $\tilde{x}_j$. Also we know that $0 \notin \tilde{x}_j$. Hence one may predict the sign of the elements of fuzzy solution vector accordingly. □

It may be noted that the fuzzy solution vector may contain non-negative, non-positive or both non-negative and non-positive elements. As such, the following theorems may be applied to handle such situations.

Theorem 3.20 If the elements of x_j for $1 \le j \le n$, are non-negative (non-positive) then the elements of fuzzy solution vector $\tilde{x}_j$ are non-negative (non-positive).

Proof. The proof of the theorem is straight forward. □

Theorem 3.21 If x_j contains both non-negative and non-positive elements, that is x_j for $\{j \in N \mid 1 \le j \le k\}$ are non-negative and for $\{j \in N \mid k+1 \le j \le n\}$ are non-positive for all i, where $1 \le i \le n$ and N is the natural number, then the fuzzy solution vector $\tilde{x}_j$ for $\{j \in N \mid 1 \le j \le k\}$ are non-negative and for $\{j \in N \mid k+1 \le j \le n\}$ are non-positive for all i.

Proof. The proof of the theorem is again straight forward. □

In general, we have three cases with respect to the sign of the elements viz.

Case A: All $\tilde{x}_j$ are non-negative,

Case B: All $\tilde{x}_j$ are non-positive,

Case C: Few $\tilde{x}_j$ are non-negative and few are non-positive.

Next we will discuss below the solution procedure for all the above cases:

Case A: In this case we have considered all $\tilde{x}_j$ are non-negative. So, Eq. (3.59) may be written as

$$\sum_{\tilde{a}_{kj}\geq 0}[\underline{a}_{kj}(\alpha),\bar{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\bar{x}_j(\alpha)]+\sum_{\tilde{a}_{kj}\leq 0}[\underline{a}_{kj}(\alpha),\bar{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\bar{x}_j(\alpha)]$$
$$+\sum_{0\in\tilde{a}_{kj}}[\underline{a}_{kj}(\alpha),\bar{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\bar{x}_j(\alpha)]=[\underline{b}_k(\alpha),\bar{b}_k(\alpha)]. \tag{3.60}$$

Applying the general rule of fuzzy multiplication we get

$$\sum_{\tilde{a}_{kj}\geq 0}[\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha),\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)]+\sum_{\tilde{a}_{kj}\leq 0}[\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha),\bar{a}_{kj}(\alpha)\underline{x}_j(\alpha)]$$
$$+\sum_{0\in\tilde{a}_{kj}}[\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha),\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)]=[\underline{b}_k(\alpha),\bar{b}_k(\alpha)]. \tag{3.61}$$

This can be written as

$$\left.\begin{aligned}&\sum_{\tilde{a}_{kj}(\alpha)\geq 0}\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\leq 0}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)=\underline{b}_k(\alpha)\\&\sum_{\tilde{a}_{kj}(\alpha)\geq 0}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\leq 0}\bar{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)=\bar{b}_k(\alpha)\end{aligned}\right\} \tag{3.62}$$

Let us now denote the system (3.62) as

$$\left.\begin{aligned}&\sum_{j=1}^{n}w_{kj}=g_k,\\&\sum_{j=1}^{n}q_{kj}=h_k,\end{aligned}\right\}\quad \text{for}\quad 1\leq k\leq n, \tag{3.63}$$

where $\sum_{k=1}^{n}w_{kj}=\sum_{\tilde{a}_{kj}(\alpha)\geq 0}\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\leq 0}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)$,

$\sum_{k=1}^{n}q_{kj}=\sum_{\tilde{a}_{kj}(\alpha)\geq 0}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\leq 0}\bar{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)$, $g_k=\underline{b}_k(\alpha)$

and $h_k=\bar{b}_k(\alpha)$.

Eq. (3.63) is converted to the following Linear Programming Problem (LPP) where we have introduced the artificial variables r_s for $s=1,2,\cdots,n,n+1,\cdots 2n$,

Minimize: $r_1+r_2+\cdots+r_{2n}$

Subject to:
$$\sum_{j=1}^{n} w_{1j} + r_1 = g_1,$$
$$\sum_{j=1}^{n} w_{2j} + r_2 = g_2,$$
$$\vdots$$
$$\sum_{j=1}^{n} w_{nj} + r_n = g_n,$$
$$\sum_{j=1}^{n} q_{1j} + r_{n+1} = h_1, \tag{3.64}$$
$$\sum_{j=1}^{n} q_{2j} + r_{n+2} = h_2,$$
$$\vdots$$
$$\sum_{j=1}^{n} q_{nj} + r_{2n} = h_n,$$

with the non-negative restrictions $\underline{x}_j(\alpha), \overline{x}_j(\alpha)$ and r_s for $s = 1,2,\cdots,n,n+1,\cdots 2n \geq 0$. Then the LPP (3.64) is solved and artificial variables are eliminated to have the optimum solution.

Case B: In this case we consider all $\tilde{x}_j$ are non-positive. Eq. (3.63) can similarly be written as

$$\begin{aligned}&\sum_{\tilde{a}_{kj} \geq 0}[\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] + \sum_{\tilde{a}_{kj} \leq 0}[\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] \\ &+ \sum_{0 \in \tilde{a}_{kj}}[\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)].\end{aligned} \tag{3.65}$$

By changing all non-positive variables to non-negative we have

$$\begin{aligned}&\sum_{\tilde{c}_{kj} \leq 0}[\underline{c}_{kj}(\alpha), \overline{c}_{kj}(\alpha)][\underline{y}_j(\alpha), \overline{y}_j(\alpha)] + \sum_{\tilde{c}_{kj} \geq 0}[\underline{c}_{kj}(\alpha), \overline{c}_{kj}(\alpha)][\underline{y}_j(\alpha), \overline{y}_j(\alpha)] \\ &+ \sum_{0 \in \tilde{c}_{kj}}[\underline{c}_{kj}(\alpha), \overline{c}_{kj}(\alpha)][\underline{y}_j(\alpha), \overline{y}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]\end{aligned} \tag{3.66}$$

where $[\underline{y}_j(\alpha), \bar{y}_j(\alpha)] = -[\underline{x}_j(\alpha), \bar{x}_j(\alpha)]$ and $[\underline{c}_{kj}(\alpha), \bar{c}_{kj}(\alpha)] = -[\underline{a}_{kj}(\alpha), \bar{a}_{kj}(\alpha)]$.

Applying the general rule of fuzzy multiplication we get

$$\begin{aligned}&\sum_{\tilde{c}_{kj}\le 0}[\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha), \bar{c}_{kj}(\alpha)\underline{y}_j(\alpha)] + \sum_{\tilde{c}_{kj}\ge 0}[\underline{c}_{kj}(\alpha)\underline{y}_j(\alpha), \bar{c}_{kj}(\alpha)\bar{y}_j(\alpha)] \\ &+ \sum_{0\in\tilde{c}_{kj}}[\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha), \bar{c}_{kj}(\alpha)\bar{y}_j(\alpha)] = [\underline{b}_k(\alpha), \bar{b}_k(\alpha)]\end{aligned} \tag{3.67}$$

Eq. (3.67) can equivalently be written as

$$\left.\begin{aligned}\sum_{\tilde{c}_{kj}\le 0}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha) + \sum_{\tilde{c}_{kj}\ge 0}\underline{c}_{kj}(\alpha)\underline{y}_j(\alpha) + \sum_{0\in\tilde{c}_{kj}}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha) = \underline{b}_k(\alpha) \\ \sum_{\tilde{c}_{kj}\le 0}\bar{c}_{kj}(\alpha)\underline{y}_j(\alpha) + \sum_{\tilde{c}_{kj}\ge 0}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha) + \sum_{0\in\tilde{c}_{kj}}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha) = \bar{b}_k(\alpha)\end{aligned}\right\} \tag{3.68}$$

As in Case A, we represent the above system as

$$\left.\begin{aligned}\sum_{j=1}^{n} w^*_{kj} = g_k, \\ \sum_{j=1}^{n} q^*_{kj} = h_k,\end{aligned}\right\} \quad \text{for} \quad 1\le k\le n, \tag{3.69}$$

where $\sum_{k=1}^{n} w^*_{kj} = \sum_{\tilde{c}_{kj}\le 0}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha) + \sum_{\tilde{c}_{kj}\ge 0}\underline{c}_{kj}(\alpha)\underline{y}_j(\alpha) + \sum_{0\in\tilde{c}_{kj}}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha)$,

$\sum_{k=1}^{n} q^*_{kj} = \sum_{\tilde{c}_{kj}\le 0}\bar{c}_{kj}(\alpha)\underline{y}_j(\alpha) + \sum_{\tilde{c}_{kj}\ge 0}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha) + \sum_{0\in\tilde{c}_{kj}}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha)$, $g_k = \underline{b}_k(\alpha)$ and

$h_k = \bar{b}_k(\alpha)$.

The following LPP from the above system may be solved to have the corresponding fuzzy solution vector.

Minimize: $r_1 + r_2 + \cdots + r_{2n}$

Subject to: $$\sum_{j=1}^{n} w^*_{1j} + r_1 = g_1,$$

$$\sum_{j=1}^{n} w^{*}{}_{2j} + r_2 = g_{2,}$$

$$\vdots$$

$$\sum_{j=1}^{n} w^{*}{}_{nj} + r_n = g_{n,}$$

$$\sum_{j=1}^{n} q^{*}{}_{1j} + r_{n+1} = h_{1,} \tag{3.70}$$

$$\sum_{j=1}^{n} q^{*}{}_{2j} + r_{n+2} = h_{2,}$$

$$\vdots$$

$$\sum_{j=1}^{n} q^{*}{}_{nj} + r_{2n} = h_n,$$

with the non-negative restrictions $\underline{y}_j(\alpha), \overline{y}_j(\alpha)$ and r_s for $s = 1,2,\cdots,n,n+1,\cdots 2n \ \geq 0$.

Case C: Finally the solution vector $\tilde{x}_j$ is assumed to contain both non-negative and non-positive fuzzy numbers. We consider $\tilde{x}_j$ for $\{j \in N \mid 1 \leq j \leq i\}$ as non-negative and for $\{j \in N \mid i+1 \leq j \leq n\}$ as non-positive for all k, where $1 \leq k \leq n$ and N is the natural number. As such from Eq. (3.58) we have

$$\sum_{j=1}^{i} \tilde{a}_{kj}(\alpha)\tilde{x}_j(\alpha) + \sum_{j=i+1}^{n} \tilde{a}_{kj}(\alpha)\tilde{x}_j(\alpha) = \tilde{b}_k(\alpha), \ \text{ for } k = 1,2,\cdots,n. \tag{3.71}$$

The above equation is expressed as

$$\sum_{j=1}^{i} [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] + \sum_{j=i+1}^{n} [\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)][\underline{x}_j(\alpha), \overline{x}_j(\alpha)] = [\underline{b}_k(\alpha), \overline{b}_k(\alpha)]. \tag{3.72}$$

Eq. (3.72) is converted to the following crisp system in the similar fashion as discussed in previous two cases

$$\left.\begin{aligned}
&\underbrace{\sum_{\tilde{a}_{kj}(\alpha)\ge 0}\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\le 0}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)}_{\text{for } 1\le j\le i}\\
&+\underbrace{\sum_{\tilde{c}_{kj}\le 0}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha)+\sum_{\tilde{c}_{kj}\ge 0}\underline{c}_{kj}(\alpha)\underline{y}_j(\alpha)+\sum_{0\in\tilde{c}_{kj}}\underline{c}_{kj}(\alpha)\bar{y}_j(\alpha)}_{\text{for } i+1\le j\le n}=\underline{b}_k(\alpha),\\
&\underbrace{\sum_{\tilde{a}_{kj}(\alpha)\ge 0}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{\tilde{a}_{kj}(\alpha)\le 0}\bar{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{0\in\tilde{a}_{kj}(\alpha)}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)}_{\text{for } 1\le j\le i}\\
&+\underbrace{\sum_{\tilde{c}_{kj}\le 0}\bar{c}_{kj}(\alpha)\underline{y}_j(\alpha)+\sum_{\tilde{c}_{kj}\ge 0}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha)+\sum_{0\in\tilde{c}_{kj}}\bar{c}_{kj}(\alpha)\bar{y}_j(\alpha)}_{\text{for } i+1\le j\le n}=\bar{b}_k(\alpha).
\end{aligned}\right\}\tag{3.73}$$

This may again be written as

$$\left.\begin{aligned}
\sum_{j=1}^{i}w_{kj}+\sum_{j=i+1}^{n}w^*{}_{kj}&=g_k,\\
\sum_{j=1}^{i}q_{kj}+\sum_{j=i+1}^{n}q^*{}_{kj}&=h_k,
\end{aligned}\right\}\ \text{for } 1\le k\le n,\tag{3.74}$$

Corresponding LPP for the above system (3.74) is

Minimize: $r_1+r_2+\cdots+r_{2n}$

Subject to:

$$\begin{aligned}
&\sum_{j=1}^{i}w_{1j}+\sum_{j=i+1}^{n}w^*{}_{1j}+r_1=g_1,\\
&\sum_{j=1}^{i}w_{2j}+\sum_{j=i+1}^{n}w^*{}_{2j}+r_2=g_2,\\
&\qquad\vdots\\
&\sum_{j=1}^{i}w_{nj}+\sum_{j=i+1}^{n}w^*{}_{nj}+r_n=g_n,\\
&\sum_{j=1}^{i}q_{1j}+\sum_{j=i+1}^{n}q^*{}_{1j}+r_{n+1}=h_1,
\end{aligned}\tag{3.75}$$

$$\sum_{j=1}^{i} q_{2j} + \sum_{j=i+1}^{n} q^{*}{}_{2j} + r_{n+2} = h_{2},$$

$$\vdots$$

$$\sum_{j=1}^{i} q_{nj} + \sum_{j=i+1}^{n} q^{*}{}_{n1j} + r_{2n} = h_{n},$$

with the non-negative restrictions that is $\underline{x}_j(\alpha), \bar{x}_j(\alpha), \underline{y}_j(\alpha), \bar{y}_j(\alpha)$ and r_s for $s = 1,2,\cdots,n,n+1,\cdots 2n \ \geq 0$.

Again the LPP (3.75) may be solved to have the required solution vector.

In order to validate the above method, the following example has been considered.

Example 3.4 Let us consider a 2×2 fully fuzzy system of linear equations

$$(-2,3,4)\tilde{x}_1 + (-2,2,3)\tilde{x}_2 = (-13,2,14)$$
$$(1,2,2)\tilde{x}_1 + (4,4,5)\tilde{x}_2 = (-14,-4,0)$$

The above system with $\alpha -$cut form may be represented as

$$[5\alpha - 2, -\alpha + 4][\underline{x}_1(\alpha), \bar{x}_1(\alpha)] + [4\alpha - 2, -\alpha + 3][\underline{x}_2(\alpha), \bar{x}_2(\alpha)] = [15\alpha - 13, -12\alpha + 14]$$
$$[\alpha + 1, 2][\underline{x}_1(\alpha), \bar{x}_1(\alpha)] + [4, -\alpha + 5][\underline{x}_2(\alpha), \bar{x}_2(\alpha)] = [10\alpha - 14, -4\alpha]$$

Using the proposed method (discussed in 3.2.1.3) we have

$$\tilde{x}_1 = (1,2,2) \text{ and } \tilde{x}_2 = (-3,-2,-1).$$

It may be noted that fuzzy arithmetic and linear programming concept are used in the solution procedure. There is no restriction on the coefficient matrix of the corresponding system. The method is found to be efficient when the elements of the fuzzy solution vector are both non-negative and non-positive.

In this chapter we propose various methods depending on FSLE or FFSLE. All the methods are validated and applied to different static problems in the next chapter.

Chapter 4

Uncertain Static Analysis of Structural Problems

The contents of this chapter have been published in:

1. Behera, D., Chakraverty, S. (2013) Fuzzy finite element analysis of imprecisely defined structures with fuzzy nodal force, Engineering Applications of Artificial Intelligence, 26 (10), 2458-2466;

2. Behera, D., Chakraverty, S. (2013) Fuzzy analysis of structures with imprecisely defined properties, Computer Modeling in Engineering & Sciences, 96 (5), 317-337;

3. Behera, D., Chakraverty, S. (2013) Fuzzy finite element based solution of uncertain static problems of structural mechanics, International Journal of Computer Applications, 69 (15), 6-11.

Chapter 4

Uncertain Static Analysis of Structural Problems

In this chapter, static analysis of imprecisely defined structures has been investigated using the proposed methodologies discussed in Chapter 3. Finite element method with interval or fuzzy uncertainties viz. Interval or Fuzzy Finite Element Method (IFEM/FFEM) has also been applied here to obtain uncertain static response of structures. Corrresponding compuutations have been done in MATLAB environment. IFEM or FFEM converts the problem into system of linear equations with interval or fuzzy uncertainties for the static analysis. In the following sections, we have considered different types of structural problems such as bar, beam, truss and rectangular sheet with interval/fuzzy material and geometric properties or external forces.

4.1. Uncertain Static Analysis of Bar

Finite element solution of stepped rectangular (non-homogeneous) bar with deterministic material properties is well established. However, the properties of the material or geometry are actually uncertain in nature due to bias or subjectivity introduced during the experiment. The type of uncertainty associated with the material properties may always be quantified using interval/fuzzy set theory. Here, numerical estimation of fuzzy static displacements of a fixed free stepped rectangular bar is considered. The material and geometric properties of the bar are taken as fuzzy with fuzzy nodal force at the ends. Triangular convex normalized fuzzy sets are used for the present analysis.

Let us now consider a three stepped bar as shown in Fig. 4.1. This was previously considered by (Balu and Rao 2012). Similar type of study has been reported in (Akpan et al. 2001a, Rao and Sawyer 1995). For the uncertain static response, three different cases have been considered here. The input variables for all the cases are shown in Table 4.1. In Case 1(a), only the load P_3 is fuzzy. In Case 1(b), the load (P_3) as well as the Young's modulus (E_i) are having fuzziness and in Case 1(c), all the properties viz., cross sectional area (A_i), length (L_i), Young's modulus for the bar elements and the load applied at free end are taken as fuzzy. Here, i varies from 1 to 3 due to three element discretization of the bar.

All the fuzzy variables are assumed as triangular fuzzy number viz. (a,b,c). Through α-cut approach, this can be represented as $[(b-a)\alpha + a, -(c-b)\alpha + c]$ where $\alpha \in [0,1]$. This defines a triangular membership function, where a and c are the lower and upper bounds of the fuzzy number at $\alpha = 0$ and b is the exact or crisp value at $\alpha = 1$.

Using the proposed methodologies obtained fuzzy translational displacement at nodes 2, 3 and 4 are depicted in Figs. 4.2, 4.3 and 4.4 respectively for all the cases. For Case 1(a), Method 5 of Section 3.1.2 and for Cases (1b) and 1(c), method proposed in Section 3.2.1.2 have been used for finding the solution. From Figs. 4.2 and 4.4, one can observe that the larger width is obtained for both the figures when fuzziness appears only in the applied external load viz. for Case 1(a). The spread in the fuzzy displacements gradually decreases when we have introduced fuzziness in the stiffness matrix viz. for Cases 1(b) and 1(c) respectively. But Fig. 4.3 shows that the translational displacement at node 3 gives weak fuzzy responses obtained for all the cases. Fig. 4.3 demonstrates the opposite behaviour compared to Figs. 4.2 and 4.4 that is the spread of the fuzzy responses are gradually increasing when we introduce fuzziness in the stiffness matrix. It also gives the smaller width for Case 1(a), that is when only fuzziness appears in the external load. Fuzzy displacements obtained at the free end gives similar behaviour as the observations reported in (Rao and Sawyer 1995). The proposed methods estimate narrow bounds for the structural responses.

Table 4.1 Data of three-stepped bar with triangular fuzzy number

Parameters	Case 1(a)	Case 1(b)	Case 1(c)
A_1 (in.^{2})	3.00	3.00	(2.99,3.00,3.01)
A_2 (in.^{2})	2.00	2.00	(1.99,2.00,2.01)
A_3 (in.^{2})	1.00	1.00	(0.99,1.00,1.01)
L_1 (in.)	12.00	12.00	(11.95,12.00,12.05)
L_2 (in.)	10.00	10.00	(9.95,10.00,10.05)
L_3 (in.)	6.00	6.00	(5.95,6.00,6.05)
E_1, E_2, E_3 (psi)	3.0e7	(2.8e7,3.0e7,3.1e7)	(2.8e7,3.0e7,3.1e7)
P_3 (lb)	(7500,10000,12500)	(7500,10000,12500)	(7500,10000,12500)

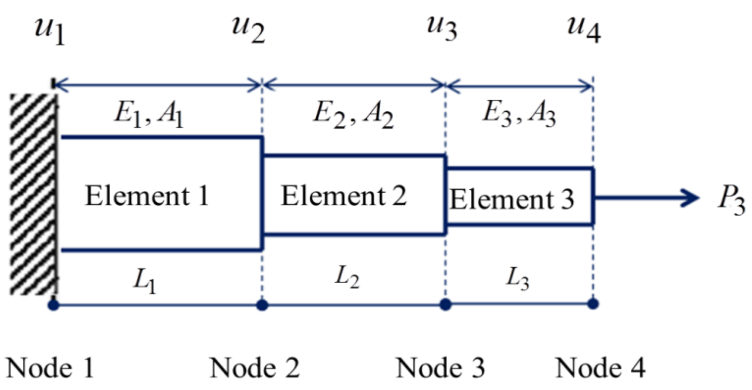


Fig. 4.1 Discretization of a stepped bar into three elements with force applied at the free end

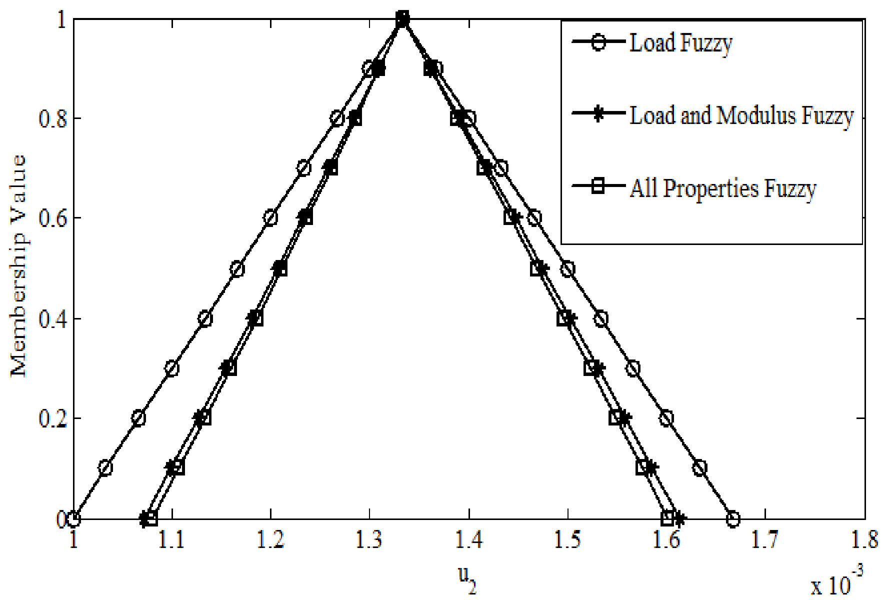


Fig. 4.2 Fuzzy translational displacement at node 2 of three stepped bar

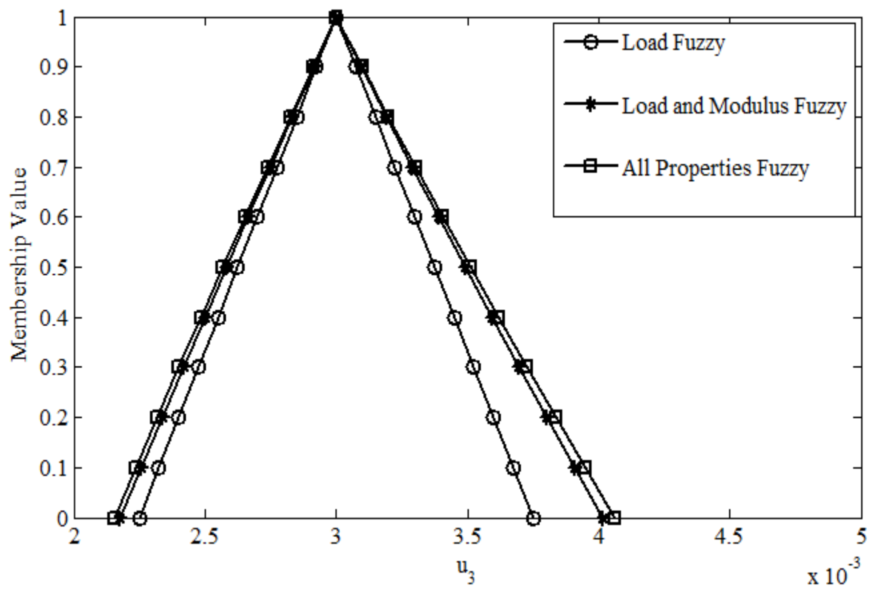


Fig. 4.3 Fuzzy translational displacement at node 3 of three stepped bar

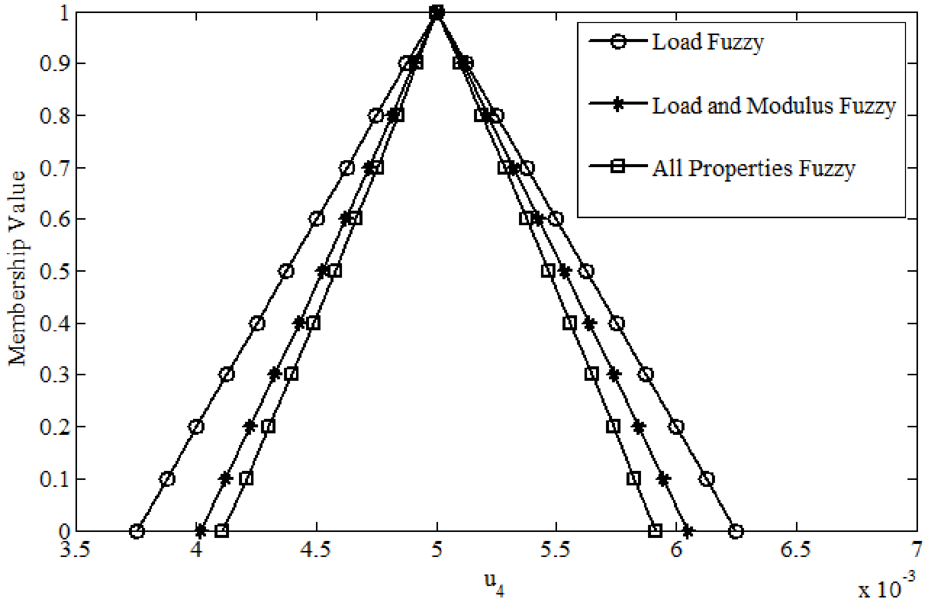


Fig. 4.4 Fuzzy translational displacement at node 4 of three stepped bar

4.2. Uncertain Static Analysis of Beam

In this section, we have considered both homogeneous and non-homogeneous beam for uncertain static response analysis. For homogeneous beam, both applied forces and material properties are assumed as uncertain. But for non-homogeneous beam, only nodal forces are taken as uncertain. Numerical results for these beam structures with various types of uncertain loads and material properties in term of crisp, interval, triangular, trapezoidal and Gaussian fuzzy numbers are presented.

4.2.1. Homogeneous beam with various type of uncertain forces

Homogeneous beam structures with respect to different types of uncertain loading conditions viz. uniform distributed load and nodal forces have been considered for the uncertain static responses. The material properties of the beam are taken as crisp. Here, the proposed methods (viz. Methods 2 and 4 of Section 3.1.2) have been used to obtain the uncertain static response.

We will demonstrate the use of developed methods for computing the fuzzy static response of beam structures as shown in the following example problems. Only the vertical displacement $\tilde{u}^j$, for $j = 1,3,5$ and angle of rotation $\tilde{u}^j$, for $j = 2,4,6$ of nodes are considered due to two element discretization of the beam structure.

- **Beam with uncertain concentrated force**

Let us consider a beam having length $L = 10$ m as shown in Fig. 4.5 (a) (Bhavikati 2005). The beam is fixed at one end and supported by a roller at the other end and carries a concentrated load at the centre of the span. Here, Young's modulus and moment of inertia of material are taken as $E = 200 \times 10^6$ kN/m^2 and $I = 24 \times 10^{-6}$ m^4 respectively. The beam is discretized into two elements as shown in Fig. 4.5 (b) and concentrated load acts at node two. For each section of the beam, Young's modulus, Moment of inertia and length are respectively considered as $E^{(i)} = 200 \times 10^6$ kN/m^2, $I^{(i)} = 24 \times 10^{-6}$ m^4 and $l^{(i)} = 5$ m for $i = 1,2$.

Due to uncertain nodal force $\tilde{p}$ at node 2, nodal force vector can be written as

$$\tilde{F} = [0 \;\; 0 \;\; \tilde{r} \;\; 0 \;\; 0 \;\; 0]^T \text{ where, } \tilde{r} = -\tilde{p}.$$

Different cases have been investigated by assuming load acting on beam as crisp, interval, and fuzzy numbers respectively as given below in Cases 2(a) to 2(c).

Case 2(a): Nodal force as crisp

As such we have considered the force at node 2 as $\tilde{p} = p = 20$ kN (Bhavikati 2005).

Case 2(b): Nodal force as interval

Here we have taken nodal force as $\tilde{p} = [p^c - \beta p^c, p^c + \beta p^c]$ N, where $p^c = 20$ kN and β is the uncertain factor and varies in some region so that we can check the static displacements bounds with changing β.

Case 2(c): Nodal force as fuzzy

Next, force at node 2 are considered as triangular, trapezoidal and Gaussian fuzzy numbers respectively as

$$\tilde{p} = (10,20,30) \text{ kN}, \; \tilde{p} = (10,15,25,30) \text{ kN and } \tilde{p} = (20,10,10) \text{ kN}.$$

Using usual finite element method for Case 2(a), one may obtain the static responses as $\tilde{u}^{(3)} = u^{(3)} = -0.03798$ m, $\tilde{u}^{(4)} = u^{(4)} = -0.00325$ m and $\tilde{u}^{(6)} = u^{(6)} = -0.01302$ m.

Obtained results for crisp parameters are compared with the crisp solution of Bhavikati (2005) and are found to be exactly same. Next, finite element method with interval uncertainty that is interval finite element method for Case 2(b) converts the problem to an interval system of linear equations. This can be solved using the proposed methods by considering $\alpha = 0$ in the considered fuzzy system of linear equations. Obtained results are shown in Figs. 4.6 to 4.8 with β varying from 0% to 1%. To be more illustrative, Table 4.2 lists the bounds of static response of beam using the presented method when $\beta = 1\%$. Lastly for Case 2(c), obtained results are shown in Figs. 4.9 to 4.17. The lower and upper bounds of fuzzy static responses are given in Tables 4.3 to 4.5.

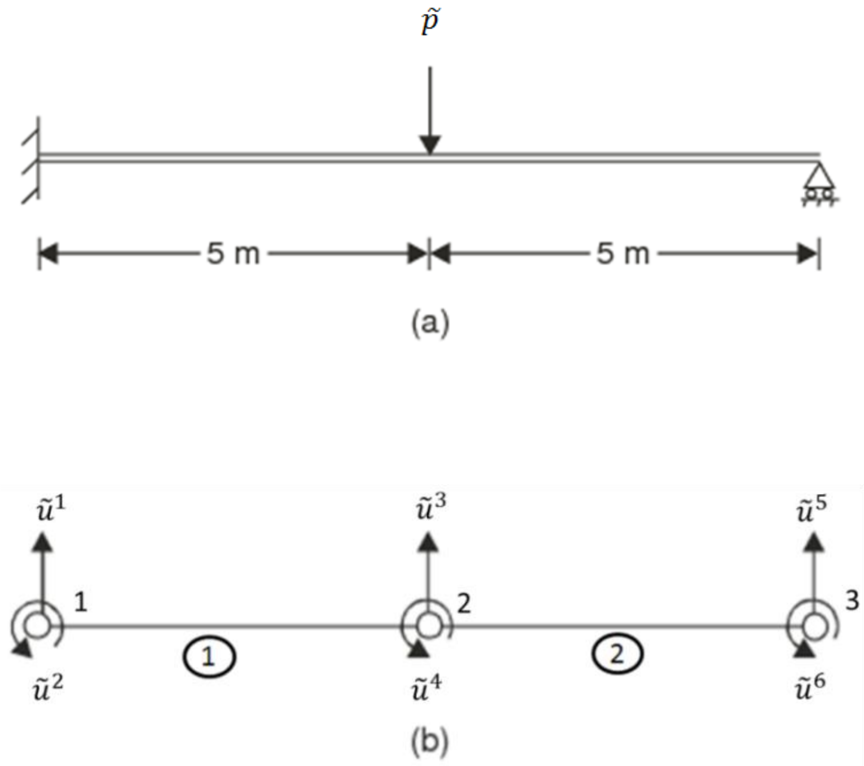


Fig. 4.5 Two element discretization of beam with concentrated force at node 2

From the results it can be noticed that maximum uncertainty width has been obtained for vertical displacements at node 2. For angle of rotation, maximum uncertainty has been obtained at node 3 and minimum at node 2. Also it is interesting to note that the uncertainty width gradually increases for all the static responses by increasing the uncertainty factor β.

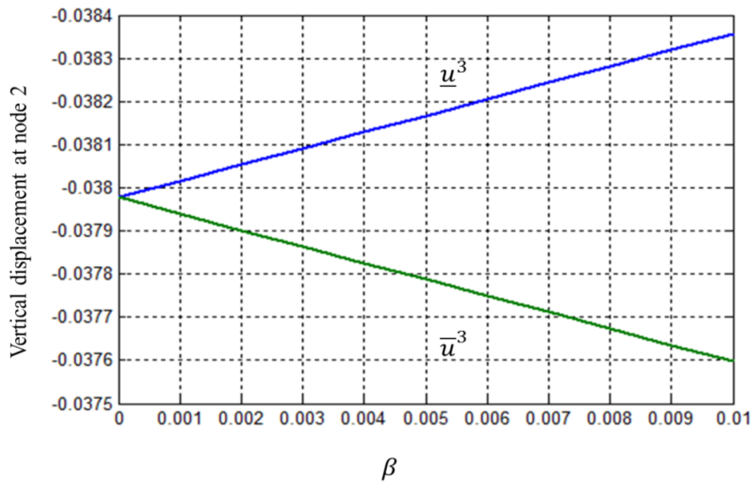


Fig. 4.6 Lower and upper bounds of the vertical displacement at node 2 versus the uncertain factor β (beam with uncertain concentrated force)

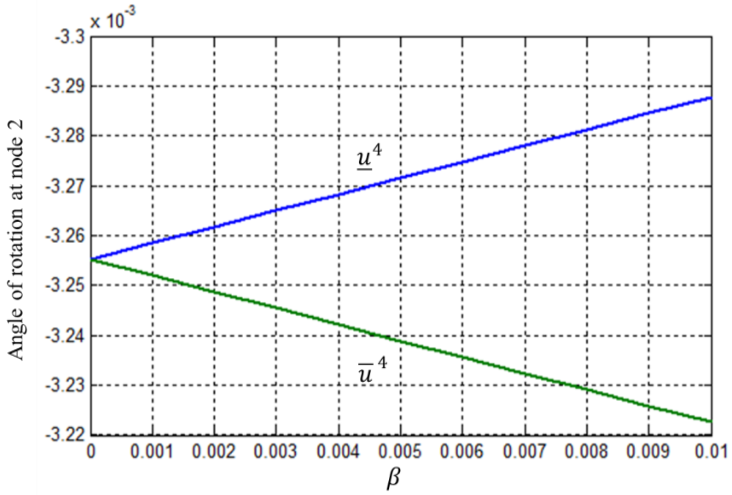


Fig. 4.7 Lower and upper bounds of the angle of rotation at node 2 versus the uncertain factor β (beam with uncertain concentrated force)

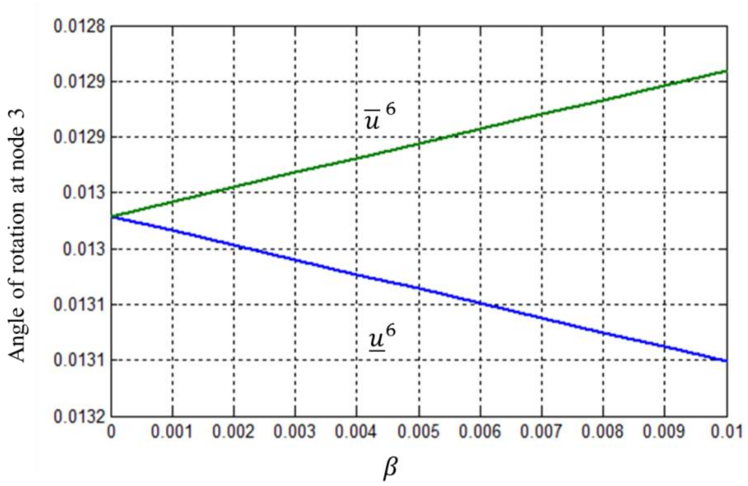


Fig. 4.8 Lower and upper bounds of the angle of rotation at node 3 versus the uncertain factor β (beam with uncertain concentrated force)

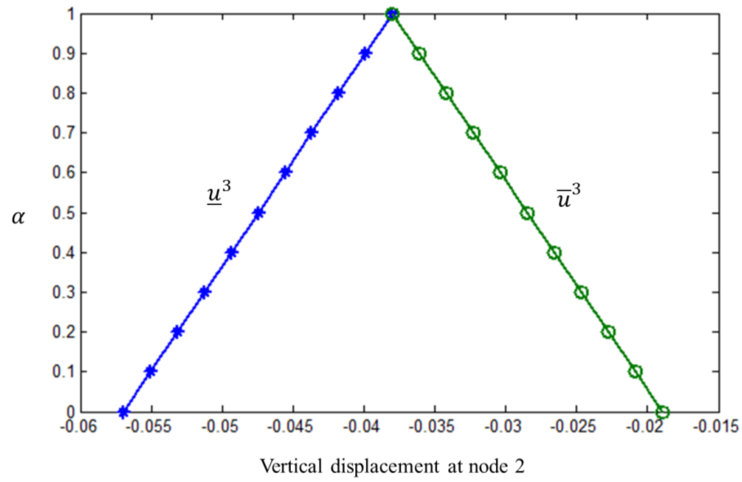


Fig. 4.9 Lower and upper bounds of vertical displacement at node 2 for triangular fuzzy forces (beam with uncertain concentrated force)

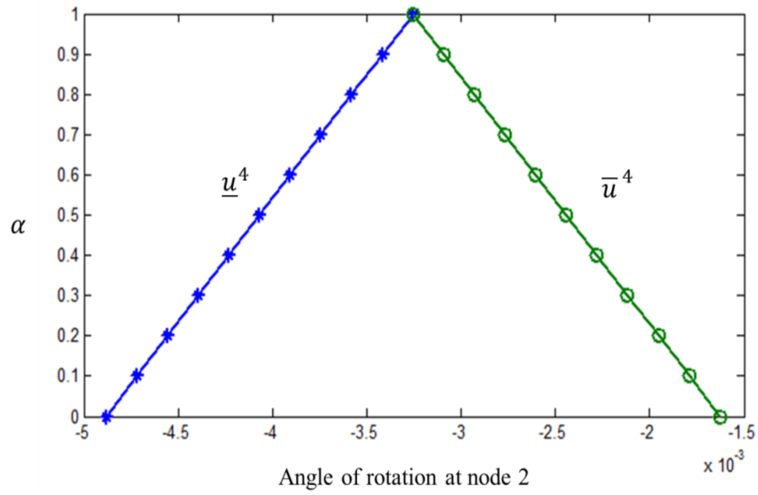


Fig. 4.10 Lower and upper bounds of angle of rotation at node 2 for triangular fuzzy forces (beam with uncertain concentrated force)

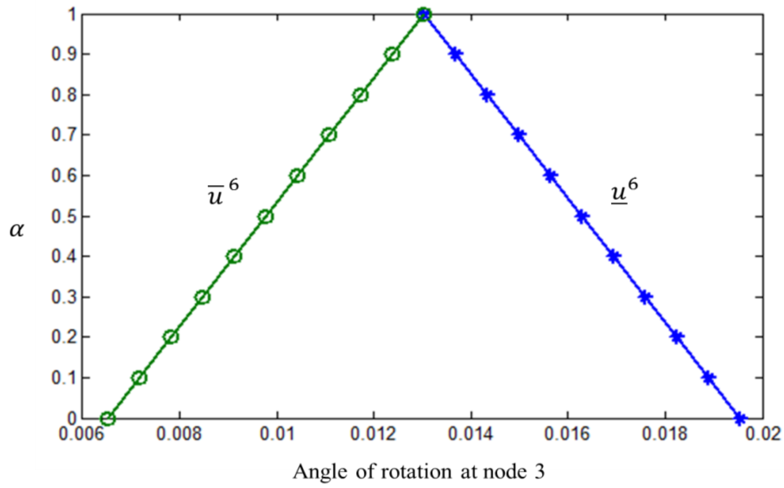


Fig. 4.11 Lower and upper bounds of angle of rotation at node 3 for triangular fuzzy forces (beam with uncertain concentrated force)

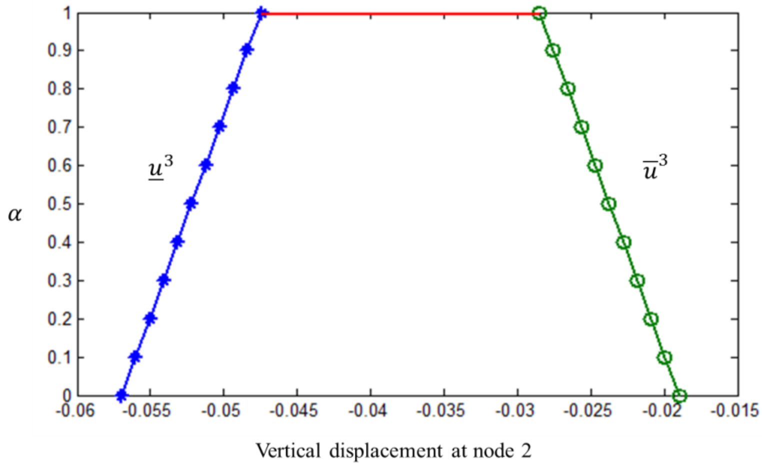


Fig. 4.12 Lower and upper bounds of vertical displacement at node 2 for trapezoidal fuzzy forces (beam with uncertain concentrated force)

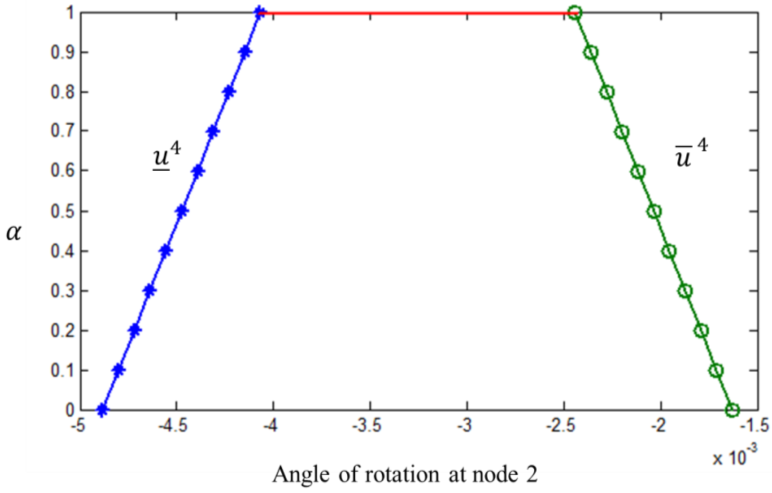


Fig 4.13 Lower and upper bounds of angle of rotation at node 2 for trapezoidal fuzzy forces (beam with uncertain concentrated force)

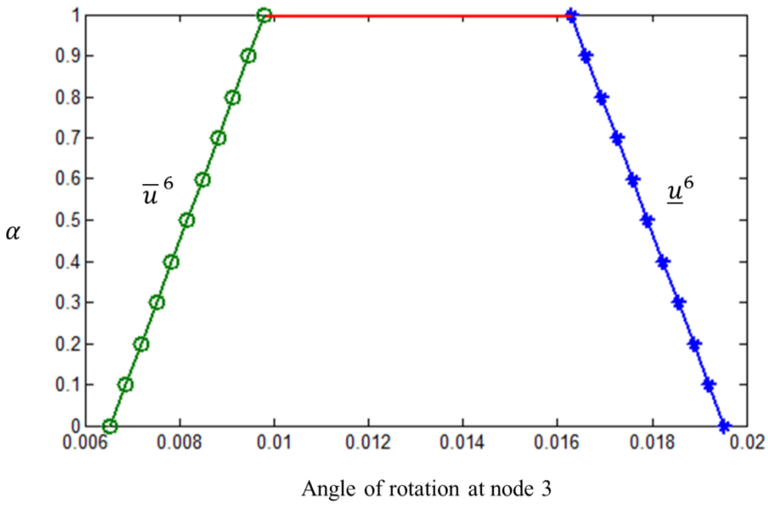


Fig. 4.14 Lower and upper bounds of angle of rotation at node 3 for trapezoidal fuzzy forces (beam with uncertain concentrated force)

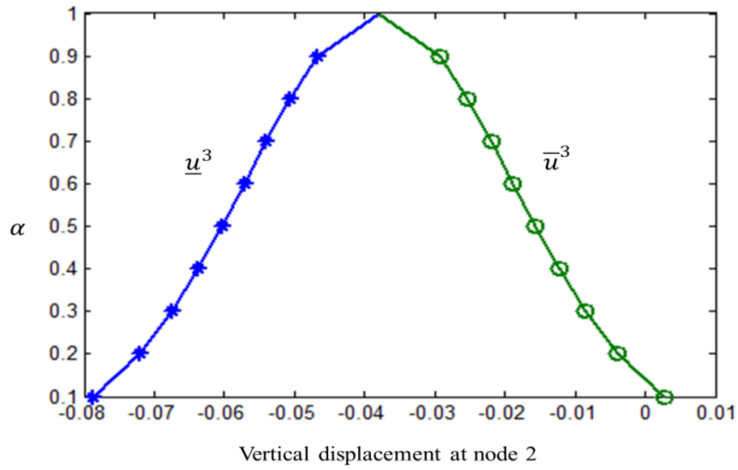


Fig. 4.15 Lower and upper bounds of vertical displacement at node 2 for Gaussian fuzzy forces (beam with uncertain concentrated force)

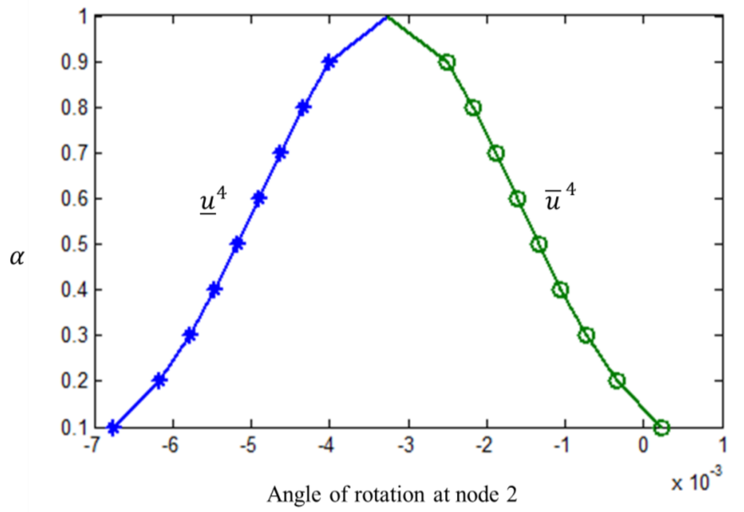


Fig. 4.16 Lower and upper bounds of angle of rotation at node 2 for Gaussian fuzzy forces (beam with uncertain concentrated force)

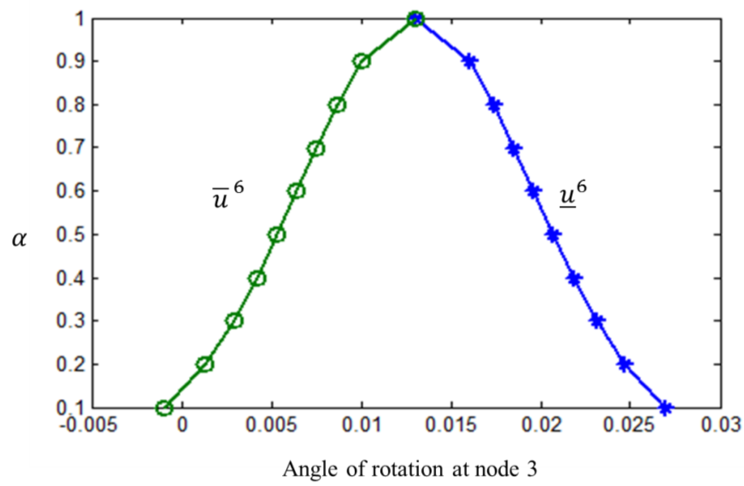


Fig. 4.17 Lower and upper bounds of angle of rotation at node 3 for Gaussian fuzzy forces (beam with uncertain concentrated force)

Table 4.2 Interval static responses (beam with uncertain concentrated force) with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
$\tilde{u}^3$	-0.0384	-0.0376
$\tilde{u}^4$	-0.0033	-0.0032
$\tilde{u}^6$	0.0132	0.0129

Table 4.3 Lower and upper bounds of fuzzy static response (beam with uncertain concentrated force) for triangular fuzzy nodal force

α	0	0.2	0.8	1
$\underline{u}^3$	-0.0570	-0.0532	-0.0418	-0.0380
$\overline{u}^3$	-0.0190	-0.0228	-0.0342	-0.0380
$\underline{u}^4$	-0.0049	-0.0046	-0.0036	-0.0033
$\overline{u}^4$	-0.0016	-0.0020	-0.0029	-0.0033
$\underline{u}^6$	0.0195	0.0182	0.0143	0.0130
$\overline{u}^6$	0.0065	0.0078	0.0117	0.0130

Table 4.4 Lower and upper bounds of fuzzy static response (beam with uncertain concentrated force) for trapezoidal fuzzy nodal force

α	0	0.2	0.8	1
$\underline{u}^3$	-0.0570	-0.0551	-0.0494	-0.0475
$\overline{u}^3$	-0.0190	-0.0209	-0.0266	-0.0285
$\underline{u}^4$	-0.0049	-0.0047	-0.0042	-0.0041
$\overline{u}^4$	-0.0016	-0.0018	-0.0023	-0.0024
$\underline{u}^6$	0.0195	0.0189	0.0169	0.0163
$\overline{u}^6$	0.0065	0.0072	0.0091	0.0098

Table 4.5 Lower and upper bounds of fuzzy static response (beam with uncertain concentrated force) for Gaussian fuzzy nodal force

α	0.1	0.4	0.7	1
$\underline{u}^3$	-0.0787	-0.0637	-0.0540	-0.0380
$\overline{u}^3$	0.0028	-0.0123	-0.0219	-0.0380
$\underline{u}^4$	-0.0067	-0.0055	-0.0046	-0.0033
$\overline{u}^4$	0.0002	-0.0011	-0.0019	-0.0033
$\underline{u}^6$	0.0270	0.0218	0.0185	0.0130
$\overline{u}^6$	-0.0010	0.0042	0.0075	0.0130

- **Beam with uncertain uniformly distributed force**

A beam structure has been considered as shown in Fig. 4.18. Two different uniformly distributed loads $\tilde{p}$ and $\tilde{q}$ act on elements 1 and 2 respectively. For each section, Young's modulus, Moment of inertia and length are assumed respectively as $E^{(i)} = 2\times10^8$ kN/m^2, $I^{(i)} = 5\times10^{-6}$ m^4 and $l^{(i)} = 5$ m for $i = 1,2$ (Bhavikati 2005). Due to uncertain uniform distributed load as defined in Cases 3(a) to 3(c), the fuzzy/interval load vector for element one and two may be written respectively as

$$\tilde{F}_1 = \begin{Bmatrix} \dfrac{\tilde{r}l^{(1)}}{2} \\ \dfrac{\tilde{r}\left(l^{(1)}\right)^2}{12} \\ \dfrac{\tilde{r}l^{(1)}}{2} \\ -\dfrac{\tilde{r}\left(l^{(1)}\right)^2}{12} \end{Bmatrix} \text{ where } \tilde{r} = -\tilde{p} \text{ and } \tilde{F}_2 = \begin{Bmatrix} \dfrac{\tilde{s}l^{(2)}}{2} \\ \dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \\ \dfrac{\tilde{s}l^{(2)}}{2} \\ -\dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \end{Bmatrix} \text{ where } \tilde{s} = -\tilde{q}\,.$$

Assembling the above, we have the load vector

$$\tilde{F} = \begin{Bmatrix} \dfrac{\tilde{r}l^{(1)}}{2} \\ \dfrac{\tilde{r}\left(l^{(1)}\right)^2}{12} \\ \dfrac{\tilde{r}l^{(1)}}{2} + \dfrac{\tilde{s}l^{(2)}}{2} \\ -\dfrac{\tilde{r}\left(l^{(1)}\right)^2}{12} + \dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \\ \dfrac{\tilde{s}l^{(2)}}{2} \\ -\dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \end{Bmatrix}$$

Case 3(a): Uniformly distributed loads as crisp

The uniformly distributed loads are considered as crisp (Bhavikati 2005) viz. $\tilde{p} = p = 12$ kN/m and $\tilde{q} = q = 24$ kN/m.

Case 3(b): Uniformly distributed loads as interval

The uniformly distributed loads as intervals are $\tilde{p} = [p^c - \beta p^c, p^c + \beta p^c]$ kN/m and $\tilde{q} = [q^c - \beta q^c, q^c + \beta q^c]$ kN/m where, $p^c = 12$kN and $q^c = 24$ kN. β is the uncertain factor as defined earlier.

Case 3(c): Uniformly distributed loads as fuzzy

The uniformly distributed loads are taken as triangular fuzzy numbers that are $\tilde{p} = (4,12,20)$ kN/m and $\tilde{q} = (20,24,28)$ kN/m and also as Gaussian fuzzy numbers such as $\tilde{p} = (12,10,10)$ kN/m and $\tilde{q} = (24,10,10)$ kN/m.

Using finite element analysis for case 3(a), one may obtain the static response as $\tilde{u}^{(4)} = u^{(4)} = -0.0357142$ m and $\tilde{u}^{(6)} = u^{(6)} = 0.0803571$ m. Obtained results for crisp parameters are found to be the same with the crisp solution of Bhavikati (2005). Results for Case 3(b) are shown in Figs. 4.19 and 4.20 with β varying from 0% to 1%. To be more illustrative, Table 4.6 lists the bounds of static response of beam using the presented method when $\beta = 1\%$. Figs. 4.21 to 4.24 represent the fuzzy static responses of structures for triangular and Gaussian fuzzy loads as defined in Case 3(c). Lower and upper bounds of Gaussian fuzzy static responses are given in Tables 4.7 and 4.8. From the results, it can be concluded that uncertainty width of the angle of rotation at node 2 is less than the angle of rotation at node 3.

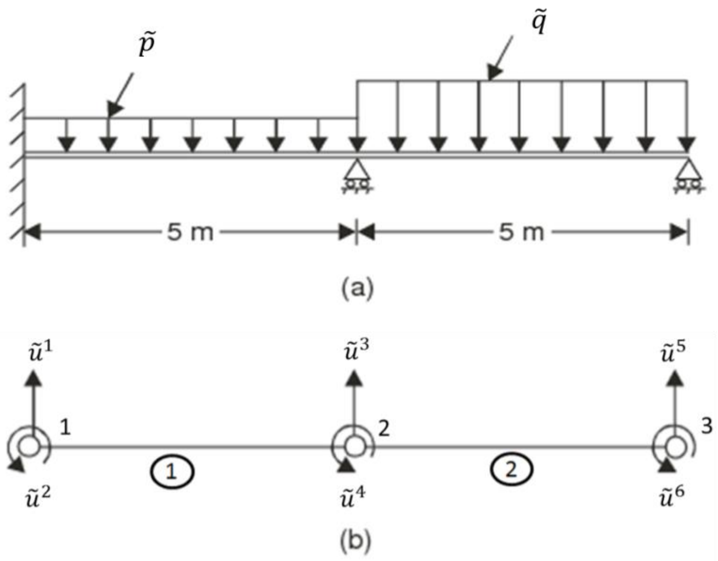


Fig. 4.18 Two element discretization of beam with uniform distributed load

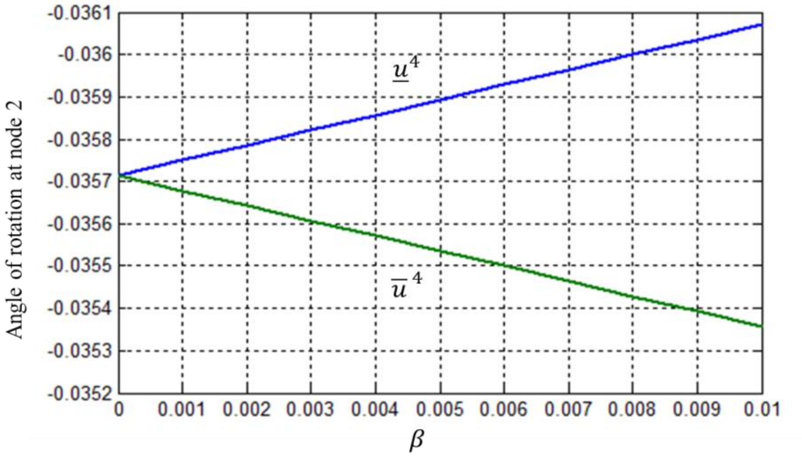


Fig. 4.19 Lower and upper bounds of the angle of rotation at node 2 versus the uncertain factor β (beam with uncertain uniformly distributed force)

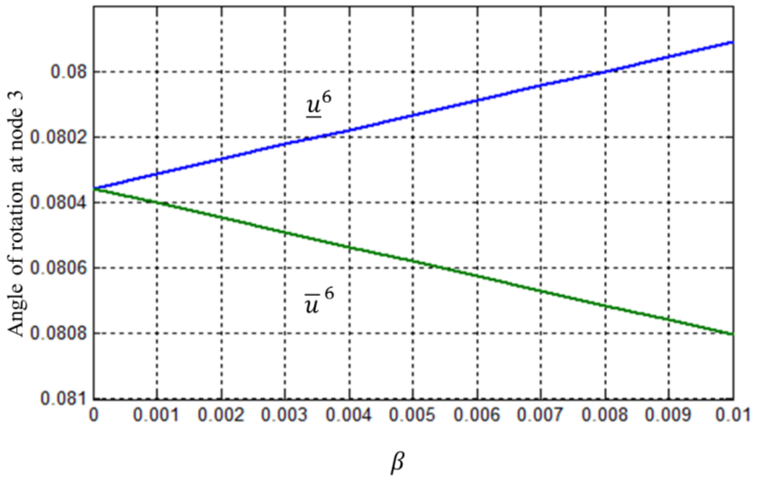


Fig. 4.20 Lower and upper bounds of the angle of rotation at node 3 versus the uncertain factor β (beam with uncertain uniformly distributed force)

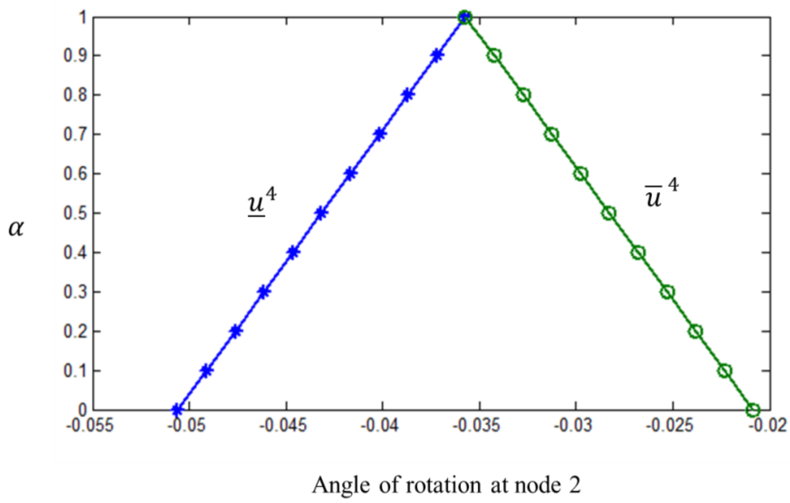


Fig. 4.21 Lower and upper bounds of angle of rotation at node 2 for triangular fuzzy forces (beam with uncertain uniformly distributed force)

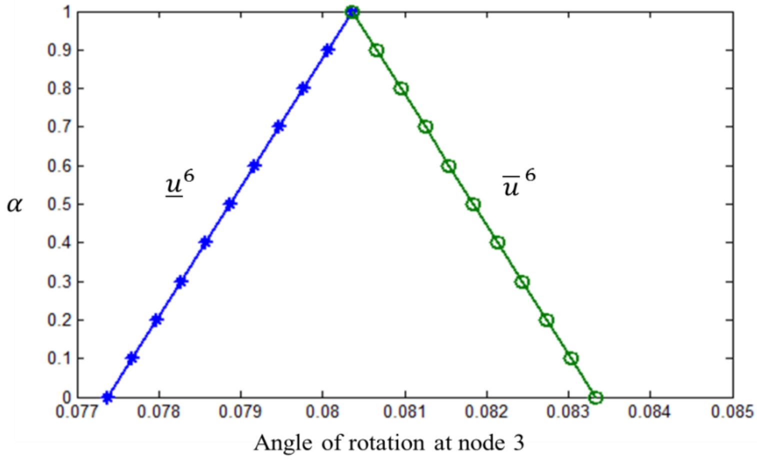


Fig. 4.22 Lower and upper bounds of angle of rotation at node 3 for triangular fuzzy forces (beam with uncertain uniformly distributed force)

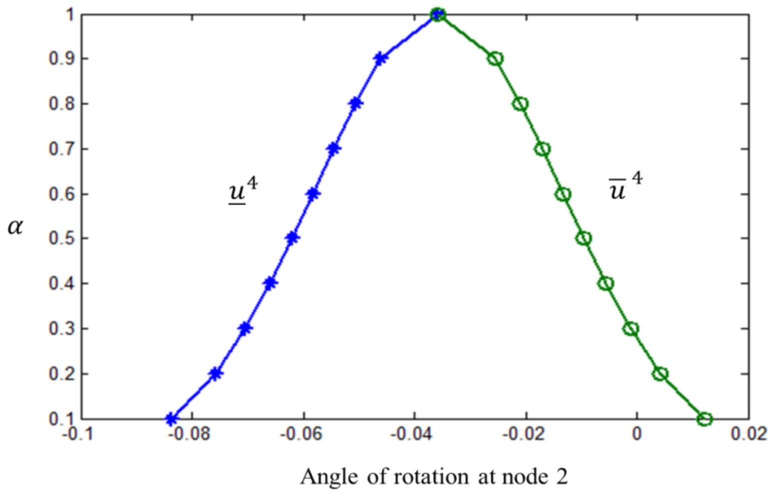


Fig. 4.23 Lower and upper bounds of angle of rotation at node 2 for Gaussian fuzzy forces (beam with uncertain uniformly distributed force)

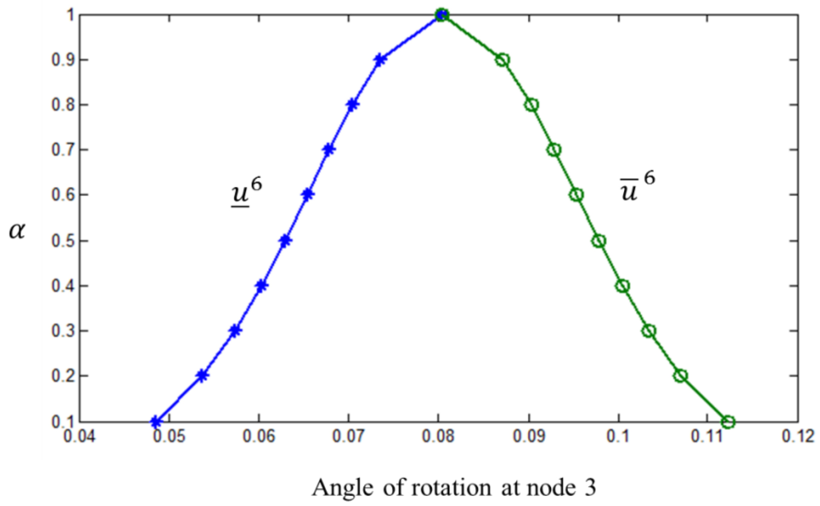


Fig. 4.24 Lower and upper bounds of angle of rotation at node 3 for Gaussian fuzzy forces (beam with uncertain uniformly distributed force)

Table 4.6 Interval static responses (beam with uncertain uniformly distributed force) with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
$\tilde{u}^4$	-0.0361	-0.0354
$\tilde{u}^6$	0.0799	0.0808

Table 4.7 Lower and upper bounds of fuzzy static response (beam with uncertain uniformly distributed force) for triangular fuzzy nodal force

α	0	0.2	0.8	1
$\underline{u}^4$	-0.0506	-0.0476	-0.0387	-0.0357
$\overline{u}^4$	-0.0208	-0.0238	-0.0327	-0.0357
$\underline{u}^6$	0.0774	0.0780	0.0798	0.0804
$\overline{u}^6$	0.0833	0.0827	0.0810	0.0804

Table 4.8 Lower and upper bounds of fuzzy static response (beam with uncertain uniformly distributed force) for Gaussian fuzzy nodal force

α	0.1	0.4	0.7	1
$\underline{u}^4$	-0.0836	-0.0659	-0.0546	-0.0357
$\overline{u}^4$	0.0122	-0.0055	-0.0169	-0.0357
$\underline{u}^6$	0.0484	0.0602	0.0678	0.0804
$\overline{u}^6$	0.1123	0.1005	0.0929	0.0804

- **Beam with uncertain nodal and uniformly distributed forces**

In this example, we have considered a beam as shown in Fig. 4.25 (Bhavikati 2005). Uncertain nodal force $\tilde{p}$ and uniformly distributed load $\tilde{q}$ are acting on the beam. Cases 4(a) to 4(c) define the value of these uncertainties. The beam has been discretized in two elements. For each section of the beam, Young's modulus and moment of inertia are assumed as $E^{(i)} = 200 \times 10^6$ kN/m^2 and $I^{(i)} = 24 \times 10^{-6}$ m^4 respectively. Length of the first and second elements respectively are $l^{(1)} = 4\text{ m}$ and $l^{(2)} = 6\text{ m}$. Due to uncertain nodal force $\tilde{p}$ at node 2, the nodal force vector can be written as

$$\tilde{F}_1 = [0 \;\; 0 \;\; \tilde{r} \;\; 0 \;\; 0 \;\; 0]^T \text{ where, } \tilde{r} = -\tilde{p}.$$

For uncertain uniform distributed load $\tilde{q}$, the load vector for element one and two may again be written respectively as

$$\tilde{F}_2 = \begin{Bmatrix} \dfrac{\tilde{s} l^{(1)}}{2} \\ \dfrac{\tilde{s}\left(l^{(1)}\right)^2}{12} \\ \dfrac{\tilde{s} l^{(1)}}{2} \\ -\dfrac{\tilde{s}\left(l^{(1)}\right)^2}{12} \end{Bmatrix} \text{ and } \tilde{F}_3 = \begin{Bmatrix} \dfrac{\tilde{s} l^{(2)}}{2} \\ \dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \\ \dfrac{\tilde{s} l^{(2)}}{2} \\ -\dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \end{Bmatrix} \text{ where } \tilde{s} = -\tilde{q}$$

and the assembled load vector are obtained as

$$\tilde{F} = \begin{Bmatrix} \dfrac{\tilde{s} l^{(1)}}{2} \\ \dfrac{\tilde{s}\left(l^{(1)}\right)^2}{12} \\ \dfrac{\tilde{s} l^{(1)}}{2} + \dfrac{\tilde{s} l^{(2)}}{2} + \tilde{r} \\ -\dfrac{\tilde{s}\left(l^{(1)}\right)^2}{12} + \dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \\ \dfrac{\tilde{s} l^{(2)}}{2} \\ -\dfrac{\tilde{s}\left(l^{(2)}\right)^2}{12} \end{Bmatrix}$$

Case 4(a): Applied forces as crisp

Loads are taken as crisp where, $\tilde{p} = p = 100\text{ kN}$ and $\tilde{q} = q = 20\text{ kN/m}$ (Bhavikati 2005).

Case 4(b): Applied forces as interval

Now we consider the loads as interval such as $\tilde{p} = [p^c - \beta p^c, p^c + \beta p^c]\text{ kN}$ and $\tilde{q} = [q^c - \beta q^c, q^c + \beta q^c]\text{ kN/m}$, where $p^c = 100\text{ kN}$ and $q^c = 20\text{ kN/m}$. β is the uncertainty factor.

Case 4(c): Applied forces as fuzzy

Again, three types of loads are taken into consideration in terms of triangular, trapezoidal and Gaussian fuzzy numbers. Thus, the values of the fuzzy loads are considered as

$\tilde{p} = (90,100,110)$ kN, $\tilde{q} = (10,20,30)$ kN/m;

$\tilde{p} = (90,95,105,110)$ kN, $\tilde{q} = (10,15,25,30)$ kN/m

and $\tilde{p} = (100,10,10)$ kN, $\tilde{q} = (20,10,10)$ kN/m respectively.

Using usual finite element analysis for Case 4(a), one may obtain the static responses as

$$\tilde{u}^{(3)} = u^{(3)} = -0.32733324 \text{ m},$$

$$\tilde{u}^{(4)} = u^{(4)} = -0.07244438 \text{ m}$$

$$\text{and } \tilde{u}^{(6)} = u^{(6)} = 0.1368055 \text{ m}.$$

Obtained results for crisp parameters are compared with the solution of Bhavikati (2005) and these are found to be in good agreement. Next, results for Case 4(b) are shown in Figs. 4.26 to 4.28 with β varying from 0% to 1%. Also, Table 4.9 lists the bounds of static responses when $\beta = 1\%$. For Case 4(c), results are shown in Figs. 4.29 to 4.37. The lower and upper bounds of fuzzy static responses are given in Tables 4.10 to 4.12. From the results, one may observe that maximum uncertainty has been obtained for vertical displacement at node 2 for all the cases. At node 2, maximum uncertainty has been obtained for angle of rotation and minimum at node 3.

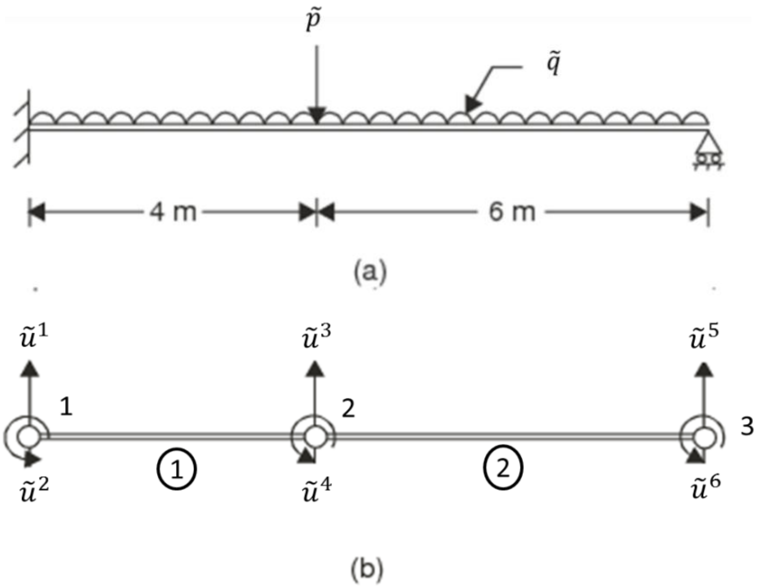


Fig. 4.25 Two element discretization of beam with both nodal force and uniform distributed load

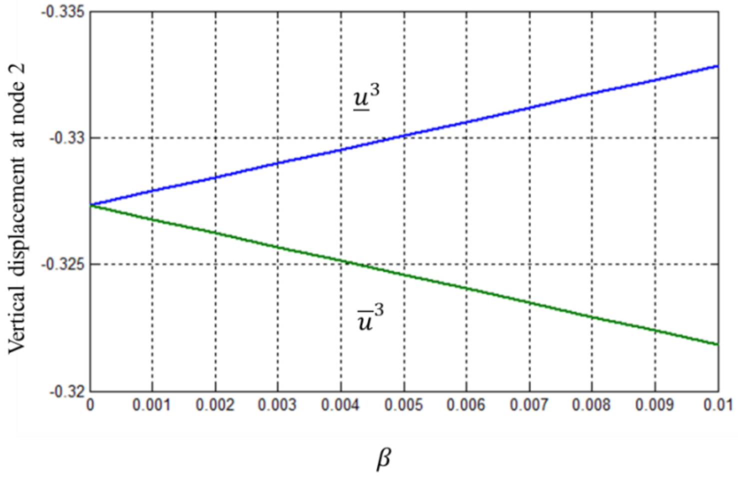


Fig. 4.26 Lower and upper bounds of the vertical displacement at node 2 versus the uncertain factor β (beam with both nodal and uniformly distributed forces)

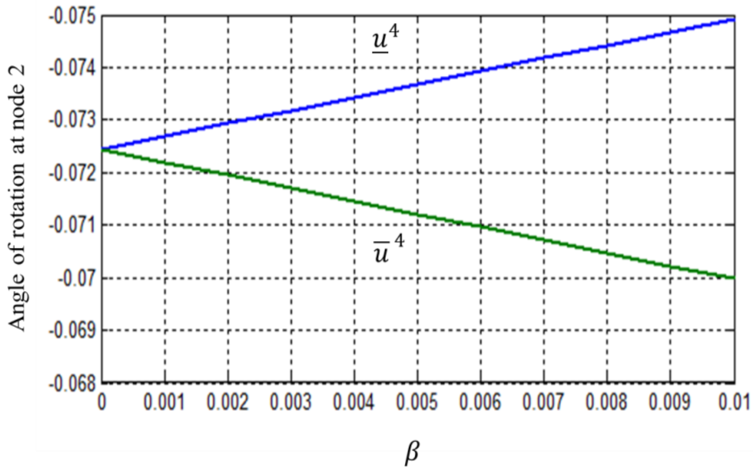


Fig. 4.27 Lower and upper bounds of the angle of rotation at node 2 versus the uncertain factor β (beam with both nodal and uniformly distributed forces)

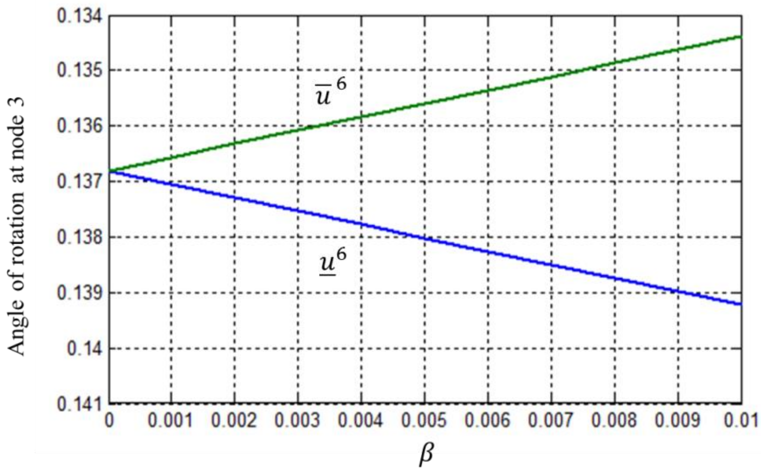


Fig. 4.28 Lower and upper bounds of the angle of rotation at node 3 versus the uncertain factor β (beam with both nodal and uniformly distributed forces)

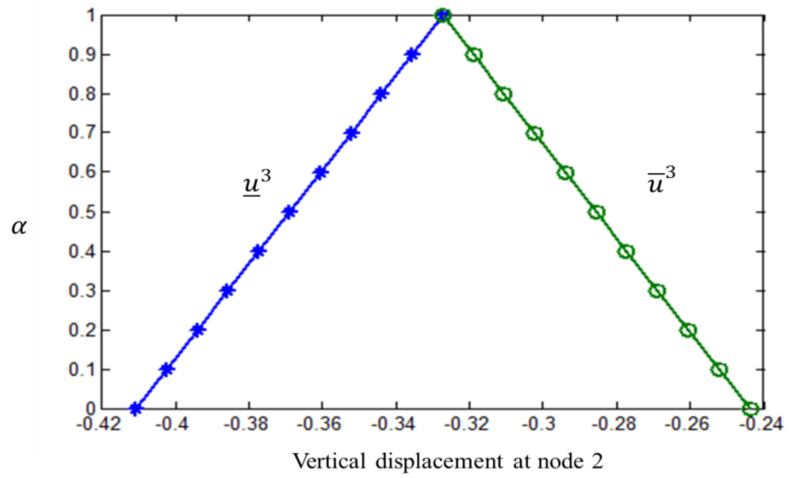


Fig. 4.29 Lower and upper bounds of vertical displacement at node 2 for triangular fuzzy forces (beam with both nodal and uniformly distributed forces)

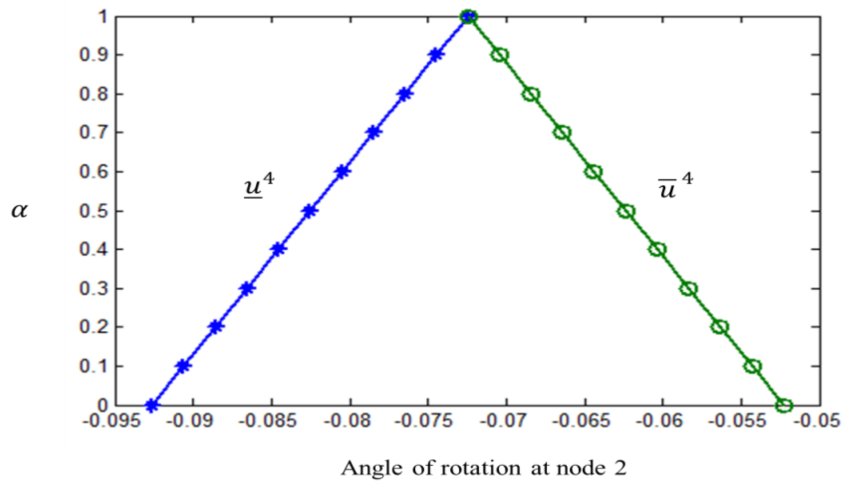


Fig. 4.30 Lower and upper bounds of angle of rotation at node 2 for triangular fuzzy forces (beam with both nodal and uniformly distributed forces)

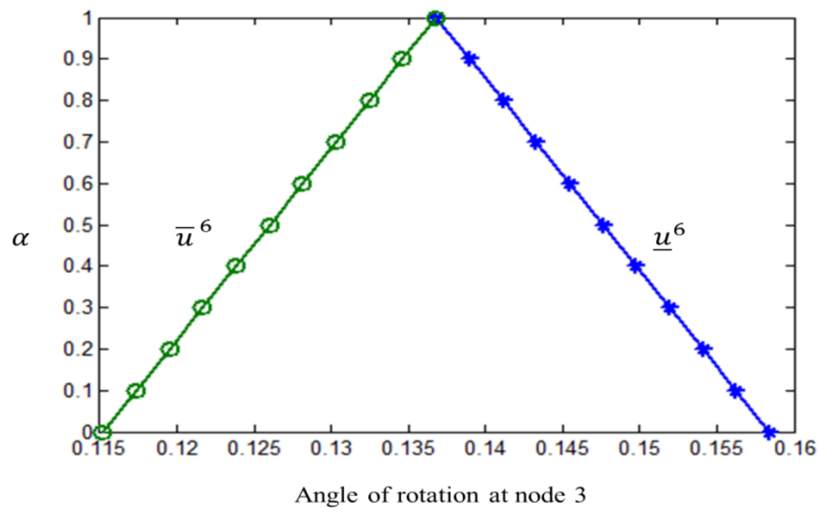


Fig. 4.31 Lower and upper bounds of angle of rotation at node 3 for triangular fuzzy forces (beam with both nodal and uniformly distributed forces)

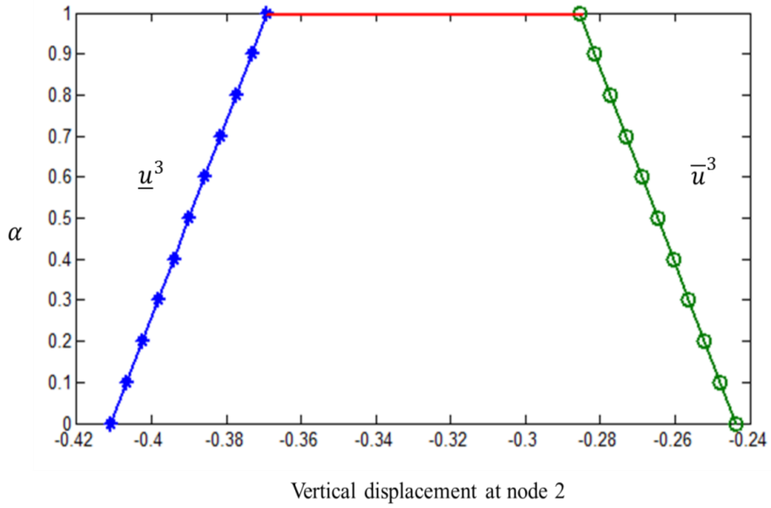


Fig. 4.32 Lower and upper bounds of vertical displacement at node 2 for trapezoidal fuzzy forces (beam with both nodal and uniformly distributed forces)

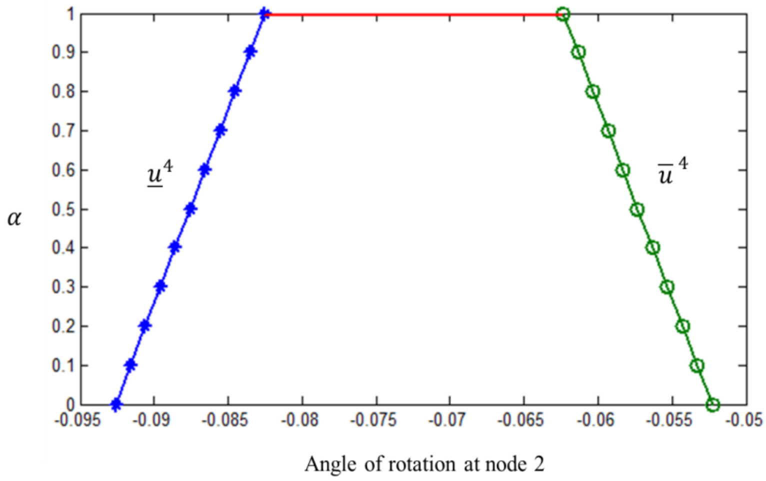


Fig. 4.33 Lower and upper bounds of angle of rotation at node 2 for trapezoidal fuzzy forces (beam with both nodal and uniformly distributed forces)

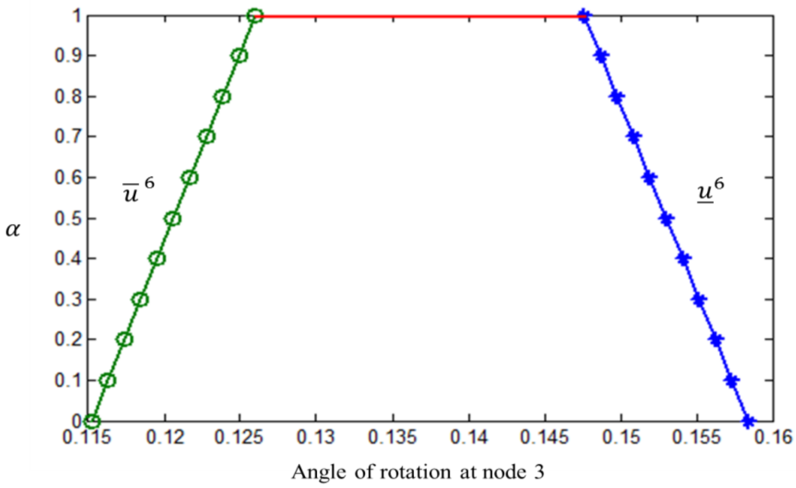


Fig. 4.34 Lower and upper bounds of angle of rotation at node 3 for trapezoidal fuzzy forces (beam with both nodal and uniformly distributed forces)

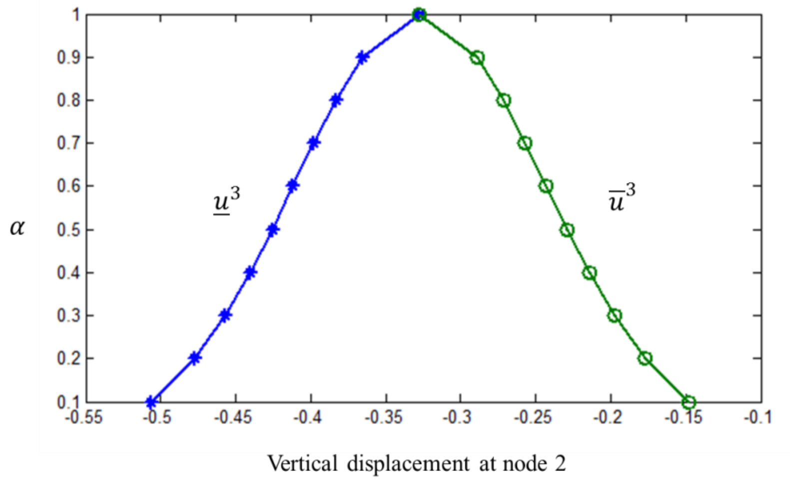


Fig. 4.35 Lower and upper bounds of vertical displacement at node 2 for Gaussian fuzzy forces (beam with both nodal and uniformly distributed forces)

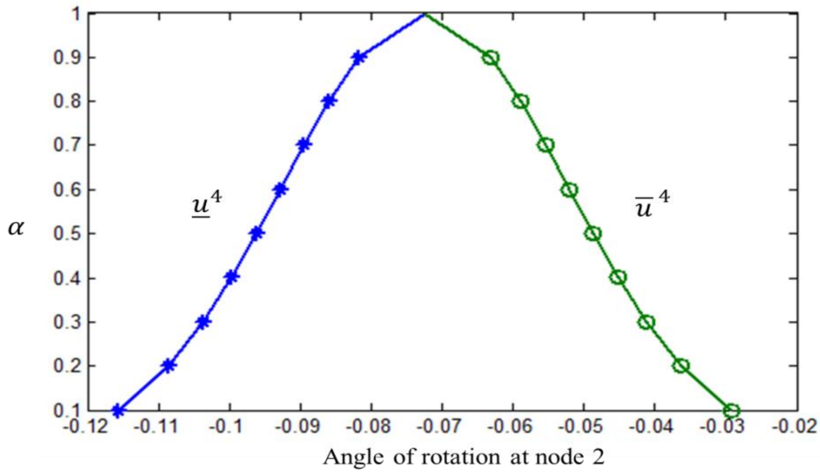


Fig. 4.36 Lower and upper bounds of angle of rotation at node 2 for Gaussian fuzzy forces (beam with both nodal and uniformly distributed forces)

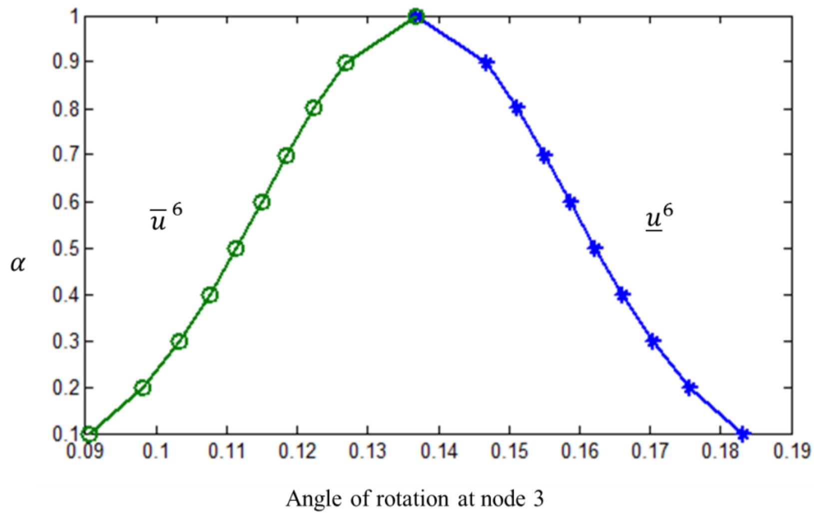


Fig. 4.37 Lower and upper bounds of angle of rotation at node 3 for Gaussian fuzzy forces (beam with both nodal and uniformly distributed forces)

Table 4.9 Interval static responses (beam with both nodal and uniformly distributed forces) with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
$\tilde{u}^3$	-0.3328	-0.3218
$\tilde{u}^4$	-0.0749	-0.0700
$\tilde{u}^6$	0.1392	0.1344

Table 4.10 Lower and upper bounds of fuzzy static response for triangular fuzzy nodal force (beam with both nodal and uniformly distributed forces)

α	0	0.2	0.8	1
$\underline{u}^3$	-0.4109	-0.3942	-0.3440	-0.3273
$\overline{u}^3$	-0.2438	-0.2605	-0.3106	-0.3273
$\underline{u}^4$	-0.0926	-0.0886	-0.0765	-0.0724
$\overline{u}^4$	-0.0523	-0.0563	-0.0684	-0.0724
$\underline{u}^6$	0.1584	0.1541	0.1411	0.1368
$\overline{u}^6$	0.1152	0.1195	0.1325	0.1368

Table 4.11 Lower and upper bounds of fuzzy static response for trapezoidal fuzzy nodal force (beam with both nodal and uniformly distributed forces)

α	0	0.2	0.8	1
$\underline{u}^3$	-0.4109	-0.4025	-0.3775	-0.3691
$\overline{u}^3$	-0.2438	-0.2522	-0.2772	-0.2856
$\underline{u}^4$	-0.0926	-0.0906	-0.0846	-0.0825
$\overline{u}^4$	-0.0523	-0.0543	-0.0603	-0.0624
$\underline{u}^6$	0.1584	0.1562	0.1498	0.1476
$\overline{u}^6$	0.1152	0.1174	0.1238	0.1260

Table 4.12 Lower and upper bounds of fuzzy static response for Gaussian fuzzy nodal force (beam with both nodal and uniformly distributed forces)

α	0.1	0.4	0.7	1
$\underline{u}^3$	-0.5066	-0.4404	-0.3831	-0.3273
$\bar{u}^3$	-0.1481	-0.2143	-0.2568	-0.3273
$\underline{u}^4$	-0.1157	-0.0998	-0.0895	-0.0724
$\bar{u}^4$	-0.0291	-0.0451	-0.0554	-0.0724
$\underline{u}^6$	0.1832	0.1660	0.1550	0.1368
$\bar{u}^6$	0.0905	0.1076	0.1186	0.1368

4.2.2. Homogeneous beam with uncertain material properties and force

A fixed-fixed beam has been considered to compute fuzzy static response as shown in Fig. 4.38 using the proposed methodologies (Method 5 of Section 3.1.2 and the method proposed in Section 3.2.1.2). This problem has been studied by Rao and Swayer (1995). Later on, Akpan et al. (2001a) and Balu and Rao (2012) also investigated the same problem. Three cases have been considered for the analysis. For Case 5(a), Method 5 of Section 3.1.2 and for Cases 5(b) and 5(c), method proposed in Section 3.2.1.2 have been used. In Case 5(a), only the load is considered as fuzzy and is represented by the triplet (360,400,440). In Case 5(b), the modulus of elasticity represented by triplet (2.94e7, 3.0e7, 3.06e7) has been considered as the only fuzzy variable. In Case 5(c), both load and modulus of elasticity have been considered as fuzzy. The model parameters for each case are listed in the form of triangular fuzzy numbers in Table 4.13. Two elements were used in each case.

Fuzzy vertical displacements and angle of rotations at the mid-span of the beam are shown in Figs. 4.39 and 4.40 respectively for all the cases. Results obtained by the proposed methods agree well with Balu and Rao (2012). Observing Fig. 4.39 it may be seen that spread of the fuzzy vertical displacements for Case 5(b) is smaller where as for Case 5(a) it is larger. The spread for Case 5(c) is smaller than Case 5(a) but greater than Case 5(b). Similar observations may be made for fuzzy angle of rotations which is depicted in Fig. 4.40. In this case, smaller width is obtained for Case 5(c) and the larger width is seen for Case 5(a). Here, width of Case 5(b) is smaller than Case 5(a) but greater than Case 5(c).

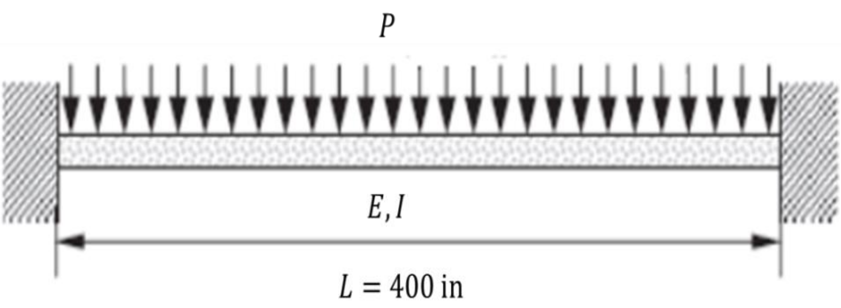


Fig. 4.38 Configuration of fixed-fixed beam

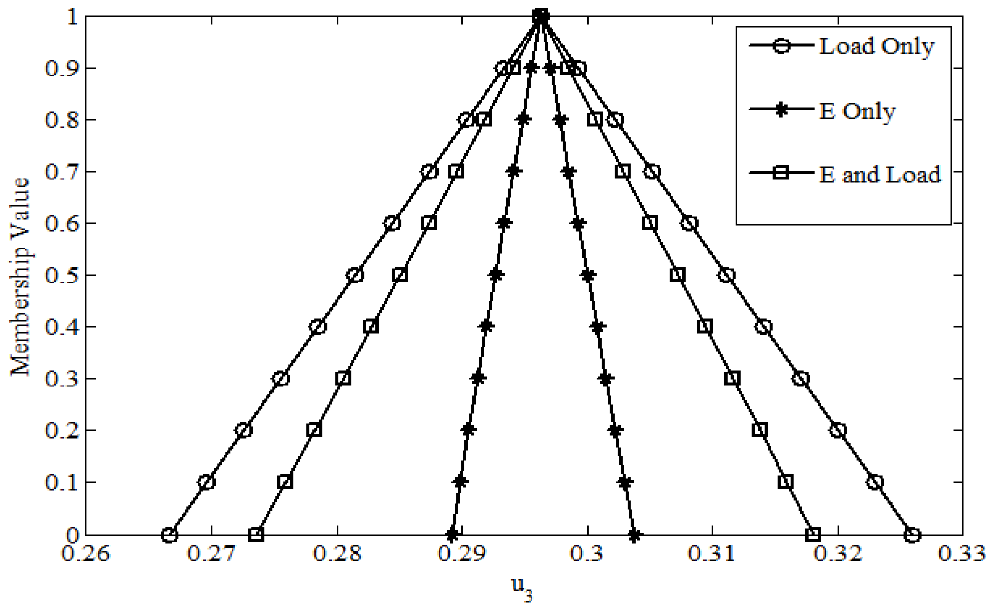


Fig. 4.39 Fuzzy vertical displacement at the mid span of fixed-fixed beam

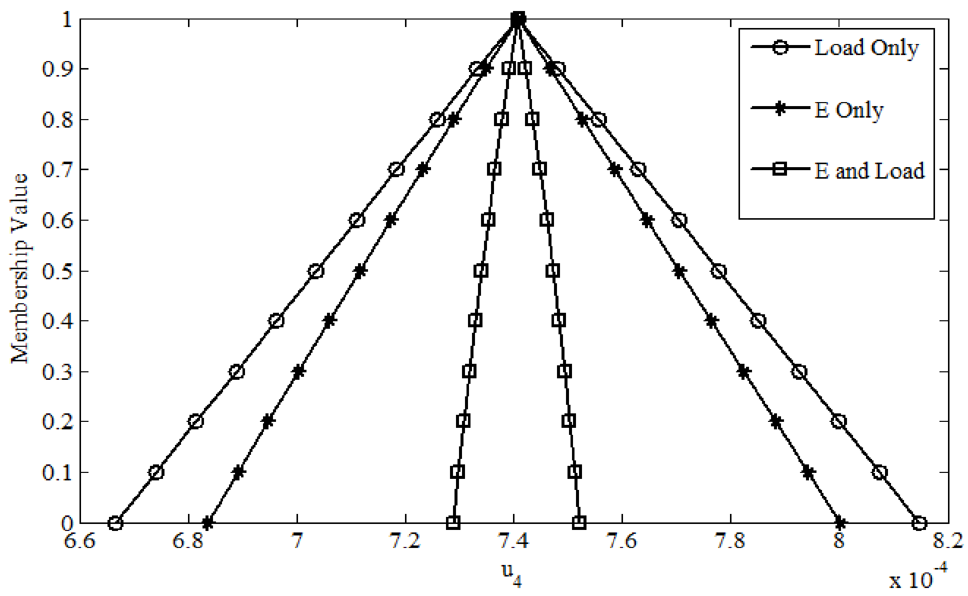


Fig. 4.40 Fuzzy angle of rotation at the mid span of fixed-fixed beam

Table 4.13 Data for beam examples as triangular fuzzy numbers

Parameters	Case 5(a)	Case 5(b)	Case 5(c)
L (in.)	400	400	400
I (in.^{4})	3.0e3	3.0e3	3.0e3
E (psi)	3.0e7	(2.94e7,3.0e7,3.06e7)	(2.94e7,3.0e7,3.06e7)
P (lb/in.)	(360,400,440)	400	(360,400,440)

4.2.3. Non-homogeneous (stepped rectangular) beam with various type of uncertain force

For uncertain static displacements, a three-stepped fixed free rectangular beam (non-homogeneous) as shown in Fig. 4.41 with uncertain nodal force has been considered. Uncertain force has been applied at the end node. Material properties of the beam have been taken as crisp. Here, Method 3 of Section 3.1.2 (Chapter 3) has been used in the solution process.

Let us denote the vertical displacement as $\tilde{u}^j$ for $j = 1,3,5,7$ and angle of rotation as $\tilde{u}^j$ for $j = 2,4,6,8$. The beam is subject to the external load acting on node 4. Here, Young's modulus and length of the three sections are assumed as crisp variables (Qiu et al. 2006) and taken as $E^{(i)} = 2.0 \times 10^{11}$ N/m^2 and $l^{(i)} = 1.0$ m for $i = 1,2,3.$ Moment of inertia for each section of the beam are also taken as crisp viz. $I^{(1)} = 2.2575 \times 10^{-4}$ m^4, $I^{(2)} = 6.7167 \times 10^{-5}$ m^4 and $I^{(3)} = 8.5835 \times 10^{-6}$ m^4. With the above material properties, the external load acting at node four is taken as crisp, interval, triangular, trapezoidal and Gaussian fuzzy number as discussed in Cases 6(a) to 6(e) (below) respectively.

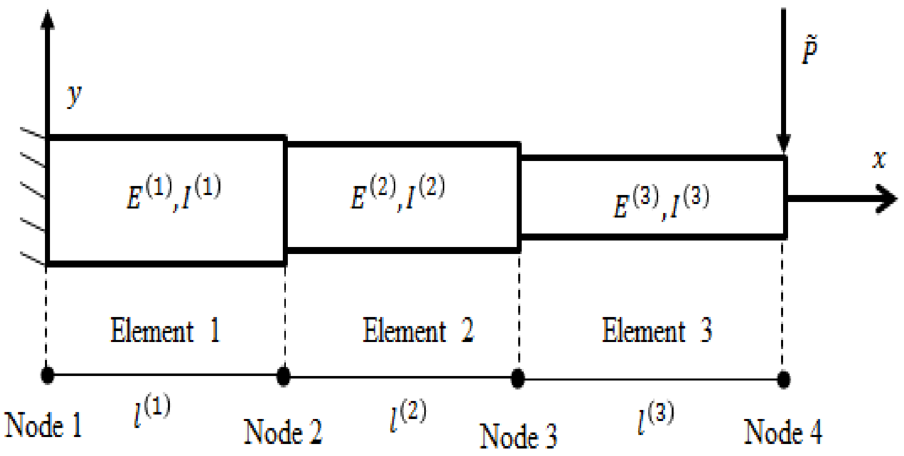


Fig. 4.41 A three-stepped beam

Case 6(a): Nodal force as crisp

The nodal force at node 4 is crisp (Qiu et al. 2006) that is $\tilde{P} = 1000$ N.

Case 6(b): Nodal force as interval

The nodal force at node 4 as interval that is $\tilde{P} = [P^c - \beta P^c, P^c + \beta P^c]$, where $P^c = 1000$ N and β is the uncertain factor.

Case 6(c): Nodal force as triangular fuzzy number

Next, we consider the nodal force at node 4 as triangular fuzzy number that is $\tilde{P} = (990,1000,1010)$ N.

Case 6(d): Nodal force as trapezoidal fuzzy number

In this case, the nodal force at node 4 is considered as trapezoidal fuzzy number that is $\tilde{P} = (990,995,1005,1010)$ N.

Case 6(e): Nodal force as Gaussian fuzzy number

Here, the nodal force is considered at node 4 as Gaussian fuzzy number that is $\tilde{P} = (1000,10,10)$ N.

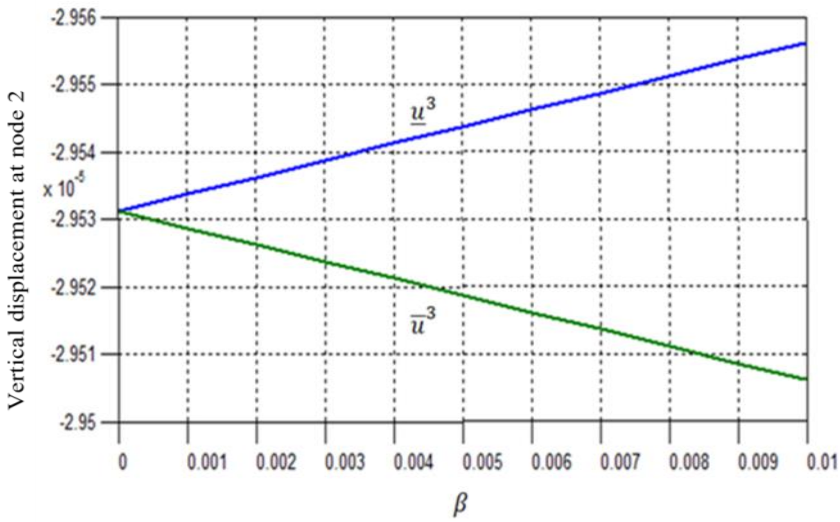
For Case 6(a), one may obtain the crisp static responses as $\tilde{u}^3 = u^3 = -.295 \times 10^{-4}$ m, $\tilde{u}^4 = u^4 = -.554 \times 10^{-4}$ m, $\tilde{u}^5 = u^5 = -1.469 \times 10^{-4}$ m, $\tilde{u}^6 = u^6 = -1.670 \times 10^{-4}$ m, $\tilde{u}^7 = u^7 = -5.081 \times 10^{-4}$ m and $\tilde{u}^8 = u^8 = -4.533 \times 10^{-4}$ m.

Results for crisp parameters are compared with the crisp solution of Qiu et al. (2006) and are found to be the same. Obtained results for Case 6(b) are shown in Figs. 4.42 to 4.44 with β varying from 0% to 1%. Table 4.14 lists the bounds of static response of beam when $\beta = 1\%$. Lower and upper bounds of fuzzy static response are given in Tables 4.15 and 4.16 for Cases 6(c) and 6(d) respectively. Lastly for Case 6(e), the responses are shown in Figs. 4.45 to 4.47. These problems have also been solved by the Method 2 of Section 3.1.2 (Chapter 3) and Friedman et al. (1998) and are found to be in good agreement.

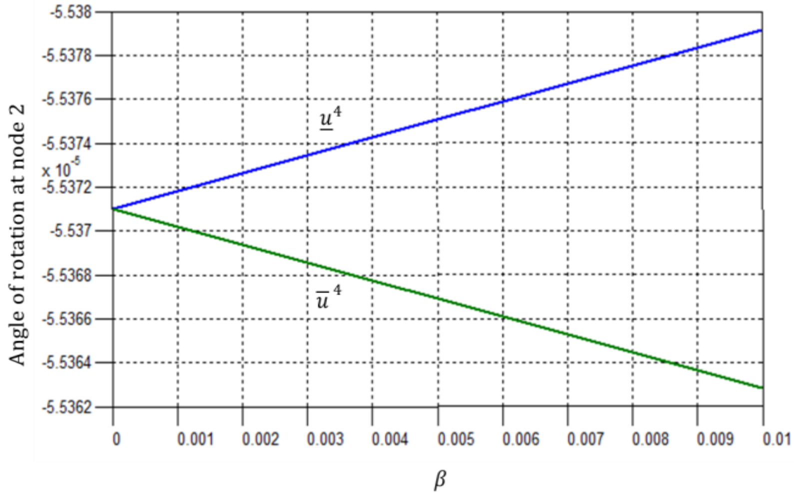
One may notice with crisp parameters that, vertical displacement and angle of rotation at node four are maximum and at node two those are minimum. For interval parameter, the lower and upper bounds of the vertical displacement and angle of rotation at the nodes four and two are maximum and minimum respectively. But the uncertainty of vertical displacement at node four is found to be maximum and the uncertainty at node three is minimum. Similarly, uncertainty for angle of rotation at node two is maximum and at node three it is minimum.

Lower and upper bounds of the vertical displacement and angle of rotation at nodes four and two are maximum and minimum respectively subject to triangular and trapezoidal nodal forces (for each α). Although the fuzzy bounds of vertical displacement and angle of rotation at nodes four and two are maximum and minimum but maximum and minimum uncertainty width for vertical displacement at node four and node three are found. Similarly, one may see that there are maximum uncertainty for angle of rotation at node two and minimum uncertainty at node three.

As pointed out earlier, Gaussian fuzzy membership functions for the uncertain parameter are also used here. However, from a computational point of view, we discard the membership of less than 0.1 as it does not affect the system much and the graph of the membership has been considered to fall straight to zero from that point. For each α, lower and upper bounds of the vertical displacement and angle of rotation at nodes four and two are maximum and minimum respectively. As in previous cases, we obtain maximum uncertainty for vertical displacement at node four and minimum uncertainty at node three. Similarly, interpretation may be done for angle of rotation. Also, it is interesting to note that the uncertainty factor $\beta = 0\%$ in interval parameter and $\alpha = 1$ for triangular and Gaussian fuzzy parameters gives the same static response as is obtained for crisp parameter.

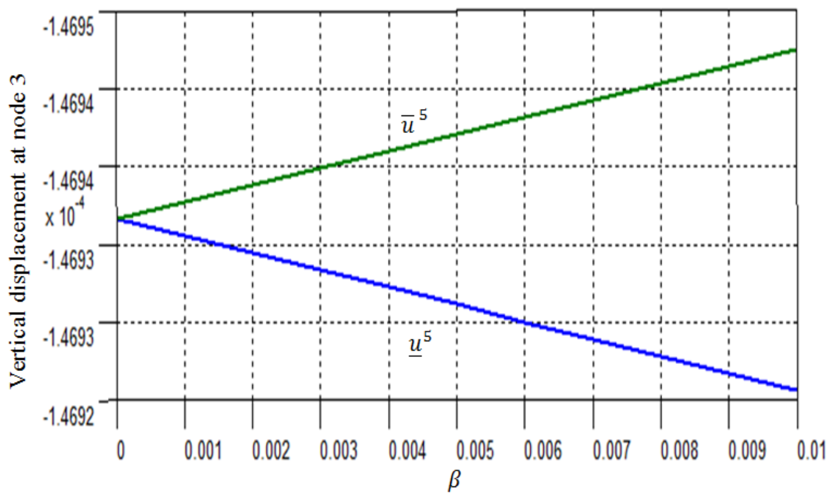


(a)

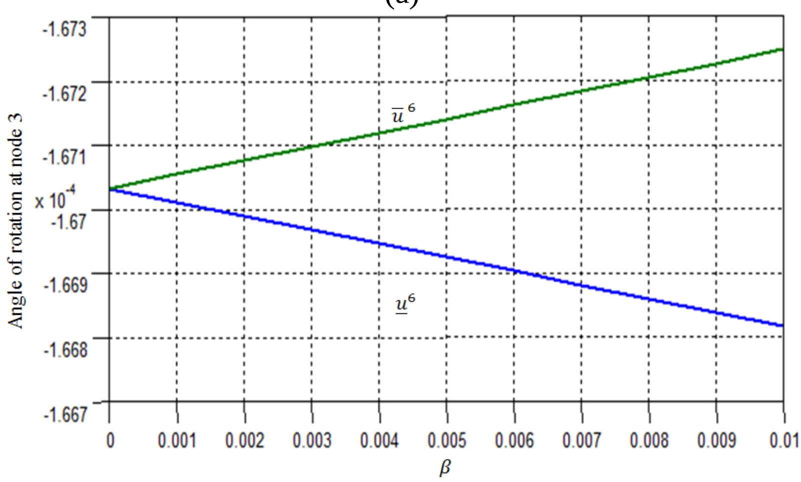


(b)

Fig. 4.42 Plot of lower and upper bounds of the (a) vertical displacement and (b) angle of rotation at node 2 versus the uncertain factor β

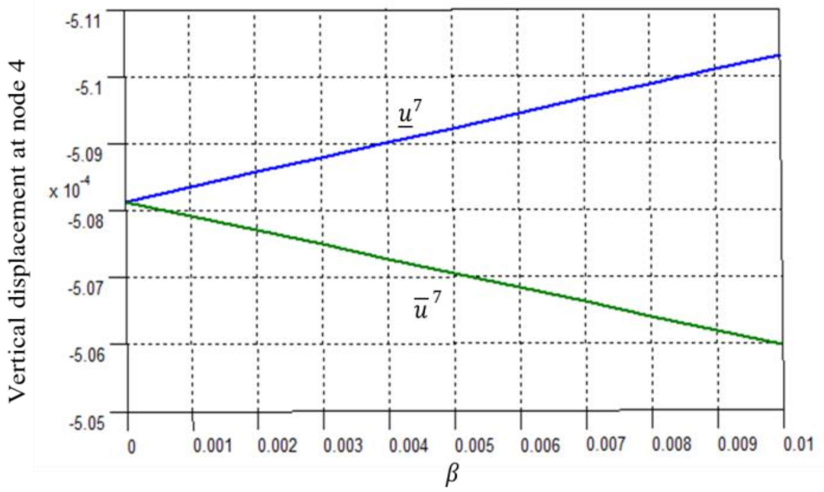


(a)

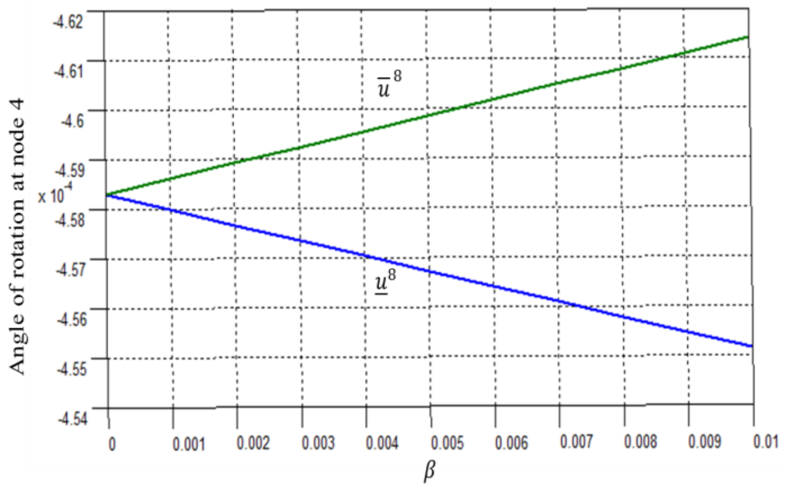


(b)

Fig. 4.43 Plot of lower and upper bounds of the (a) vertical displacement and (b) angle of rotation at node 3 versus the uncertain factor β



(a)



(b)

Fig. 4.44 Plot of lower and upper bounds of the (a) vertical displacement and (b) angle of rotation at node 4 versus the uncertain factor β

Table 4.14 Interval static responses of three-stepped beam with uncertain factor $\beta = 1\%$

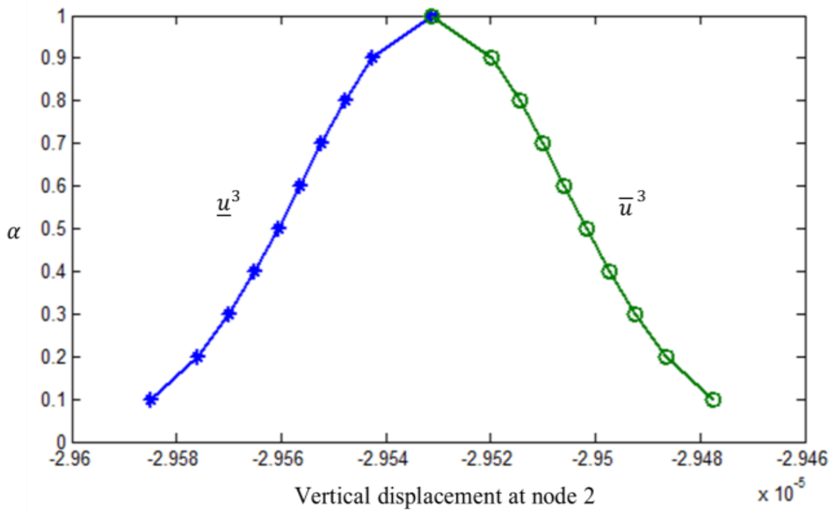
u	$\underline{u}^i$	$\overline{u}^i$
u^3	-.2955619257e-4	-.2950619207e-4
u^4	-.5537914010e-4	-.5536283111e-4
u^5	-.1469257133e-3	-.1469475057e-3
u^6	-.1668174485e-3	-.1672484764e-3
u^7	-.5103086790e-3	-.5059724389e-3
u^8	-.4551613641e-3	-.4614175217e-3

Table 4.15 Lower and upper bounds of fuzzy static response for triangular fuzzy nodal force

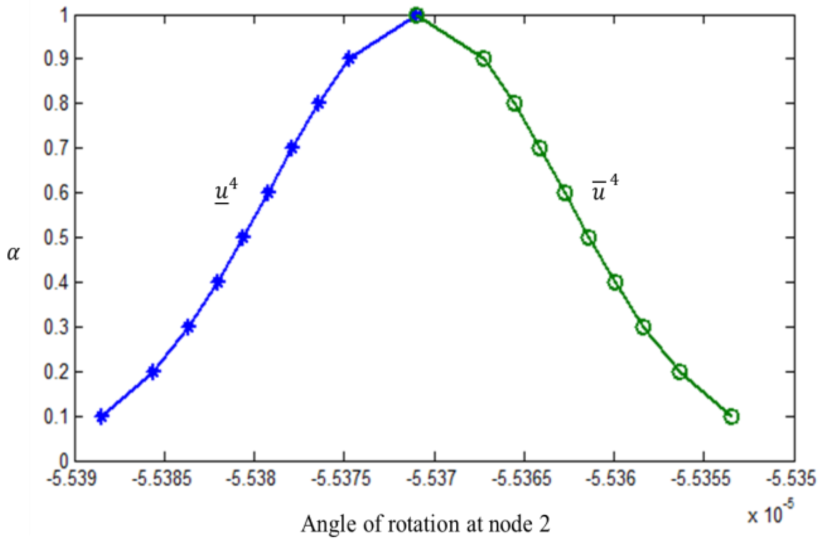
α	0	0.2	0.8	1
$\underline{u}^3$	-.2955619257e-4	-.2955119252e-4	-.2953619237e-4	-.2953119232e-4
$\overline{u}^3$	-.2950619207e-4	-.2951119212e-4	-.2952619227e-4	-.2953119232e-4
$\underline{u}^4$	-.5537914010e-4	-.5537750920e-4	-.5537261650e-4	-.5537098560e-4
$\overline{u}^4$	-.5536283111e-4	-.5536446201e-4	-.5536935471e-4	-.5537098560e-4
$\underline{u}^5$	-.1469257133e-3	-.1469278926e-3	-.1469344303e-3	-.1469366095e-3
$\overline{u}^5$	-.1469475057e-3	-.1469453264e-3	-.1469387887e-3	-.1469366095e-3
$\underline{u}^6$	-.1668174485e-3	-.1668605513e-3	-.1669898597e-3	-.1670329625e-3
$\overline{u}^6$	-.1672484764e-3	-.1672053736e-3	-.1670760652e-3	-.1670329625e-3
$\underline{u}^7$	-.5103086790e-3	-.5098750550e-3	-.5085741830e-3	-.5081405590e-3
$\overline{u}^7$	-.5059724389e-3	-.5064060629e-3	-.5077069349e-3	-.5081405590e-3
$\underline{u}^8$	-.4551613641e-3	-.4557869799e-3	-.4576638272e-3	-.4582894429e-3
$\overline{u}^8$	-.4614175217e-3	-.4607919059e-3	-.4589150586e-3	-.4582894429e-3

Table 4.16 Lower and upper bounds of fuzzy static response for trapezoidal fuzzy nodal force

α	0	0.2	0.8	1
$\underline{u}^3$	-.2955619257e-4	-.2955369254e-4	-.2954619247e-4	-.2954369244e-4
$\overline{u}^3$	-.2950619207e-4	-.2950869210e-4	-.2951619217e-4	-.2951869220e-4
$\underline{u}^4$	-.5537914010e-4	-.5537832465e-4	-.5537587830e-4	-.5537506285e-4
$\overline{u}^4$	-.5536283111e-4	-.5536364656e-4	-.5536609291e-4	-.5536690836e-4
$\underline{u}^5$	-.1469257133e-3	-.1469268029e-3	-.1469300718e-3	-.1469311614e-3
$\overline{u}^5$	-.1469475057e-3	-.1469464161e-3	-.1469431472e-3	-.1469420576e-3
$\underline{u}^6$	-.1668174485e-3	-.1668389999e-3	-.1669036541e-3	-.1669252055e-3
$\overline{u}^6$	-.1672484764e-3	-.1672269250e-3	-.1671622708e-3	-.1671407194e-3
$\underline{u}^7$	-.5103086790e-3	-.5100918670e-3	-.5094414310e-3	-.5092246190e-3
$\overline{u}^7$	-.5059724389e-3	-.5061892509e-3	-.5068396869e-3	-.5070564989e-3
$\underline{u}^8$	-.4551613641e-3	-.4554741720e-3	-.4564125956e-3	-.4567254035e-3
$\overline{u}^8$	-.4614175217e-3	-.4611047138e-3	-.4601662902e-3	-.4598534823e-3

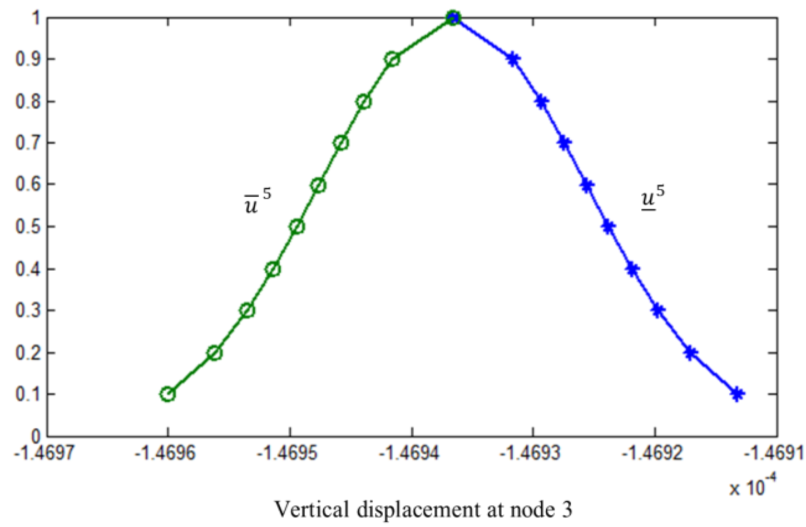


(a)

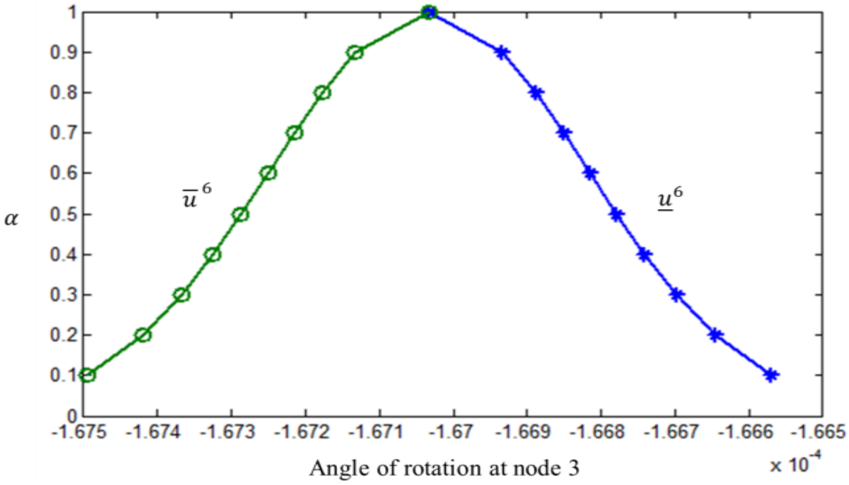


(b)

Fig. 4.45 Lower and upper bounds of (a) vertical displacement and (b) angle of rotation at node 2 for Gaussian fuzzy force

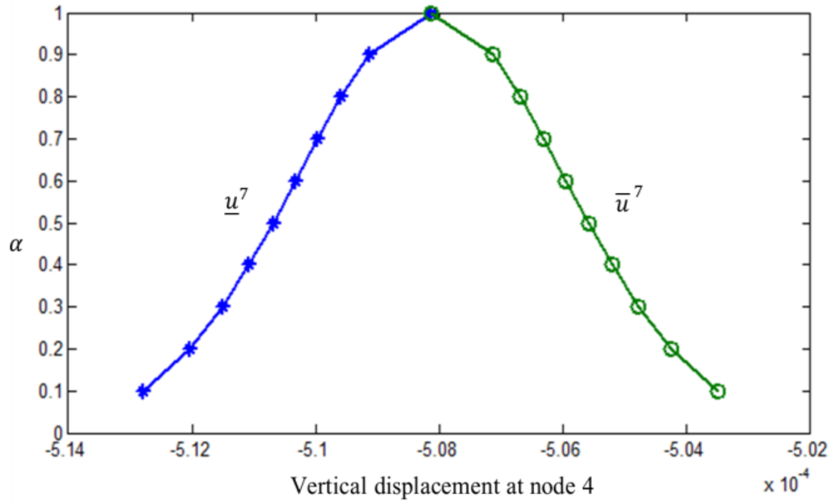


(a)

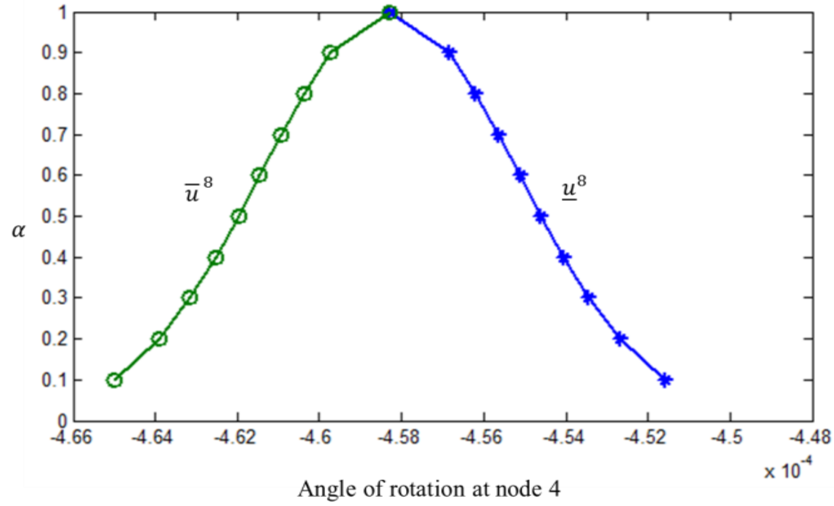


(b)

Fig. 4.46 Lower and upper bounds of (a) vertical displacement and (b) angle of rotation at node 3 for Gaussian fuzzy force



(a)



(b)

Fig. 4.47 Lower and upper bounds of (a) vertical displacement and (b) angle of rotation at node 4 for Gaussian fuzzy force

4.3. Uncertain Static Analysis of Truss

In this section, different type of truss viz. three bar truss, six bar truss and fifteen bar truss structures have been analysed for the uncertain static responses. In the following paragraphs, first a three bar truss structure with uncertain nodal forces have been investigated, where material and geometrical properties are taken as crisp. Next, a six bar truss structure with material and geometric properties along with the nodal force are all taken as fuzzy. Lastly, a fifteen bar truss structure with uncertain nodal forces have been considered where material and geometric properties are taken as crisp .

4.3.1. A three-bar truss with uncertain nodal force

A three-bar truss as shown in Fig. 4.48 has been taken into consideration. Only the horizontal ($\tilde{u}^j$ for $j = 1,3,5$) and vertical displacements ($\tilde{u}^j$ for $j = 2,4,6$) of nodes are considered. The truss is subject to the external load acting on node 3. Young's modulus and length of the three sections are considered here as crisp variables and are taken as $E^{(i)} = 200\ \text{Gpa}$ for $i = 1, 2, 3$ and $l^{(1)} = 800\ \text{mm}$, $l^{(2)} = l^{(3)} = 400\sqrt{2}\ \text{mm}$ (Bhavikati, 2005). The cross sectional area of the three members of truss is also taken as crisp viz. $A^{(1)} = 1500\ \text{mm}^2$, $A^{(2)} = 2000\ \text{mm}^2$ and $A^{(3)} = 2000\ \text{mm}^2$. With these material and geometric properties, the external load acting at node three are considered as crisp, interval and trapezoidal fuzzy numbers in Cases 7(a) to 7(c) respectively. Here, uncertain responses have been computed by Method 3 of Section 3.1.2.

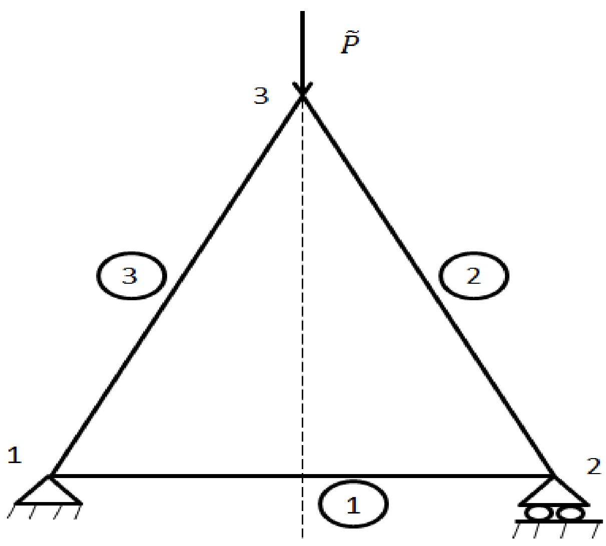


Fig. 4.48 A three-bar truss

Case 7(a): Nodal force as crisp

Let us consider the force at node 3 as crisp that is $\tilde{P} = 150$ kN.

Case 7(b): Nodal force as interval

Here, the nodal force has been taken as interval that is $\tilde{P} = [P^c - \beta P^c, P^c + \beta P^c]$ kN, where $P^c = 150$ kN and β is the uncertain factor.

Case 7(c): Nodal force as trapezoidal fuzzy number

In this case, the nodal force at node 3 has been considered as trapezoidal fuzzy number that is $\tilde{P} = (140, 145, 155, 160)$ kN.

For Case 7(a) one may obtain the static responses as $\tilde{u}^3 = u^3 = 0.2$ mm, $u^5 = \tilde{u}^5 = 0.1$ mm and $\tilde{u}^6 = u^6 = -0.312132$ mm using usual finite element analysis. Obtained results are compared with the results of Bhavikati (2005) and found to be equal. Computed results for Case 7(b) are shown in Figs. 4.49 and 4.50 with β changing from 0% to 25%. This problem has also been solved using Method 2 of Section 3.1.2 (Chapter 3) and Friedman et al. (1998) with uncertainty factor $\beta = 10\%$. Results obtained by these methods are compared in Table 4.17. Fuzzy static responses for Case 7(c) are depicted in Figs. 4.51 and 4.52.

Table 4.17 Interval static responses of three-bar truss with uncertain factor $\beta = 10\%$

$\tilde{u}$(mm)	$\underline{u}^i$			$\overline{u}^i$		
	Method 3 of Section 3.1.2	Method 2 of Section 3.1.2	Friedman et al. (1998)	Method 3 of Section 3.1.2	Method 2 of Section 3.1.2	Friedman et al. (1998)
$\tilde{u}^3$	0.22	0.22	0.22	0.18	0.18	0.18
$\tilde{u}^5$	0.09	0.09	0.09	0.11	0.11	0.11
$\tilde{u}^6$	-0.3433	-0.3433	-0.3433	-0.2809	-0.2809	-0.2809

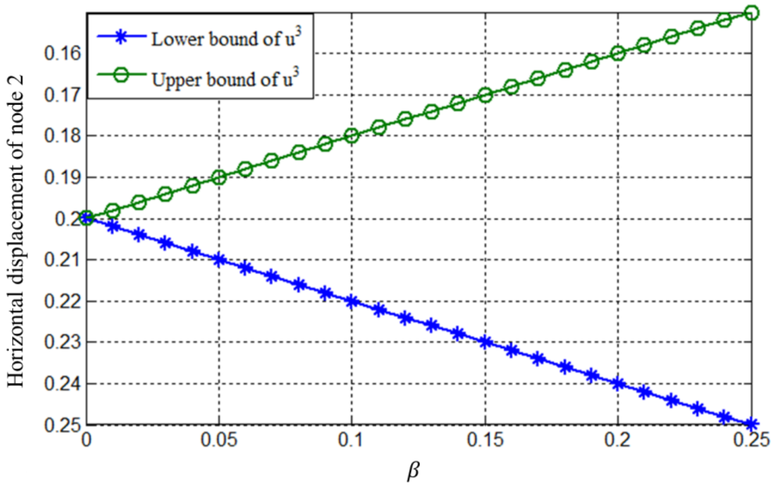
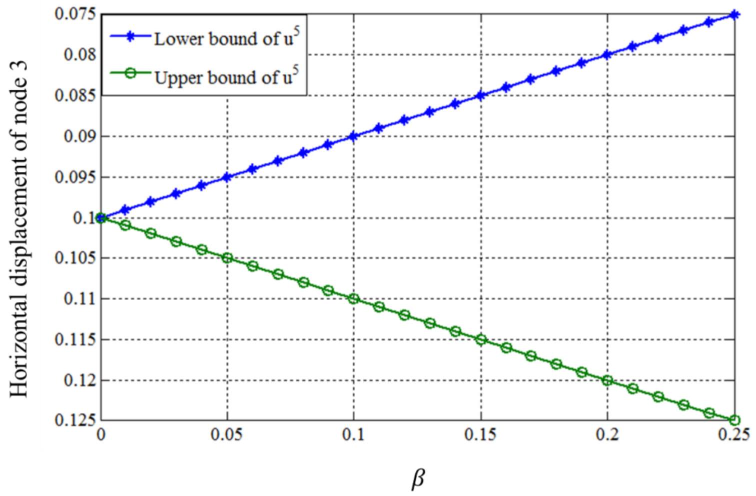
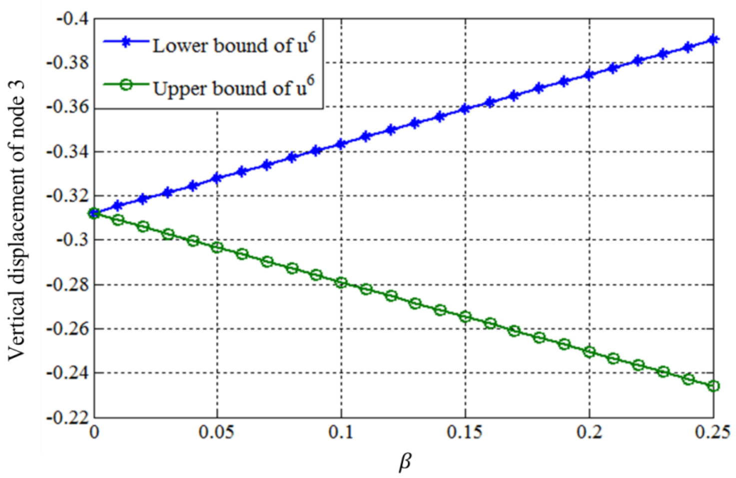


Fig. 4.49 Lower and upper bounds of the horizontal displacement at node 2 versus the uncertain factor β for three-bar truss



(a)



(b)

Fig. 4.50 Lower and upper bounds of (a) horizontal displacement and (b) vertical displacement at node 3 versus the uncertain factor β for three-bar truss

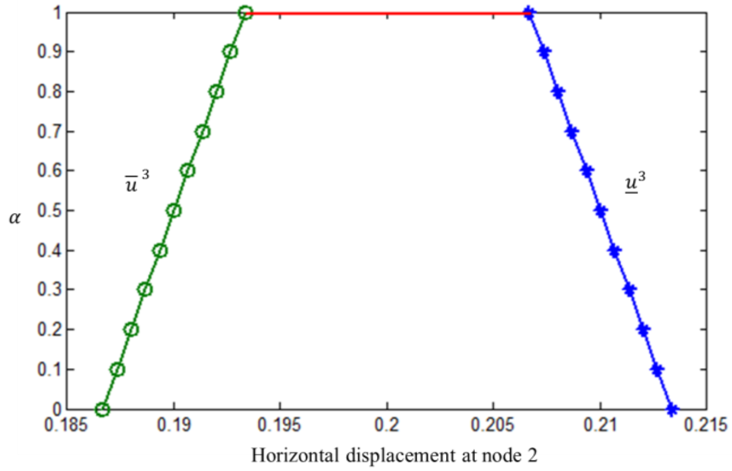
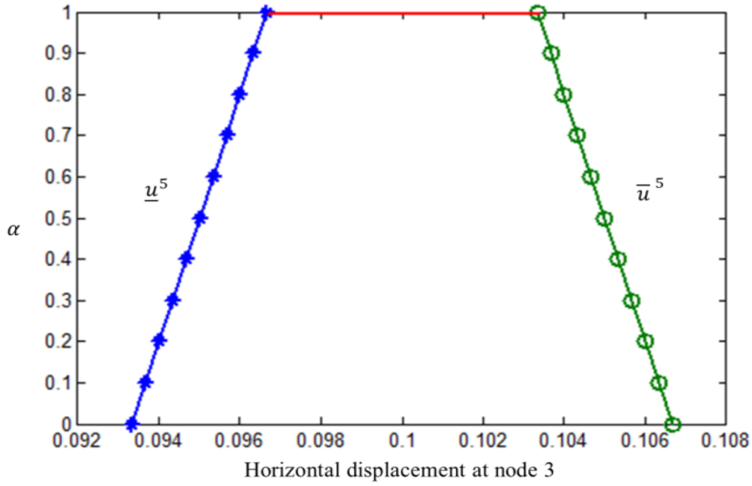
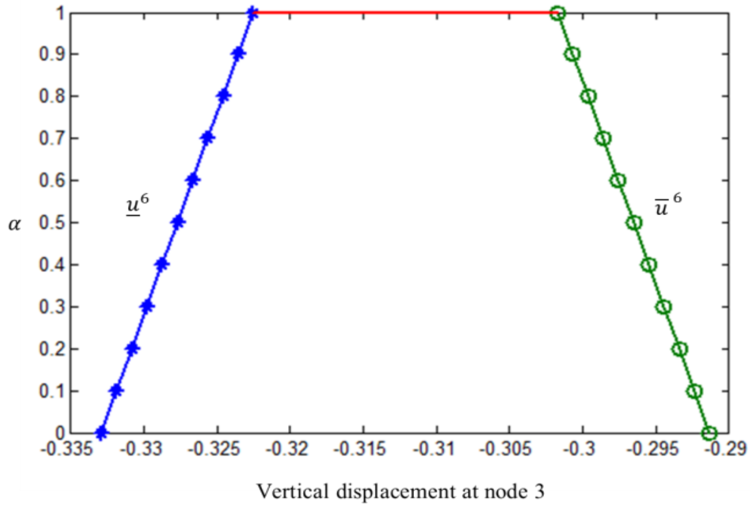


Fig. 4.51 Lower and upper bounds of horizontal displacement at node 2 for trapezoidal fuzzy force of three-bar truss



(a)



(b)

Fig. 4.52 Lower and upper bounds of (a) horizontal displacement and (b) vertical displacement at node 3 for trapezoidal fuzzy force of three-bar truss

In this example, one may notice (for crisp parameters) that horizontal displacement at node two is maximum. For trapezoidal nodal force, the lower and upper bounds of the horizontal displacement at node two are maximum. Here, maximum uncertainty at node two is obtained for horizontal displacements.

4.3.2. A six-bar truss with uncertain material, geometric properties and applied load

In this example, a 6-bar truss structure shown in Fig. 4.53 has been considered. The structure consists of 6 elements. Material, geometric properties and the applied load parameter $\tilde{P}$ are assumed as uncertain. The values of the input variables for the present analysis are shown in Table 4.18 for different cases. Here, the proposed method (Section 3.2.1.3 of Chapter 3) has been used to obtain the uncertain static responses of the structure.

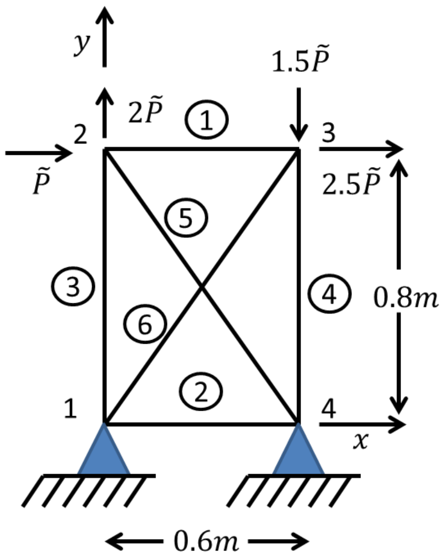


Fig. 4.53 Six-bar truss structure

In Case 8(a), the input parameters are considered as crisp value. Applied load parameter is only assumed as uncertain in Case 8(b). Cross sectional area for elements 5 and 6 along with load parameter are considered as fuzzy in Case 8(c). Lastly in Case 8(d), all the input variables are assumed as uncertain.

Usual finite element method for static analysis of structures with crisp parameters converts the problem into an algebraic system of linear equations. Hence, the corresponding equilibrium equation for the present problem, for Case 8(a) of Table 4.18 is represented as

$$K\delta = F\ ,$$

where K, F and δ are the reduced stiffness matrix, load and displacements vector respectively.

Here,

$$K = \begin{bmatrix} \frac{E_1A_1}{L_1} + 0.36\frac{E_5A_5}{L_5} & -0.48\frac{E_5A_5}{L_5} & -\frac{E_1A_1}{L_1} & 0 \\ -0.48\frac{E_5A_5}{L_5} & \frac{E_3A_3}{L_3} + 0.64\frac{E_5A_5}{L_5} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & 0 & \frac{E_1A_1}{L_1} + 0.36\frac{E_6A_6}{L_6} & 0.48\frac{E_6A_6}{L_6} \\ 0 & 0 & 0.48\frac{E_6A_6}{L_6} & \frac{E_4A_4}{L_4} + 0.64\frac{E_6A_6}{L_6} \end{bmatrix},$$

$F = \begin{bmatrix} P \\ 2P \\ 2.5P \\ -1.5P \end{bmatrix}$ and $\delta = \begin{bmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$ to be determined. We have assumed, $\tilde{A}_i = A_i$, $\tilde{E}_i = E_i$ and $\tilde{P} = P$ for all $i = 1$ to 6.

Substituting the corresponding values in the above expression as defined in Case 8(a) one may have

$$K = \begin{bmatrix} 429380 & -105840 & -350000 & 0 \\ -105840 & 403620 & 0 & 0 \\ -350000 & 0 & 429380 & 105840 \\ 0 & 0 & 105840 & 403620 \end{bmatrix} \text{(N/mm)},\ F = \begin{bmatrix} 20500 \\ 41000 \\ 51250 \\ -30750 \end{bmatrix} \text{(N)}.$$

As such, horizontal and vertical displacements at nodes 2 and 3 for Case 8(a) along with the comparison of Shu-xiang and Zhen-zhou (2001) and Qiu and Elishakoff (1998) are shown in Table 4.19. Obtained results are found to be in good agreement. Next, for Cases 8(b), 8(c) and 8(d), fuzzy finite element method is used with the proposed methodology of Section 3.2.1.3 to compute the uncertain static displacements. Corresponding results are depicted in Figs. 4.54 to 4.57. For special case $\alpha = 0$, obtained results for Case 8(c) are also compared with Shu-xiang and Zhen-zhou (2001) and Qiu and Elishakoff (1998). These are depicted in Table 4.20 and again a good agreement may be seen.

Table 4.18 Input data of 6 bar truss structure

Parameters	Case 8(a)	Case 8(b)	Case 8(c)	Case 8(d)
Modulus of elasticity $\tilde{E}_i$ for $i =$ 1 to 6 all elements (kN/m^2)	2.1×10^8	2.1×10^8	2.1×10^8	$(2\times10^8, 2.1\times10^8, 2.2\times10^8)$
Cross sectional area $\tilde{A}_i$, for $i = 1$ to 4 (m^2)	1.0×10^{-3}	1.0×10^{-3}	1.0×10^{-3}	$(0.9\times10^{-3}, 1.0\times10^{-3}, 1.1\times10^{-3})$
Cross sectional area of all other elements viz. $\tilde{A}_5$ and $\tilde{A}_6$ (m^2)	1.05×10^{-3}	1.05×10^{-3}	$(1\times10^{-3}, 1.05\times10^{-3}, 1.1\times10^{-3})$	$(1\times10^{-3}, 1.05\times10^{-3}, 1.1\times10^{-3})$
$\tilde{P}$ (kN)	20.5	(20,20.5,21)	(20,20.5,21)	(20,20.5,21)

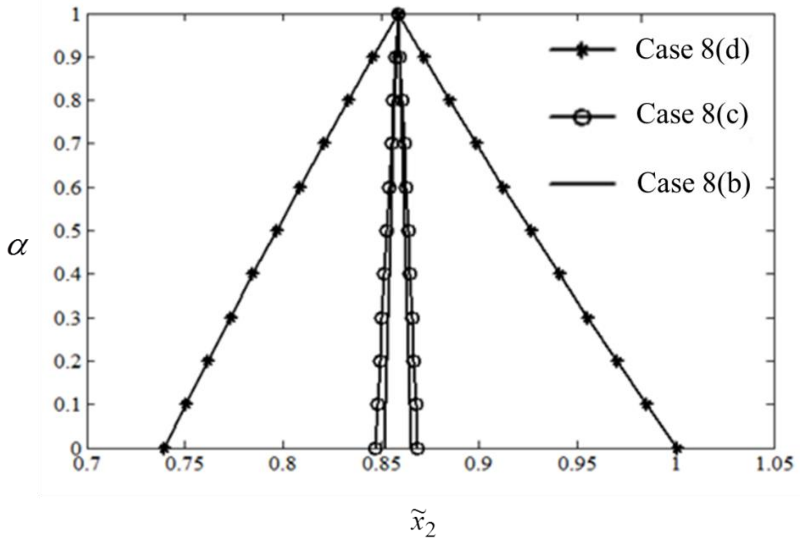


Fig. 4.54 Horizontal displacement at node 2 for 6 bar truss structure

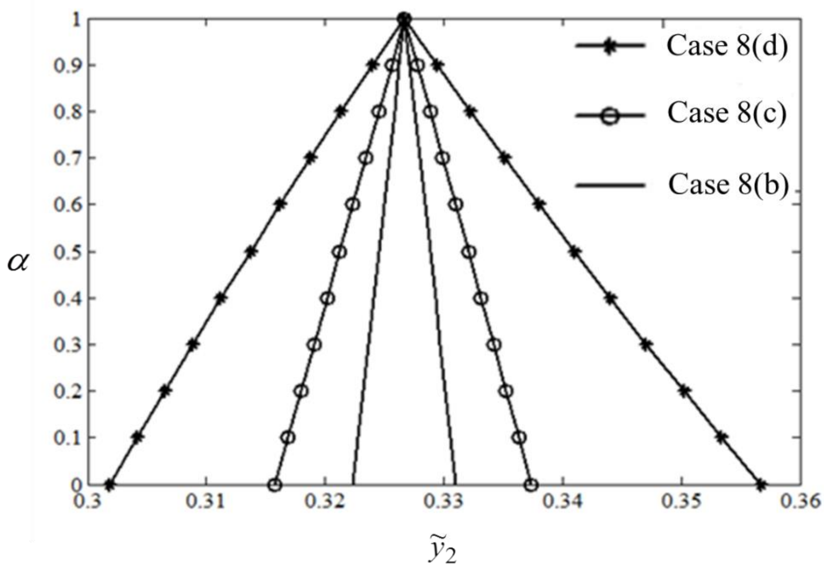


Fig. 4.55 Vertical displacement at node 2 for 6 bar truss structure

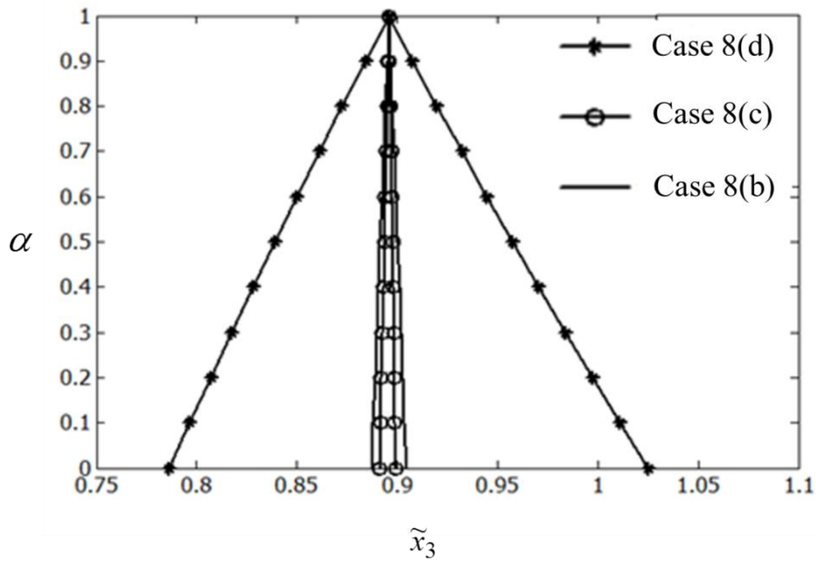


Fig. 4.56 Horizontal displacement at node 3 for 6 bar truss structure

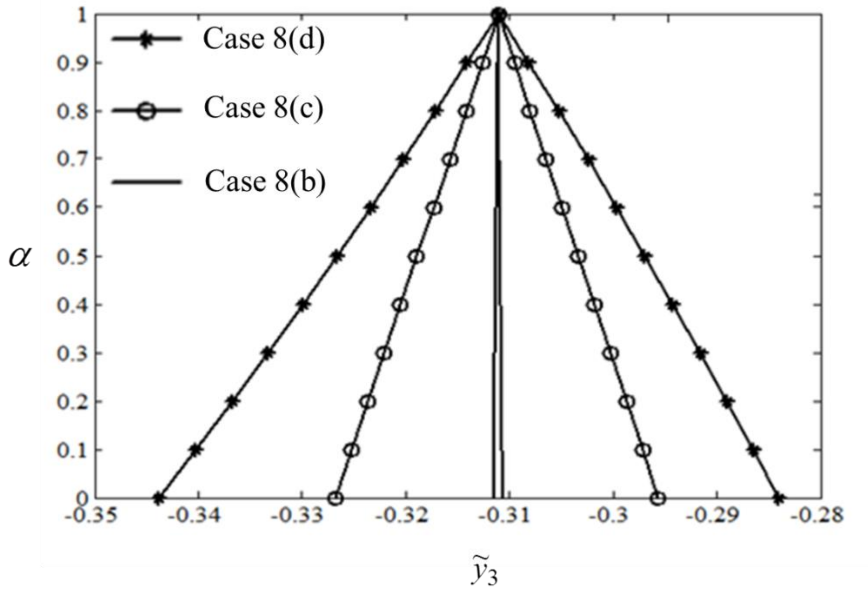


Fig. 4.57 Vertical displacement at node 3 for 6 bar truss structure

Table 4.19 Horizontal and vertical displacement of six- bar truss structure for Case 8(a)

Displacements (mm)	Shu-xiang and Zhen-zhou (2001)	Qiu and Elishakoff (1998)	Present method
x_2	0.8778	0.86	0.8585
y_2	0.3345	0.33	0.3267
x_3	0.9148	0.90	0.8958
y_3	-0.3171	-0.31	-0.3111

Table 4.20 Uncertain but bounded displacements of 6 bar truss structure of Case 8(c) for $\alpha = 0$

Displacements (mm)	Shu-xiang and Zhen-zhou (2001)	Qiu and Elishakoff (1998)	Present method
$\tilde{x}_2$	[0.7355,1.0201]	[0.69, 1.03]	[0.8469,0.8687]
$\tilde{y}_2$	[0.2782,0.3908]	[0.26,0.40]	[0.3158,0.3374]
$\tilde{x}_3$	[0.7730,1.0566]	[0.73,1.07]	[0.8910,0.8992]
$\tilde{y}_3$	[-0.3623,-0.2719]	[-0.38,-0.24]	[-0.3268, -0.2956]

One may notice from the results of six bar truss for Case 8(a) as depicted in Table 4.19 that horizontal and vertical displacements at node three are maximum and minimum respectively. From Table 4.20, for interval parameter (special case of Case 8(c) for $\alpha = 0$), uncertainty in the horizontal displacement at node two is found to be maximum and minimum at node three. Uncertainty in the vertical displacement at node three is maximum and minimum at node two. From Figs. 4.54 to 4.57, one can observe that larger width is obtained when fuzziness appears for all the parameters (Case 8(d)). It can clearly be seen from Figs. 4.54, 4.55 and 4.57 that minimum width is obtained for the displacements when fuzziness appears only in external load (Case 8(b)), and spread in the fuzzy displacements gradually increases when we have introduced fuzziness in the cross sectional area along with other parameters viz. for Cases 8(c) and 8(d) respectively. But in Fig. 4.56, minimum spread is obtained for Case 8(c). As discussed above, obtained results are compared in Table 4.20 with the results of existing methods (Shu-xiang and Zhen-zhou (2001) and Qiu and Elishakoff (1998)) in special cases to show the effectiveness of the proposed method. Moreover, we found that the proposed solution method estimates narrow bounds for the structural responses.

4.3.3. Truss with fifteen elements with uncertain nodal forces

A 15 bar truss structure (simplified bridge) as shown in Fig. 4.58 is considered. The structure consists of fifteen elements. Horizontal and vertical loads are applied at nodes 3 and 5 respectively. Material and geometric properties of the structure are considered as deterministic (crisp value). Forces applied at the nodes are taken as uncertain and let us first assume these as interval. Two different Cases 9(a) and 9(b) have been considered depending upon the input variables as shown in Table 4.21.

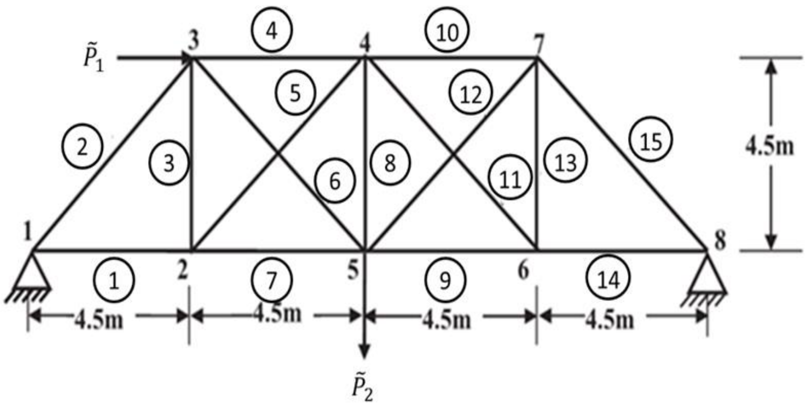


Fig. 4.58 Truss with fifteen elements

Using finite element method for Case 9(a), maximum horizontal and vertical displacements at nodes 3 and 5 are obtained as $0.1459\,\text{m}$ and $-0.4070\,\text{m}$ respectively.

For Case 9(b), varying β from 0% to 15%, bounds for horizontal and vertical displacements at nodes 3 and 5 are computed using Method 3 of Section 3.1.2 and are depicted in Figs. 4.59 and 4.60 respectively. The problem is also solved for $\beta = 30\%$ using Methods 2 and 3 of Section 3.1.2 and Friedman et al. (1998) which are shown in Table 4.22.

Table 4.21 Data of 15 bar truss with forces as crisp and interval value

Parameters	Case 9(a)	Case 9(b)
Cross sectional area A_1, A_2, A_3, A_{13}, A_{14}, A_{15} (m^2)	10×10^{-5}	10×10^{-5}
Cross sectional area of all other elements (m^2)	6×10^{-5}	6×10^{-5}
$\tilde{P}_1$ (N)	150×10^3	$[150\times10^3-\beta(150\times10^3), 150\times10^3+\beta(150\times10^3)]$
$\tilde{P}_2$ (N)	250×10^3	$[250\times10^3-\beta(250\times10^3), 250\times10^3+\beta(250\times10^3)]$
Modulus of elasticity of all elements $(\mathrm{N/m}^2)$	2×10^{11}	2×10^{11}

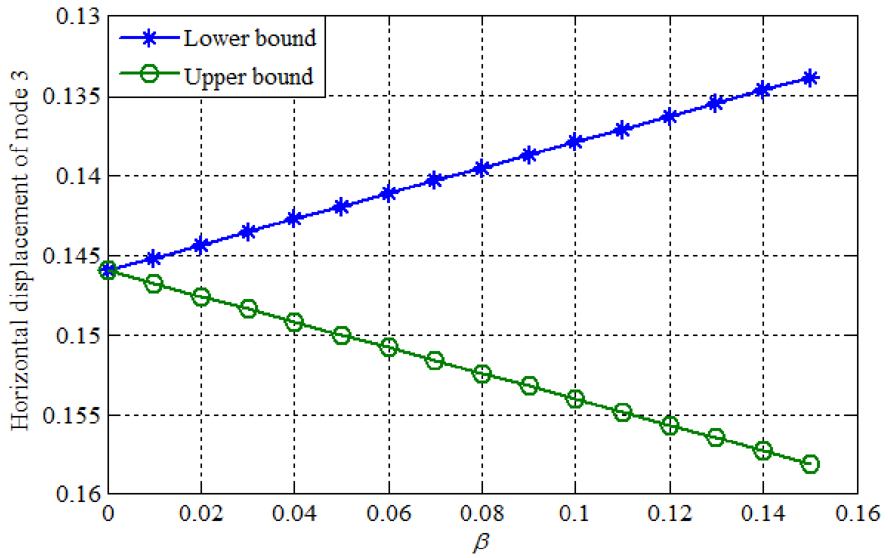


Fig. 4.59 Lower and upper bounds of horizontal displacement at node 3 for interval force of 15-bar truss

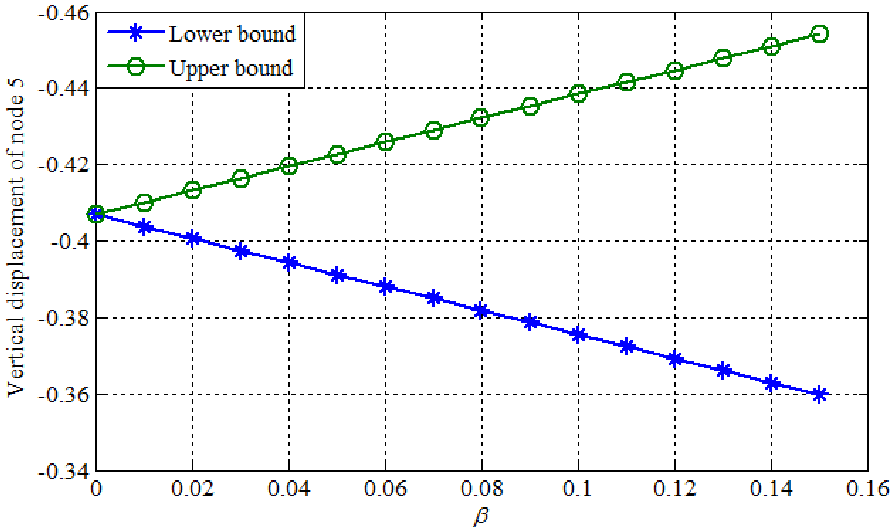


Fig. 4.60 Lower and upper bounds of vertical displacement at node 5 for interval force of 15-bar truss

Table 4.22 Interval static responses of 15-bar truss with uncertain factor $\beta = 30\%$

$\tilde{u}$(m)	$\underline{u}^i$			$\overline{u}^i$		
	Method 2 of Section 3.1.2	Method 3 of Section 3.1.2	Friedman et al. (1998)	Method 2 of Section 3.1.2	Method 3 of Section 3.1.2	Friedman et al. (1998)
$\tilde{u}^3$	0.1217	0.1217	0.1217	0.1702	0.1702	0.1702
$\tilde{u}^5$	-0.3126	-0.3126	-0.3126	-0.5014	-0.5014	-0.5014

Next, forces applied at the nodes are taken as Gaussian and trapezoidal fuzzy numbers. Three different cases have been considered for both trapezoidal and Gaussian fuzzy nodal force varying the uncertainty width. The input variables for various cases are shown in Tables 4.23 and 4.24 respectively for Gaussian and trapezoidal fuzzy nodal force.

For Gaussian fuzzy forces as given in Cases 10(a), 10(b) and 10(c) of Table 4.23, fuzzy finite element method has been used with the Methods 1 and 4 of Section 3.1.2. Solution bounds for horizontal and vertical displacements at nodes 3 and 5 are computed and depicted in Figs. 4.61 and 4.62 respectively. Similarly for trapezoidal fuzzy, horizontal and vertical displacements at nodes 3 and 5 are obtained using the Methods 2 and 4 of Section 3.1.2 and are shown in Figs. 4.63 and 4.64 respectively.

Table 4.23 Data of 15 bar truss element with forces as Gaussian fuzzy numbers

Parameters	Case 10(a)	Case 10(b)	Case 10(c)
Cross sectional area A_1, A_2, A_3, A_{13}, A_{14}, A_{15} (m^2)	10×10^{-5}	10×10^{-5}	10×10^{-5}
Cross sectional area of all other element (m^2)	6×10^{-5}	6×10^{-5}	6×10^{-5}
$\tilde{P}_1$ (N)	$(150\times10^3, 5\times10^3, 5\times10^3)$	$(150\times10^3, 10\times10^3, 10\times10^3)$	$(150\times10^3, 15\times10^3, 15\times10^3)$
$\tilde{P}_2$ (N)	$(250\times10^3, 5\times10^3, 5\times10^3)$	$(250\times10^3, 10\times10^3, 10\times10^3)$	$(250\times10^3, 15\times10^3, 15\times10^3)$
Modulus of elasticity of all elements (N/m^2)	2×10^{11}	2×10^{11}	2×10^{11}

Table 4.24 Data of 15 bar truss element with forces as trapezoidal fuzzy numbers

Parameters	Case 11(a)	Case 11(b)	Case 11(c)
Cross sectional area A_1, A_2, A_3, A_{13}, A_{14}, A_{15} (m^2)	10×10^{-5}	10×10^{-5}	10×10^{-5}
Cross sectional area of all other element (m^2)	6×10^{-5}	6×10^{-5}	6×10^{-5}
$\tilde{P}_1$ (N)	$[(2\times10^3)\alpha+(145\times10^3), -(2\times10^3)\alpha+(155\times10^3)]$	$[(5\times10^3)\alpha+(140\times10^3), -(5\times10^3)\alpha+(160\times10^3)]$	$[(5\times10^3)\alpha+(135\times10^3), -(5\times10^3)\alpha+(165\times10^3)]$
$\tilde{P}_2$ (N)	$[(2\times10^3)\alpha+(245\times10^3), -(2\times10^3)\alpha+(255\times10^3)]$	$[(5\times10^3)\alpha+(240\times10^3), -(5\times10^3)\alpha+(260\times10^3)]$	$[(5\times10^3)\alpha+(235\times10^3), -(5\times10^3)\alpha+(265\times10^3)]$
Modulus of elasticity of all elements (N/m^2)	2×10^{11}	2×10^{11}	2×10^{11}

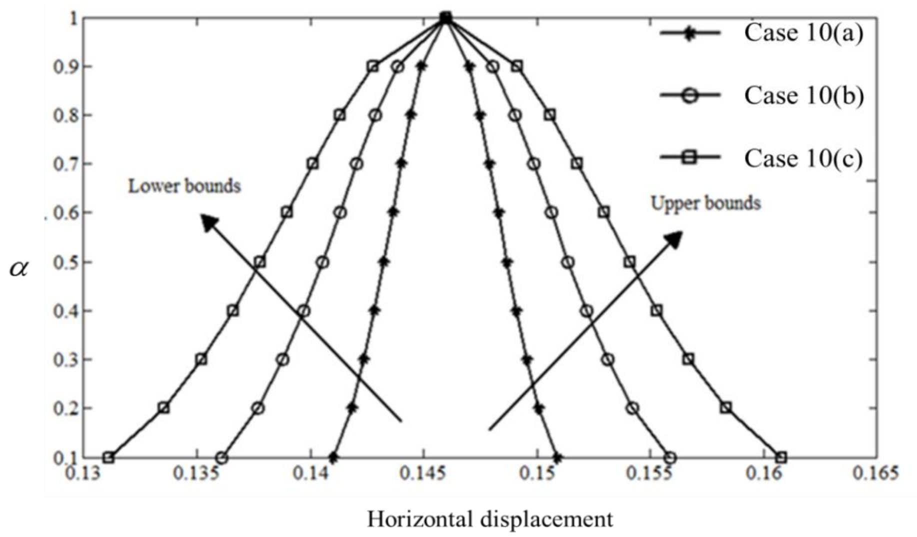


Fig. 4.61 Gaussian fuzzy horizontal displacement at node 3 for 15 bar truss

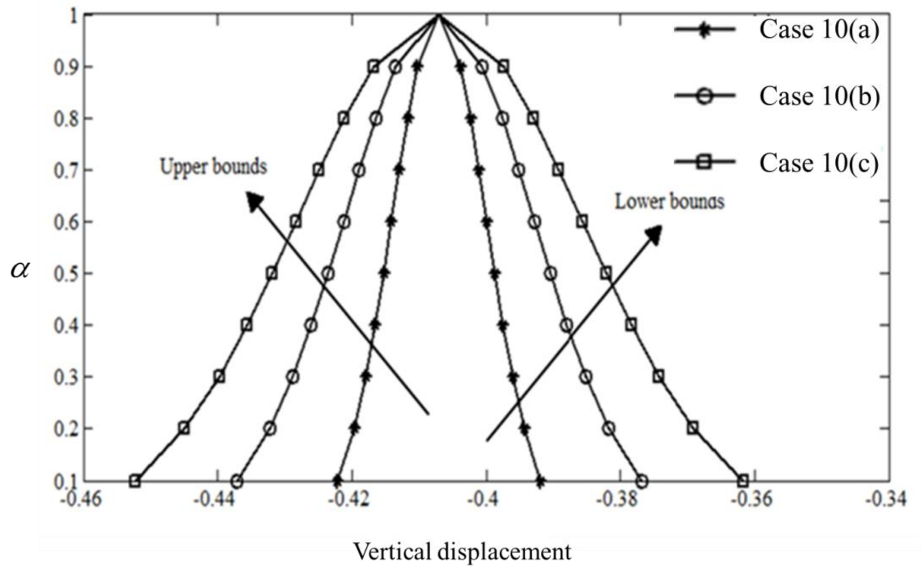


Fig. 4.62 Gaussian fuzzy vertical displacements at node 5 for 15 bar truss

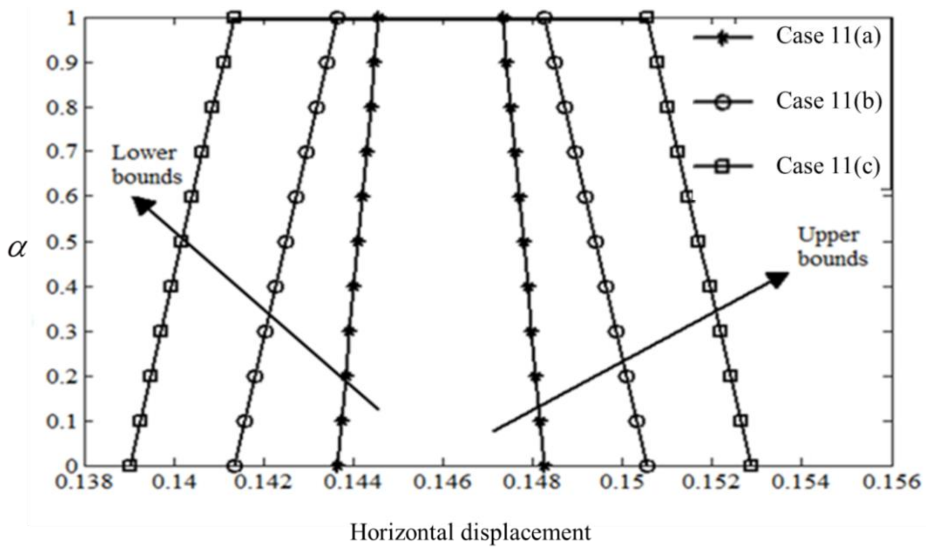


Fig. 4.63 Trapezoidal fuzzy horizontal displacements at node 3 for 15 bar truss

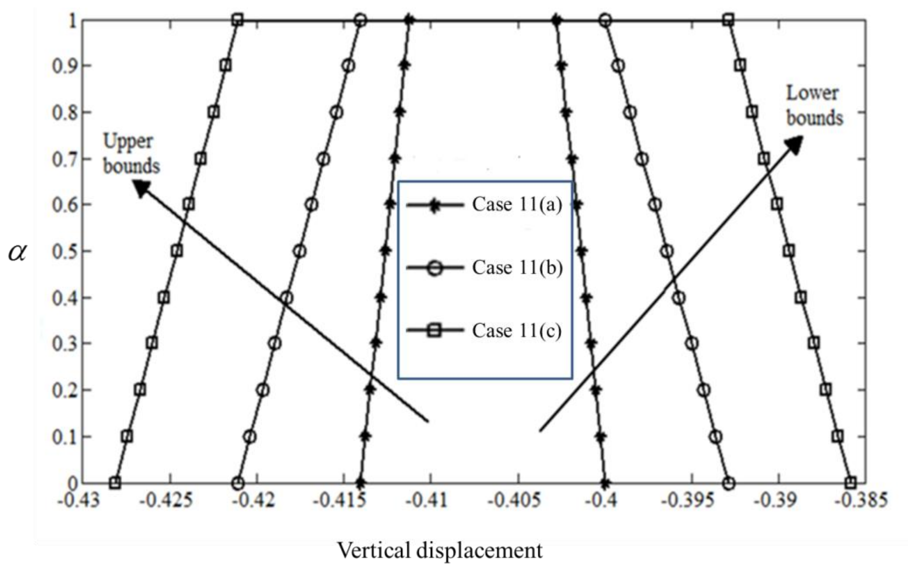


Fig. 4.64 Trapezoidal fuzzy vertical displacements at node 5 for 15 bar truss

It is worth mentioning that the results obtained by the Methods 1 and 4 are found to be exactly the same for Gaussian fuzzy load. Similarly, results obtained by Methods 2 and 4 for trapezoidal fuzzy load are also the same. Here, one may see that the uncertainty spread gradually increases by increasing the uncertainty spread of input parameter viz. fuzzy nodal forces.

4.4. Uncertain Static Analysis of Rectangular Sheet

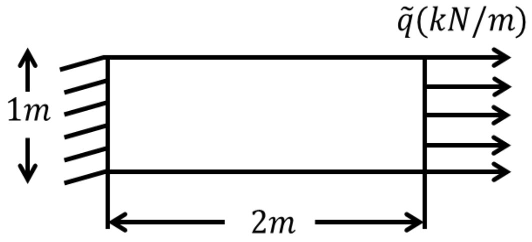
A uniform rectangular sheet as shown in Fig. 4.65 is considered. One of the ends is fixed and in the other end, uniform force $\tilde{q}$ is applied. Applied force and elastic modulus are considered as uncertain. Input for all the variables and parameters are shown in Table 4.25 for different cases viz. Cases 12(a) to 12(e).

Applying usual finite element method with crisp parameters with the boundary conditions, the reduced equilibrium equation for the above structure may be obtained as

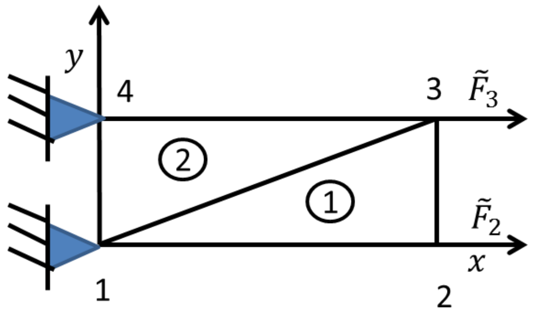
$$K\delta = F\,,$$

where $K = \dfrac{3Eh}{32}\begin{bmatrix} 7 & -4 & -4 & 2 \\ -4 & 13 & 2 & -12 \\ -4 & 2 & 7 & 0 \\ 2 & -12 & 0 & 13 \end{bmatrix}$, $F = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{q}{2} \\ 0 \\ \frac{q}{2} \\ 0 \end{bmatrix}$ and $\delta = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$.

Here we have assumed $\tilde{E} = E$ and $\tilde{q} = q$.



(a)



(b)

Fig. 4.65 (a) The applied force and (b) finite elements of the rectangular sheet

Solving the corresponding system with the input values as defined in Case 12(a), crisp displacements along with the comparison of existing results are given in Table 4.26. Next, for Cases 12(b) to 12(d), Methods 2 to 5 of Section 3.1.2 have been used. For Case 12(b), obtained interval displacements are shown in Table 4.27. For Case 12(c), we have supposed the uniformly applied force $\tilde{q} = [q^c - \beta q^c, q^c + \beta q^c]$ N/m as intervals where $q^c = 1000$ N/m and β is the uncertainty factor so that we can check the static displacement bounds with changing β. Obtained results for this are shown in Figs. 4.66 to 4.69, varying β from 0% to 25%. To be more illustrative, Table 4.28 lists the bounds of static response of rectangular sheet when $\beta = 5\%$, 10% and $\beta = 15\%$. It is worth mentioning that Case 12(c) has also been solved by Friedman et al. (1998). The results obtained by these methods are compared with the present results and are shown in Table 4.29 for uncertainty factor $\beta = 20\%$. It can be observed that the uncertainty widths of the displacements gradually increase with the increase of uncertainty factor β. Results for Case 12(d) are depicted in Table 4.30. Lastly, method described in Section 3.2.1.3 has been applied for Case 12(e) and corresponding results are given in Table 4.31.

Table 4.25 Input data of rectangular sheet

Parameters	Case 12(a)	Case 12(b)	Case 12(c)	Case 12(d)	Case 12(e)
Thickness h (mm)	1	1	1	1	1
Length (m)	2	2	2	2	2
Width (m)	1	1	1	1	1
Modulus of elasticity $\tilde{E}$ (N/m^2)	206×10^9	206×10^9	206×10^9	206×10^9	$(204,206,208)\times10^9$
Poison ratio μ	1/3	1/3	1/3	1/3	1/3
$\tilde{q}$ (kN/m)	1	[0.98,1.02]	$[q^c-\beta q^c, q^c+\beta q^c]$	(0.99,1,1.01)	(0.99,1,1.01)

Table 4.26 Horizontal and vertical displacement of rectangular sheet for Case 12(a)

Displacements (m)	Huang and Li (2005)	Li et al. (2003)	Present method
x_2	9.6295e-006	9.6295e-006	0.9629e-005
y_2	1.7298e-006	1.7298e-006	0.1730e-005
x_3	8.7069e-006	8.7069e-006	0.8707e-005
y_3	0.1153e-006	0.1153e-006	0.0115e-005

Table 4.27 Interval displacements of rectangular sheet structure (Case 12(b)) with forces as interval value

Interval displacements (m)	
$\tilde{x}_2$	[9.538e-006, 9.721 e-006]
$\tilde{y}_2$	[1.907 e-006,1.552 e-006]
$\tilde{x}_3$	[8.634 e-006, 8.780 e-006]
$\tilde{y}_3$	[-0.364e-006, 2.652 e-006]

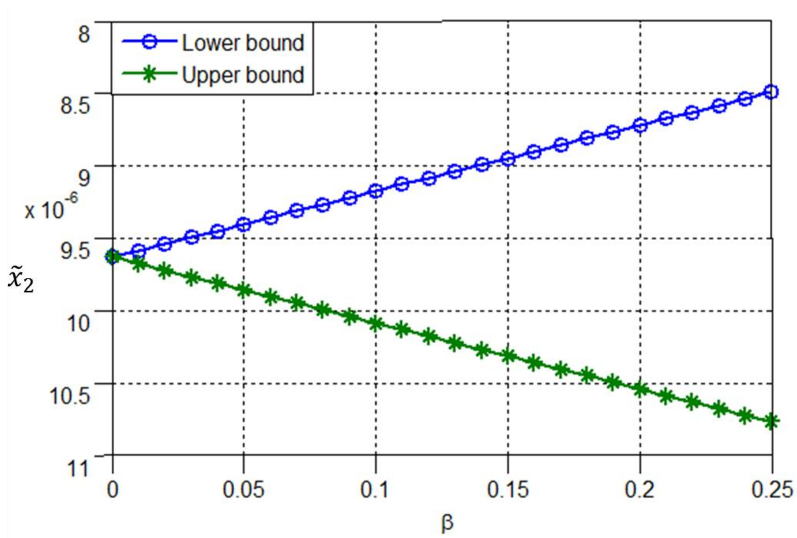


Fig. 4.66 Solution bound for $\tilde{x}_2$ (Case 12(c))

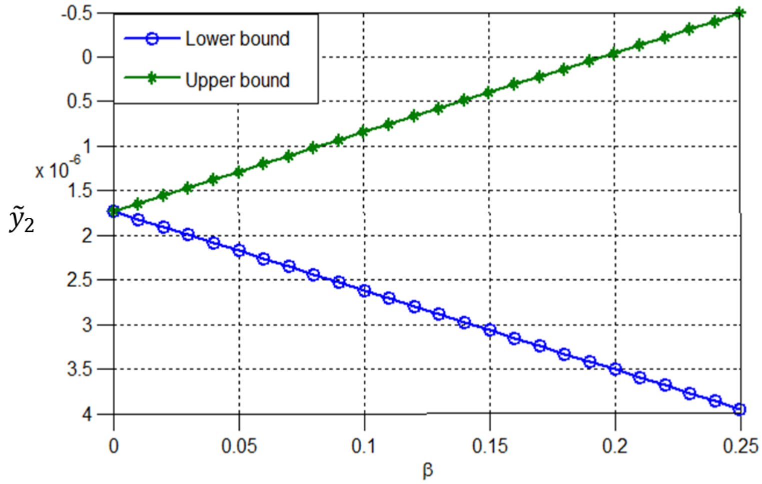


Fig. 4.67 Solution bound for $\tilde{y}_2$ (Case 12(c))

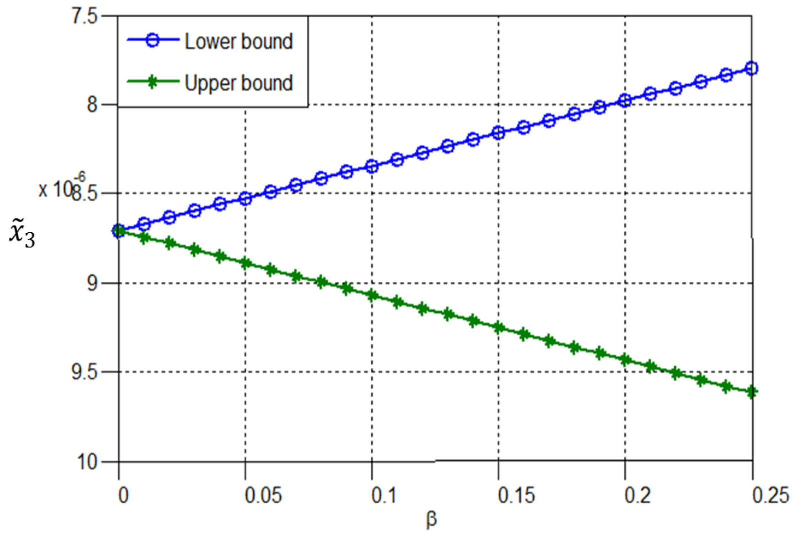


Fig. 4.68 Solution bound for $\tilde{x}_3$ (Case 12(c))

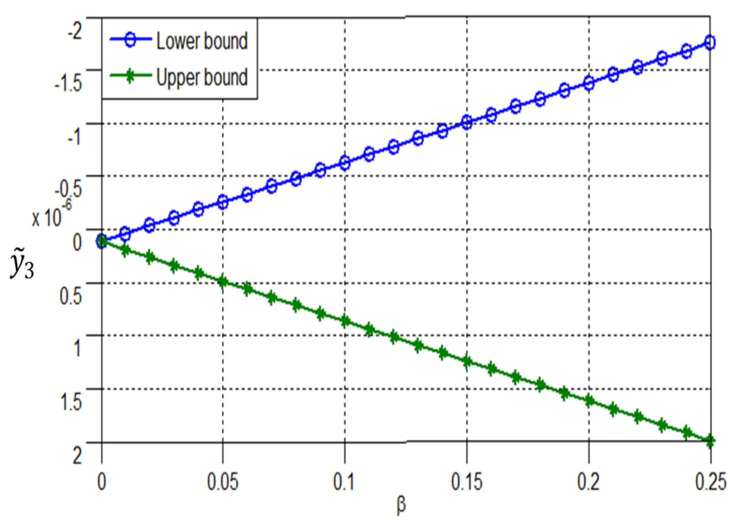


Fig. 4.69 Solution bound for $\tilde{y}_3$ (Case 12(c))

Table 4.28 Interval static responses for rectangular sheet (Case 12(c)) with uncertain factor $\beta = 5\%$, 10% and $\beta =$ 15%

Interval displacements	$\beta = 5\%$	$\beta = 10\%$	$\beta =$ 15%
$\tilde{x}_2$	[9.4017e-006, 9.8572e-006]	[9.1739e-006, 1.0085e-005]	[8.9462e-006, 1.0313e-005]
$\tilde{y}_2$	[2.1738e-006, 1.2858e-006]	[2.6178e-006, 8.4186e-007]	[3.0618e-006, 3.9786e-007]
$\tilde{x}_3$	[8.5252e-006, 8.8885e-006]	[8.3436e-006, 9.0701e-006]	[8.1620e-006, 9.2518e-006]
$\tilde{y}_3$	[-2.5948e-007, 4.9012e-007]	[-6.3428e-007, 8.6492e-007]	[-1.0091e-006, 1.2397e-006]

Table 4.29 Interval static responses for rectangular sheet (Case 12(c)) with uncertain factor $\beta =$ 20%

Interval displacements	$\beta =$ 20%	
	Present method(s)	Friedman et al. (1998)
$\tilde{x}_2$	[8.7184e-006, 1.0541e-005]	[8.7184e-006, 1.0541e-005]
$\tilde{y}_2$	[3.5058e-006, -4.6129e-008]	[3.5058e-006, -4.6129e-008]
$\tilde{x}_3$	[7.9803e-006, 9.4334e-006]	[7.9803e-006, 9.4334e-006]
$\tilde{y}_3$	[-1.3839e-006, 1.6145e-006]	[-1.3839e-006, 1.6145e-006]

Table 4.30 Lower and upper bounds of fuzzy static responses of rectangular sheet for Case 12(d)

α	0	0.2	0.6	0.8	1
$\underline{x}_2$	0.9584e-005	0.9593e-005	0.9611e-005	0.9620e-005	0.9629e-005
$\overline{x}_2$	0.9675e-005	0.9666e-005	0.9648e-005	0.9639e-005	0.9629e-005
$\underline{y}_2$	0.1641e-005	0.1659e-005	0.1694e-005	0.1712e-005	0.1730e-005
$\overline{y}_2$	0.1819e-005	0.1801e-005	0.1765e-005	0.1748e-005	0.1730-005
$\underline{x}_3$	0.8671e-005	0.8678e-005	0.8692e-005	0.8700e-005	0.8707e-005
$\overline{x}_3$	0.8743e-005	0.8736e-005	0.8721e-005	0.8714e-005	0.8707e-005
$\underline{y}_3$	0.0040e-005	0.0055e-005	0.0085e-005	0.0100e-005	0.0115e-005
$\overline{y}_3$	0.0190e-005	0.0175e-005	0.0145e-005	0.0130e-005	0.0115e-005

Table 4.31 Lower and upper bounds of fuzzy static responses of rectangular sheet for Case 12(e)

α	0	0.2	0.6	0.8	1
$\underline{x}_2$	0.9582e-005	0.9591e-005	0.9610e-005	0.9620e-005	0.9629e-005
$\overline{x}_2$	0.9678e-005	0.9668e-005	0.9649e-005	0.9639e-005	0.9629e-005
$\underline{y}_2$	0.1625e-005	0.1646e-005	0.1688e-005	0.1709e-005	0.1730e-005
$\overline{y}_2$	0.1836e-005	0.1815e-005	0.1772e-005	0.1751e-005	0.1730-005
$\underline{x}_3$	0.8659e-005	0.8669e-005	0.8688e-005	0.8697e-005	0.8707e-005
$\overline{x}_3$	0.8756e-005	0.8746e-005	0.8726e-005	0.8717e-005	0.8707e-005
$\underline{y}_3$	0.0041e-005	0.0056e-005	0.0086e-005	0.0101e-005	0.0115e-005
$\overline{y}_3$	0.0188e-005	0.0174e-005	0.0145e-005	0.0130e-005	0.0115e-005

From Table 4.26 one may observe that horizontal and vertical displacements of rectangular sheet for crisp parameters at node two and three are maximum and minimum respectively. Obtained results are compared with Huang and Li (2005) and Li et al. (2003) and are found to be same. One may note from Tables 4.30 and 4.31 that horizontal and vertical displacements at node two and horizontal displacement at node 3 suffers more uncertainty for Case 12(e). Also vertical displacement at node 3 for Case 12(d) suffers more uncertainty. Similarly horizontal and vertical displacements at node two and horizontal displacement at node 3 suffers less uncertainty for Case 12(d). Vertical displacement at node 3 for Case 12(e) suffers less uncertainty. It is worth mentioning that in special case for $\alpha = 1$, fuzzy displacements completely agree with the deterministic results as shown in Table 4.26.

Chapter 5

Uncertain Dynamic Analysis of Structural Systems

The content of this chapter has been published in:

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2. Chakraverty, S., Behera, D. (2014) Uncertain static and dynamic analysis of imprecisely-defined structural systems, Mathematics of Uncertainty Modeling in the Analysis of Engineering and Science Problems, Editor: S. Chakraverty, IGI Global Publication, USA, 357-382 (Book Chapter).

Chapter 5

Uncertain Dynamic Analysis of Structural Systems

For dynamic analysis of structures with uncertain material and geometric properties, one may obtain the generalized uncertain eigenvalue problem. Uncertainties are assumed to be present here in the material and geometric properties which are modeled through triangular and trapezoidal convex normalized fuzzy sets. It is an important and challenging issue to deal with this type of uncertain eigenvalue problems. From the literature review, it reveals that interval eigenvalue problems are handled mostly for standard and generalized eigenvalue problems. But fuzzy eigenvalue problems are analysed by few authors only for standard eigenvalue problems. Accordingly, this chapter proposes an algorithm to solve generalized fuzzy eigenvalue problem by extending the method of Chen et al. (1995). For verification and validation of the algorithm, vibration analysis of multistorey shear building, spring mass mechanical system and stepped beam are taken into consideration. Obtained results are compared with the existing results in special cases. Computed frequency parameters and corresponding mode shapes are depicted in term of plots and tables.

This chapter also includes the parameter identification procedure of a multistory frame structure from uncertain dynamic data. In this regard, an iterative scheme has been proposed using fuzzy Taylor series expansion. Numerical examples related to this have been incorporated to show the efficiency of the proposed method. As such, we first discuss below about the generalized fuzzy eigenvalue problem.

5.1. Generalized Fuzzy Eigenvalue Problem

Let us now consider the generalized fuzzy eigenvalue problem as

$$[\tilde{K}]\{\tilde{W}\} = \tilde{\lambda}[\tilde{M}]\{\tilde{W}\} \tag{5.1}$$

where, $[\tilde{K}]$ and $[\tilde{M}]$ are square matrices with elements as triangular fuzzy number. Here $[\tilde{K}] = [\underline{K}, K, \overline{K}]$ and $[\tilde{M}] = [\underline{M}, M, \overline{M}]$ in which $\underline{K} = \underline{k}_{ij}$, $K = k_{ij}$, $\overline{K} = \bar{k}_{ij}$ are $n \times n$

symmetric matrices and $\underline{M} = \underline{m}_{ij}$, $M = m_{ij}$, $\overline{M} = \overline{m}_{ij}$ are $n \times n$ symmetric positive definite matrices.

$\underline{K}$, $\overline{K}$, $\underline{M}$ and $\overline{M}$ are known matrices which have the elements for membership, $\alpha = 0$ of $[\tilde{K}]$ and $[\tilde{M}]$. Also, K and M contains the entries for membership $\alpha = 1$.

Similarly, for trapezoidal fuzzy number matrices in Eq. (1), $[\tilde{K}] = [\underline{K}, \underline{\underline{K}}, \overline{\overline{K}}, \overline{K}]$ and $[\tilde{M}] = [\underline{M}, \underline{\underline{M}}, \overline{\overline{M}}, \overline{M}]$ where $\underline{K} = \underline{k}_{ij}$, $\underline{\underline{K}} = \underline{\underline{k}}_{ij}$, $\overline{\overline{K}} = \overline{\overline{k}}_{ij}$, $\overline{K} = \overline{k}_{ij}$ are $n \times n$ symmetric matrices and $\underline{M} = \underline{m}_{ij}$, $\underline{\underline{M}} = \underline{\underline{m}}_{ij}$, $\overline{\overline{M}} = \overline{\overline{m}}_{ij}$, $\overline{M} = \overline{m}_{ij}$ are $n \times n$ symmetric positive definite matrices.

Here, $\underline{K}$, $\overline{K}$, $\underline{M}$ and $\overline{M}$ are matrices obtained for membership $\alpha = 0$ and $\underline{\underline{K}}$, $\overline{\overline{K}}$, $\underline{\underline{M}}$ and $\overline{\overline{M}}$ are the matrices for membership $\alpha = 1$ for $[\tilde{K}]$ and $[\tilde{M}]$.

So for triangular and trapezoidal fuzzy matrices, the generalized fuzzy eigenvalue problem viz. Eq. (5.1), can be written respectively as follows

$$[\underline{K}, K, \overline{K}]\{\tilde{W}\} = \tilde{\lambda}[\underline{M}, M, \overline{M}]\{\tilde{W}\} \tag{5.2}$$

and

$$[\underline{K}, \underline{\underline{K}}, \overline{\overline{K}}, \overline{K}]\{\tilde{W}\} = \tilde{\lambda}[\underline{M}, \underline{\underline{M}}, \overline{\overline{M}}, \overline{M}]\{\tilde{W}\} \tag{5.3}$$

where, $\tilde{\lambda}$ and $\{\tilde{W}\}$ are the corresponding fuzzy eigenvalue and eigenvector.

Using $\alpha -$cut approach, one may write Eqs. (5.2) and (5.3) as

$$[\tilde{K}(\alpha)]\{\tilde{W}(\alpha)\} = \tilde{\lambda}(\alpha)[\tilde{M}(\alpha)]\{\tilde{W}(\alpha)\} \tag{5.4}$$

where, $\tilde{K}(\alpha) = [\underline{K}(\alpha), \overline{K}(\alpha)]$ and $\tilde{M}(\alpha) = [\underline{M}(\alpha), \overline{M}(\alpha)]$.

Let all eigenvalues $\tilde{\lambda}_i(\alpha)$ for $i = 1,2,3,\cdots,n$ of Eq. (5.4) be arranged in an ascending order defined by $\tilde{\lambda}_i(\alpha) = [\underline{\lambda}_i(\alpha), \overline{\lambda}_i(\alpha)]$ and the corresponding eigenvectors also be defined as $\tilde{W}_i(\alpha) = [\underline{W}_i(\alpha), \overline{W}_i(\alpha)]$ for $i = 1,2,3,\cdots,n$.

One may notice that the fuzzy matrices in $\alpha -$cut representation are in interval form. So, we may write the interval fuzzy matrices as

$$\tilde{K}(\alpha) = [\underline{K}(\alpha), \overline{K}(\alpha)] = [\tilde{K}^c(\alpha) - \Delta\tilde{K}(\alpha), \tilde{K}^c(\alpha) + \Delta\tilde{K}(\alpha)]$$

and

$$\tilde{M}(\alpha)=[\underline{M}(\alpha),\overline{M}(\alpha)]=[\tilde{M}^{c}(\alpha)-\Delta\tilde{M}(\alpha),\tilde{M}^{c}(\alpha)+\Delta\tilde{M}(\alpha)]$$

where $\tilde{K}^{c}(\alpha)=\dfrac{\underline{K}(\alpha)+\overline{K}(\alpha)}{2}$, $\Delta\tilde{K}(\alpha)=\dfrac{\overline{K}(\alpha)-\underline{K}(\alpha)}{2}$, $\tilde{M}^{c}(\alpha)=\dfrac{\underline{M}(\alpha)+\overline{M}(\alpha)}{2}$ and

$\Delta\tilde{M}(\alpha)=\dfrac{\overline{M}(\alpha)-\underline{M}(\alpha)}{2}$.

In order to solve the fuzzy eigenvalue problem, an algorithm is developed for finding the uncertain eigenvalues of Eq. (5.1) by extending the method of Chen et al. (1995).

5.1.1. Algorithm for computing fuzzy eigenvalues

Step 1 : Initialize the fuzzy stiffness and mass matrices $[\tilde{K}(\alpha)]$ and $[\tilde{M}(\alpha)]$ respectively.

Step 2 : Find central as well as width of stiffness and mass matrices respectively as

$\tilde{K}^{c}(\alpha)$, $\Delta\tilde{K}(\alpha)$, $\tilde{M}^{c}(\alpha)$ and $\Delta\tilde{M}(\alpha)$.

Step 3 : Solve the central generalized eigenvalue problem

$[\tilde{K}^{c}(\alpha)]\{\tilde{W}^{c}(\alpha)\}=\tilde{\lambda}^{c}(\alpha)[\tilde{M}^{c}(\alpha)]\{\tilde{W}^{c}(\alpha)\}$.

Step 4 : Compute $S_i=diag(sign(w^{1}{}_{i}),sign(w^{2}{}_{i}),\cdots sign(w^{n}{}_{i}))$ where $w^{j}{}_{i}\neq 0$. $w^{j}{}_{i}$ are the components of $i-$the eigenvector obtained in Step 3 for $j=1,2,\cdots,n$.

Step 5 : Obtain the left bound of fuzzy eigenvalue by solving

$[\tilde{K}^{c}(\alpha)-S_i\Delta\tilde{K}(\alpha)S_i]\{\underline{W}_i(\alpha)\}=\underline{\lambda}_i(\alpha)[\tilde{M}^{c}(\alpha)+S_i\Delta\tilde{M}(\alpha)S_i]\{\underline{W}_i(\alpha)\}$

for $i=1,2,3,\cdots,n$.

For each i, one has n number of eigenvalues viz. $\underline{\lambda}_i^{j}(\alpha)$ for $j=1,2,\cdots,n$.

Step 6 : Obtain the right bound of fuzzy eigenvalue by solving

$[\tilde{K}^{c}(\alpha)+S_i\Delta\tilde{K}(\alpha)S_i]\{\overline{W}_i(\alpha)\}=\overline{\lambda}_i(\alpha)[\tilde{M}^{c}(\alpha)-S_i\Delta\tilde{M}(\alpha)S_i]\{\overline{W}_i(\alpha)\}$

for, $i=1,2,3,\cdots,n$.

Again for each i, one has n number of eigenvalues viz. $\overline{\lambda}_i^{j}(\alpha)$ for $j=1,2,\cdots,n$.

Step 7 : Fuzzy eigenvalue can be written as $\tilde{\lambda}_i(\alpha)=[\underline{\lambda}_i(\alpha),\overline{\lambda}_i(\alpha)]$ where

$$\underline{\lambda}_i(\alpha) = \min(\underline{\lambda}_i^j(\alpha), \text{for } i = 1,2,\cdots,n),$$

$$\overline{\lambda}_i(\alpha) = \max(\overline{\lambda}_i^j(\alpha), \text{for } i = 1,2,\cdots,n).$$

To compute left and right bounds of fuzzy eigenvalues for a particular $\tilde{\lambda}_i(\alpha)$, Deif's assumption (Deif et al. (1991)), that is the sign components of the associated eigenvector $\tilde{W}^c(\alpha)$ viz. in term of S_i where $\tilde{K}(\alpha) = [\underline{K}(\alpha), \overline{K}(\alpha)]$ and $\tilde{M}(\alpha) = [\underline{M}(\alpha), \overline{M}(\alpha)]$, are introduced for fuzzy parameters in Steps 4 to 6 of the algorithm.

In the following sections, uncertain eigenvalue problems for vibration of multistorey shear building, spring mass system and stepped beam structure have been considered.

5.2. Multistorey Shear Building Structure with Fuzzy Parameters

Let us consider $n-$storey shear building structure as shown in Fig. 5.1. Stiffness and mass parameters are assumed of two types viz.

i) triangular fuzzy number $\tilde{k}_i = (\underline{k}_i, k_i, \overline{k}_i)$, $\tilde{m}_i = (\underline{m}_i, m_i, \overline{m}_i)$ and ii) trapezoidal fuzzy number $\tilde{k}_i = (\underline{k}_i, \underline{\underline{k}}_i, \overline{\overline{k}}_i, \overline{k}_i)$, $\tilde{m}_i = (\underline{m}_i, \underline{\underline{m}}_i, \overline{\overline{m}}_i, \overline{m}_i)$ for $i = 1,2,3,\cdots,n$.

Equation of motion for n-storey shear building structure subject to ambient vibration may be written as

$$[\tilde{K}]\begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \vdots \\ \tilde{y}_n \end{Bmatrix} = -[\tilde{M}]\begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \vdots \\ \vdots \\ \ddot{\tilde{y}}_n \end{Bmatrix} \tag{5.5}$$

where $[\tilde{M}]$ and $[\tilde{K}]$ are global fuzzy mass and stiffness matrices given by

$$[\tilde{M}] = \begin{bmatrix} \tilde{m}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{m}_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \tilde{m}_{n-1} & \cdots \\ 0 & \cdots & \cdots & 0 & \tilde{m}_n \end{bmatrix}$$

and

$$[\tilde{K}]=\begin{bmatrix} \tilde{k}_1+\tilde{k}_2 & -\tilde{k}_2 & 0 & \cdots & 0 \\ -\tilde{k}_2 & \tilde{k}_2+\tilde{k}_3 & -\tilde{k}_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\tilde{k}_{n-1} & \tilde{k}_{n-1}+\tilde{k}_n & -\tilde{k}_n \\ 0 & \cdots & \cdots & -\tilde{k}_n & \tilde{k}_n \end{bmatrix}$$

whereas $\begin{Bmatrix} \ddot{\tilde{y}}_1 \\ \ddot{\tilde{y}}_2 \\ \vdots \\ \vdots \\ \ddot{\tilde{y}}_n \end{Bmatrix}$ and $\begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \vdots \\ \tilde{y}_n \end{Bmatrix}$ are the vectors of acceleration and deflection, respectively.

For simple harmonic motion, we substitute $\begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \vdots \\ \tilde{y}_n \end{Bmatrix}=\begin{Bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \vdots \\ \tilde{w}_n \end{Bmatrix}e^{i\tilde{\omega}t}$ in Eq. (5.5), which gives

$$[\tilde{K}]\{\tilde{W}\}=\tilde{\lambda}[\tilde{M}]\{\tilde{W}\}. \tag{5.6}$$

For $i-$th eigenvalue and vector, one can write

$$[\tilde{K}]\{\tilde{W}_i\}=\tilde{\lambda}_i[\tilde{M}]\{\tilde{W}_i\}, \tag{5.7}$$

where $\tilde{\lambda}_i=\tilde{\omega}_i^{\,2}$ is the $i-$th eigenvalue and $\{\tilde{W}_i\}=\begin{Bmatrix}\tilde{w}_1^{\,i} & \tilde{w}_2^{\,i} & \cdots & \cdots & \tilde{w}_n^{\,i}\end{Bmatrix}^T$ designates the $i-$th mode shape or eigenvector for $i=1,2,3,\cdots,n$.

Through $\alpha-$cut approach, Eq. (5.6) can be written as

$$[\tilde{K}(\alpha)]\{\tilde{W}(\alpha)\}=\tilde{\lambda}(\alpha)[\tilde{M}(\alpha)]\{\tilde{W}(\alpha)\}, \tag{5.8}$$

where

$$[\tilde{K}(\alpha)]=\begin{bmatrix} \tilde{k}_1(\alpha)+\tilde{k}_2(\alpha) & -\tilde{k}_2(\alpha) & 0 & \cdots & 0 \\ -\tilde{k}_2(\alpha) & \tilde{k}_2(\alpha)+\tilde{k}_3(\alpha) & -\tilde{k}_3(\alpha) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\tilde{k}_{n-1}(\alpha) & \tilde{k}_{n-1}(\alpha)+\tilde{k}_n(\alpha) & -\tilde{k}_n(\alpha) \\ 0 & \cdots & \cdots & -\tilde{k}_n(\alpha) & \tilde{k}_n(\alpha) \end{bmatrix},$$

$$[\tilde{M}(\alpha)] = \begin{bmatrix} \tilde{m}_1(\alpha) & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{m}_2(\alpha) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \tilde{m}_{n-1}(\alpha) & \cdots \\ 0 & \cdots & \cdots & 0 & \tilde{m}_n(\alpha) \end{bmatrix},$$

$$\{\tilde{W}(\alpha)\} = \begin{Bmatrix} \tilde{w}_1(\alpha) \\ \tilde{w}_2(\alpha) \\ \vdots \\ \vdots \\ \tilde{w}_n(\alpha) \end{Bmatrix} \text{ and } \tilde{\lambda}(\alpha) = \tilde{\lambda}_i(\alpha).$$

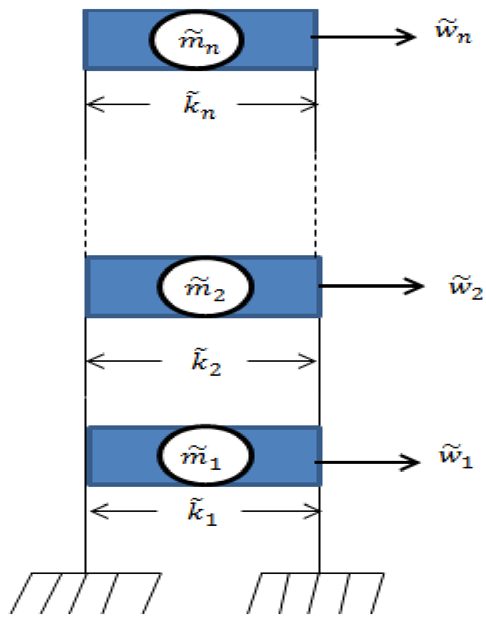

Fig. 5. 1 n-storey shear building structure

Eq. (5.8) can now be solved using the proposed algorithm to find the fuzzy vibration characteristics of n storey shear building structure.

5.2.1. Numerical examples for multistorey shear building

For validation of the proposed algorithm, a 5 storey shear building structure is taken into consideration. Three different cases have been investigated by assuming stiffness and mass parameters as crisp, triangular and trapezoidal fuzzy numbers respectively as given below in Cases 1(a) to 1(c).

Case 1(a): Stiffness and mass parameters are crisp

This is the well-known case and it is incorporated to have the comparison in special cases. The stiffness parameters are taken as $k_1 = 2010$ N/m, $k_2 = 1825$ N/m, $k_3 = 1615$ N/m, $k_4 = 1410$ N/m and $k_5 = 1205$ N/m and the mass parameters are $m_1 = 30$ Kg, $m_2 = 27$ Kg, $m_3 = 27$ Kg, $m_4 = 25$ Kg and $m_5 = 18$ Kg.

Frequency parameters and corresponding eigenmodes are computed by standard procedure for the crisp case and are given in Table 5.1.

Table 5.1 Frequency parameters and corresponding eigenmodes for crisp material properties

λ_1	λ_2	λ_3	λ_4	λ_5
6.1662	44.0780	103.5670	165.5908	219.4201
W_1	W_2	W_3	W_4	W_5
0.0318	-0.0810	0.1043	0.0890	-0.0834
0.0636	-0.1115	0.0416	-0.0552	0.1256
0.0930	-0.0638	-0.1013	-0.0653	-0.0989
0.1157	0.0447	-0.0641	0.1303	0.0596
0.1274	0.1308	0.1172	-0.0884	-0.0262

Case 1(b): Stiffness and mass parameters are triangular fuzzy numbers

Material parameters are considered as:

$\tilde{k}_1 = (2000, 2010, 2020)$ N/m,

$\tilde{k}_2 = (1800, 1825, 1850)$ N/m,

$\tilde{k}_3 = (1600, 1615, 1630)$ N/m,

$\tilde{k}_4 = (1400, 1410, 1420)$ N/m,

$\tilde{k}_5 = (1200, 1205, 1210)$ N/m

and

$\tilde{m}_1 = (29,30,31)$ Kg,

$\tilde{m}_2 = (26,27,28)$ Kg,

$\tilde{m}_3 = (26,27,28)$ Kg,

$\tilde{m}_4 = (24,25,26)$ Kg,

$\tilde{m}_5 = (17,18,19)$ Kg.

Through $\alpha-$cut approach, triangular fuzzy stiffness and mas parameters in this case may be written as

$\tilde{k}_1(\alpha) = [10\alpha + 2000, -10\alpha + 2020]$ N/m,

$\tilde{k}_2(\alpha) = [25\alpha + 1800, -25\alpha + 1850]$ N/m,

$\tilde{k}_3(\alpha) = [15\alpha + 1600, -15\alpha + 1630]$ N/m,

$\tilde{k}_4(\alpha) = [10\alpha + 1400, -10\alpha + 1420]$ N/m,

$\tilde{k}_5(\alpha) = [5\alpha + 1200, -5\alpha + 1210]$ N/m

and

$\tilde{m}_1(\alpha) = [\alpha + 29, -\alpha + 31]$ Kg,

$\tilde{m}_2(\alpha) = [\alpha + 26, -\alpha + 28]$ Kg,

$\tilde{m}_3(\alpha) = [\alpha + 26, -\alpha + 28]$ Kg,

$\tilde{m}_4 = [\alpha + 24, -\alpha + 26]$ Kg,

$\tilde{m}_5(\alpha) = [\alpha + 17, -\alpha + 19]$ Kg.

Results for left bound of frequency parameters with left bound of eigenmodes and right bounds of frequency parameters with right bounds of eigenmodes are incorporated in Tables 5.2(a) and 5.2(b) respectively for different α. The triangular fuzzy eigenvalue plots are depicted in Figs. 5.2(a) to 5.2(e).

Table 5.2(a) Left bounds of the frequency parameters and corresponding eigenmodes for triangular fuzzy material properties

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\underline{\lambda}_1$	4.6166	4.7669	4.9182	5.0705	5.2238	5.3782	5.5336	5.6901	5.8477	6.0064	6.1662
$\underline{W}_1$	0.0326 0.0643 0.0922 0.1125 0.1224	0.0325 0.0643 0.0922 0.1128 0.1229	0.0324 0.0642 0.0923 0.1131 0.1234	0.0323 0.0641 0.0924 0.1134 0.1239	0.0322 0.0640 0.0925 0.1137 0.1244	0.0322 0.0640 0.0926 0.1141 0.1249	0.0321 0.0639 0.0927 0.1144 0.1254	0.0320 0.0638 0.0927 0.1147 0.1259	0.0320 0.0638 0.0928 0.1150 0.1264	0.0319 0.0637 0.0929 0.1154 0.1269	0.0318 0.0636 0.0930 0.1157 0.1274
$\underline{\lambda}_2$	40.6428	40.9735	41.3070	41.6433	41.9824	42.3243	42.6691	43.0169	43.3676	43.7213	44.0780
$\underline{W}_2$	-0.0793 -0.1089 -0.0611 0.0455 0.1287	-0.0795 -0.1091 -0.0614 0.0454 0.1289	-0.0796 -0.1094 -0.0616 0.0453 0.1291	-0.0798 -0.1097 -0.0619 0.0452 0.1293	-0.0800 -0.1099 -0.0622 0.0451 0.1295	-0.0801 -0.1102 -0.0625 0.0451 0.1297	-0.0803 -0.1104 -0.0627 0.0450 0.1299	-0.0805 -0.1107 -0.0630 0.0449 0.1301	-0.0806 -0.1110 -0.0633 0.0448 0.1303	-0.0808 -0.1112 -0.0635 0.0447 0.1305	-0.0810 -0.1115 -0.0638 0.0447 0.1308
$\underline{\lambda}_3$	98.1800	98.7001	99.2272	99.7523	100.2846	100.8210	101.3616	101.9065	102.4556	103.0091	103.5670
$\underline{W}_3$	0.1007 0.0412 -0.0997 -0.0640 0.1154	0.1011 0.0412 -0.0998 -0.0640 0.1156	0.1014 0.0413 -0.1000 -0.0640 0.1158	0.1018 0.0413 -0.1002 -0.0641 0.1159	0.1021 0.0414 -0.1003 -0.0641 0.1161	0.1025 0.0414 -0.1005 -0.0641 0.1163	0.1028 0.0415 -0.1006 -0.0641 0.1166	0.1032 0.0415 -0.1008 -0.0641 0.1168	0.1036 0.0415 -0.1010 -0.0641 0.1170	0.1039 0.0416 -0.1011 -0.0641 0.1170	0.1043 0.0416 -0.1013 -0.0641 0.1172
$\underline{\lambda}_4$	157.8433	158.5905	159.3436	160.1027	160.8679	161.6393	162.4168	163.2007	163.9909	164.7876	165.5908
$\underline{W}_4$	0.0885 -0.0538 -0.0641 0.1276 -0.0851	0.0886 -0.0539 -0.0642 0.1279 -0.0854	0.0886 -0.0540 -0.0644 0.1281 -0.0857	0.0887 -0.0542 -0.0645 0.1284 -0.0861	0.0887 -0.0543 -0.0646 0.1287 -0.0864	0.0888 -0.0545 -0.0647 0.1289 -0.0867	0.0888 -0.0546 -0.0649 0.1292 -0.0870	0.0889 -0.0548 -0.0650 0.1294 -0.0874	0.0889 -0.0549 -0.0651 0.1297 -0.0877	0.0890 -0.0551 -0.0652 0.1300 -0.0880	0.0890 -0.0552 -0.0653 0.1303 -0.0884
$\underline{\lambda}_5$	209.5148	210.4735	211.4390	212.4116	213.3911	214.3777	215.3715	216.3726	217.3810	218.3968	219.4201
$\underline{W}_5$	-0.0822 0.1231 -0.0973 0.0585 -0.0252	-0.0824 0.1234 -0.0974 0.0586 -0.0253	-0.0825 0.1236 -0.0976 0.0587 -0.0254	-0.0826 0.1239 -0.0978 0.0588 -0.0255	-0.0827 0.1241 -0.0979 0.0589 -0.0256	-0.0828 0.1243 -0.0981 0.0590 -0.0257	-0.0829 0.1246 -0.0983 0.0591 -0.0258	-0.0831 0.1248 -0.0984 0.0592 -0.0259	-0.0832 0.1251 -0.0986 0.0593 -0.0260	-0.0833 0.1253 -0.0988 0.0595 -0.0261	-0.0834 0.1256 -0.0989 0.0596 -0.0262

Table 5.2(b) Right bounds of the frequency parameters and corresponding eigenmodes for triangular fuzzy material properties

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\bar{\lambda}_1$	7.8303	7.6582	7.4875	7.3180	7.1498	6.9828	6.8171	6.6526	6.4893	6.3272	6.1662
$\bar{W}_1$	0.0311	0.0312	0.0313	0.0313	0.0314	0.0315	0.0315	0.0316	0.0317	0.0317	0.0318
	0.0630	0.0630	0.0631	0.0632	0.0632	0.0633	0.0634	0.0634	0.0635	0.0636	0.0636
	0.0940	0.0939	0.0938	0.0937	0.0936	0.0935	0.0934	0.0933	0.0932	0.0931	0.0930
	0.1190	0.1187	0.1184	0.1180	0.1177	0.1173	0.1170	0.1167	0.1163	0.1160	0.1157
	0.1327	0.1321	0.1316	0.1311	0.1305	0.1300	0.1295	0.1290	0.1285	0.1279	0.1274
$\bar{\lambda}_2$	47.8200	47.4309	47.0453	46.6630	46.2840	45.9083	45.5359	45.1667	44.8007	44.4378	44.0780
$\bar{W}_2$	-0.0828	-0.0826	-0.0825	-0.0822	-0.0820	-0.0819	-0.0817	-0.0815	-0.0813	-0.0812	-0.0810
	-0.1142	-0.1139	-0.1136	-0.1134	-0.1131	-0.1128	-0.1126	-0.1123	-0.1120	-0.1118	-0.1115
	-0.0665	-0.0662	-0.0660	-0.0657	-0.0654	-0.0652	-0.0649	-0.0646	-0.0644	-0.0641	-0.0638
	0.0440	0.0441	0.0441	0.0442	0.0443	0.0443	0.0444	0.0444	0.0445	0.0446	0.0447
	0.1331	0.1328	0.1326	0.1323	0.1321	0.1319	0.1316	0.1314	0.1312	0.1310	0.1308
$\bar{\lambda}_3$	109.3993	108.7946	108.1947	107.5998	107.0097	106.4243	105.8437	105.2677	104.6963	104.1294	103.5670
$\bar{W}_3$	0.1081	0.1077	0.1074	0.1070	0.1066	0.1062	0.1058	0.1054	0.1051	0.1047	0.1043
	0.0419	0.0419	0.0419	0.0418	0.0418	0.0418	0.0417	0.0417	0.0417	0.0416	0.0416
	-0.1031	-0.1029	-0.1027	-0.1025	-0.1023	-0.1022	-0.1020	-0.1018	-0.1016	-0.1015	-0.1013
	-0.0639	-0.0640	-0.0640	-0.0640	-0.0640	-0.0640	-0.0641	-0.0641	-0.0641	-0.0641	-0.0641
	0.1190	0.1188	0.1186	0.1185	0.1183	0.1181	0.1179	0.1177	0.1175	0.1173	0.1172
$\bar{\lambda}_4$	174.0029	173.1292	172.2630	171.4042	170.5527	169.7083	168.8710	168.0407	167.2173	166.4007	165.5908
$\bar{W}_4$	0.0895	0.0895	0.0894	0.0894	0.0893	0.0893	0.0892	0.0892	0.0891	0.0891	0.0890
	-0.0569	-0.0567	-0.0566	-0.0564	-0.0562	-0.0561	-0.0559	-0.0557	-0.0556	-0.0554	-0.0552
	-0.0664	-0.0663	-0.0662	-0.0661	-0.0660	-0.0659	-0.0658	-0.0656	-0.0655	-0.0654	-0.0653
	0.1330	0.1327	0.1325	0.1322	0.1319	0.1316	0.1313	0.1311	0.1308	0.1305	0.1303
	-0.0921	-0.0917	-0.0913	-0.0909	-0.0906	-0.0902	-0.0898	-0.0895	-0.0891	-0.0887	-0.0884
$\bar{\lambda}_5$	230.0845	228.9814	227.8867	226.8003	225.7221	224.6520	223.5899	222.5358	221.4895	220.4510	219.4201
$\bar{W}_5$	-0.0846	-0.0845	-0.0844	-0.0843	-0.0841	-0.0840	-0.0839	-0.0838	-0.0837	-0.0835	-0.0834
	0.1282	0.1279	0.1276	0.1274	0.1271	0.1269	0.1266	0.1264	0.1261	0.1258	0.1256
	-0.1007	-0.1005	-0.1004	-0.1002	-0.1000	-0.0998	-0.0996	-0.0995	-0.0993	-0.0991	-0.0989
	0.0609	0.0607	0.0606	0.0605	0.0603	0.0602	0.0601	0.0599	0.0598	0.0597	0.0596
	-0.0273	-0.0271	-0.0270	-0.0269	-0.0268	-0.0267	-0.0266	-0.0265	-0.0264	-0.0263	-0.0262

Case 1(c): Stiffness and mass parameters are trapezoidal fuzzy number

Corresponding material properties are taken as

$\tilde{k}_1 = (2000, 2005, 2015, 2020)$ N/m, $\tilde{k}_2 = (1800, 1820, 1830, 1850)$ N/m,

$\tilde{k}_3 = (1600, 1610, 1620, 1630)$ N/m, $\tilde{k}_4 = (1400, 1405, 1415, 1420)$ N/m,

$\tilde{k}_5 = (1200, 1202, 1208, 1210)$ N/m

and

$\tilde{m}_1 = (29, 29.5, 30.5, 31)$ Kg, $\tilde{m}_2 = (26, 26.5, 27.5, 28)$ Kg,

$\tilde{m}_3 = (26, 26.5, 27.5, 28)$ Kg, $\tilde{m}_4 = (24, 24.5, 25.5, 26)$ Kg,

$\tilde{m}_5 = (17, 17.5, 18.5, 19)$ Kg.

Through $\alpha -$cut approach, trapezoidal fuzzy stiffness and mass parameters may again be written as

$\tilde{k}_1(\alpha) = [5\alpha + 2000, -5\alpha + 2020]$ N/m, $\tilde{k}_2(\alpha) = [20\alpha + 1800, -20\alpha + 1850]$ N/m,

$\tilde{k}_3(\alpha) = [10\alpha + 1600, -10\alpha + 1630]$ N/m, $\tilde{k}_4(\alpha) = [5\alpha + 1400, -5\alpha + 1420]$ N/m,

$\tilde{k}_5(\alpha) = [2\alpha + 1200, -2\alpha + 1210]$ N/m

and

$\tilde{m}_1(\alpha) = [0.5\alpha + 29, -0.5\alpha + 31]$ Kg, $\tilde{m}_2(\alpha) = [0.5\alpha + 26, -0.5\alpha + 28]$ Kg,

$\tilde{m}_3(\alpha) = [0.5\alpha + 26, -0.5\alpha + 28]$ Kg, $\tilde{m}_4(\alpha) = [0.5\alpha + 24, -0.5\alpha + 26]$ Kg,

$\tilde{m}_5(\alpha) = [0.5\alpha + 17, -0.5\alpha + 19]$ Kg.

Using the proposed algorithm for the trapezoidal fuzzy parameters, the frequency parameters and the eigenmodes are computed. Corresponding results for left bound of the frequency parameters with the left bound of the eigenmodes and right bounds of the frequency parameters with the right bounds of the eigenmodes are given in Tables 5.3(a) and 5.3(b) respectively for each α. The trapezoidal fuzzy eigenvalue plots are cited in Figs. 5.3(a) to 5.3(e). By comparing the results shown in Figs. 5.2 (a) and 5.3(a), one may clearly see that the results for $\alpha = 0$ both are equal to each other where as for $\alpha = 1$, first natural frequency for triangular fuzzy inputs exactly comes inside the first natural frequency for trapezoidal fuzzy inputs.

Table 5.3(a) Left bounds of the frequency parameters and corresponding eigenmodes for trapezoidal fuzzy material properties

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\underline{\lambda}_1$	4.6166	4.7020	4.7876	4.8733	4.9592	5.0454	5.1317	5.2182	5.3049	5.3917	5.4788
$\underline{W}_1$	0.0326 0.0643 0.0922 0.1125 0.1224	0.0325 0.0642 0.0922 0.1127 0.1227	0.0324 0.0641 0.0922 0.1129 0.1230	0.0323 0.0640 0.0922 0.1130 0.1233	0.0322 0.0639 0.0922 0.1132 0.1236	0.0321 0.0638 0.0922 0.1134 0.1240	0.0320 0.0637 0.0922 0.1136 0.1243	0.0319 0.0636 0.0922 0.1138 0.1246	0.0318 0.0635 0.0923 0.1140 0.1249	0.0317 0.0634 0.0923 0.1142 0.1252	0.0316 0.0632 0.0923 0.1144 0.1256
$\underline{\lambda}_2$	40.6428	40.8402	41.0384	41.2375	41.4375	41.6383	41.8401	42.0427	42.2462	42.4506	42.6560
$\underline{W}_2$	-0.0793 -0.1089 -0.0611 0.0455 0.1287	-0.0794 -0.1090 -0.0613 0.0454 0.1288	-0.0794 -0.1092 -0.0615 0.0452 0.1288	-0.0795 -0.1093 -0.0618 0.0451 0.1289	-0.0796 -0.1095 -0.0620 0.0450 0.1289	-0.0797 -0.1096 -0.0622 0.0448 0.1290	-0.0797 -0.1098 -0.0625 0.0447 0.1291	-0.0798 -0.1099 -0.0627 0.0446 0.1291	-0.0799 -0.1101 -0.0629 0.0444 0.1292	-0.0800 -0.1102 -0.0632 0.0443 0.1292	-0.0800 -0.1104 -0.0634 0.0442 0.1293
$\underline{\lambda}_3$	98.1800	98.4581	98.7372	99.0175	99.2989	99.5814	99.8650	100.1498	100.4358	100.7229	101.0111
$\underline{W}_3$	0.1007 0.0412 -0.0997 -0.0640 0.1154	0.1009 0.0412 -0.0997 -0.0641 0.1155	0.1011 0.0413 -0.0997 -0.0641 0.1156	0.1013 0.0414 -0.0998 -0.0641 0.1156	0.1016 0.0414 -0.0998 -0.0642 0.1157	0.1018 0.0415 -0.0998 -0.0642 0.1157	0.1020 0.0416 -0.0999 -0.0642 0.1158	0.1022 0.0416 -0.0999 -0.0642 0.1158	0.1024 0.0417 -0.0999 -0.0643 0.1159	0.1026 0.0418 -0.0999 -0.0643 0.1159	0.1028 0.0418 -0.1000 -0.0643 0.1160
$\underline{\lambda}_4$	157.8433	158.2326	158.6233	159.0156	159.4094	159.8047	160.2016	160.6000	161.0000	161.4015	161.8046
$\underline{W}_4$	0.0885 -0.0538 -0.0641 0.1276 -0.0851	0.0885 -0.0538 -0.0643 0.1278 -0.0853	0.0885 -0.0538 -0.0644 0.1279 -0.0854	0.0885 -0.0538 -0.0645 0.1280 -0.0856	0.0885 -0.0538 -0.0647 0.1282 -0.0857	0.0885 -0.0539 -0.0648 0.1283 -0.0858	0.0885 -0.0539 -0.0649 0.1285 -0.0860	0.0885 -0.0539 -0.0650 0.1286 -0.0861	0.0885 -0.0539 -0.0652 0.1288 -0.0863	0.0885 -0.0539 -0.0653 0.1289 -0.0864	0.0885 -0.0540 -0.0654 0.1290 -0.0866
$\underline{\lambda}_5$	209.5148	210.0368	210.5607	211.0864	211.6141	212.1437	212.6752	213.2087	213.7441	214.2815	214.8207
$\underline{W}_5$	-0.0822 0.1231 -0.0973 0.0585 -0.0252	-0.0823 0.1233 -0.0973 0.0585 -0.0252	-0.0824 0.1234 -0.0974 0.0585 -0.0252	-0.0825 0.1236 -0.0974 0.0585 -0.0253	-0.0826 0.1237 -0.0975 0.0585 -0.0253	-0.0827 0.1239 -0.0975 0.0585 -0.0253	-0.0828 0.1240 -0.0976 0.0585 -0.0253	-0.0829 0.1242 -0.0976 0.0585 -0.0253	-0.0830 0.1243 -0.0977 0.0585 -0.0253	-0.0831 0.1245 -0.0977 0.0585 -0.0253	-0.0832 0.1246 -0.0978 0.0585 -0.0253

Table 5.3(b) Right bounds of the frequency parameters and corresponding eigenmodes for trapezoidal fuzzy material properties

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\overline{\lambda}_1$	7.8303	7.7344	7.6388	7.5435	7.4484	7.3536	7.2590	7.1647	7.0706	6.9768	6.8832
$\overline{W}_1$	0.0311 0.0630 0.0940 0.1190 0.1327	0.0312 0.0631 0.0939 0.1188 0.1323	0.0313 0.0632 0.0939 0.1186 0.1320	0.0314 0.0633 0.0939 0.1184 0.1317	0.0315 0.0634 0.0939 0.1182 0.1313	0.0316 0.0635 0.0939 0.1180 0.1310	0.0316 0.0636 0.0938 0.1178 0.1307	0.0317 0.0637 0.0938 0.1176 0.1303	0.0318 0.0638 0.0938 0.1174 0.1300	0.0319 0.0639 0.0938 0.1172 0.1297	0.0320 0.0640 0.0938 0.1170 0.1293
$\overline{\lambda}_2$	47.8200	47.5895	47.3600	47.1315	46.9041	46.6777	46.4524	46.2280	46.0047	45.7824	45.5611
$\overline{W}_2$	-0.0828 -0.1142 -0.0665 0.0440 0.1331	-0.0827 -0.1140 -0.0663 0.0441 0.1330	-0.0826 -0.1139 -0.0660 0.0443 0.1329	-0.0825 -0.1137 -0.0658 0.0444 0.1328	-0.0824 -0.1136 -0.0656 0.0445 0.1327	-0.0824 -0.1134 -0.0654 0.0446 0.1327	-0.0823 -0.1133 -0.0651 0.0447 0.1326	-0.0822 -0.1131 -0.0649 0.0448 0.1325	-0.0821 -0.1129 -0.0647 0.0450 0.1324	-0.0820 -0.1128 -0.0645 0.0451 0.1323	-0.0820 -0.1126 -0.0642 0.0452 0.1323
$\overline{\lambda}_3$	109.3993	109.0759	108.7539	108.4332	108.1139	107.7959	107.4792	107.1639	106.8498	106.5379	106.2257
$\overline{W}_3$	0.1081 0.0419 -0.1031 -0.0639 0.1190	0.1079 0.0418 -0.1030 -0.0639 0.1190	0.1077 0.0418 -0.1030 -0.0639 0.1189	0.1074 0.0417 -0.1030 -0.0639 0.1188	0.1072 0.0417 -0.1029 -0.0639 0.1188	0.1070 0.0416 -0.1029 -0.0639 0.1187	0.1068 0.0416 -0.1028 -0.0638 0.1186	0.1065 0.0415 -0.1028 -0.0638 0.1186	0.1063 0.0415 -0.1028 -0.0638 0.1185	0.1061 0.0414 -0.1027 -0.0638 0.1184	0.1058 0.0414 -0.1027 -0.0638 0.1184
$\overline{\lambda}_4$	174.0029	173.5478	173.0946	172.6433	172.1939	171.7464	171.3008	170.8570	170.4150	169.9741	169.5366
$\overline{W}_4$	0.0895 -0.0569 -0.0664 0.1330 -0.0921	0.0895 -0.0569 -0.0663 0.1329 -0.0919	0.0895 -0.0568 -0.0661 0.1327 -0.0917	0.0895 -0.0568 -0.0660 0.1326 -0.0915	0.0895 -0.0568 -0.0659 0.1324 -0.0914	0.0895 -0.0567 -0.0658 0.1322 -0.0912	0.0895 -0.0567 -0.0656 0.1321 -0.0910	0.0895 -0.0567 -0.0655 0.1319 -0.0908	0.0895 -0.0566 -0.0654 0.1318 -0.0906	0.0895 -0.0566 -0.0653 0.1316 -0.0905	0.0895 -0.0566 -0.0651 0.1315 -0.0903
$\overline{\lambda}_5$	230.0845	229.4853	228.8884	228.2938	227.7015	227.1115	226.5237	225.9381	225.3548	224.7737	224.1948
$\overline{W}_5$	-0.0846 0.1282 -0.1007 0.0609 -0.0273	-0.0845 0.1280 -0.1007 0.0608 -0.0272	-0.0844 0.1278 -0.1006 0.0608 -0.0272	-0.0843 0.1277 -0.1005 0.0608 -0.0272	-0.0842 0.1275 -0.1005 0.0608 -0.0272	-0.0841 0.1274 -0.1004 0.0608 -0.0271	-0.0840 0.1272 -0.1004 0.0608 -0.0271	-0.0839 0.1271 -0.1003 0.0608 -0.0271	-0.0839 0.1269 -0.1003 0.0608 -0.0271	-0.0838 0.1267 -0.1002 0.0608 -0.0270	-0.0837 0.1266 -0.1001 0.0608 -0.0270

Special Case: We consider the fuzzy stiffness and mass parameters for $\alpha = 0$. For this, the parameters are in interval form and the material parameters of the 5 storey building may be written as

$\tilde{k}_1 = [2000,2020]$ N/m, $\tilde{k}_2 = [1800,1850]$ N/m, $\tilde{k}_3 = [1600,1630]$ N/m,

$\tilde{k}_4 = [1400,1420]$ N/m, $\tilde{k}_5 = [1200,1210]$ N/m

and

$\tilde{m}_1 = [29,31]$ Kg, $\tilde{m}_2 = [26,28]$ Kg, $\tilde{m}_3 = [26,28]$ Kg, $\tilde{m}_4 = [24,26]$ Kg,

$\tilde{m}_5 = [17,19]$ Kg.

As discussed earlier, the proposed algorithm converts into a generalized interval eigenvalue problem for $\alpha = 0$. The left and right bounds of the frequency parameters and respective left and right bound of the eigenmodes are obtained using the method of interval case. Left bounds of the frequency parameters with the left bounds of the eigenmodes and the right bound of the frequency parameters with right bounds of the eigenmodes are included in Tables 5.4(a) and 5.4(b) respectively.

Table 5.4(a) Left bounds of the frequency parameters and corresponding eigenmodes for interval material properties

$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$	$\underline{\lambda}_5$
4.6166	40.6428	98.1800	157.8433	209.5148
$\underline{W}_1$	$\underline{W}_2$	$\underline{W}_3$	$\underline{W}_4$	$\underline{W}_5$
0.0326	-0.0793	0.1007	0.0885	-0.0822
0.0643	-0.1089	0.0412	-0.0538	0.1231
0.0922	-0.0611	-0.0997	-0.0641	-0.0973
0.1125	0.0455	-0.0640	0.1276	0.0585
0.1224	0.1287	0.1154	-0.0851	-0.0252

Table 5.4(b) Right bounds of the frequency parameters and corresponding eigenmodes for interval material properties

$\overline{\lambda}_1$	$\overline{\lambda}_2$	$\overline{\lambda}_3$	$\overline{\lambda}_4$	$\overline{\lambda}_5$
7.8303	47.8200	109.3993	174.0029	230.0845
$\overline{W}_1$	$\overline{W}_2$	$\overline{W}_3$	$\overline{W}_4$	$\overline{W}_5$
0.0311	-0.0828	0.1081	0.0895	-0.0846
0.0630	-0.1142	0.0419	-0.0569	0.1282
0.0940	-0.0665	-0.1031	-0.0664	-0.1007
0.1190	0.0440	-0.0639	0.1330	0.0609
0.1327	0.1331	0.1190	-0.0921	-0.0273

Deformations for first to fifth modes with interval material properties as well as crisp material properties are shown in Figs. 5.4(a) to 5.4(e) respectively. In these figures, the solid line with star represents the crisp deformations, dashed line with square and dotted line with circle represents the left and right bounds of the deformations for interval material properties. It may be seen from Tables 5.1, 5.4(a) and 5.4(b) that the interval widths are very small and so the plots in Figs. 5.4(a) to 5.4(e) do not distinguish the crisp, left and right mode shapes.

In case of triangular fuzzy material properties, the first deformations for $\alpha = 0, 1$ and $\alpha = 0.6, 1$ are depicted in Figs. 5.5(a) and 5.5(b) respectively. Similarly, the fifth deformations for $\alpha = 0, 1$ and $\alpha = 0.6, 1$ are shown in Figs. 5.6(a) and 5.6(b) respectively. Here, the solid line with star represents the deformations for $\alpha = 1$, dashed line with square and dotted line with circle represent the left and right bounds of the deformations for $\alpha = 0$ and 0.6 respectively.

The first deformations are depicted in terms of plots in Figs. 5.7(a) and 5.7(b) for trapezoidal fuzzy material properties for $\alpha = 0$ and 0.6 respectively along with membership 1. Here, the solid line with dot and solid line with triangle represents the deformations for $\alpha = 1$, dashed line with square and dotted line with circle represents the left and right bounds of the deformations for $\alpha = 0$ and 0.6.

Results obtained by the proposed algorithm for the special cases are compared with the existing results of Sim et al. (2007) in Table 5.5. Also for Case 2, the present results are compared with the results obtained by Chiao (1998) for different α and the results are tabulated in Tables 5.6 and 5.7 for left and right bounds respectively. For both the cases the results are found to be in good agreement.

Table 5.5 Interval eigenvalues and comparison with Sim et al. (2007)

Sim et al. (2007)		Present method	
$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\underline{\lambda}_i$	$\overline{\lambda}_i$
4.6166	7.8303	4.6166	7.8303
40.754	47.702	40.6428	47.8200
98.572	108.99	98.1800	109.3993
158.86	172.89	157.8433	174.0029
211.51	227.51	209.5148	230.0845

Table 5.6 Left bounds of fuzzy eigenvalues for Case 2 with the comparison of Chiao (1998)

α	Chiao (1998)					Present Method				
	$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$	$\underline{\lambda}_5$	$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$	$\underline{\lambda}_5$
0	4.6166	40.6428	98.1800	157.8433	209.5148	4.6166	40.6428	98.1800	157.8433	209.5148
0.2	4.9182	41.3070	99.2272	159.3436	211.4390	4.9182	41.3070	99.2272	159.3436	211.4390
0.8	5.8477	43.3676	102.4556	163.9909	217.3810	5.8477	43.3676	102.4556	163.9909	217.3810
1	6.1662	44.0780	103.5670	165.5908	219.4201	6.1662	44.0780	103.5670	165.5908	219.4201

Table 5.7 Right bounds of fuzzy eigenvalues for Case 2 with the comparison of Chiao (1998)

α	Chiao (1998)					Present Method				
	$\overline{\lambda}_1$	$\overline{\lambda}_2$	$\overline{\lambda}_3$	$\overline{\lambda}_4$	$\overline{\lambda}_5$	$\overline{\lambda}_1$	$\overline{\lambda}_2$	$\overline{\lambda}_3$	$\overline{\lambda}_4$	$\overline{\lambda}_5$
0	7.8303	47.8200	109.3993	174.0029	230.0845	7.8303	47.8200	109.3993	174.0029	230.0845
0.2	7.4875	47.0453	108.1947	172.2630	227.8867	7.4875	47.0453	108.1947	172.2630	227.8867
0.8	6.4893	44.8007	104.6963	167.2173	221.4895	6.4893	44.8007	104.6963	167.2173	221.4895
1	6.1662	44.0780	103.5670	165.5908	219.4201	6.1662	44.0780	103.5670	165.5908	219.4201

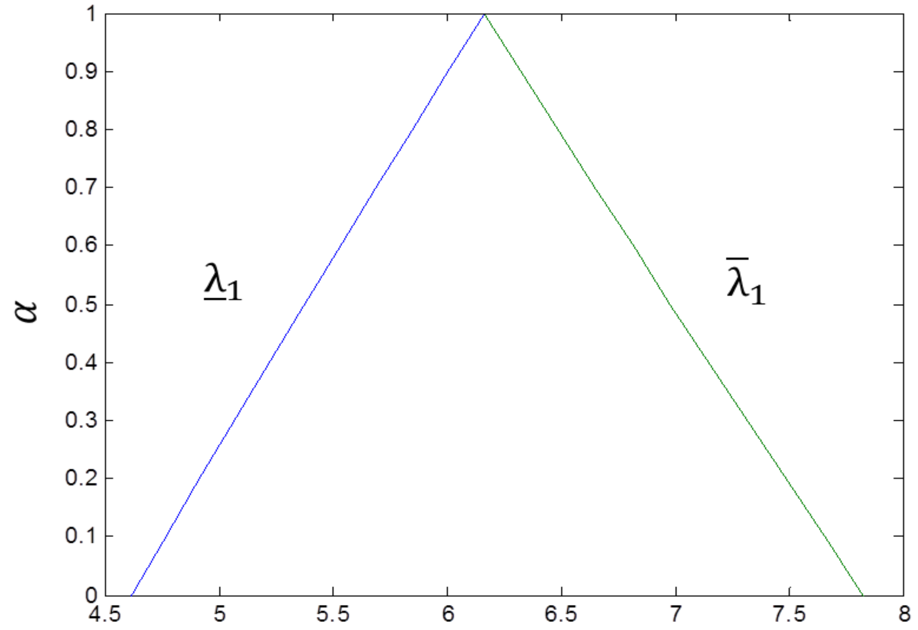

Fig. 5.2(a) First natural frequency for triangular parameters

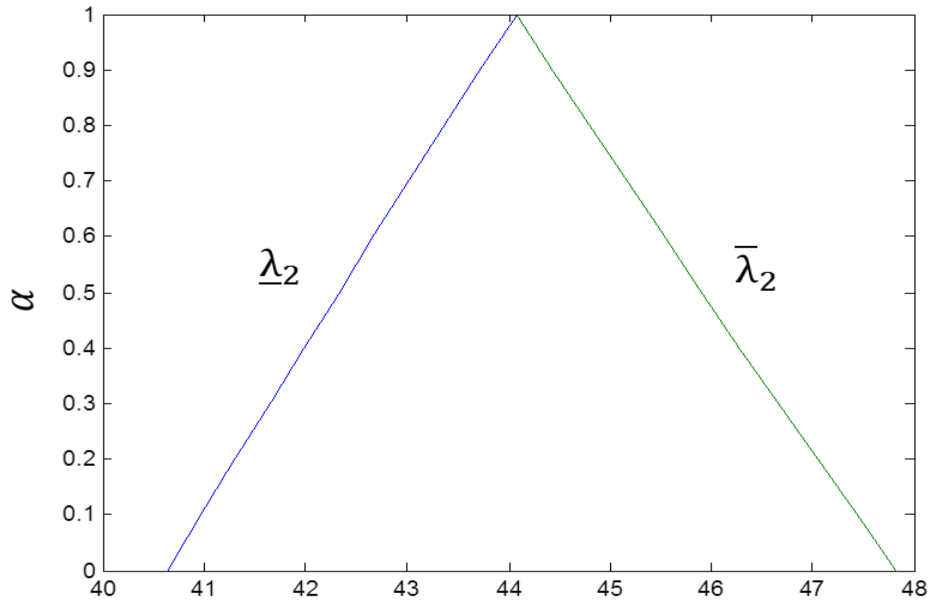

Fig. 5.2(b) Second natural frequency for triangular parameters

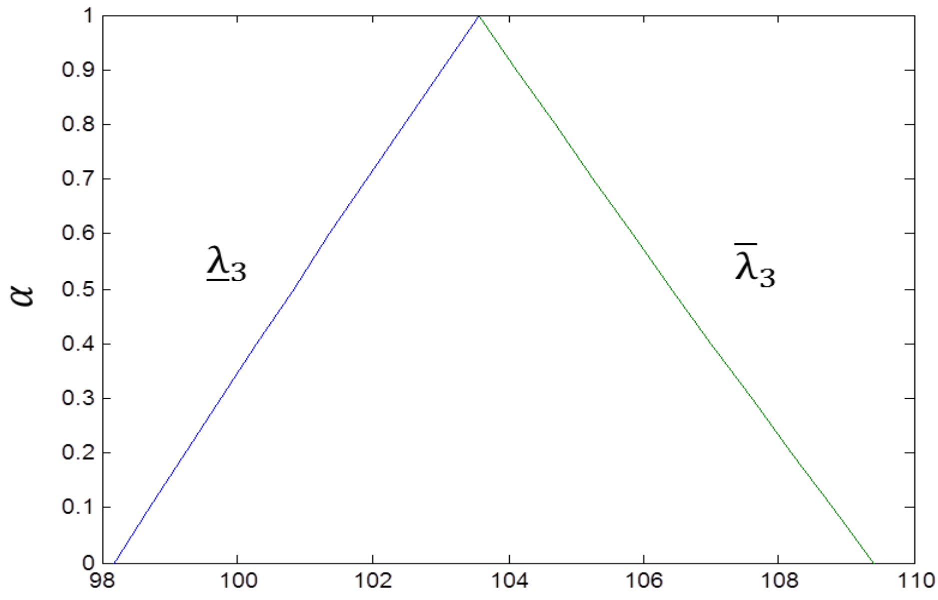

Fig. 5.2(c) Third natural frequency for triangular parameters

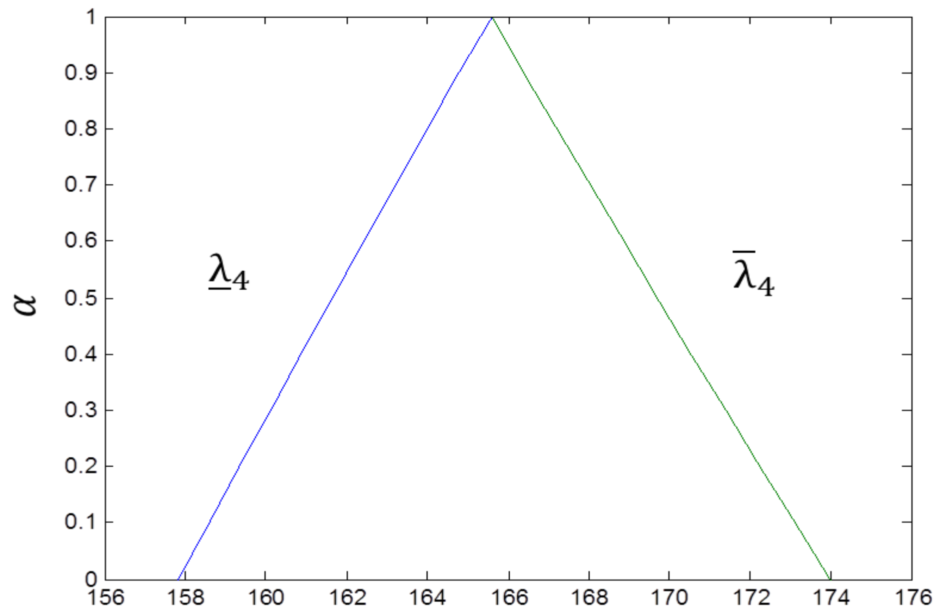

Fig. 5.2(d) Fourth natural frequency for triangular parameters

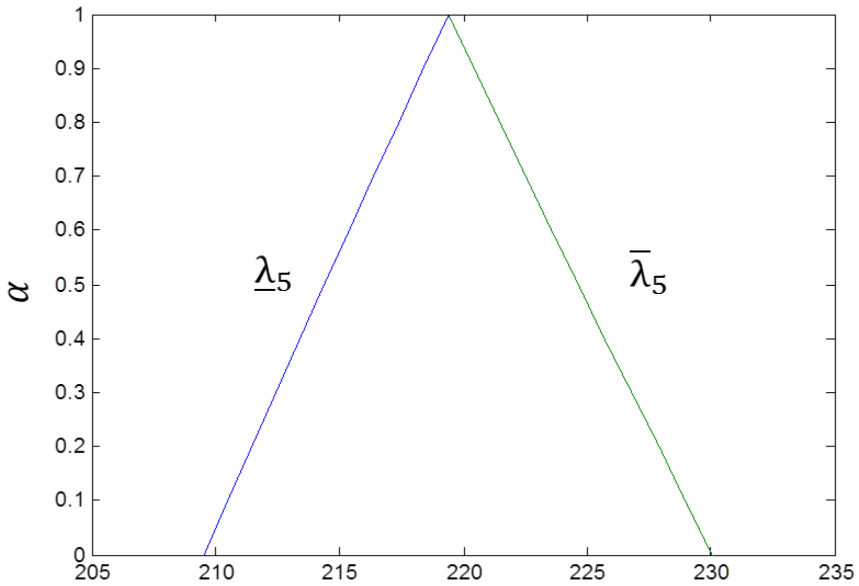

Fig. 5.2(e) Fifth natural frequency for triangular parameters

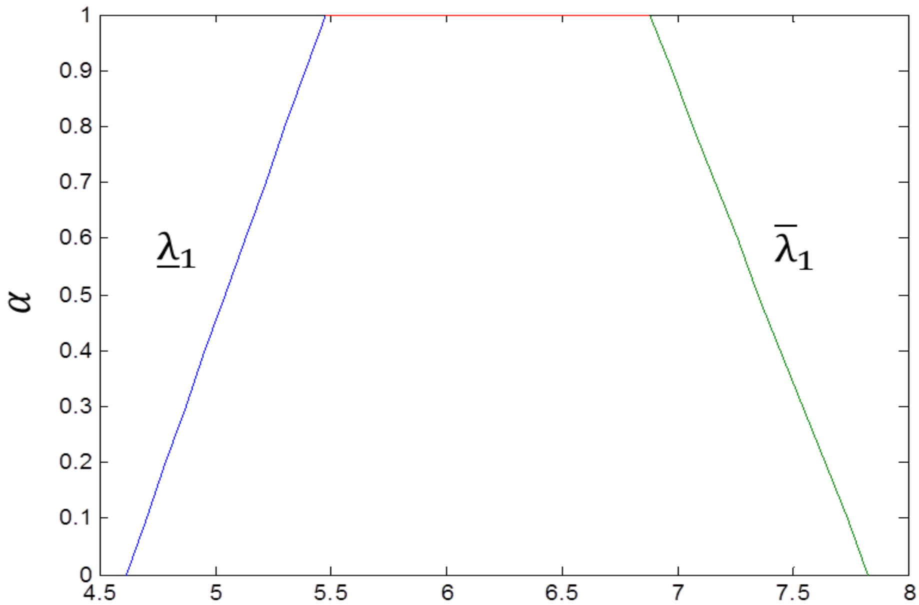

Fig. 5.3(a) First natural frequency for trapezoidal parameters

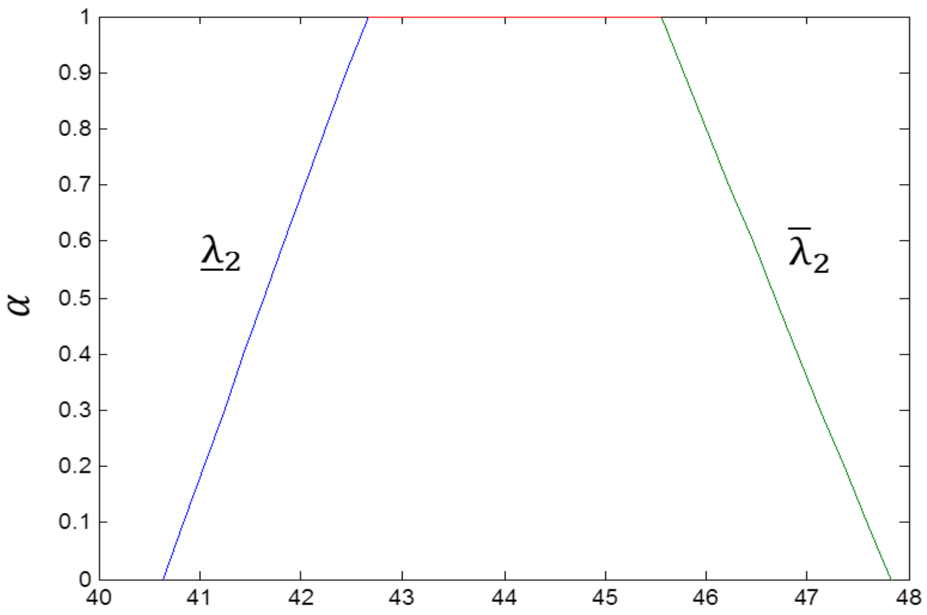

Fig. 5.3(b) Second natural frequency for trapezoidal parameters

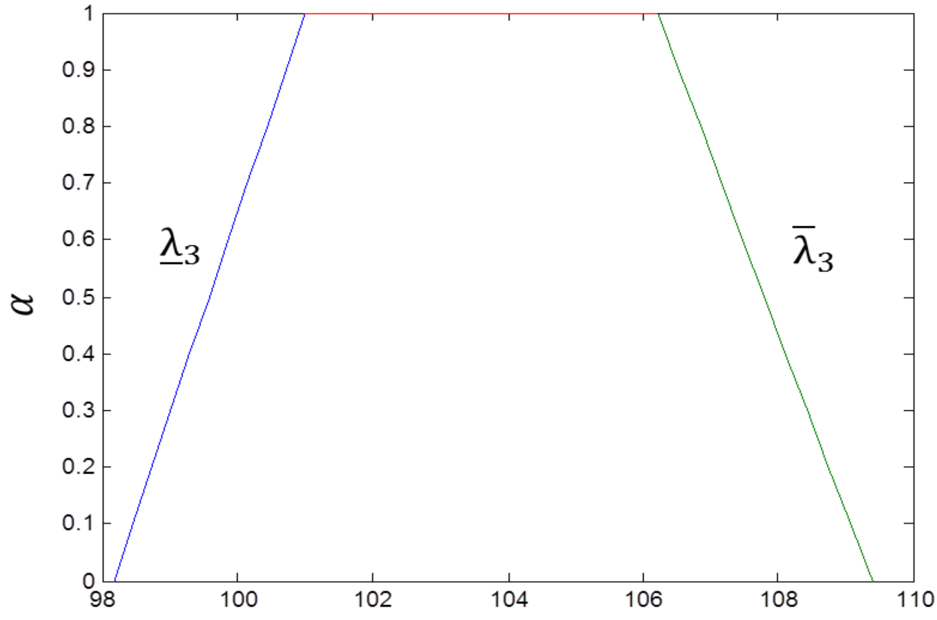

Fig. 5.3(c) Third natural frequency for trapezoidal parameters

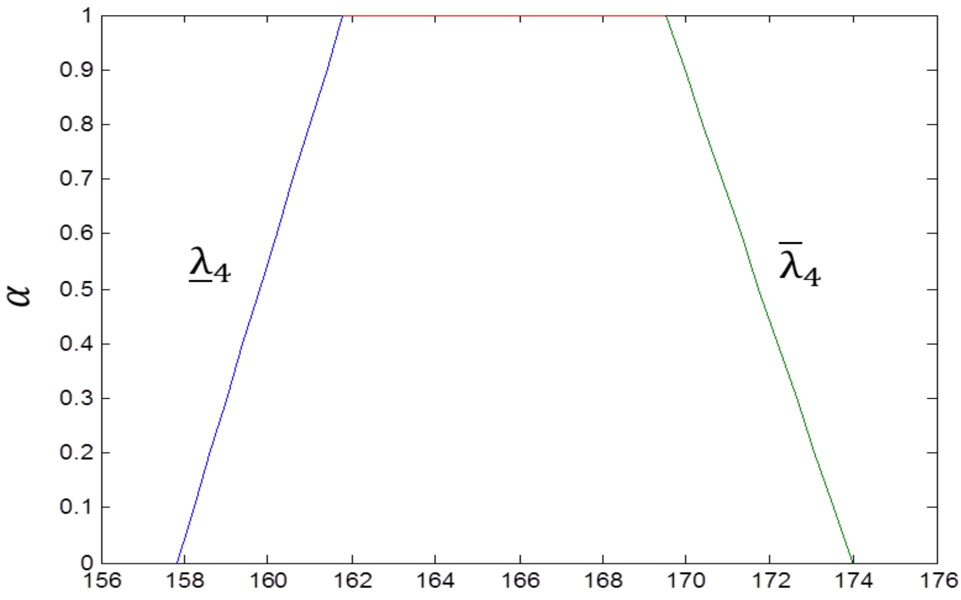

Fig. 5.3(d) Fourth natural frequency for trapezoidal parameters

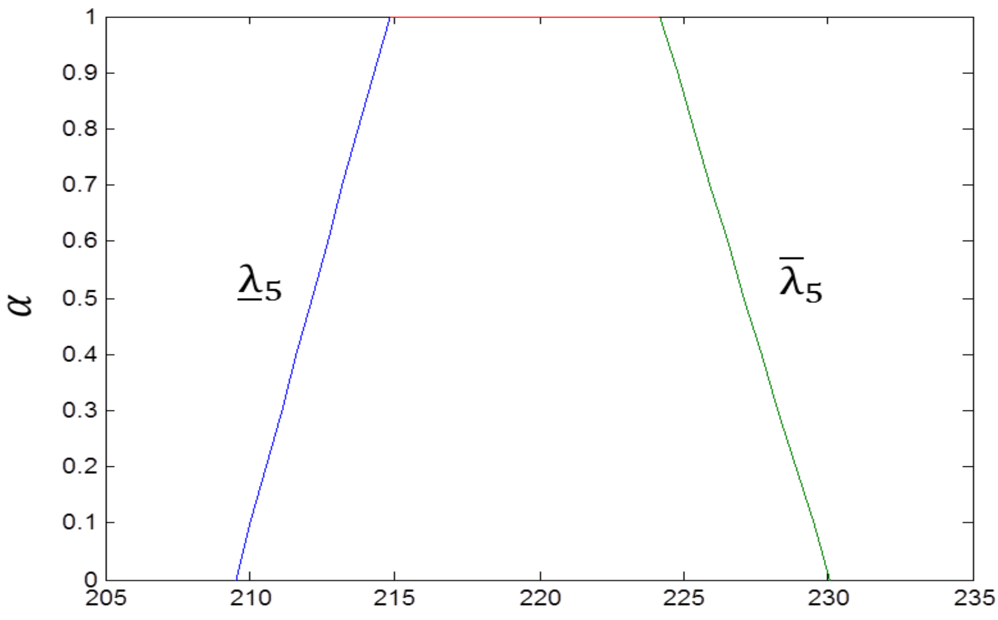

Fig. 5.3(e) Fifth natural frequency for trapezoidal parameters

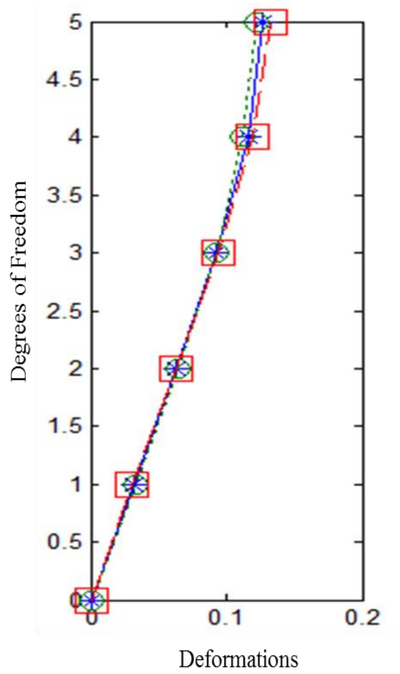

Fig. 5.4(a) First mode for crisp and interval parameters

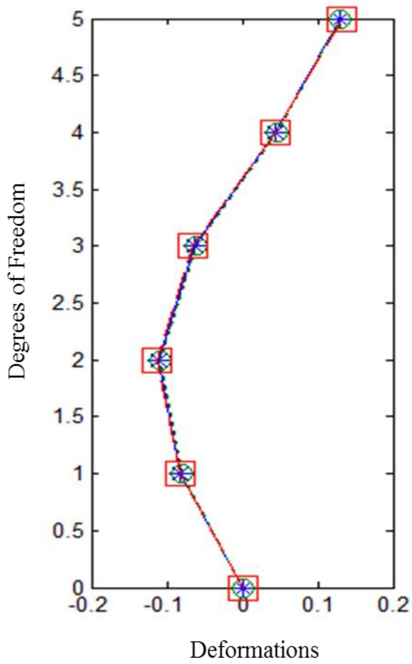

Fig. 5.4(b) Second mode for crisp and interval parameters

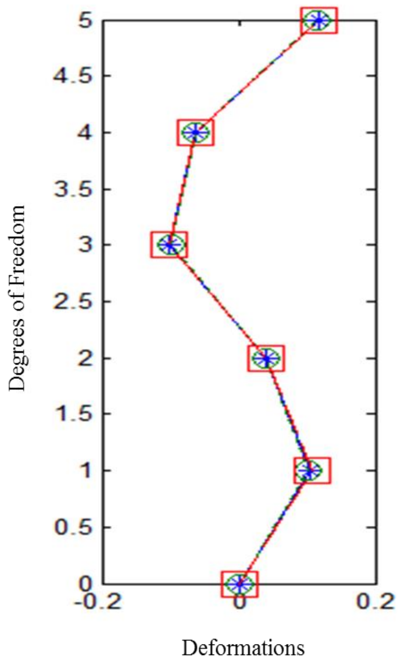

Fig. 5.4(c) Third mode for crisp and interval parameters

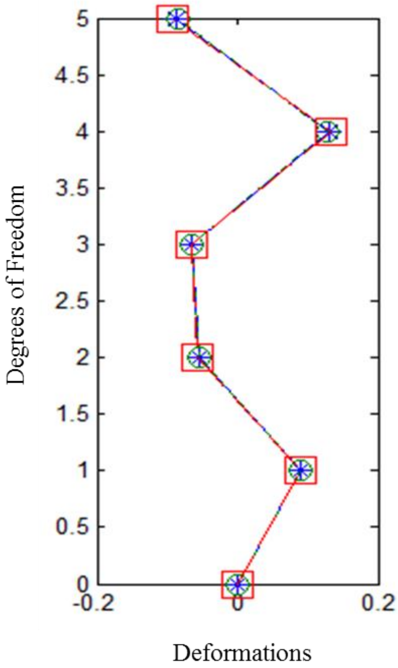

Fig. 5.4(d) Fourth mode for crisp and interval parameters

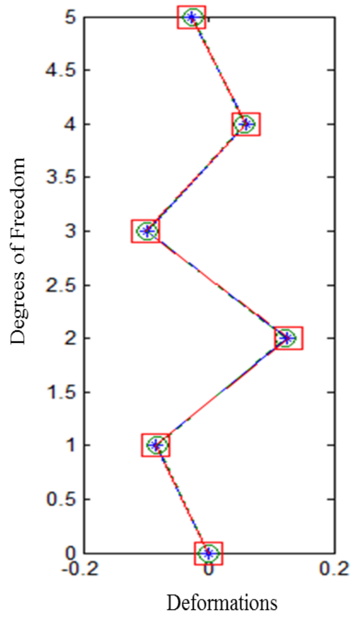

Fig. 5.4(e) Fifth mode for crisp and interval parameters

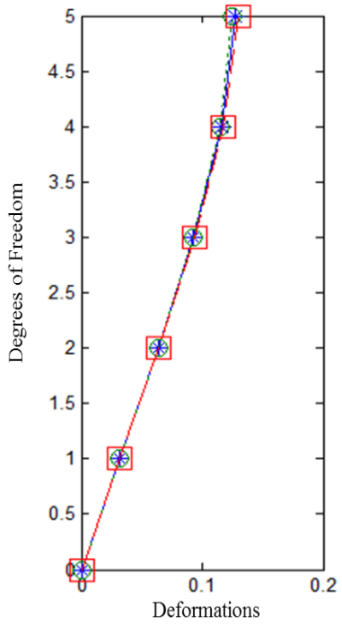

Fig. 5.5(b) First mode for $\alpha = 0.6$ and 1 (triangular fuzzy parameters)

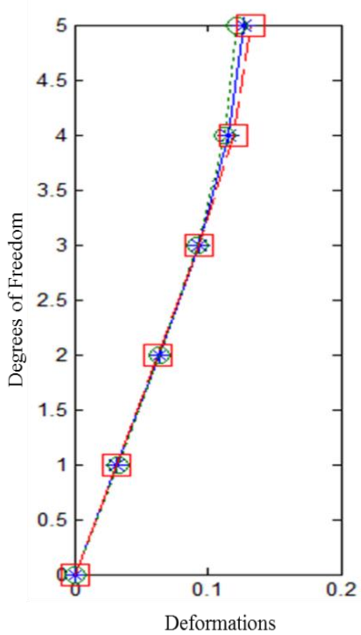

Fig. 5.5(a) First mode for $\alpha = 0$ and 1 (triangular fuzzy parameters)

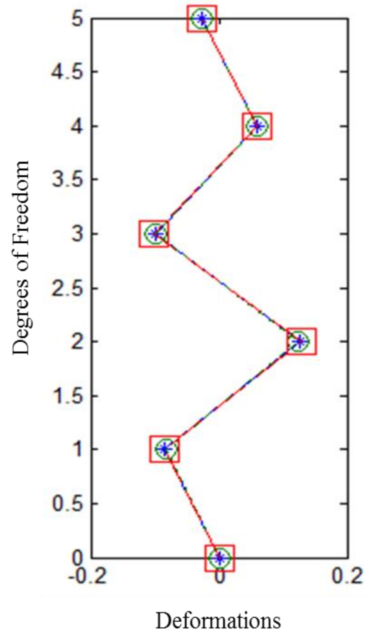

Fig. 5.6(a) Fifth mode for $\alpha = 0$ and 1 (triangular fuzzy parameters)

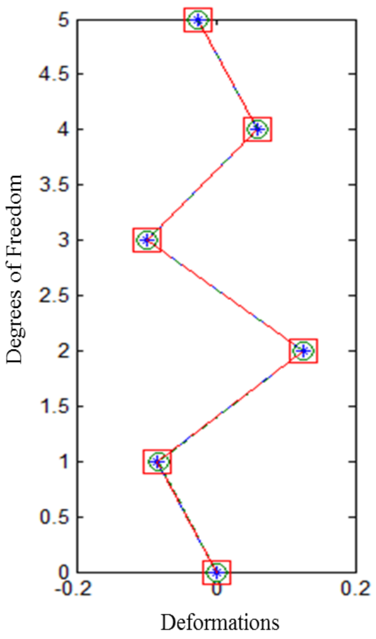

Fig. 5.6(b) Fifth mode for $\alpha = 0.6$ and 1 (triangular fuzzy parameters)

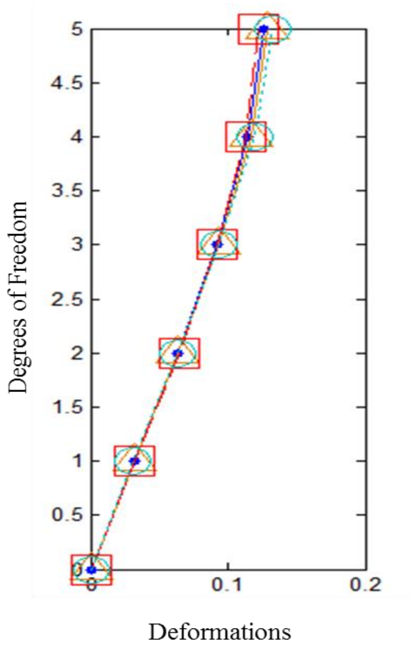

Fig. 5.7(a) First mode for $\alpha = 0$ and 1 (trapezoidal fuzzy parameters)

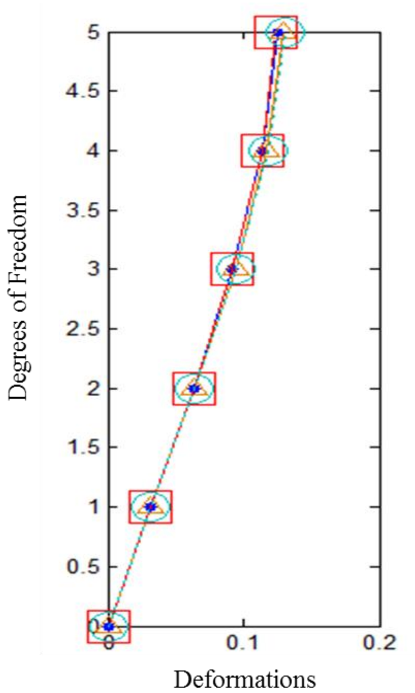

Fig. 5.7(b) First mode for $\alpha = 0.6$ and 1 (trapezoidal fuzzy parameters)

From the results, one may notice that first and fifth natural frequencies are minimum and maximum respectively for crisp parameters. Accordingly, left and right bounds of the first and fifth natural frequencies are maximum and minimum respectively for fuzzy parameters. Also for fuzzy material properties, one may find minimum and maximum uncertainty width for first and fifth natural frequencies respectively. And the uncertainty width gradually increases from first to fifth natural frequencies.

5.3. Spring Mass Mechanical System with Fuzzy Parameters

Let us first consider n Degree of Freedom (DOF) spring–mass mechanical system as shown in Fig. 5.8(a). Here, the spring stiffness and mass are taken as triangular fuzzy number that is $\tilde{k}_i = (\underline{k}_i, k_i, \overline{k}_i)$ for $i = 1, 2, \cdots, n, n+1$ and $\tilde{m}_i = (\underline{m}_i, m_i, \overline{m}_i)$ for $i = 1, 2, \cdots, n$. Next, the trapezoidal fuzzy number spring stiffness and mass are considered as $\tilde{k}_i = (\underline{k}_i, \underline{\underline{k}}_i, \overline{\overline{k}}_i, \overline{k}_i)$ for $i = 1, 2, \cdots, n, n+1$ and $\tilde{m}_i = (\underline{m}_i, \underline{\underline{m}}_i, \overline{\overline{m}}_i, \overline{m}_i)$ for $i = 1, 2, \cdots, n$.

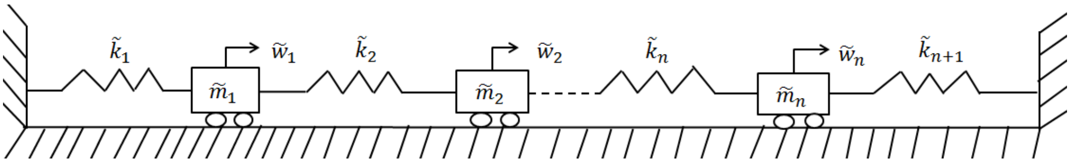

Fig. 5.8(a) The $n-$th degrees of freedom spring - mass system with fuzzy parameters

Then free vibration equation of motion for n DOF fuzzy spring-mass system can be written as

$$[\tilde{K}]\{\tilde{W}\} = \tilde{\lambda}[\tilde{M}]\{\tilde{W}\} \tag{5.9}$$

where,

$$[\tilde{K}] = \begin{bmatrix} \tilde{k}_1 + \tilde{k}_2 & -\tilde{k}_2 & 0 & \cdots & 0 \\ -\tilde{k}_2 & \tilde{k}_2 + \tilde{k}_3 & -\tilde{k}_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\tilde{k}_{n-1} & \tilde{k}_{n-1} + \tilde{k}_n & -\tilde{k}_n \\ 0 & \cdots & \cdots & -\tilde{k}_n & \tilde{k}_n + \tilde{k}_{n+1} \end{bmatrix},$$

$$[\tilde{M}] = \begin{bmatrix} \tilde{m}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{m}_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \tilde{m}_{n-1} & \cdots \\ 0 & \cdots & \cdots & 0 & \tilde{m}_n \end{bmatrix},$$

$$\{\tilde{W}\} = \{\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{n-1}, \tilde{w}_n\} \text{ and } \tilde{\lambda} = \tilde{\lambda}_i \text{ for } i = 1, 2, \cdots, n.$$

Through $\alpha-$cut approach Eq. (5.9) can be written as

$$[\tilde{K}(\alpha)]\{\tilde{W}(\alpha)\} = \tilde{\lambda}(\alpha)[\tilde{M}(\alpha)]\{\tilde{W}(\alpha)\}, \tag{5.10}$$

where

$$[\tilde{K}(\alpha)] = \begin{bmatrix} \tilde{k}_1(\alpha)+\tilde{k}_2(\alpha) & -\tilde{k}_2(\alpha) & 0 & \cdots & 0 \\ -\tilde{k}_2(\alpha) & \tilde{k}_2(\alpha)+\tilde{k}_3(\alpha) & -\tilde{k}_3(\alpha) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\tilde{k}_{n-1}(\alpha) & \tilde{k}_{n-1}(\alpha)+\tilde{k}_n(\alpha) & -\tilde{k}_n(\alpha) \\ 0 & \cdots & \cdots & -\tilde{k}_n(\alpha) & \tilde{k}_n(\alpha)+\tilde{k}_{n+1}(\alpha) \end{bmatrix},$$

$$[\tilde{M}(\alpha)] = \begin{bmatrix} \tilde{m}_1(\alpha) & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{m}_2(\alpha) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \tilde{m}_{n-1}(\alpha) & \cdots \\ 0 & \cdots & \cdots & 0 & \tilde{m}_n(\alpha) \end{bmatrix},$$

$$\{\tilde{W}(\alpha)\} = \begin{Bmatrix} \tilde{w}_1(\alpha) \\ \tilde{w}_2(\alpha) \\ \vdots \\ \vdots \\ \tilde{w}_n(\alpha) \end{Bmatrix} \text{ and } \tilde{\lambda}(\alpha) = \tilde{\lambda}_i(\alpha).$$

Now Eq. (5.10) can be solved using proposed algorithm to find the fuzzy natural frequencies of n DOF fuzzy spring - mass system.

5.3.1. Numerical examples for spring mass system

Let us consider a spring-mass system with 5 degrees of freedom as shown in Fig. 5.8(b). For this system, four cases have been studied by taking stiffness and mass parameters as crisp, triangular fuzzy, trapezoidal fuzzy and non-symmetric triangular fuzzy number respectively as given below in cases 2(a) to 2(c).

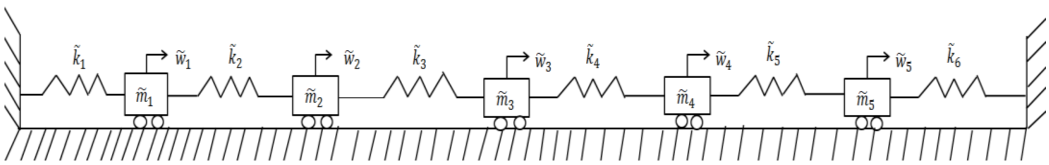

Fig. 5.8(b) The 5 degrees of freedom spring - mass system with fuzzy parameters

Case 2(a): Stiffness and mass parameters are crisp

This is the well known case and it is incorporated here to understand the methodology and comparing the results with fuzzy solution. The stiffness parameters are taken as $k_1 = 2050$ N/m, $k_2 = 1825$ N/m, $k_3 = 1615$ N/m, $k_4 = 1410$ N/m, $k_5 = 1205$ N/m, $k_6 = 1004$ N/m.

The mass parameters are taken as $m_1 = 11$ Kg, $m_2 = 13$ Kg, $m_3 = 15$ Kg, $m_4 = 17$ Kg and $m_5 = 19$ Kg. Eigenvalues are computed for the crisp parameters and those are given in Table 5.8.

Table 5.8 Eigenvalues for crisp material properties

λ_1	λ_2	λ_3	λ_4	λ_5
24.7903	93.5129	185.6226	297.0207	487.6949

Case 2(b): Stiffness and mass parameters are triangular fuzzy number

The material parameters are now considered as

$\tilde{k}_1 = (2000,2050,2100)$ N/m, $\tilde{k}_2 = (1800,1825,1850)$ N/m, $\tilde{k}_3 = (1600,1615,1630)$ N/m, $\tilde{k}_4 = (1400,1410,1420)$ N/m, $\tilde{k}_5 = (1200,1205,1210)$ N/m, $\tilde{k}_6 = (1000,1004,1008)$ N/m

and

$\tilde{m}_1 = (10,11,12)$ Kg, $\tilde{m}_2 = (12,13,14)$ Kg, $\tilde{m}_3 = (14,15,16)$ Kg, $\tilde{m}_4 = (16,17,18)$ Kg, $\tilde{m}_5 = (18,19,20)$ Kg.

Through $\alpha -$cut approach, trapezoidal fuzzy stiffness and mass parameters may be written as

$\tilde{k}_1(\alpha) = [50\alpha + 2000, -50\alpha + 2100]$ N/m, $\tilde{k}_2(\alpha) = [25\alpha + 1800, -25\alpha + 1850]$ N/m,

$\tilde{k}_3(\alpha) = [15\alpha + 1600, -15\alpha + 1630]$ N/m, $\tilde{k}_4(\alpha) = [10\alpha + 1400, -10\alpha + 1420]$ N/m,

$\tilde{k}_5(\alpha) = [5\alpha + 1200, -5\alpha + 1210]$ N/m, $\tilde{k}_6(\alpha) = [4\alpha + 1000, -4\alpha + 1008]$ N/m.

and

$\tilde{m}_1(\alpha) = [\alpha + 10, -\alpha + 12]$ Kg, $\tilde{m}_2(\alpha) = [\alpha + 12, -\alpha + 14]$ Kg,

$\tilde{m}_3(\alpha) = [\alpha + 14, -\alpha + 16]$ Kg, $\tilde{m}_4(\alpha) = [\alpha + 16, -\alpha + 18]$ Kg,

$\tilde{m}_5(\alpha) = [\alpha + 18, -\alpha + 20]$ Kg.

Using the proposed method in the corresponding fuzzy eigenvalue problem, the fuzzy eigenvalues are computed. Obtained results for left and right bounds of fuzzy eigenvalues are incorporated in Tables 5.9(a) and 5.9(b) respectively for each α. Corresponding triangular fuzzy eigenvalue plots are depicted in Figs. 5.9(a) to 5.9(e).

Table 5.9(a) Left bounds of eigenvalues for triangular fuzzy material properties

α	0	0.2	0.4	0.5	0.6	0.8	1
$\underline{\lambda}_1$	20.6734	21.4665	22.2743	22.6838	23.0972	23.9356	24.7903
$\underline{\lambda}_2$	84.8674	86.5127	88.1986	89.0572	89.9264	91.6974	93.5129
$\underline{\lambda}_3$	171.3316	174.0497	176.8355	178.2545	179.6913	182.6195	185.6226
$\underline{\lambda}_4$	274.3046	278.5820	282.9864	285.2380	287.5235	292.1994	297.0207
$\underline{\lambda}_5$	443.7522	451.9731	460.4637	464.8144	469.2377	478.3096	487.6949

Table 5.9(b) Right bounds of eigenvalues for triangular fuzzy material properties

α	0	0.2	0.4	0.5	0.6	0.8	1
$\overline{\lambda}_1$	29.3269	28.3823	27.4570	27.0014	26.5504	25.6617	24.7903
$\overline{\lambda}_2$	103.3138	101.2518	99.2424	98.2569	97.2839	95.3746	93.5129
$\overline{\lambda}_3$	201.8605	198.4399	195.1090	193.4762	191.8645	188.7034	185.6226
$\overline{\lambda}_4$	323.5612	317.9018	312.4265	309.7552	307.1267	301.9940	297.0207
$\overline{\lambda}_5$	539.9518	528.7230	517.9045	512.6420	507.4738	497.4104	487.6949

Case 2(c): Stiffness and mass parameters are trapezoidal fuzzy number

Corresponding material properties are taken as

$\tilde{k}_1 = (2000,2025,2075,2100)$ N/m, $\tilde{k}_2 = (1800,1820,1830,1850)$ N/m,

$\tilde{k}_3 = (1600,1610,1620,1630)$ N/m, $\tilde{k}_4 = (1400,1405,1415,1420)$ N/m,

$\tilde{k}_5 = (1200,1202,1208,1210)$ N/m, $\tilde{k}_6 = (1000,1002,1006,1008)$ N/m

and

$\tilde{m}_1 = (10,10.5,11.5,12)$ Kg, $\tilde{m}_2 = (12,12.5,13.5,14)$ Kg, $\tilde{m}_3 = (14,14.5,15.5,16)$ Kg,

$\tilde{m}_4 = (16,16.5,17.5,18)$ Kg, $\tilde{m}_5 = (18,18.5,19.5,20)$ Kg.

Through $\alpha-$cut approach, trapezoidal fuzzy stiffness and mass parameters may again be written as

$\tilde{k}_1(\alpha) = [25\alpha + 2000, -25\alpha + 2100]$ N/m, $\tilde{k}_2(\alpha) = [20\alpha + 1800, -20\alpha + 1850]$ N/m,

$\tilde{k}_3(\alpha) = [10\alpha + 1600, -10\alpha + 1630]$ N/m, $\tilde{k}_4(\alpha) = [5\alpha + 1400, -5\alpha + 1420]$ N/m,

$\tilde{k}_5(\alpha) = [2\alpha + 1200, -2\alpha + 1210]$ N/m, $\tilde{k}_6(\alpha) = [2\alpha + 1000, -2\alpha + 1008]$ N/m.

and

$\tilde{m}_1(\alpha) = [0.5\alpha + 10, -0.5\alpha + 12]$ Kg, $\tilde{m}_2(\alpha) = [0.5\alpha + 12, -0.5\alpha + 14]$ Kg,

$\tilde{m}_3(\alpha) = [0.5\alpha + 14, -0.5\alpha + 16]$ Kg, $\tilde{m}_4(\alpha) = [0.5\alpha + 16, -0.5\alpha + 18]$ Kg,

$\tilde{m}_5(\alpha) = [0.5\alpha + 18, -0.5\alpha + 20]$ Kg.

Again, using the proposed method for the trapezoidal fuzzy parameters, the fuzzy eigenvalues are computed. Corresponding results for left and right bounds of fuzzy eigenvalues are given in Tables 5.10(a) and 5.10(b) respectively for each α. The trapezoidal fuzzy eigenvalue plots are cited in Figs. 5.10(a) to 5.10(e).

Table 5.10(a) Left bounds of fuzzy eigenvalues for trapezoidal fuzzy material properties

α	0	0.2	0.4	0.5	0.6	0.8	1
$\underline{\lambda}_1$	20.6734	21.1287	21.5870	21.8172	22.0482	22.5124	22.9798
$\underline{\lambda}_2$	84.8674	85.7835	86.7111	87.1792	87.6502	88.6011	89.5639
$\underline{\lambda}_3$	171.3316	172.7613	174.2090	174.9397	175.6750	177.1594	178.6627
$\underline{\lambda}_4$	274.3046	276.4919	278.7131	279.8366	280.9688	283.2598	285.5871
$\underline{\lambda}_5$	443.7522	448.0719	452.4615	454.6830	456.9225	461.4568	466.0662

Table 5.10(b) Right bounds of fuzzy eigenvalues for trapezoidal fuzzy material properties

α	0	0.2	0.4	0.5	0.6	0.8	1
$\overline{\lambda}_1$	29.3269	28.7952	28.2676	28.0053	27.7441	27.2247	26.7092
$\overline{\lambda}_2$	103.3138	102.1654	101.0317	100.4703	99.9125	98.8076	97.7168
$\overline{\lambda}_3$	201.8605	200.0486	198.2609	197.3759	196.4968	194.7560	193.0381
$\overline{\lambda}_4$	323.5612	320.6160	317.7220	316.2938	314.8779	312.0823	309.3341
$\overline{\lambda}_5$	539.9518	534.0016	528.1621	525.2829	522.4303	516.8030	511.2776

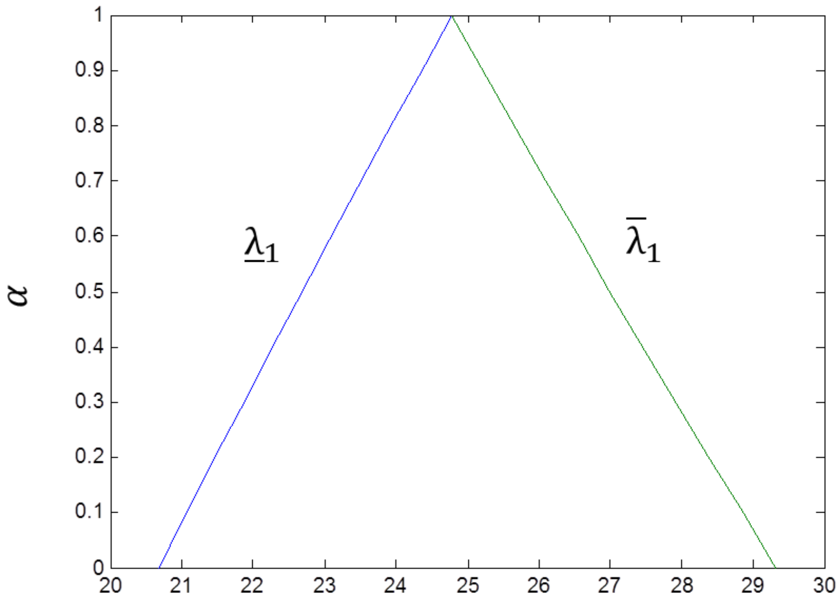

Fig. 5.9(a) First natural frequency for triangular parameters

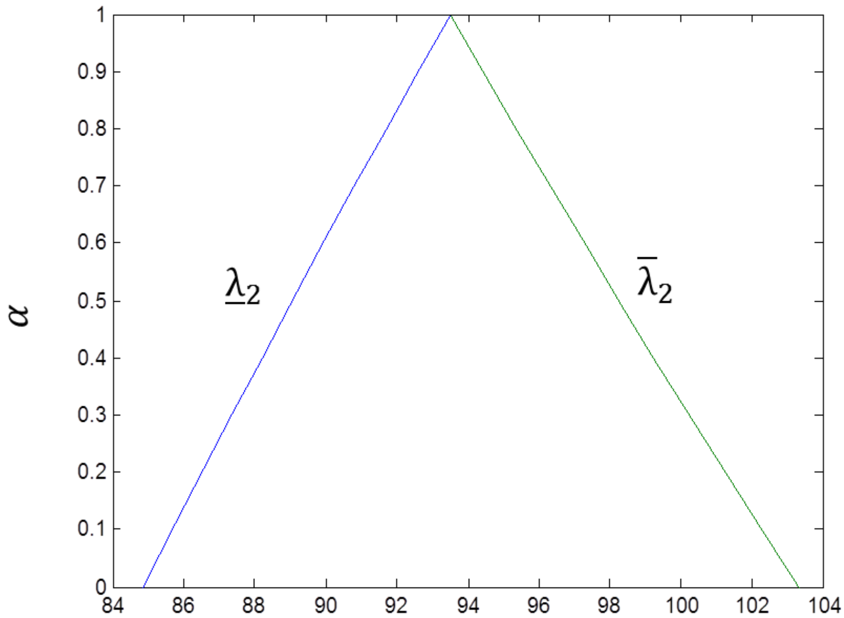

Fig. 5.9(b) Second natural frequency for triangular parameters

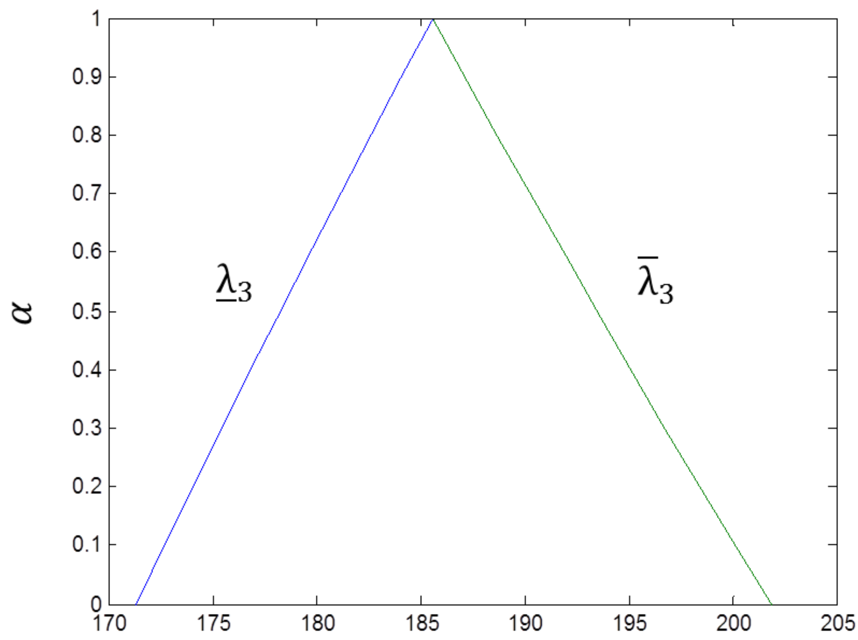

Fig. 5.9(c) Third natural frequency for triangular parameters

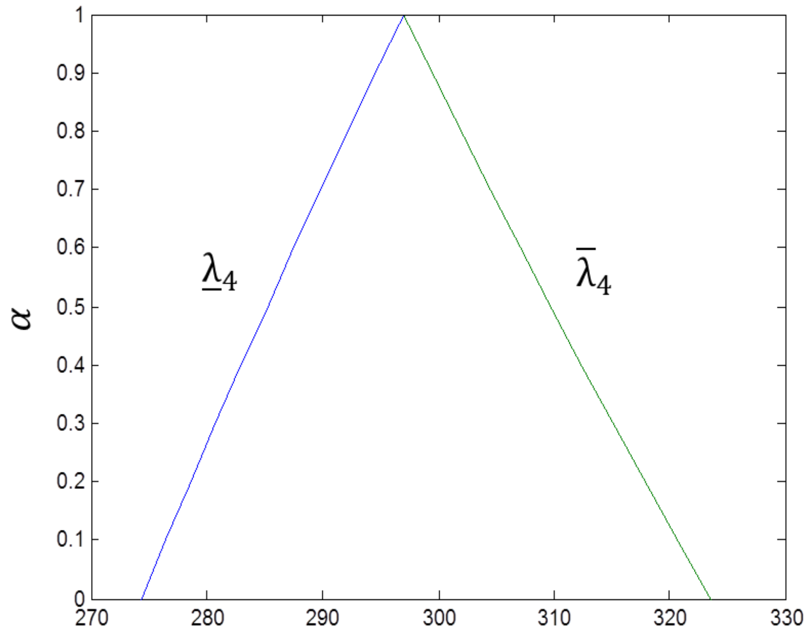

Fig. 5.9(d) Fourth natural frequency for triangular parameters

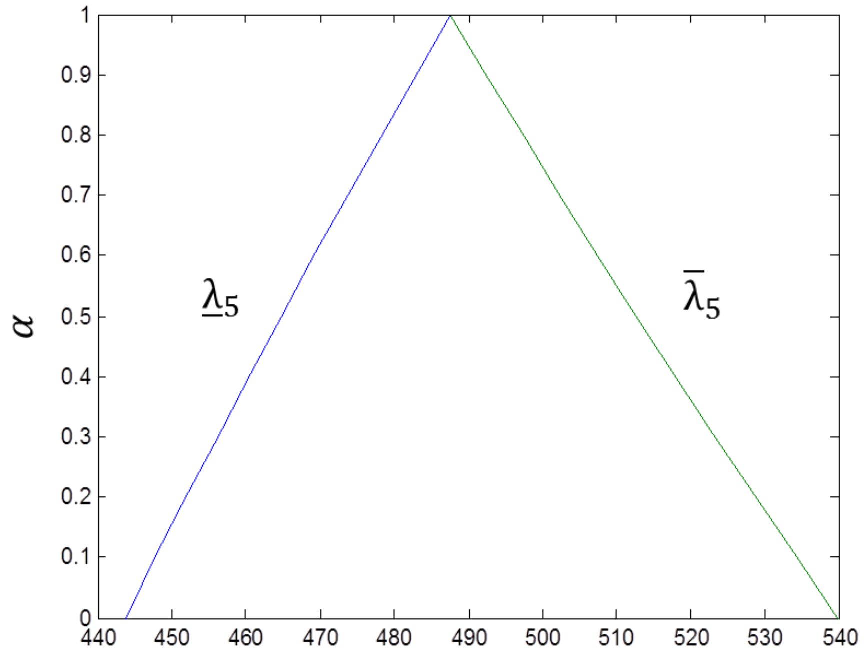

Fig. 5.9(e) Fifth natural frequency for triangular parameters

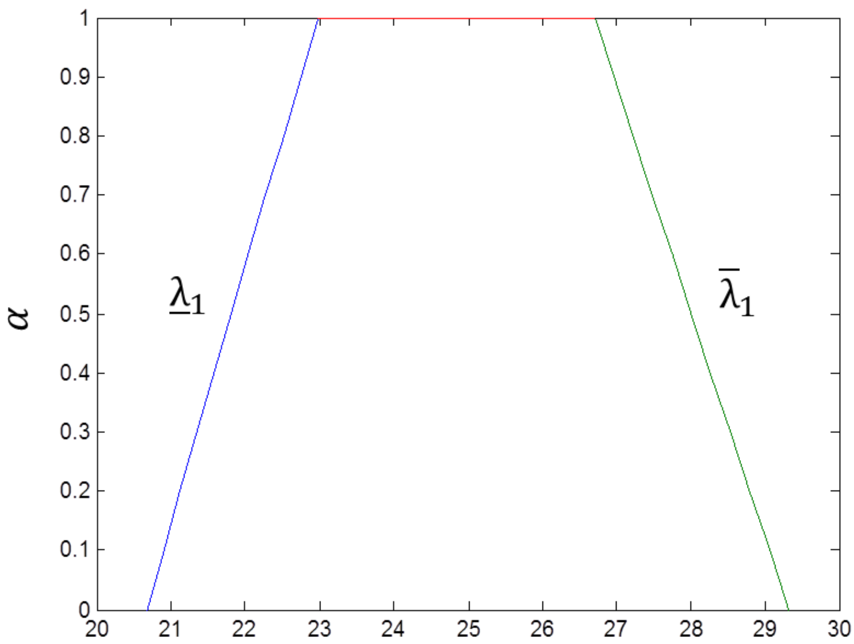

Fig. 5.10(a) First natural frequency for trapezoidal parameters

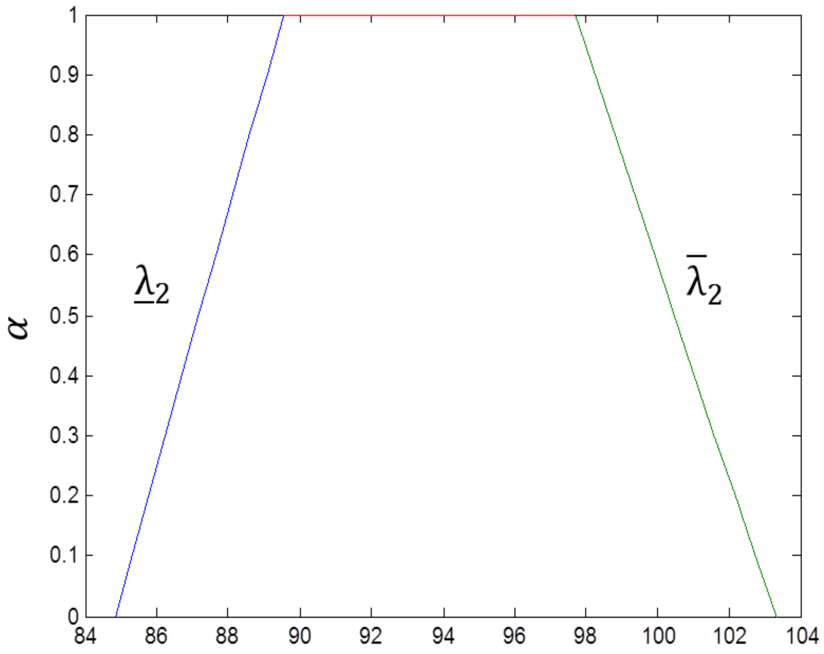

Fig. 5.10(b) Second natural frequency for trapezoidal parameters

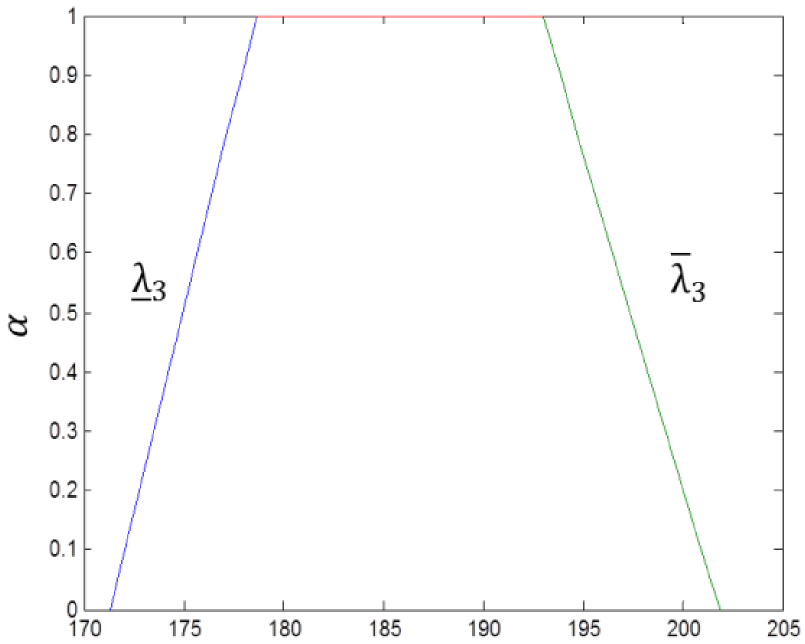

Fig. 5.10(c) Third natural frequency for trapezoidal parameters

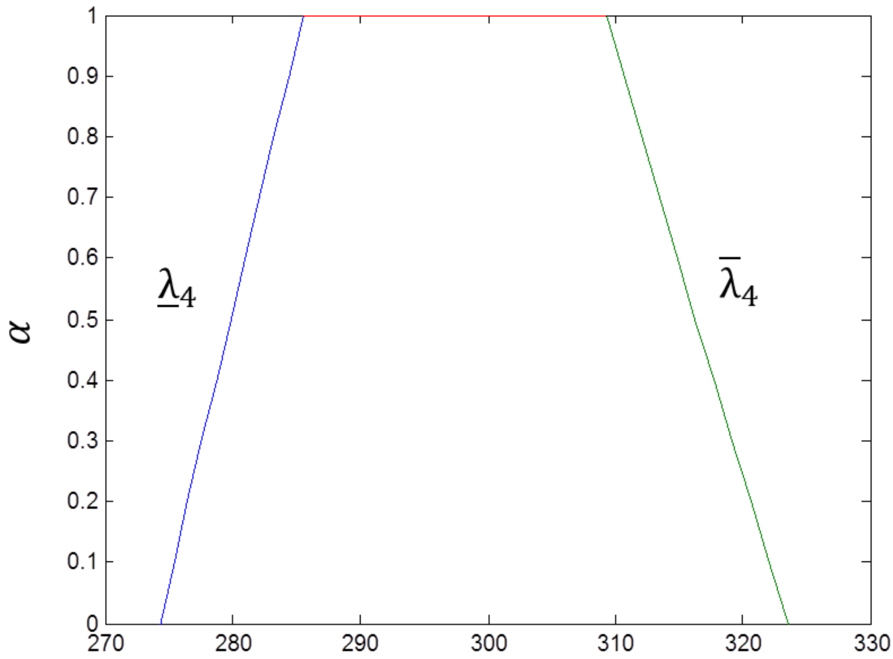

Fig. 5.10(d) Fourth natural frequency for trapezoidal parameters

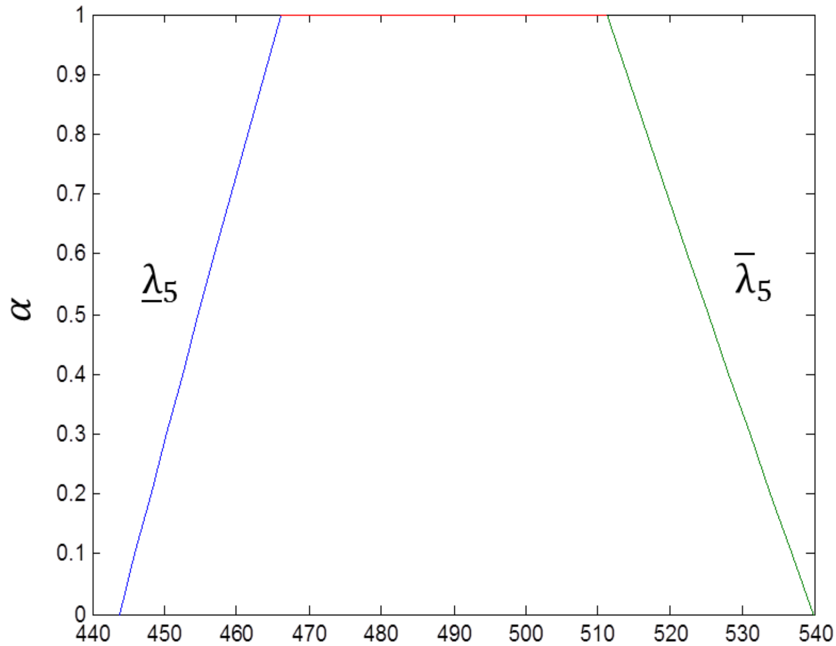

Fig. 5.10(e) Fifth natural frequency for trapezoidal parameters

Special Case: Next, let us consider the fuzzy stiffness and mass parameters for $\alpha = 0$. In this case we get the parameters in interval form. So the material parameters of the system will be as follows.

$\tilde{k}_1 = [2000, 2100]$ N/m, $\tilde{k}_2 = [1800, 1850]$ N/m, $\tilde{k}_3 = [1600, 1630]$ N/m,

$\tilde{k}_4 = [1400, 1420]$ N/m, $\tilde{k}_5 = [1200, 1210]$ N/m, $\tilde{k}_5 = [1000, 1008]$ N/m

and

$\tilde{m}_1 = [10, 12]$ Kg, $\tilde{m}_2 = [12, 14]$ Kg, $\tilde{m}_3 = [14, 16]$ Kg, $\tilde{m}_4 = [16, 18]$ Kg,

$\tilde{m}_5 = [18, 20]$ Kg.

As mentioned earlier, $\alpha = 0$ in the proposed method converts into a generalized interval eigenvalue problem. The left and right bounds of interval eigenvalues are obtained using the method in interval case. Left and the right bounds of the interval eigenvalues are depicted in Tables 5.11(a) and 5.11(b) respectively.

Table 5.11(a) Left bounds of interval eigenvalues for interval material properties

$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$	$\underline{\lambda}_5$
20.6734	84.8674	171.3316	274.3046	443.7522

Table 5.11(b) Right bounds of interval eigenvalues for interval material properties

$\overline{\lambda}_1$	$\overline{\lambda}_2$	$\overline{\lambda}_3$	$\overline{\lambda}_4$	$\overline{\lambda}_5$
29.3269	103.3138	201.8605	323.5612	539.9518

Obtained interval eigenvaules are compared with the results of Chen et al. (1995) in Table 5.12 and are found to be in good agreement.

Table 5.12 Interval eigenvalues for the special case and comparison with Chen et al. (1995)

Chen et al. (1995)		Present method	
$\underline{\lambda}_i$	$\overline{\lambda}_i$	$\underline{\lambda}_i$	$\overline{\lambda}_i$
20.6733690	29.3269368	20.6734	29.3269
84.8673933	103.3137982	84.8674	103.3138
171.3316032	201.8605276	171.3316	201.8605
274.3045861	323.5612202	274.3046	323.5612
443.7521711	539.9518289	443.7522	539.9518

5.4. Stepped Beam with Fuzzy Parameters

A typical beam element viz. ith element of a stepped rectangular beam has been shown in Fig. 5.11(a) with fuzzy parameters. Fuzzy finite element method is used to analyze the vibration of the beam. Here, Young's modulus, density and length of the typical beam element are considered as crisp and denoted respectively by $E^{(i)}$, $\rho^{(i)}$ and $l^{(i)}$. Area of cross section and moment of inertia viz. $\tilde{A}^{(i)}$ and $\tilde{I}^{(i)}$ are taken as fuzzy. Here $\tilde{w}_1^{(i)}$, $\tilde{w}_3^{(i)}$ and $\tilde{w}_2^{(i)}$, $\tilde{w}_4^{(i)}$ are the vertical fuzzy displacement and fuzzy slope at node 1 and node 2 respectively.

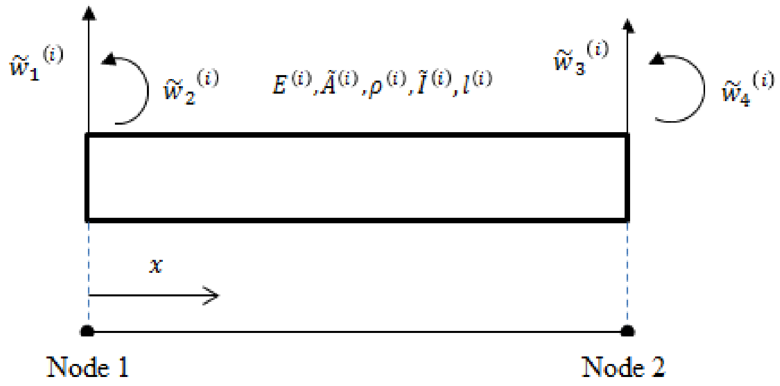

Fig. 5.11(a) A typical beam element corresponding to $i-$th element

Using Lagrange's equation, one may obtain fuzzy equation of motion for vibration of beam element as

$$[\tilde{M}^{(i)}]\{\ddot{\tilde{U}}\}+[\tilde{K}^{(i)}]\{\tilde{U}\}=\{0\} \tag{5.11}$$

where

$$\tilde{M}^{(i)}=\frac{\rho^{(i)}\tilde{A}^{(i)}l^{(i)}}{420}\begin{bmatrix}156 & 22l^{(i)} & 54 & -13l^{(i)}\\ 22l^{(i)} & 4(l^{(i)})^2 & 13l^{(i)} & -3(l^{(i)})^2\\ 54 & 13l^{(i)} & 156 & -22l^{(i)}\\ -13l^{(i)} & -3(l^{(i)})^2 & -22l^{(i)} & 4(l^{(i)})^2\end{bmatrix}$$

and

$$\tilde{K}^{(i)}=\frac{E^{(i)}\tilde{I}^{(i)}}{(l^{(i)})^3}\begin{bmatrix}12 & 6l^{(i)} & -12 & 6l^{(i)}\\ 6l^{(i)} & 4(l^{(i)})^2 & -6l^{(i)} & 2(l^{(i)})^2\\ -12 & -6l^{(i)} & 12 & -6l^{(i)}\\ 6l^{(i)} & 2(l^{(i)})^2 & -6l^{(i)} & 4(l^{(i)})^2\end{bmatrix}$$

are the mass and stiffness matrices of the i-th element. Now writing Eq. (5.11) in the form of fuzzy generalized eigenvalue problem, one may have

$$[\tilde{K}^{(i)}]\{\tilde{W}\} = \tilde{\lambda}[\tilde{M}^{(i)}]\{\tilde{W}\} \tag{5.12}$$

where, $\tilde{\lambda} = \tilde{\omega}^2$ and $\{\tilde{W}\} = \{\tilde{w}_1^{(i)}, \tilde{w}_2^{(i)}, \tilde{w}_3^{(i)}, \tilde{w}_4^{(i)}\}$ are the eigenvalue and corresponding eigenvector of Eq. (5.12) respectively. This can be solved by using the proposed algorithm.

5.4.1. Numerical examples for stepped beam

A three stepped beam as shown in Fig. 5.11(b) has been taken into consideration. The data for this example are considered as

$\tilde{A}^1 = [1.426, 1.44, 1.454] \times 10^{-2}\ \mathrm{m}^2$, $\tilde{A}^2 = [0.99, 1, 1.01] \times 10^{-2}\ \mathrm{m}^2$,

$\tilde{A}^3 = [0.634, 0.64, 0.646] \times 10^{-2}\ \mathrm{m}^2$ and $\tilde{I}^1 = [0.1998, 0.2, 0.2002] \times 10^{-4}\ \mathrm{m}^4$,

$\tilde{I}^2 = [0.0999, 0.1, 0.1001] \times 10^{-4}\ \mathrm{m}^4$, $\tilde{I}^3 = [0.04995, 0.05, 0.05005] \times 10^{-4}\ \mathrm{m}^4$.

Other structural parameters viz. Young's modulus, densities and lengths are considered crisp and these are taken as $E^i = 2 \times 10^{11}\ \mathrm{N/m}^2$, $\rho^i = 7800\ \mathrm{kg/m}^3$ and $l^i = 0.4\ \mathrm{m}$ for $i = 1, 2, 3$.

The beam structure is discretized into 3 elements. For each element, stiffness and mass matrices have been obtained as above. Assembling these and applying the boundary conditions at node 1, one may have the eigenvalue problem with reduced stiffness and mass matrices. Left and right bounds of fuzzy eigenvalue are obtained by utilising the present algorithm. Here, left and right bounds of generalised fuzzy eigenvalue problems have been denoted by $\underline{\lambda}_j$ and $\overline{\lambda}_j$ for $j = 1, 2, \cdots, 6$. Computed results are shown in Tables 5.13(a) and 5.13(b).

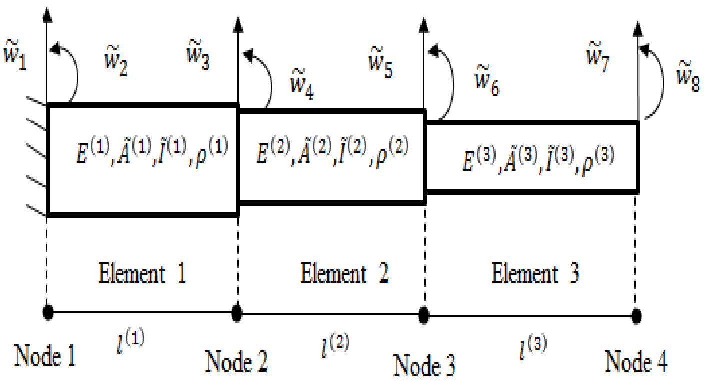

Fig. 5.11(b) A stepped beam element discretized into three finite elements corresponding to four nodes

Table 5.13(a) Left bounds of fuzzy eigenvalues of stepped beam for triangular parameters

α	0	0.2	0.4	0.5	0.6	0.8	1
$\underline{\lambda}_1$	283498.26	299685.15	315936.53	324086.51	332252.74	348634.13	365081.05
$\underline{\lambda}_2$	7032296.7	7092122.5	7152355.7	7282626.2	7212999.8	7274058.4	7335534.9
$\underline{\lambda}_3$	46616830	46931252	47248630	47408434	47568986	47892344	48218723
$\underline{\lambda}_4$	2.483555×10^8	2.5021725×10^8	2.5211726×10^8	2.5308194×10^8	2.5405658×10^8	2.5603625×10^8	2.5805738×10^8
$\underline{\lambda}_5$	8.309656×10^8	8.4229255×10^8	8.5403749×10^8	8.6007244×10^8	8.6621885×10^8	8.7885574×10^8	8.9196795×10^8
$\underline{\lambda}_6$	2.2545007×10^9	2.3344124×10^9	2.4217226×10^9	2.4684761×10^9	2.5174927×10^9	2.6229966×10^9	2.739776×10^9

Table 5.13(b) Right bounds of fuzzy eigenvalues of stepped beam for triangular parameters

α	0	0.2	0.4	0.5	0.6	0.8	1
$\overline{\lambda}_1$	448310.91	431531.05	414818.49	406487.34	398172.87	381593.85	365081.05
$\overline{\lambda}_2$	7649314.1	7585693.2	7522508.5	7491078.6	7459756.3	7697433	7335534.9
$\overline{\lambda}_3$	49896671	49554876	49216202	49048030	48880632	48548145	48218723
$\overline{\lambda}_4$	2.6882503×10^8	2.6657926×10^8	2.6438084×10^8	2.6329899×10^8	2.6222851×10^8	2.6012109×10^8	2.5805738×10^8
$\overline{\lambda}_5$	9.6539483×10^8	9.4958776×10^8	9.3436392×10^8	9.2696363×10^8	9.197007×10^8	9.0557592×10^8	8.9196795×10^8
$\overline{\lambda}_6$	3.5776134×10^9	3.3648781×10^9	3.178968×10^9	3.0945461×10^9	3.0151427×10^9	2.8697154×10^9	2.739776×10^9

5.5. Parameter Identification of Multistorey Frame Structure from Uncertain Dynamic Data

In this section, we have investigated the identification procedure of the column stiffness of multistorey frame structures by using the prior known uncertain parameters and dynamic data. Uncertainties are modelled through triangular convex normalized fuzzy sets. Bounds of the identified uncertain stiffness are obtained by using a proposed fuzzy based iteration algorithm associated with the Taylor series expansion. Example problems are solved to demonstrate the reliability and efficiency of the identification process.

5.5.1. Mathematical modelling and method of identification

To investigate the present method, a two-storeyed frame structure, as shown in Fig. 5.12 is considered. The uncertain floor masses, $\tilde{m}$ are assumed to be the same and the uncertain column stiffnesses $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ and $\tilde{k}_4$ (as labelled in Fig. 5.12) are the structural parameters which are to be identified. Corresponding uncertain dynamic equation of motion may be written as

$$[\tilde{M}]\{\ddot{\tilde{x}}\}+[\tilde{K}]\{\tilde{x}\}=\{0\} \tag{5.13}$$

where

$[\tilde{M}]==\begin{bmatrix}\tilde{m} & 0\\ 0 & \tilde{m}\end{bmatrix}$ and $[\tilde{K}]==\begin{bmatrix}(\tilde{k}_1+\tilde{k}_2+\tilde{k}_3+\tilde{k}_4) & -(\tilde{k}_3+\tilde{k}_4)\\ -(\tilde{k}_3+\tilde{k}_4) & (\tilde{k}_3+\tilde{k}_4)\end{bmatrix}$ are 2×2 fuzzy mass matrix and fuzzy stiffness matrices and $\{\tilde{X}\}=2\times 1$ fuzzy vector of displacements.

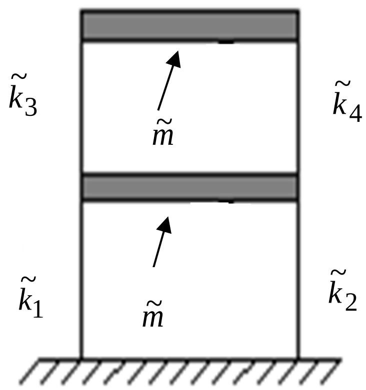

Fig. 5.12 Two storey frame structure

Considering the simple harmonic motion, Eq.(5.13) can be written as a fuzzy eigenvalue problem as

$$[\tilde{K}]\{\tilde{X}\}=\tilde{\lambda}[\tilde{M}]\{\tilde{X}\}. \tag{5.14}$$

Using the parametric form of fuzzy numbers, Eq. (5.14) will be

$$[\underline{K}(\alpha),\overline{K}(\alpha)]\{\underline{X}(\alpha),\overline{X}(\alpha)\}=[\underline{\lambda}(\alpha),\overline{\lambda}(\alpha)][\underline{M}(\alpha),\overline{M}(\alpha)]\{\underline{X}(\alpha),\overline{X}(\alpha)\}$$

Now our aim is to solve the above fuzzy eigenvalue problem to get the lower and upper bounds of the fuzzy eigenvalues.

With the above in mind, let us proceed now with the identification procedure which can handle the uncertain data. Let us assume that the uncertain structural parameters to be identified are denoted by $\tilde{P}_i$, for $i=1,2,3,4$. The uncertain value of the structural parameters of the prior original structure given initially are denoted by $\hat{\tilde{P}}_i$, for $i=1,2,3,4$ and the corresponding fuzzy eigenvalues are symbolized as, $\hat{\tilde{\lambda}}_i(\hat{\tilde{P}})$.

Next, the well-known Taylor's series expansion of the fuzzy modal parameters about the initial estimates of the parameters gives

$$\{\tilde{\lambda}(\tilde{P})\}=\{\hat{\tilde{\lambda}}(\hat{\tilde{P}})\}+[\tilde{S}]\left(\{\tilde{P}\}-\{\hat{\tilde{P}}\}\right) \tag{5.15}$$

where, $\{\tilde{P}\}=[\tilde{P}_1,\tilde{P}_2,\tilde{P}_3,\tilde{P}_4]^T$,$\{\hat{\tilde{P}}\}=[\hat{\tilde{P}}_1,\hat{\tilde{P}}_2,\hat{\tilde{P}}_3,\hat{\tilde{P}}_4]^T$ and $[\tilde{S}]$ is the fuzzy eigenvalue partial derivative matrix, $[\partial(\tilde{\lambda})/\partial(\tilde{P})]$.

Let us now denote experimentally measured uncertain eigenvalues by $\{\tilde{\lambda}_E\}$. It is interesting to note here that if the values of the initial and experimental parameters are equal, then no modification is done. But if the values are different, then we denote this difference by

$$\{\delta\tilde{\lambda}\}=\{\tilde{\lambda}_E\}-\{\hat{\tilde{\lambda}}\}. \tag{5.16}$$

We denote the modified parameters as

$$\{\overline{\tilde{P}}\}=[\overline{\tilde{P}}_1,\overline{\tilde{P}}_2,\overline{\tilde{P}}_3,\overline{\tilde{P}}_4]^T \tag{5.17}$$

and in general, for $n-$degrees of freedom system, the expression for the uncertain modified parameters from Eq. (5.15) can be written as

$$\{\overline{\tilde{P}}\}=\{\hat{\tilde{P}}\}+[\tilde{Q}]\{\delta\tilde{\lambda}\} \tag{5.18}$$

where

$$[\tilde{Q}]=\left([\tilde{S}]^T[\tilde{S}]\right)^{-1}[\tilde{S}]^T.$$

To have the uncertain bounds of the identified parameters with acceptable accuracy. After finding the modified parameters from Eq. (5.18), these are substituted in Eq. (5.14) to get revised uncertain vibration characteristics viz. $\{\overline{\overline{\tilde{\lambda}}}\}$.

New fuzzy eigenvalue partial derivative matrix $\{\overline{\overline{\tilde{S}}}\}$ is then obtained using the current values of $\{\overline{\overline{\tilde{P}}}\}$ and $\{\overline{\overline{\tilde{\lambda}}}\}$. From Eq. (5.18), the modified parameters $\{\overline{\overline{\tilde{P}}}_t\}$ are again found by utilizing the above values and then the new (revised) estimates of fuzzy eigenvalue are obtained as $\{\overline{\overline{\tilde{\lambda}}}_t\}$.

If the vector norm of $\{\overline{\overline{\tilde{\lambda}}}\}$ and $\{\overline{\overline{\tilde{\lambda}}}_t\}$ is less than some specified accuracy, then the procedure is stopped and the revised parameter viz. $\{\overline{\overline{\tilde{P}}}_t\}$ is identified, otherwise the next iteration is to be followed.

5.5.2. Numerical examples for parameter idefication

As mentioned earlier, the procedure is demonstrated for a two storeyed frame structure. Implementing the above procedure with the proposed iterative cycle for the revised uncertain frequencies and parameters, computer programs have been written and tested for the above problem.

In this example, floor masses $m = (3550, 3600, 3650)$ kg and the column stiffnesses $\tilde{k}_1 = \tilde{k}_2 = (5350, 5400, 5450)$ N/m, $\tilde{k}_3 = \tilde{k}_4 = (3550, 3600, 3650)$ N/m have been taken as triangular fuzzy number.

Through $\alpha-$cut, these may represented as $m = [50\alpha + 3550, -50\alpha + 3650]$ kg, $\tilde{k}_1 = \tilde{k}_2 = [50\alpha + 5350, -50\alpha + 5450]$ N/m and $\tilde{k}_3 = \tilde{k}_4 = [50\alpha + 3550, -50\alpha + 3650]$N/m.

From these prior mass and stiffness parameters, the uncertain vibration characteristics may be computed from Eq. (5.14) as $\tilde{\lambda}_1 = (0.9314, 1, 1.0703)$ and $\tilde{\lambda}_2 = (5.8906, 6, 6.1128)$.

Using the above sets of initial data of the fuzzy parameters with different uncertain experimental (hypothetical) test data for the frequencies, viz. $\tilde{\lambda}_{1E} = (0.65, 0.7, 0.75)$ and $\tilde{\lambda}_{2E} = (5.3, 5.5, 5.7)$ (that is first and second experimental eigenvalues of the system) the bounds of the stiffness parameters of the structure have been identified and these are depicted in Figs. 5.13 and 5.14.

Similarly, for another set of experimental (hypothetical) fuzzy data of the natural frequencies $\tilde{\lambda}_{1E} = (0.88, 0.9, 0.92)$ and $\tilde{\lambda}_{2E} = (5.3, 5.5, 5.7)$, the identified bounds of the stiffness parameters are shown in Figs. 5.15 and 5.16.

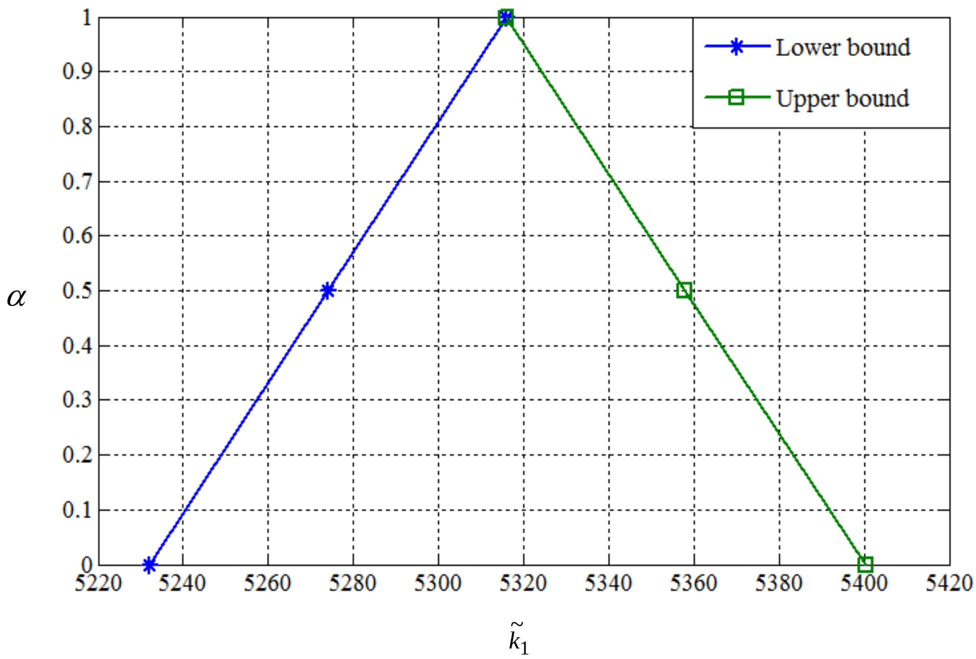

Fig. 5.13 Identified lower and upper bounds of stiffness parameter $\tilde{k}_1$ (N/m)

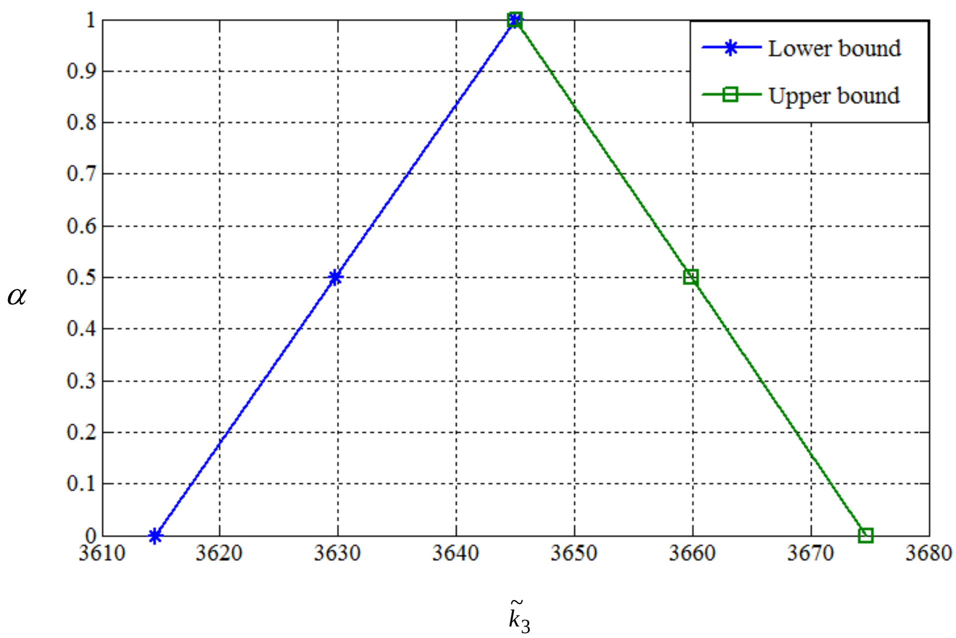

Fig. 5.14 Identified lower and upper bounds of stiffness parameter $\tilde{k}_3$ (N/m)

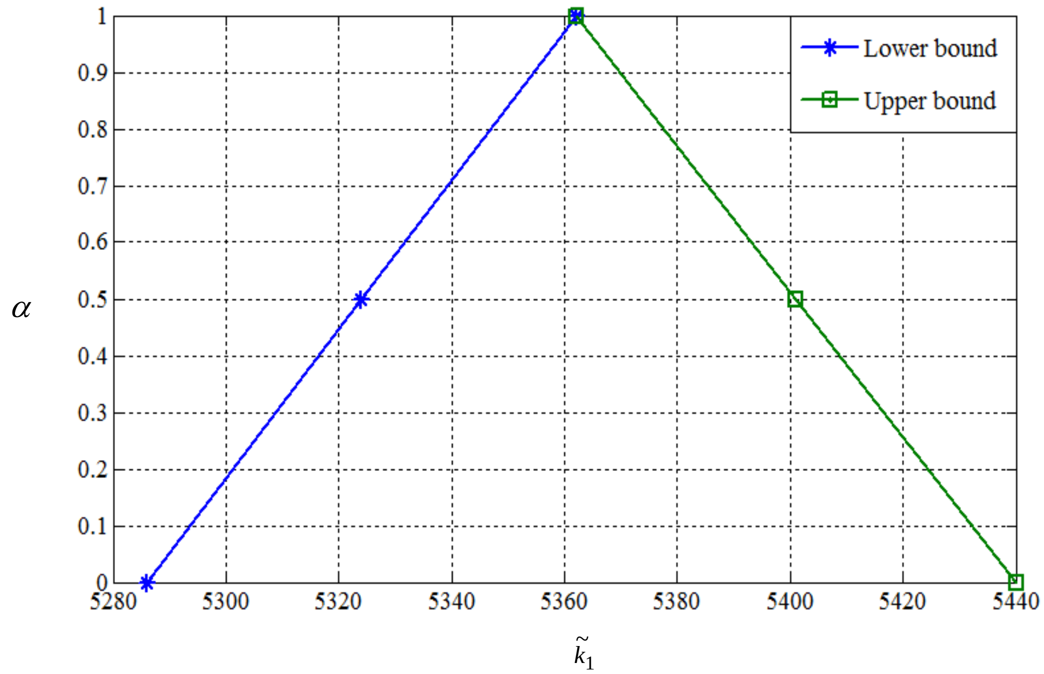

Fig. 5.15 Identified lower and upper bounds of stiffness parameter $\tilde{k}_1$(N/m)

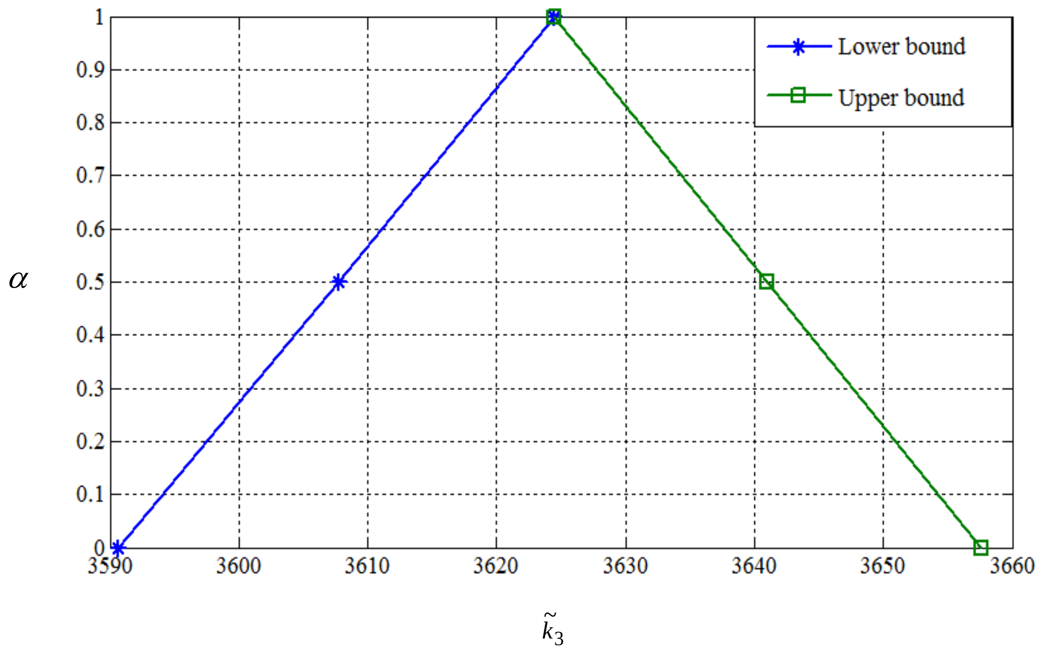

Fig. 5.16 Identified lower and upper bounds of stiffness parameter $\tilde{k}_3$ (N/m)

It is worth mentioning that if the input data set viz. the design frequency is near to the experimental frequency data then the modified stiffness data bound has less width. This is expected as the design and experimental frequency being close means that the structure has not deteriorated much. On the other hand, when the experimental data is taken bit far from the designed one then the estimated stiffness parameters give larger bound. These effects may be seen from the Figs. 5.13 to 5.16. It may be noted that the accuracy of the results depends upon many factors viz. on the uncertain bound of the experimental data, initial design values of the parameters, the fuzzy computation, norm as defined etc. Although the method has been demonstrated for a simple problem of two storey, the method may very well be extended to higher storey frames and other structures in a similar fashion.

Chapter 6

Fractionally Damped Discrete System

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Chapter 6

Fractionally Damped Discrete System

This chapter investigates the semi analytical solution of a fractionally damped dynamic system. A single degree of freedom spring-mass mechanical system with fractional damping of order 1/2 is considered for the analysis. Homotopy Perturbation Method (HPM) is used to compute the dynamic responses of the system subject to unit step and unit impulse loads. Obtained results are depicted in terms of plots. Comparisons are made with the analytical solutions obtained by using fractional green function of Podlunby (1999) and numerical solution of Suarez and Shokooh (1997) and Yuan and Agrawal (2002) in the special cases to show the effectiveness and validation of the present analysis. In this chapter, we have investigated the problem with deterministic parameter and then in Chapter 8 we have considered this problem with uncertainty.

6.1. Fractionally Damped Spring Mass system

To estimate the dynamic response of a fractionally damped discrete system, let us consider a single degree-of-freedom spring-mass-damper system (Suarez and Shokooh 1997; Yuan and Agrawal 2002) which may be described by the following differential equation

$$mD^2 x(t) + cD^{\lambda} x(t) + kx(t) = f(t) \tag{6.1}$$

where, m, c and k represent the mass, damping and stiffness coefficients respectively. $f(t)$ is the externally applied force, and $D^{\lambda} x(t), 0 < \lambda < 1,$ is the derivative of order λ of the displacement function $x(t)$. Although the coefficient λ (known as the memory parameter), may take any value between 0 to 1, the value $1/2$ has been adopted (Suarez and Shokooh 1997) here for this study because it has been shown that it describes the frequency dependence of the damping materials quite satisfactorily (Bagley 1979; Bagley and Torvik 1983; Torvik and Bagley 1984). We have considered the initial conditions as the initial displacement $x(0) = 0$ and the initial velocity $v(0) = \dot{x}(0) = 0.$

Now Eq. (6.1) can be written as

$$D^2 x(t) + \frac{c}{m} D^{1/2} x(t) + \frac{k}{m} x(t) = \frac{f(t)}{m}. \tag{6.2}$$

According to HPM, we may construct a simple homotopy for an embedding parameter $p \in [0,1]$ as follows

$$(1-p)D^2 x(t) + p\left(D^2 x(t) + \frac{c}{m} D^{1/2} x(t) + \frac{k}{m} x(t) - \frac{f(t)}{m} \right) = 0, \tag{6.3}$$

or

$$D^2 x(t) + p\left(\frac{c}{m} D^{1/2} x(t) + \frac{k}{m} x(t) - \frac{f(t)}{m} \right) = 0. \tag{6.4}$$

As discussed earlier, p is considered as a small homotopy parameter $0 \le p \le 1$. So in the changing process from 0 to 1, for $p = 0$, Eqs. (6.3) and (6.4) become a linear equation that is $D^2 x(t) = 0$, which is easy to solve. For $p = 1$, Eqs. (6.3) and (6.4) turns out to be same as the original Eq. (6.1) or (6.2). This is called deformation in topology. $D^2 x(t)$ and $\frac{c}{m} D^{1/2} x(t) + \frac{k}{m} x(t) - \frac{f(t)}{m}$ are called homotopic.

We assume the solution of Eq. (6.3) or (6.4) as a power series expansion in p as

$$x(t) = x_0(t) + p x_1(t) + p^2 x_2(t) + p^3 x_3(t) + \cdots, \tag{6.5}$$

where $x_i(t)$, $\text{i} = 0, 1, 2, \ldots$ are functions yet to be determined. Substituting Eq. (6.5) into Eq. (6.3) or (6.4), and equating the terms with the identical power of p, we can obtain a series of equations of the form

$$\begin{aligned}
p^0 &: D^2 x_0(t) = 0, \\
p^1 &: D^2 x_1(t) + \frac{c}{m} D^{1/2} x_0(t) + \frac{k}{m} x_0(t) - \frac{f(t)}{m} = 0, \\
p^2 &: D^2 x_2(t) + \frac{c}{m} D^{1/2} x_1(t) + \frac{k}{m} x_1(t) = 0, \\
p^3 &: D^2 x_3(t) + \frac{c}{m} D^{1/2} x_2(t) + \frac{k}{m} x_2(t) = 0, \qquad (6.6) \\
p^4 &: D^2 x_4(t) + \frac{c}{m} D^{1/2} x_3(t) + \frac{k}{m} x_3(t) = 0, \\
p^5 &: D^2 x_5(t) + \frac{c}{m} D^{1/2} x_4(t) + \frac{k}{m} x_4(t) = 0, \\
p^6 &: D^2 x_6(t) + \frac{c}{m} D^{1/2} x_5(t) + \frac{k}{m} x_5(t) = 0,
\end{aligned}$$

and so on.

Applying the operator L_{tt}^{-1} (which is the inverse of the operator $L_{tt} = D^2$) on both sides of Eq. (6.6), one may obtain the following equations

$$\begin{aligned}
x_0(t) &= 0,\\
x_1(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_0(t) - \frac{k}{m}x_0(t) + \frac{f(t)}{m}\right)\\
&= \frac{c}{m}D^{-3/2}x_0(t) - \frac{k}{m}D^{-2}x_0(t) + D^{-2}\frac{f(t)}{m},\\
x_2(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_1(t) - \frac{k}{m}x_1(t)\right)\\
&= -\frac{c}{m}D^{-3/2}x_1(t) - \frac{k}{m}D^{-2}x_1(t),\\
x_3(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_2(t) - \frac{k}{m}x_2(t)\right)\\
&= -\frac{c}{m}D^{-3/2}x_2(t) - \frac{k}{m}D^{-2}x_2(t),\\
x_4(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_3(t) - \frac{k}{m}x_3(t)\right)\\
&= -\frac{c}{m}D^{-3/2}x_3(t) - \frac{k}{m}D^{-2}x_3(t),\\
x_5(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_4(t) - \frac{k}{m}x_4(t)\right)\\
&= -\frac{c}{m}D^{-3/2}x_4(t) - \frac{k}{m}D^{-2}x_4(t),\\
x_6(t) &= L_{tt}^{-1}\left(-\frac{c}{m}D^{1/2}x_5(t) - \frac{k}{m}x_5(t)\right)\\
&= -\frac{c}{m}D^{-3/2}x_5(t) - \frac{k}{m}D^{-2}x_5(t),
\end{aligned} \tag{6.7}$$

and so on.

Now substituting these terms in Eq. (6.5), one may get the approximate solution of Eq. (6.1) as

$$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) + x_6(t) + \cdots$$

The solution series converge very rapidly. The proof of convergence of the above series may be found in He (2009, 2010). The rapid convergence means that only few terms are sufficient to get the approximate solutions.

6.2. Step Function Response

We will now consider a stationary oscillator subject to an excitation of the form $f(t) = u(t)$, where $u(t)$ is the Heaviside function with unit step load in Eq. (6.1). By using HPM we have

$x_0(t) = 0$,

$x_1(t) = \frac{1}{2m} t^2 u(t)$,

$x_2(t) = \left(-\frac{c}{m^2} \frac{t^{7/2}}{\Gamma(9/2)} - \frac{k}{m^2} \frac{t^4}{\Gamma(5)} \right) u(t)$,

$x_3(t) = \left(\frac{c^2}{m^3} \frac{t^5}{\Gamma(6)} + \frac{2kc}{m^3} \frac{t^{11/2}}{\Gamma(13/2)} + \frac{k^2}{m^3} \frac{t^6}{\Gamma(7)} \right) u(t)$,

$$x_4(t) = \left(-\frac{c^3}{m^4} \frac{t^{13/2}}{\Gamma(15/2)} - \frac{3kc^2}{m^4} \frac{t^7}{\Gamma(8)} - \frac{3k^2c}{m^4} \frac{t^{15/2}}{\Gamma(17/2)} - \frac{k^3}{m^4} \frac{t^8}{\Gamma(9)} \right) u(t), \tag{6.8}$$

$x_5(t) = \left(\frac{c^4}{m^5} \frac{t^8}{\Gamma(9)} + \frac{4kc^3}{m^5} \frac{t^{17/2}}{\Gamma(19/2)} + \frac{6k^2c^2}{m^5} \frac{t^9}{\Gamma(10)} + \frac{4k^3c}{m^5} \frac{t^{19/2}}{\Gamma(21/2)} + \frac{k^4}{m^5} \frac{t^{10}}{\Gamma(11)} \right) u(t)$,

and so on.

In the similar manner, the rest of the components can be obtained. Therefore, the solution can be written in general form as

$$x(t) = \frac{u(t)}{m} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m} \right)^r t^{2(r+1)} \sum_{j=0}^{\infty} \left(\frac{-c}{m} \right)^j \frac{(j+r)!\, t^{3j/2}}{j!\, \Gamma\left(\frac{3j}{2} + 2r + 3 \right)} \tag{6.9}$$

$$= \frac{u(t)}{m} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m} \right)^r t^{2(r+1)} E^r_{3/2,\, r/2+3} \left(\frac{-c}{m} t^{3/2} \right). \tag{6.10}$$

Now Eq. (6.10) can be rewritten as

$$x(t) = \frac{u(t)}{m} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\omega_n^2 \right)^r t^{2(r+1)} E^r_{3/2,\, r/2+3} \left(-2\eta \omega_n^{3/2} t^{3/2} \right)$$

where, $\omega_n = \sqrt{k/m}$ and $\eta = c/2m\omega_n^{3/2}$ are the natural frequency and damping ratio respectively.

6.3. Impulse Function Response

In this section, we consider response subject to a unit impulse load that is $f(t)=\delta(t)$, where $\delta(t)$ is the unit impulse function. Again by using HPM we obtain

$$x_0(t)=0\,,$$

$$x_1(t)=\frac{t}{m}\,,$$

$$x_2(t)=-\frac{c}{m^2}\frac{t^{5/2}}{\Gamma(7/2)}-\frac{k}{m^2}\frac{t^3}{\Gamma(4)}\,,$$

$$x_3(t)=\frac{c^2}{m^3}\frac{t^4}{\Gamma(5)}+\frac{2kc}{m^3}\frac{t^{9/2}}{\Gamma(11/2)}+\frac{k^2}{m^3}\frac{t^5}{\Gamma(6)}\,, \tag{6.11}$$

$$x_4(t)=-\frac{c^3}{m^4}\frac{t^{11/2}}{\Gamma(13/2)}-\frac{3kc^2}{m^4}\frac{t^6}{\Gamma(7)}-\frac{3k^2c}{m^4}\frac{t^{13/2}}{\Gamma(15/2)}-\frac{k^3}{m^4}\frac{t^7}{\Gamma(8)}\,,$$

$$x_5(t)=\frac{c^4}{m^5}\frac{t^7}{\Gamma(8)}+\frac{4kc^3}{m^5}\frac{t^{15/2}}{\Gamma(17/2)}+\frac{6k^2c^2}{m^5}\frac{t^8}{\Gamma(9)}+\frac{4k^3c}{m^5}\frac{t^{17/2}}{\Gamma(19/2)}+\frac{k^4}{m^5}\frac{t^9}{\Gamma(10)}\,,$$

and so on.

Accordingly, the general solution may be written as

$$x(t)=\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!\,t^{3j/2}}{j!\,\Gamma\left(\frac{3j}{2}+2r+2\right)} \tag{6.12}$$

$$=\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}E^{r}_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right). \tag{6.13}$$

Substituting $\omega_n=\sqrt{k/m}$ and $\eta=c/2m\omega_n^{3/2}$ in Eq. (6.12) we get

$$x(t)=\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\omega_n^2\right)^r t^{2r+1}E^{r}_{3/2,r/2+2}\left(-2\eta\omega_n^{3/2}t^{3/2}\right).$$

6.4. Analytical Solution Using Fractional Greens Function

Analytical solution of Eq. (6.1) can be obtained by using fractional Green's function for a three-term fractional differential equation with constant coefficients (Section 5.4 of Podlunby (1999)) as

$$x(t)=\int_0^t G_3(t-\tau)f(\tau)d\tau\,,$$

with the above discussed homogeneous initial condition. For unit step function (when $f(t)=u(t)$), the solution may be obtained as

$$x(t)=\frac{u(t)}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right) \tag{6.14}$$

and for unit impulse function (i.e. when $f(t)=\delta(t)$), the solution may be computed as

$$x(t)=\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right). \tag{6.15}$$

Now, one may see that the analytical solution obtained for both the cases (unit step and impulse function) in Eqs. (6.14) and (6.15) are exactly same as the solution obtained by HPM given in Eqs. (6.10) and (6.13) respectively.

In Eqs. (6.10), (6.13), (6.14) and (6.15), $E^r_{\lambda,\mu}(y)$ is called the Mittage-Leffler function of two parameters γ and μ where,

$$E^r_{\gamma,\mu}(y)\equiv\frac{d^r}{dy^r}E_{\gamma,\mu}(y)$$

$$=\sum_{j=0}^{\infty}\frac{(j+r)!\,y^j}{j!\,\Gamma(\gamma j+\gamma r+\mu)},\ (r=0,1,2,\ldots)\,.$$

For unit step response, $\gamma=3/2$, $\mu=(r/2)+3$ and for impulse response, $\gamma=3/2$, $\mu=r/2+(2)$.

6.5. Numerical Results

As discussed above, here two response functions viz. unit step and impulse response function have been considered for the present analysis. Obtained results are compared with the existing solution of Suarez and Shokooh (1997) and Yuan and Agrawal (2002) for the validation. Computed results are depicted in terms of plots.

6.5.1. Case studies for unit step response function

Depending upon the values of natural frequency ω_n and damping ratio η, four cases have been considered subject to a unit step load as follows. In the first case, the numerical values of the parameters are taken as $\omega_n = 5\,\text{rad/s}$, $\eta = 0.05, 0.5$ and 1. Next, in Case 2, three oscillators with natural frequency $\omega_n = 10\,\text{rad/s}$, $\eta = 0.05, 0.5$ and 1 are considered. Similarly, the values of natural frequencies are taken as $\omega_n = 5\,\text{rad/s}$ and $10\,\text{rad/s}$ respectively for the third and fourth cases with damping ratios $\eta = \sqrt{\pi}, 3$ and 5.

For first, second, third and fourth cases, computed displacements with respect to time are depicted in Figs. 6.1 to 6.4 using HPM respectively. The solutions for $\eta = 0.5$ and 1 in Fig. 6.1 do not show any oscillations around the static equilibrium response $1/\omega_n^2 = 0.04$. For the second case as shown in Fig. 6.2 exhibits the same behaviour as the first case. These displacement curves for $\eta = 0.5$ and 1 also do not show any oscillations around the static equilibrium response 0.01. But one may notice from Figs. 6.3 and 6.4 for the third and fourth cases, which demonstrates that the three oscillators for different damping ratios also do not show any oscillations around the static equilibrium responses 0.04 and 0.01. In special case (only for first case), we have compared the results obtained by the present analysis with the existing solution (Suarez and Shokooh 1997, Yuan and Agrawal 2002) and are found to be in good agreement.

6.5.2. Case studies for impulse response function

Impulse response functions have been computed using Eq. (6.13) for different values of natural frequency and damping ratios as follows. In the first numerical example, natural frequency $\omega_n = 5\,\text{rad/s}$ and damping ratios $\eta = 0.05, 0.5$ and 1 are taken. Next, the same damping ratios with $\omega_n = 10\,\text{rad/s}$ are considered for the oscillations. Obtained displacements are shown in terms of Figs. 6.5 and 6.6 for the above two cases. The impulse response for $\eta = 1$ has an oscillatory character for both the Figs. 6.5 and 6.6. Figs. 6.7 and 6.8 show the oscillation for the natural frequencies $\omega_n = 5\text{rad/s}$ and $10\,\text{rad/s}$ respectively for damping ratios $\eta = \sqrt{\pi}, 3$ and 5. Similarly, one may see from Figs. 6.7 and 6.8 that when the damping ratio is equal to $\sqrt{\pi}$, the curves are tangent to the axis of zero displacement. Again, for values of greater than $\sqrt{\pi}$, the curves tend to zero without crossing the zero axis. In this regard, $\sqrt{\pi}$ has been considered as critical damping. Now it is interesting to note that for all the above cases, impulse responses give the oscillations above the equilibrium position (i.e. at $1/\omega_n^2$). It may be noted that present results (Fig. 6.6) exactly coincide with the solution of Suarez and Shokooh (1997).

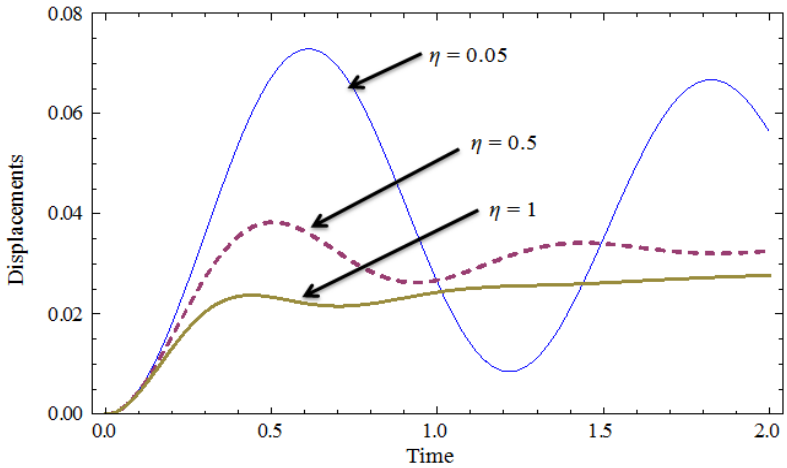


Fig. 6.1 Unit step response function for oscillators with natural frequency $\omega_n = 5$ rad/s and damping ratios $\eta = 0.05$, 0.5 and 1

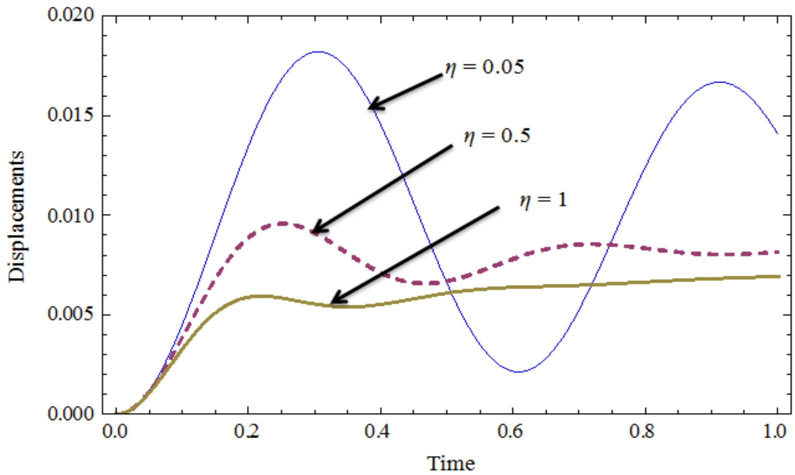


Fig. 6.2 Unit step response function for oscillators with natural frequency $\omega_n = 10$ rad/s and damping ratios $\eta = 0.05$, 0.5 and 1

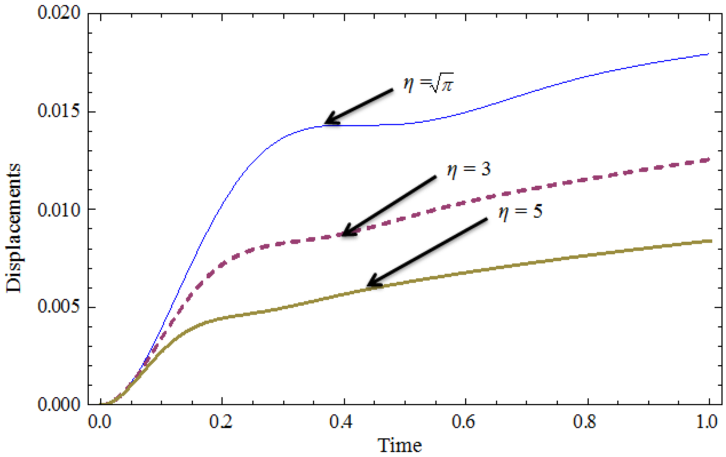


Fig. 6.3 Unit step response function for oscillators with natural frequency $\omega_n = 5$ rad/s and damping ratios $\eta = \sqrt{\pi}$, 3 and 5

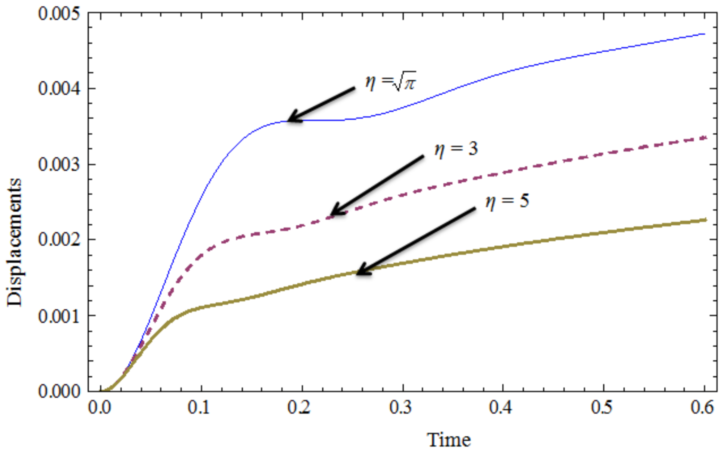


Fig. 6.4 Unit step response function for oscillators with natural frequency $\omega_n = 10$ rad/s and damping ratios $\eta = \sqrt{\pi}$, 3 and 5

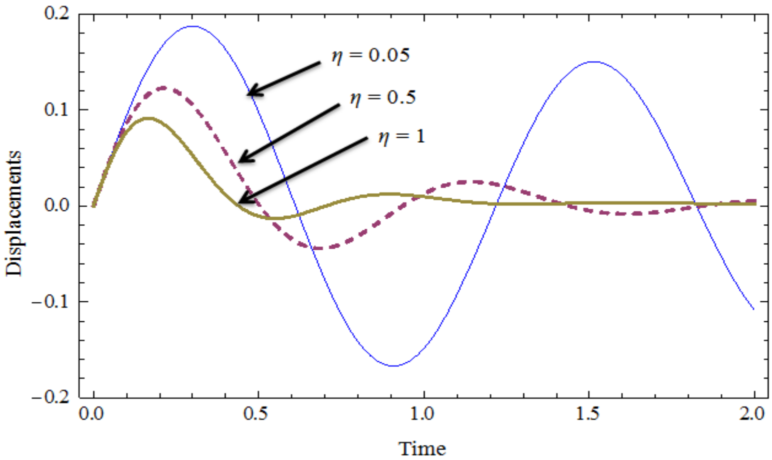


Fig. 6.5 Impulse response function for oscillators with natural frequency $\omega_n = 5$ rad/s and damping ratios $\eta = 0.05$, 0.5 and 1

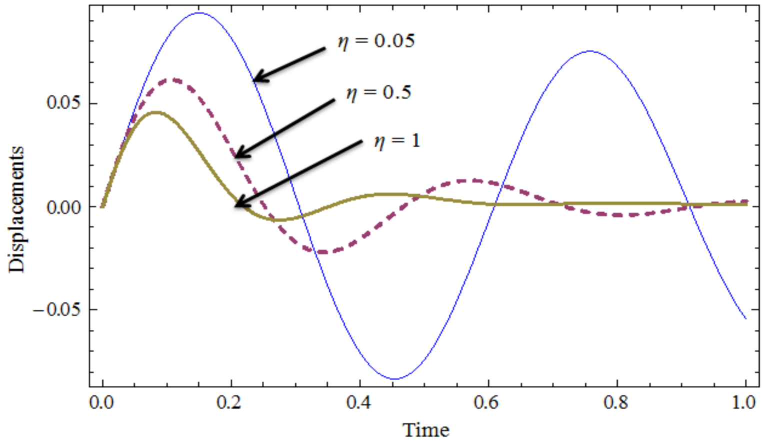


Fig. 6.6 Impulse response function for oscillators with natural frequency $\omega_n = 10$ rad/s and damping ratios $\eta = 0.05$, 0.5 and 1

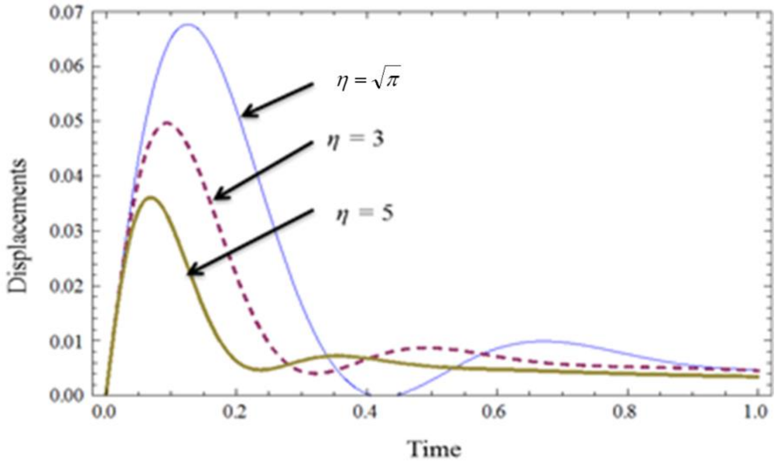


Fig. 6.7 Impulse response function for oscillators with natural frequency $\omega_n = 5$ rad/s and damping ratios $\eta = \sqrt{\pi}$, 3 and 5

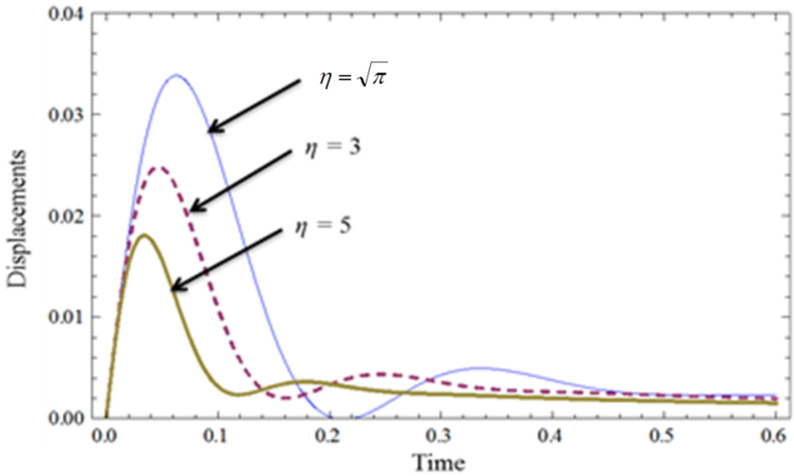


Fig. 6.8 Impulse response function for oscillators with natural frequency $\omega_n = 10$ rad/s and damping ratios $\eta = \sqrt{\pi}$, 3 and 5

Homotopy perturbation method has been applied successfully to the solution of a fractionally damped viscoelastic system, where the fractional derivative is considered as of order $1/2$. The unit step and impulse response functions with initial conditions are chosen to illustrate the proposed method. It is interesting to note that the results obtained by present method exactly matches with that of the solution obtained by Podlunby (1999), Suarez and Shokooh (1997) and Yuan and Agrawal (2002) in special cases. In all the cases increments of damping ratios affect the displacement of the oscillation (decrease the oscillation) which is clearly visible in the depicted figures.

Chapter 7

Fractionally Damped Continuous System

The content of this chapter has been published in:

1. Behera, D., Chakraverty, S. (2013) Numerical solution of fractionally damped beam by homotopy perturbation method, Central European Journal of Physics, 11 (6), 792-798.

Chapter 7
Fractionally Damped Continuous System

This chapter investigates the semi analytical solution of a viscoelastic continuous beam whose damping behaviours are defined in terms of fractional derivatives of arbitrary order. Homotopy Perturbation Method (HPM) has been used to obtain the dynamic responses. Unit step function and impulse function responses are considered for the analysis. Obtained results are depicted in terms of plots. Comparisons are made with the analytic solutions obtained by Zu-feng and Xiao-yan (2007) to show the effectiveness and validation of the present method. Again in this chapter, we have investigated the above problem with deterministic parameter and then in Chapter 9 we have considered this problem with uncertainty.

7.1. Fractional Damped Viscoelastic Beam

The governing differential equation for a fractionally damped viscoelastic beam with an arbitrary fractional derivative of order λ may be written as

$$\rho A \frac{\partial^2 v}{\partial t^2} + c \frac{\partial^\lambda v}{\partial t^\lambda} + EI \frac{\partial^4 v}{\partial x^4} = F(x,t) \tag{7.1}$$

where, ρ, A, c, E and I represents the mass density, cross sectional area, damping coefficient per unit length, Young's modulus of elasticity and moment of inertia of the beam respectively. $F(x,t)$ is the externally applied force and $v(x,t)$ is the transverse displacement. $\frac{\partial^\lambda}{\partial t^\lambda}$ is the fractional derivative of order $\lambda \in (0,1)$ of the displacement function $v(x,t)$. The present authors considered the initial conditions as $v(x,0) = 0$ and $\dot{v}(x,0) = 0$. The homogeneous initial conditions are taken here to compare the present solution with the solution of Zu-feng and Xiao-yan (2007). Eq. (7.1) can be written as

$$\frac{\partial^2 v}{\partial t^2} + \frac{c}{\rho A} \frac{\partial^\lambda v}{\partial t^\lambda} + \frac{EI}{\rho A} \frac{\partial^4 v}{\partial x^4} = \frac{F(x,t)}{\rho A}. \tag{7.2}$$

According to HPM, we may construct a simple homotopy for an embedding parameter $p \in [0,1]$ as follows

$$(1-p)\frac{\partial^2 v}{\partial t^2}+p\left(\frac{\partial^2 v}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}\right)=0, \tag{7.3}$$

or

$$\frac{\partial^2 v}{\partial t^2}+p\left(\frac{c}{\rho A}\frac{\partial^\lambda v}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}\right)=0. \tag{7.4}$$

Here, p is considered as a small homotopy parameter $0\le p\le 1$. For $p=0$, Eqs. (7.3) and (7.4) become a linear equation i.e. $\frac{\partial^2 v}{\partial t^2}=0$, which is easy to solve. For $p=1$, Eqs. (7.3) and (7.4) turns out to be same as the original Eq. (7.1) or (7.2). This is called deformation in topology. $\frac{\partial^2 v}{\partial t^2}$ and $\frac{c}{\rho A}\frac{\partial^\lambda v}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}$ are called homotopic.

We then assume the solution of Eq. (7.3) or (7.4) as a power series in p as

$$v(x,t)=v_0(x,t)+pv_1(x,t)+p^2v_2(x,t)+p^3v_3(x,t)+\cdots, \tag{7.5}$$

where $v_i(x,t)$ for $i=0,1,2,\ldots$ are functions yet to be determined. Substituting Eq. (7.5) in Eq. (7.3) or (7.4), and equating the terms with the identical power of p, we can obtain a series of equations of the form

$$\begin{aligned}
&p^0:\frac{\partial^2 v_0}{\partial t^2}=0,\\
&p^1:\frac{\partial^2 v_1}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v_0}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v_0}{\partial x^4}-\frac{F(x,t)}{\rho A}=0,\\
&p^2:\frac{\partial^2 v_2}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v_1}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v_1}{\partial x^4}=0,\\
&p^3:\frac{\partial^2 v_3}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v_2}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v_2}{\partial x^4}=0,\\
&p^4:\frac{\partial^2 v_4}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v_3}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v_3}{\partial x^4}=0,
\end{aligned} \tag{7.6}$$

and so on.

Choosing initial approximation $v_0(x,0)=0$ and applying the operator L_{tt}^{-1} (which is the inverse of the operator $L_{tt}=\frac{\partial^2}{\partial t^2}$) on both sides of Eq. (7.6), one may obtain the following equations

$v_0(x,t)=0,$

$v_1(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\lambda v_0}{\partial t^\lambda}-\frac{EI}{\rho A}\frac{\partial^4 v_0}{\partial x^4}+\frac{F(x,t)}{\rho A}=0\right),$

$v_2(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\lambda v_1}{\partial t^\lambda}-\frac{EI}{\rho A}\frac{\partial^4 v_1}{\partial x^4}\right),$ (7.7)

$v_3(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\lambda v_2}{\partial t^\lambda}-\frac{EI}{\rho A}\frac{\partial^4 v_2}{\partial x^4}\right),$

$v_4(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\lambda v_3}{\partial t^\lambda}-\frac{EI}{\rho A}\frac{\partial^4 v_3}{\partial x^4}\right)$ and so on.

Substituting the above in Eq. (7.5), one may get the approximate solution of Eq. (7.1) as

$$v(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+v_4(x,t)+\cdots$$

The solution series converge very rapidly. The proof of convergence of the above series may be found in He (2009, 2010). The rapid convergence means that only few terms are required to get the approximate solutions.

7.2. Response Analysis

Similar to Zu-feng and Xiao-yan (2007), the external applied force defined by $F(x,t)$ has been considered as

$$F(x,t)=f(x)g(t),$$

where, $f(x)$ is a specified space dependent deterministic function, and $g(t)$ is a time dependent process.

7.2.1. Unit step response

The unit step load has been considered of the form $g(t)=Bu(t)$, where $u(t)$ is the Heaviside function and B is a constant. By using HPM we have

$v_0(x,t)=0,$

$v_1(x,t)=\frac{fB}{\rho A}\frac{t^2}{2},$

$$v_2(x,t) = -\frac{cfB}{\rho^2 A^2}\frac{t^{4-\lambda}}{\Gamma(5-\lambda)} - \frac{EIBf^{(4)}}{\rho^2 A^2}\frac{t^4}{\Gamma(5)}, \tag{7.8}$$

$$v_3(x,t) = \frac{c^2 fB}{\rho^3 A^3}\frac{t^{6-2\lambda}}{\Gamma(7-2\lambda)} + \frac{2cEIBf^{(4)}}{\rho^3 A^3}\frac{t^{6-\lambda}}{\Gamma(7-\lambda)} + \frac{E^2 I^2 Bf^{(8)}}{\rho^3 A^3}\frac{t^6}{\Gamma(7)},$$

$$v_4(x,t) = -\frac{c^3 fB}{\rho^4 A^4}\frac{t^{8-3\lambda}}{\Gamma(9-3\lambda)} - \frac{3c^2 EIBf^{(4)}}{\rho^4 A^4}\frac{t^{8-2\lambda}}{\Gamma(9-2\lambda)} - \frac{3cE^2 I^2 Bf^{(8)}}{\rho^4 A^4}\frac{t^{8-\lambda}}{\Gamma(9-\lambda)} - \frac{E^3 I^3 Bf^{(12)}}{\rho^4 A^4}\frac{t^8}{\Gamma(9)},$$

and so on, where $f^{(i)} = \frac{\partial^i f}{\partial x^i}$.

Therefore, the solution can be written in the general form as

$$v(x,t) = \frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2(r+1)} \sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j \frac{(j+r)!\, t^{(2-\lambda)j}}{j!\,\Gamma((2-\lambda)j+2r+3)}. \tag{7.9}$$

Eq. (7.9) can be rewritten as follows

$$v(x,t) = \frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2(r+1)} E^{r}_{2-\lambda,\,\lambda r+2}\left(\frac{-c}{\rho A}t^{2-\lambda}\right). \tag{7.10}$$

7.2.2. Unit impulse response

Unit impulsive load has been taken as $g(t) = \delta(t)$, where $\delta(t)$ is the unit impulse function. Again by using HPM we have

$$v_0(x,t) = 0,$$

$$v_1(x,t) = \frac{f}{\rho A}t,$$

$$v_2(x,t) = -\frac{cf}{\rho^2 A^2}\frac{t^{3-\lambda}}{\Gamma(4-\lambda)} - \frac{EIf^{(4)}}{\rho^2 A^2}\frac{t^3}{\Gamma(4)}, \tag{7.11}$$

$$v_3(x,t) = \frac{c^2 f}{\rho^3 A^3}\frac{t^{5-2\lambda}}{\Gamma(6-2\lambda)} + \frac{2cEIf^{(4)}}{\rho^3 A^3}\frac{t^{5-\lambda}}{\Gamma(6-\lambda)} + \frac{E^2 I^2 f^{(8)}}{\rho^3 A^3}\frac{t^5}{\Gamma(6)},$$

$$v_4(x,t) = -\frac{c^3 f}{\rho^4 A^4}\frac{t^{7-3\lambda}}{\Gamma(8-3\lambda)} - \frac{3c^2 EIf^{(4)}}{\rho^4 A^4}\frac{t^{7-2\lambda}}{\Gamma(8-2\lambda)} - \frac{3cE^2 I^2 f^{(8)}}{\rho^4 A^4}\frac{t^{7-\lambda}}{\Gamma(8-\lambda)}$$
$$-\frac{E^3 I^3 f^{(12)}}{\rho^4 A^4}\frac{t^7}{\Gamma(8)}$$

and so on. Where, $f^i = \dfrac{\partial^i f}{\partial x^i}$.

Hence for this case, the solution can be written in the general form as

$$v(x,t) = \frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2(r+1)} \sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j \frac{(j+r)!\, t^{(2-\lambda)j}}{j!\,\Gamma\big((2-\lambda)j+2r+2\big)}. \quad (7.12)$$

Eq. (7.12) can be rewritten as follows

$$v(x,t) = \frac{1}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2r+1} E_{2-\lambda,\lambda r+2}^{(r)}\left(\frac{-c}{\rho A}t^{2-\lambda}\right). \quad (7.13)$$

In Eqs. (7.10) and (7.13), $E_{\gamma,\mu}^{r}(y)$ is called the Mittage-Leffler function of two parameters γ and μ where,

$$E_{\gamma,\mu}^{r}(y) = \sum_{j=0}^{\infty}\frac{(j+r)!\, y^j}{j!\,\Gamma(\gamma j+\gamma r+\mu)}, \quad r = 0,1,2,\ldots$$

For both the responses, we have $\gamma = 2-\lambda$ and $\mu = \lambda r + 2$.

7.3. Numerical Results

In order to show the responses in a precise way, some numerical results are presented in this section. Eqs. (7.10) and (7.13) provide the desired expressions for the considered loading condition. As we have considered a simply supported beam, one may have the expression for the force distribution as

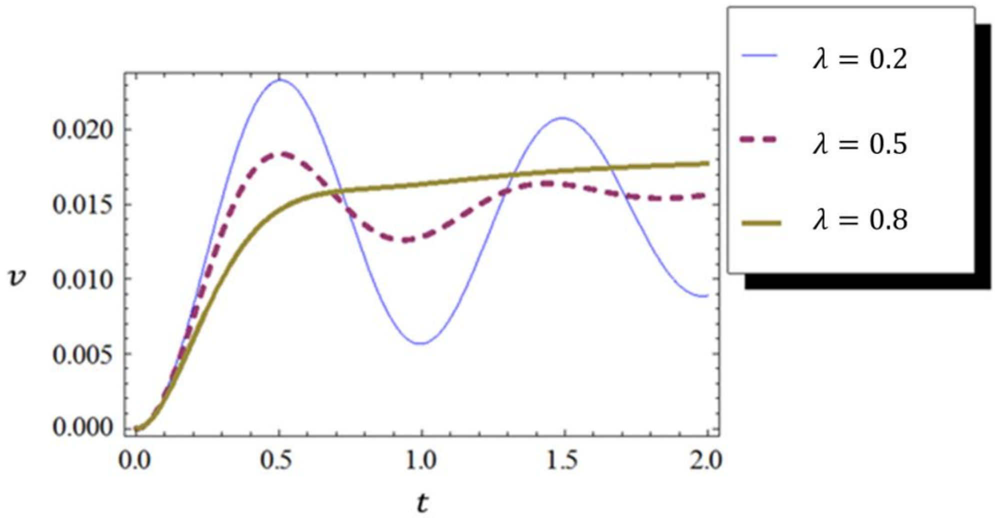
$$f(x) = \sin\left(\tfrac{\pi x}{L}\right).$$

The numerical computation has been done by truncating the infinite series (7.10) and (7.13) to a finite number of terms. Let us denote c/m and $EI/\rho A$ as $2\eta\omega^{3/2}$ and ω^2 respectively where, ω is the natural frequency and η is the damping ratio. The values of the parameters are taken as $\rho A = 1$, $L = \pi$ and $m = 1$.

7.3.1. Case studies for unit step response

Obtained results for various parameters are depicted in Figs. 7.1 to 7.4. Fig. (7.1) gives the effect of displacement against time for various values of $\lambda = (0.2, 0.5, 0.8)$. Here, x and η are taken as 1/2. Figs. 7.1(a) and 7.1(b) present the plot for $\omega = 5\,\text{rad/s}$ and $\omega = 10\,\text{rad/s}$ respectively. A similar simulation has been done with damping ratio $\eta = 0.05$ and the results are depicted in Fig. 7.2. Figs. 7.2 (a) and 7.2 (b) show the plot for $\omega = 5\,\text{rad/s}$ and $\omega = 10\,\text{rad/s}$ respectively. The dynamic responses versus time for different values of $\eta = (0.05, 0.5, 1)$ are given in Fig. 7.3. In this computation, $\lambda = 0.2$ and $x = 0.5$ are considered. Finally, Fig. 7.4 cites the results as above with $\lambda = 0.5$.

It is interesting to note from Figs. (7.1) and (7.2) that if we increase the order of the fractional derivative λ, the beam suffers less oscillation. That means the beam suffers more oscillations for smaller value of λ. Similar observations may be made by keeping the order of the fractional derivative constant and varying the damping ratios as shown in Figs. (7.3) and (7.4). It can be seen clearly that increasing the value of the damping ratios decreases the oscillations.



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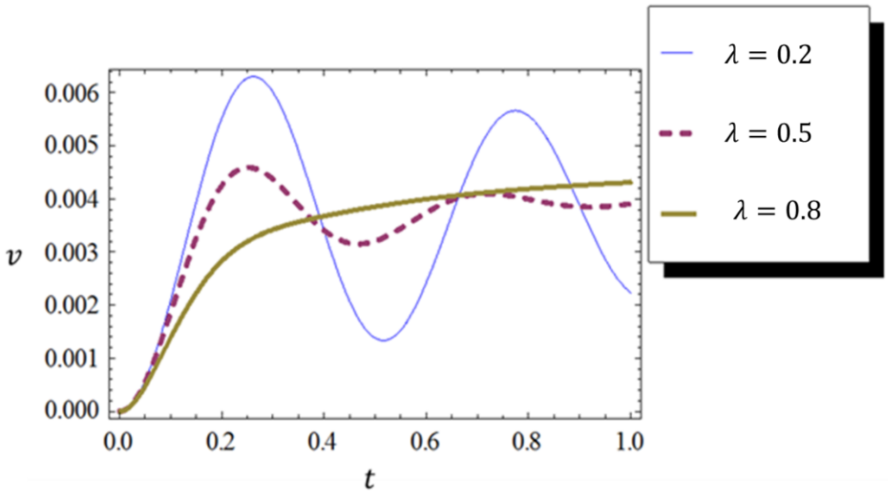
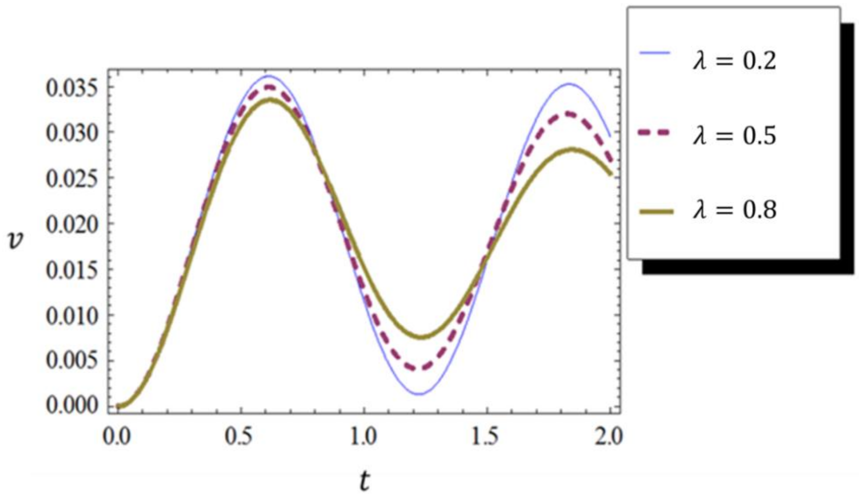


(b)

Fig. 7.1 Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s, (b) $\omega = 10$ rad/s and damping ratio $\eta = 0.5$



(a)

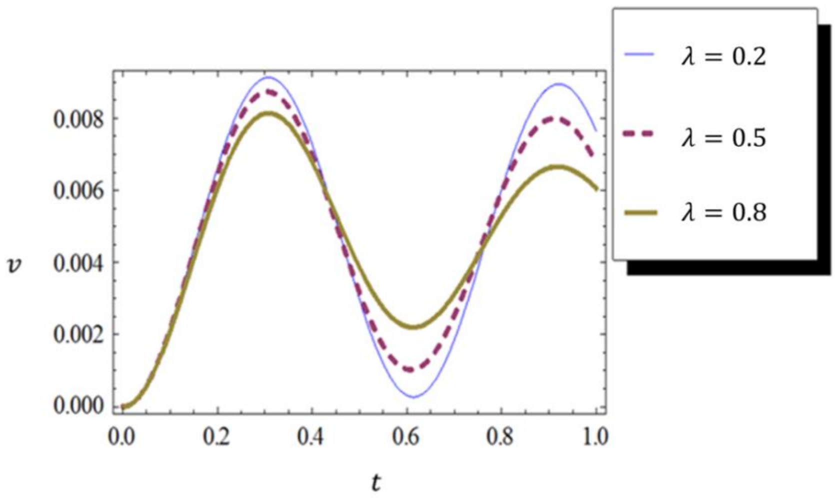


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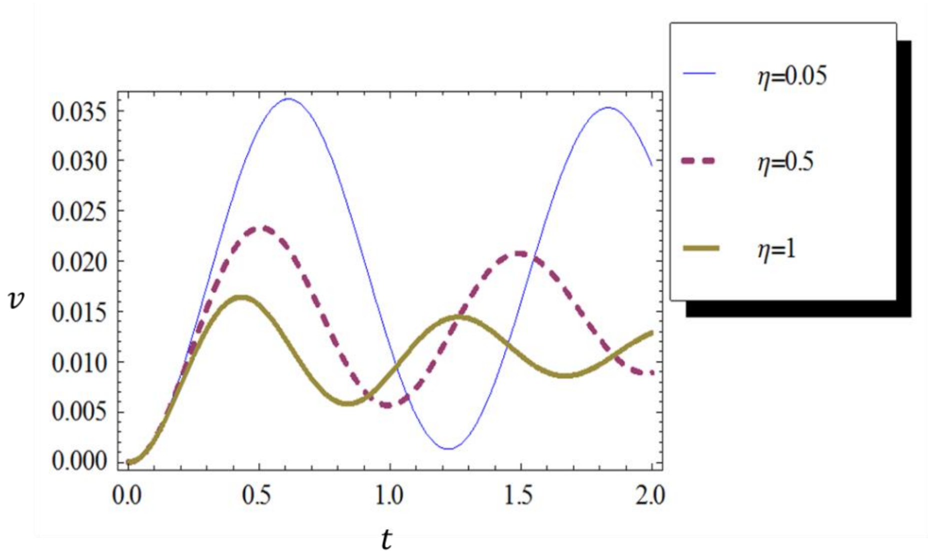
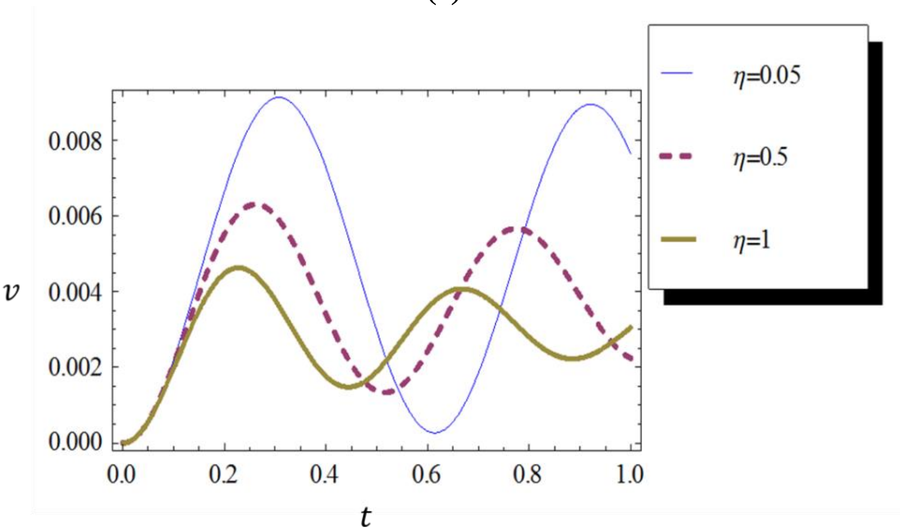
Fig. 7.2 Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s, (b) $\omega = 10$ rad/s and damping ratio $\eta = 0.05$

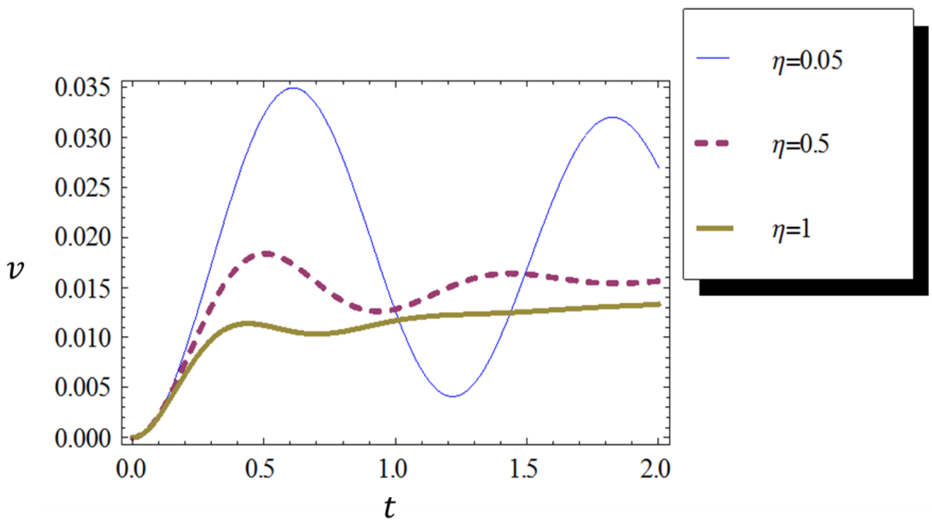


(a)

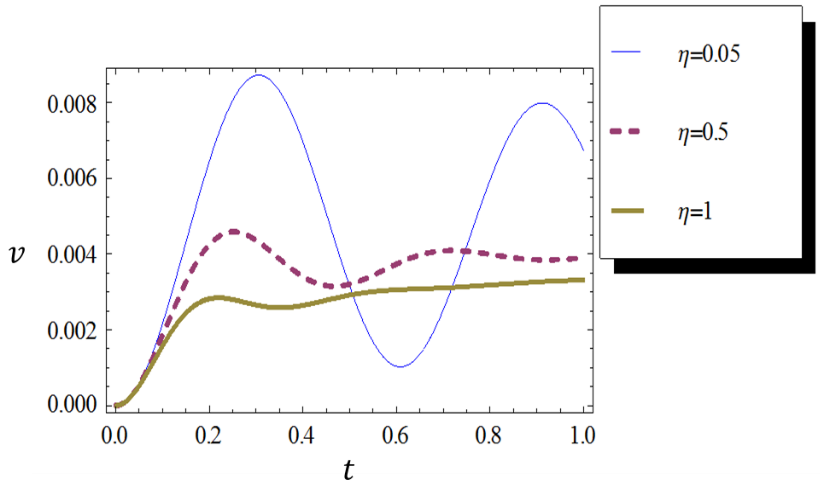


(b)

Fig. 7.3 Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s , (b) $\omega = 10$ rad/s and damping ratios $\eta = 0.05,\ 0.5$ and 1 for $\lambda = 0.2$



(a)



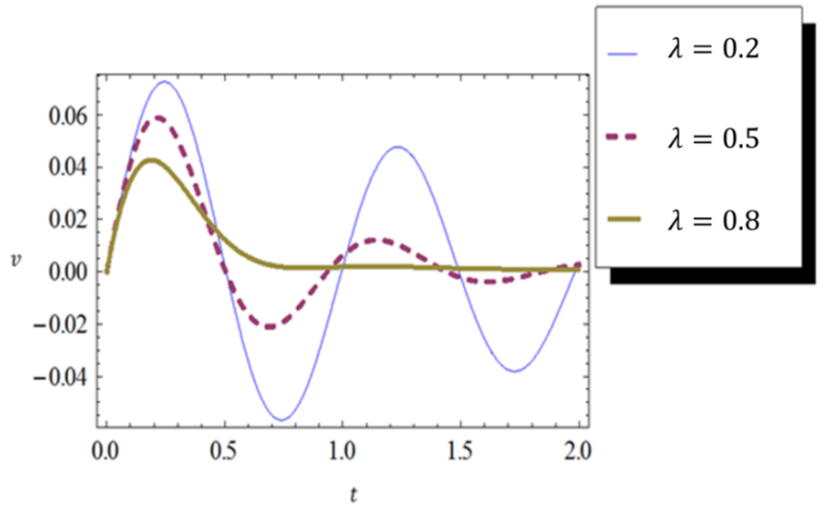
(b)

Fig. 7.4 Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s , (b) $\omega = 10$ rad/s and damping ratios $\eta = 0.05, 0.5$ and 1 for $\lambda = 0.5$

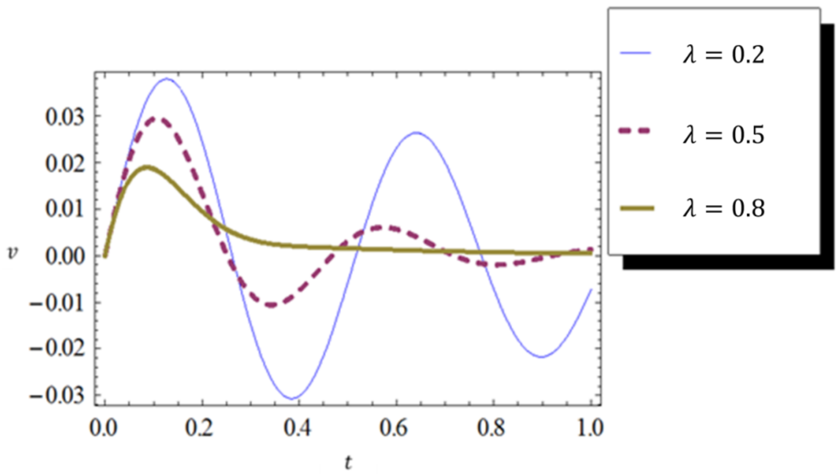
7.3.2. Case studies for impulse response

Eq. (7.13) provides the desired expressions for the considered loading condition. In order to show the responses more clearly, some numerical results are presented in this section. Obtained results for this have been incorporated in Figs. 7.5 to 7.8. Fig. (7.5) gives the effect of displacement on time for various values of $\lambda (= 0.2, 0.5, 0.8)$. In this computation, x and η are taken as $1/2$. Figs. 7.5(a) and 7.5(b) present the plot for $\omega = 5$ rad/s and 10 rad/s respectively. Similar simulation has been done for damping ratio $\eta = 0.05$ and obtained results are depicted in Fig. 7.6. Next, for different values of $\eta (= 0.05, 0.5, 1)$, dynamic responses versus time are given in Fig. 7.7. There, $\lambda = 0.2$ and $x = 1/2$ have been considered. Again Figs. 7.7(a) and 7.7(b) depict the plot for $\omega = 5$ rad/s and 10 rad/s respectively. Finally, Fig. 7.8 cites the results as above (Fig. 7.7) with $\lambda = 0.5$.

It is interesting to note from Figs. (7.5) and (7.6) that if we increase the order of the fractional derivative λ, the beam suffers less oscillation. That means the beam suffers more oscillations for smaller value of λ. Similar observations may be made by keeping the order of the fractional derivative constant and varying the damping ratios as shown in Figs. (7.7) and (7.8). It may clearly be seen that increase of the value of the damping ratios decrease the oscillations.

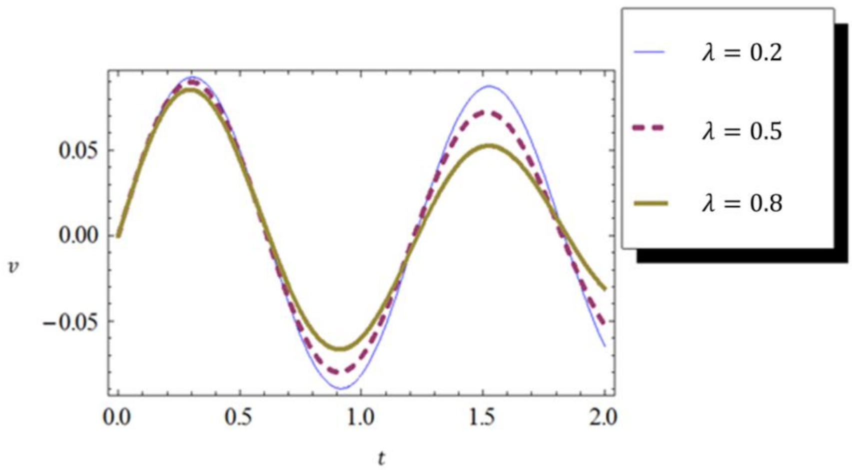


(a)

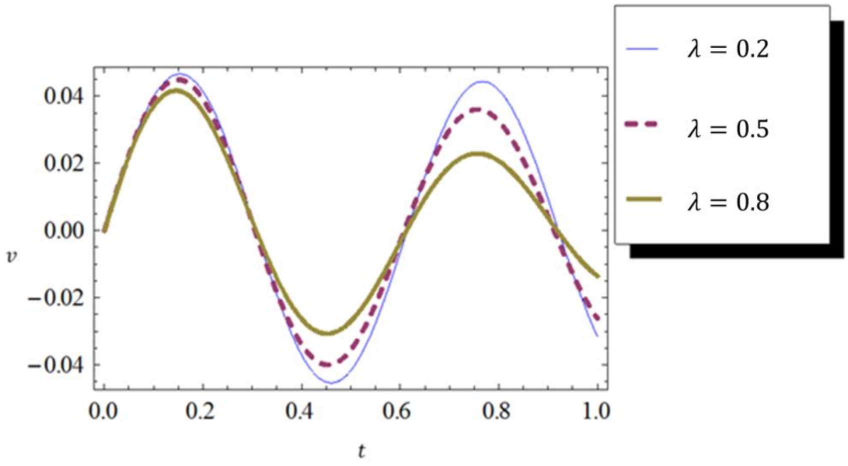


(b)

Fig. 7.5 Impulse responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s (b) $\omega = 10$ rad/s and damping ratio $\eta = 0.5$

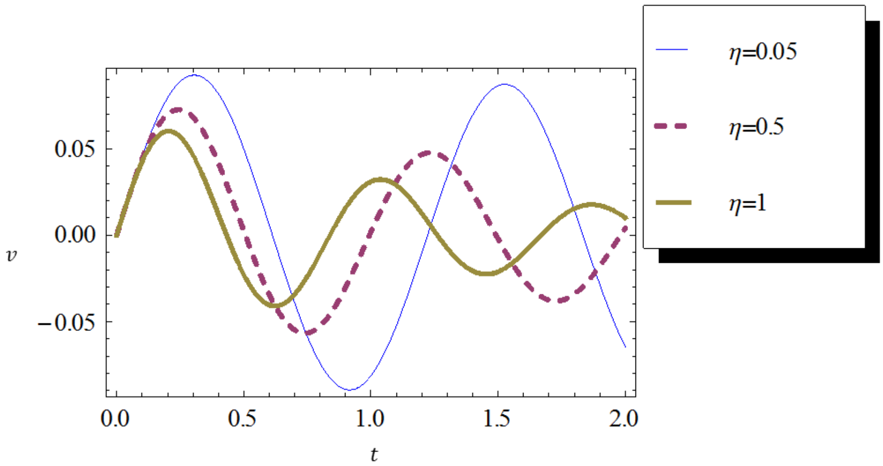


(a)

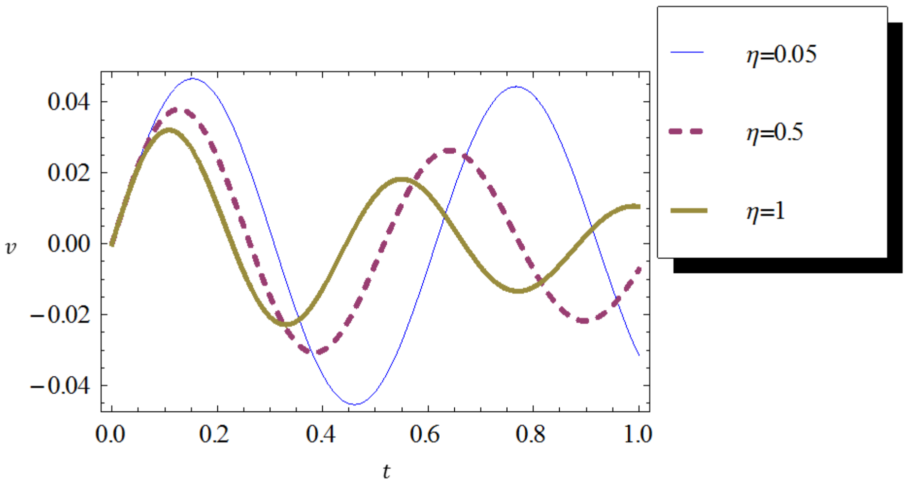


(b)

Fig. 7.6 Impulse responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s (b) $\omega = 10$ rad/s and damping ratio $\eta = 0.05$

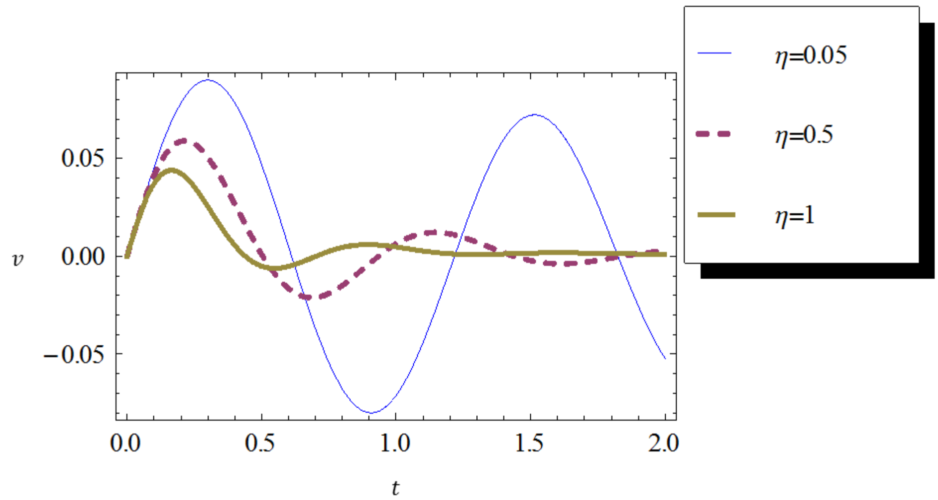


(a)

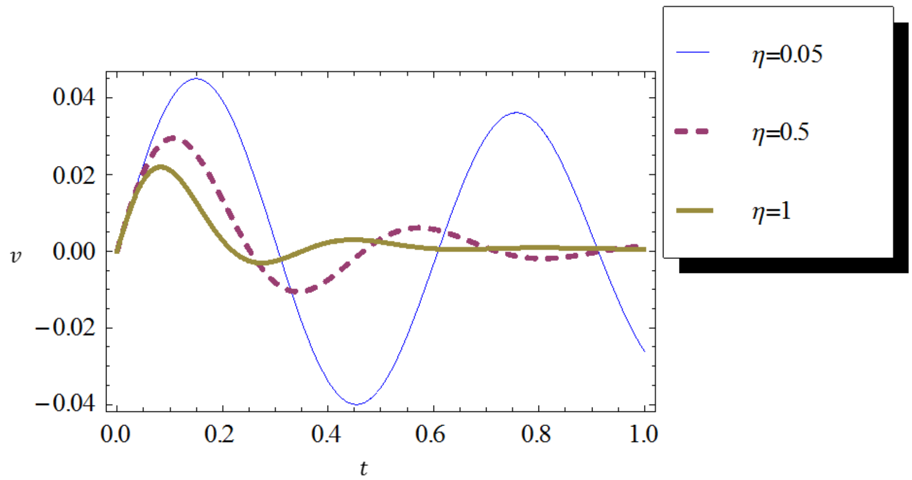


(b)

Fig. 7.7 Impulse responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s (b) $\omega = 10$ rad/s and $\lambda = 0.2$



(a)



(b)

Fig. 7.8 Impulse responses along $x = 1/2$ with natural frequency (a) $\omega = 5$ rad/s (b) $\omega = 10$ rad/s and $\lambda = 0.5$

HPM has successfully been applied to the solution of a fractionally damped viscoelastic beam. The unit step and impulse response functions with homogeneous initial conditions are chosen to illustrate the proposed method. The obtained results are compared with the analytical solution of Zu-feng and Xiao-yan (2007) and those are found to be in good agreement.

Chapter 8

Uncertain Fractionally Damped Discrete System

The content of this chapter has been published in:

1. Behera, D., Chakraverty, S. (2014) Uncertain impulse response of imprecisely defined half order mechanical system, Annals of Fuzzy Mathematics and Informatics, 7 (3), 401-419.

Chapter 8

Uncertain Fractionally Damped Discrete System

This chapter investigates the semi analytical solution of imprecisely defined fractional order discrete system, subject to unit step and impulse loads. In this regard, a mechanical spring mass system having fractional damping of order $1/2$ with fuzzy initial condition is taken into consideration. Fuzziness appeared in the initial conditions are modelled through different types of convex normalised fuzzy sets viz. triangular, trapezoidal and Gaussian fuzzy numbers. Homotopy Perturbation Method (HPM) is used with fuzzy based approach to obtain the uncertain response.

8.1. Fuzzy Fractionally Damped Spring Mass System

A fuzzy fractionally damped single degree of freedom spring mass system may be written as

$$mD^2\tilde{x}(t;\alpha)+cD^{\lambda}\tilde{x}(t;\alpha)+k\tilde{x}(t;\alpha)=f(t) \tag{8.1}$$

where, m, c and k represent the mass, damping and stiffness coefficients respectively. $f(t)$ is the externally applied force, and $D^{\lambda}\tilde{x}(t;\alpha)$ for $0<\lambda<1$, is the derivative of order λ of the fuzzy displacement function $\tilde{x}(t;\alpha)=[\underline{x}(t;\alpha),\overline{x}(t;\alpha)]$. Here, $\tilde{x}(t;\gamma)$ is represented by $\alpha-$cut form of fuzzy displacements. Although the coefficient λ (known as the memory parameter), may take any value between 0 to 1, but the value of $1/2$ has been adopted here. This is because it has been shown that it describes the frequency dependence of the damping materials quite satisfactorily in the crisp fractional dynamic systems (Suarez and Shokooh 1997; Yuan and Agrawal 2002). Fuzzy initial displacements $\tilde{x}(0)$ and initial velocity $v(0)=\dot{\tilde{x}}(0)$ are taken as triangular, trapezoidal and Gaussian fuzzy number respectively in Cases 1 to 3 as depicted in Table 8.1.

Table 8.1 Data for fuzzy initial conditions

Initial conditions	Case 1	Case 2	Case 3
$\tilde{x}(0)$	$(-0.1, 0, 0.1)$	$(-0.1, -0.050, 0.05, 0.1)$	$(0, 0.1, 0.1)$
$v(0)=\dot{\tilde{x}}(0)$	$(-0.1, 0, 0.1)$	$(-0.1, -0.050, 0.05, 0.1)$	$(0, 0.1, 0.1)$

Through $\alpha-$cut approach, the fuzzy initial conditions for Cases 1 to 3 given in Table 8.1 are then expressed as given in Table 8.2.

Table 8.2 $\alpha-$cut representations of fuzzy initial conditions

Initial conditions	Case 1	Case 2	Case 3
$\tilde{x}(0;\alpha)$	$[0.1\alpha-0.1,$ $0.1-0.1\alpha]$	$[0.05\alpha-0.1,$ $0.1-0.05\alpha]$	$[-0.1\sqrt{-2\log_e(\alpha)},$ $0.1\sqrt{-2\log_e(\alpha)}]$
$v(0;\alpha)$ $=\tilde{\dot{x}}(0;\alpha)$	$[0.1\alpha-0.1,$ $0.1-0.1\alpha]$	$[0.05\alpha-0.1,$ $0.1-0.05\alpha]$	$[-0.1\sqrt{-2\log_e(\alpha)},$ $0.1\sqrt{-2\log_e(\alpha)}]$

Eq. (8.1) may now be written as

$$D^2\tilde{x}(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{x}(t;\alpha)+\frac{k}{m}\tilde{x}(t;\alpha)=\frac{f(t)}{m}\ . \tag{8.2}$$

According to HPM, we may construct a simple homotopy for an embedding parameter $p\in[0,1]$ as follows

$$\begin{aligned}&(1-p)\left(D^2\tilde{X}(t;\alpha)-D^2\tilde{x}_0(t;\alpha)\right)\\&+p\left(D^2\tilde{X}(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}(t;\alpha)+\frac{k}{m}\tilde{X}(t;\alpha)-\frac{f(t)}{m}\right)=0,\end{aligned} \tag{8.3}$$

or

$$\begin{aligned}&D^2\tilde{X}(t;\alpha)-D^2\tilde{x}_0(t;\alpha)\\&+p\left(D^2\tilde{x}_0(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}(t;\alpha)+\frac{k}{m}\tilde{X}(t;\alpha)-\frac{f(t)}{m}\right)=0.\end{aligned} \tag{8.4}$$

In the changing process from 0 to 1, for $p=0,$ Eq. (8.3) or (8.4) gives

$$\left(D^2\tilde{X}(t;\alpha)-D^2\tilde{x}_0(t;\alpha)\right)=0$$

and for $p=1,$ we have the original system

$$\left(D^2\tilde{X}(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}(t;\alpha)+\frac{k}{m}\tilde{X}(t;\alpha)-\frac{f(t)}{m}\right)=0\,.$$

This is called deformation in topology.

$\left(D^2\tilde{X}(t;\alpha)-D^2\tilde{x}_0(t;\alpha)\right)$ and $\left(D^2\tilde{X}(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}(t;\alpha)+\frac{k}{m}\tilde{X}(t;\alpha)-\frac{f(t)}{m}\right)$ are called homotopic.

Next, we assume solution of Eq. (8.3) or (8.4) as a power series expansion in p as

$$\tilde{X}(t;\alpha)=\tilde{X}_0(t;\alpha)+p\tilde{X}_1(t;\alpha)+p^2\tilde{X}_2(t;\alpha)+p^3\tilde{X}_3(t;\alpha)+\cdots, \qquad (8.5)$$

where $\tilde{X}_i(t;\alpha)$, $\mathrm{i}=0, 1, 2, \ldots$ are functions yet to be determined. As per HPM, substituting Eq. (8.5) into Eq. (8.3) or (8.4), and equating the terms with the identical power of p, we can obtain a series of equations of the form

$$
\begin{aligned}
&p^0: D^2\tilde{X}_0(t;\alpha)-D^2\tilde{x}_0(t;\alpha)=0,\\
&p^1: D^2\tilde{X}_1(t;\alpha)+D^2\tilde{x}_0(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_0(t;\alpha)+\frac{k}{m}\tilde{X}_0(t;\alpha)-\frac{f(t)}{m}=0,\\
&p^2: D^2\tilde{X}_2(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_1(t;\alpha)+\frac{k}{m}\tilde{X}_1(t;\alpha)=0,\\
&p^3: D^2\tilde{X}_3(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_2(t;\alpha)+\frac{k}{m}\tilde{X}_2(t;\alpha)=0,\\
&p^4: D^2\tilde{X}_4(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_3(t;\alpha)+\frac{k}{m}\tilde{X}_3(t;\alpha)=0, \qquad (8.6)\\
&p^5: D^2\tilde{X}_5(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_4(t;\alpha)+\frac{k}{m}\tilde{X}_4(t;\alpha)=0,\\
&p^6: D^2\tilde{X}_6(t;\alpha)+\frac{c}{m}D^{1/2}\tilde{X}_5(t;\alpha)+\frac{k}{m}\tilde{X}_5(t;\alpha)=0,
\end{aligned}
$$

and so on.

Applying the operator L_{tt}^{-1} (the inverse operator of $D^2=\dfrac{d^2}{dt^2}$) on both sides of Eq. (8.6), one may get the approximate solution $\tilde{x}(t;\alpha)=\lim\limits_{p\to 1}\tilde{X}(t;\alpha)$ which can be expressed as

$$\tilde{x}(t;\alpha)=\tilde{X}_0(t;\alpha)+\tilde{X}_1(t;\alpha)+\tilde{X}_2(t;\alpha)+\tilde{X}_3(t;\alpha)+\cdots. \qquad (8.7)$$

We can write the above expression equivalently as

$$[\underline{x}(t;\alpha),\overline{x}(t;\alpha)]=\sum_{n=0}^{\infty}\tilde{X}_n(t;\alpha), \text{ where } \tilde{X}_n(t;\alpha)=[\underline{X}_n(t;\alpha),\overline{X}_n(t;\alpha)] .$$

Hence, the lower and upper bounds of the solution in parametric form are given as $\underline{x}(t;\gamma)=\sum_{n=0}^{\infty}\underline{X}_n(t;\alpha)$ and $\overline{x}(t;\gamma)=\sum_{n=0}^{\infty}\overline{X}_n(t;\alpha)$ respectively.

8.2. Uncertain Response Analysis

In this section, the beam has been analysed with respect to unit step and impulse loading as follows for the different cases as given in Tables 8.1 or 8.2.

8.2.1. Uncertain step function response

The unit step load has been considered as $f(t)=u(t)$, where $u(t)$ is the Heaviside function.

- **Solution for Case 1**

For triangular fuzzy initial conditions we have

$\underline{X}_0(t;\alpha)=(0.1\alpha-0.1)$,

$\overline{X}_0(t;\alpha)=(0.1-0.1\alpha)$,

$$\underline{X}_1(t;\alpha)=-\frac{t^2}{2}\left(\frac{k}{m}(0.1\alpha-0.1)-\frac{u(t)}{m}\right),$$

$$\overline{X}_1(t;\alpha)=-\frac{t^2}{2}\left(\frac{k}{m}(0.1-0.1\alpha)-\frac{u(t)}{m}\right),$$

$$\underline{X}_2(t;\alpha)=(0.1\alpha-0.1)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)}+\frac{k}{m}\frac{t^4}{\Gamma(5)}\right)+u(t)\left(-\frac{c}{m^2}\frac{t^{7/2}}{\Gamma(9/2)}+\frac{k}{m^2}\frac{t^4}{\Gamma(5)}\right),$$

$$\overline{X}_2(t;\alpha)=(0.1-0.1\alpha)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)}+\frac{k}{m}\frac{t^4}{\Gamma(5)}\right)+u(t)\left(-\frac{c}{m^2}\frac{t^{7/2}}{\Gamma(9/2)}+\frac{k}{m^2}\frac{t^4}{\Gamma(5)}\right),$$

$$\underline{X}_3(t;\alpha)=(0.1\alpha-0.1)\frac{k}{m}\left(-\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)}-\frac{2kc}{m^2}\frac{t^{11/2}}{\Gamma(13/2)}-\frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)$$
$$+u(t)\left(\frac{c^2}{m^3}\frac{t^5}{\Gamma(6)}+\frac{2kc}{m^3}\frac{t^{11/2}}{\Gamma(13/2)}+\frac{k^2}{m^3}\frac{t^6}{\Gamma(7)}\right),$$

$$\overline{X}_3(t;\alpha)=(0.1-0.1\alpha)\frac{k}{m}\left(-\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)}-\frac{2kc}{m^2}\frac{t^{11/2}}{\Gamma(13/2)}-\frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)$$
$$+u(t)\left(\frac{c^2}{m^3}\frac{t^5}{\Gamma(6)}+\frac{2kc}{m^3}\frac{t^{11/2}}{\Gamma(13/2)}+\frac{k^2}{m^3}\frac{t^6}{\Gamma(7)}\right),$$

and so on.

Substituting these in Eq. (8.7), we may get the approximate solution of $\tilde{x}(t)$. Accordingly, the bounds of the general solution may be written as

$$\underline{x}(t;\alpha) = (0.1\alpha - 0.1) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} \sum_{j=0}^{\infty} \left(\frac{-c}{m}\right)^j \frac{(j+r)!\, t^{3j/2}}{j!\,\Gamma\left(\frac{3j}{2}+2r+3\right)} \right) \cdot \left(\frac{u(t)}{m} - \frac{k}{m}(0.1\alpha - 0.1) \right)$$

$$= (0.1\alpha - 0.1) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} E^{r}_{3/2, r/2+3}\left(\frac{-c}{m} t^{3/2}\right) \right) \left(\frac{u(t)}{m} - \frac{k}{m}(0.1\alpha - 0.1) \right) \tag{8.8}$$

and

$$\overline{x}(t;\alpha) = (0.1 - 0.1\alpha) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} \sum_{j=0}^{\infty} \left(\frac{-c}{m}\right)^j \frac{(j+r)!\, t^{3j/2}}{j!\,\Gamma\left(\frac{3j}{2}+2r+3\right)} \right) \cdot \left(\frac{u(t)}{m} - \frac{k}{m}(0.1 - 0.1\alpha) \right)$$

$$= (0.1 - 0.1\alpha) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} E^{r}_{3/2, r/2+3}\left(\frac{-c}{m} t^{3/2}\right) \right) \left(\frac{u(t)}{m} - \frac{k}{m}(0.1 - 0.1\alpha) \right). \tag{8.9}$$

- **Solution for Case 2**

For trapezoidal fuzzy initial condition, the general solution can be represented as

$$\underline{x}(t;\alpha) = (0.05\alpha - 0.1) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} \sum_{j=0}^{\infty} \left(\frac{-c}{m}\right)^j \frac{(j+r)!\, t^{3j/2}}{j!\,\Gamma\left(\frac{3j}{2}+2r+3\right)} \right) \cdot \left(\frac{u(t)}{m} - \frac{k}{m}(0.05\alpha - 0.1) \right)$$

$$= (0.05\alpha - 0.1) + \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{k}{m}\right)^r t^{2(r+1)} E^{r}_{3/2, r/2+3}\left(\frac{-c}{m} t^{3/2}\right) \right) \left(\frac{u(t)}{m} - \frac{k}{m}(0.05\alpha - 0.1) \right) \tag{8.10}$$

and

$$\overline{x}(t;\alpha) = (0.1-0.05\alpha) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\left(\frac{u(t)}{m}-\frac{k}{m}(0.1-0.05\alpha)\right)$$

$$= (0.1-0.05\alpha) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\left(\frac{u(t)}{m}-\frac{k}{m}(0.1-0.05\alpha)\right).$$

(8.11)

- **Solution for Case 3**

Similarly for Gaussian fuzzy initial condition, one may have the general solution as

$$\underline{x}(t;\alpha) = \left(-0.1\sqrt{-2\log_e(\alpha)}\right) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\left(\frac{u(t)}{m}-\frac{k}{m}\left(-0.1\sqrt{-2\log_e(\alpha)}\right)\right)$$

$$= \left(-0.1\sqrt{-2\log_e(\alpha)}\right) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\left(\frac{u(t)}{m}-\frac{k}{m}\left(-0.1\sqrt{-2\log_e(\alpha)}\right)\right)$$

(8.12)

and

$$\overline{x}(t;\alpha) = \left(0.1\sqrt{-2\log_e(\alpha)}\right) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\left(\frac{u(t)}{m}-\frac{k}{m}\left(0.1\sqrt{-2\log_e(\alpha)}\right)\right)$$

$$= \left(0.1\sqrt{-2\log_e(\alpha)}\right) + \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\left(\frac{u(t)}{m}-\frac{k}{m}\left(0.1\sqrt{-2\log_e(\alpha)}\right)\right).$$

(8.13)

8.2.2. Uncertain impulse function response

We have considered the response subject to unit impulsive load viz. $f(t) = \delta(t)$, where $\delta(t)$ is the unit impulse function.

- **Solution for Case 1**

For triangular fuzzy initial conditions we have

$$\underline{X}_0(t;\alpha) = 0.1\alpha - 0.1,$$

$$\overline{X}_0(t;\alpha) = 0.1 - 0.1\alpha,$$

$$\underline{X}_1(t;\alpha) = -(0.1\alpha - 0.1)\frac{k}{m}\frac{t^2}{2} + \frac{t}{m},$$

$$\overline{X}_1(t;\alpha) = -(0.1 - 0.1\alpha)\frac{k}{m}\frac{t^2}{2} + \frac{t}{m},$$

$$\underline{X}_2(t;\alpha) = (0.1\alpha - 0.1)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)} + \frac{k}{m}\frac{t^4}{\Gamma(5)}\right) - \frac{c}{m^2}\frac{t^{5/2}}{\Gamma(7/2)} - \frac{k}{m^2}\frac{t^3}{\Gamma(4)},$$

$$\overline{X}_2(t;\alpha) = (0.1 - 0.1\alpha)\frac{k}{m}\left(\frac{c}{m}\frac{t^{7/2}}{\Gamma(9/2)} + \frac{k}{m}\frac{t^4}{\Gamma(5)}\right) - \frac{c}{m^2}\frac{t^{5/2}}{\Gamma(7/2)} - \frac{k}{m^2}\frac{t^3}{\Gamma(4)},$$

$$\underline{X}_3(t;\alpha) = -(0.1\alpha - 0.1)\frac{k}{m}\left(\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)} + \frac{2kc}{m^2}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)\frac{c^2}{m^3}\frac{t^4}{\Gamma(5)}$$
$$+ \frac{2kc}{m^3}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^3}\frac{t^5}{\Gamma(6)},$$

$$\overline{X}_3(t;\alpha) = -(0.1 - 0.1\gamma)\frac{k}{m}\left(\frac{c^2}{m^2}\frac{t^5}{\Gamma(6)} + \frac{2kc}{m^2}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^2}\frac{t^6}{\Gamma(7)}\right)\frac{c^2}{m^3}\frac{t^4}{\Gamma(5)}$$
$$+ \frac{2kc}{m^3}\frac{t^{9/2}}{\Gamma(11/2)} + \frac{k^2}{m^3}\frac{t^5}{\Gamma(6)},$$

and so on.

Substituting these in Eq. (8.7), we may get the approximate solution of $\tilde{x}(t)$. Accordingly, the general solution may be written as

$$\underline{x}(t;\alpha) = (0.1\alpha - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.1\alpha - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right)$$

(8.14)

and

$$\bar{x}(t;\gamma) = (0.1 - 0.1\alpha)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.1 - 0.1\alpha)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right).$$

(8.15)

- **Solution for Case 2**

For trapezoidal fuzzy initial condition, the general solution can be represented as

$$\underline{x}(t;\alpha) = (0.05\alpha - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.05\alpha - 0.1)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$
$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right) \quad (8.16)$$

and

$$\overline{x}(t;\alpha) = (0.1 - 0.05\alpha)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j \frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= (0.1 - 0.05\alpha)\left(1 - \frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)} E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$
$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1} E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right). \quad (8.17)$$

- **Solution for Case 3**

Similarly for Gaussian fuzzy initial condition, one may have the general solution as

$$\underline{x}(t;\alpha) = -0.1\sqrt{-2\log_e(\alpha)}\left(1-\frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= -0.1\sqrt{-2\log_e(\alpha)}\left(1-\frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right) \tag{8.18}$$

and

$$\overline{x}(t;\alpha) = 0.1\sqrt{-2\log_e(\alpha)}\left(1-\frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+3\right)}\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{m}\right)^j\frac{(j+r)!t^{3j/2}}{j!\Gamma\left(\frac{3j}{2}+2r+2\right)}$$

$$= 0.1\sqrt{-2\log_e(\alpha)}\left(1-\frac{k}{m}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2(r+1)}E^r_{3/2,r/2+3}\left(\frac{-c}{m}t^{3/2}\right)\right)\right)$$

$$+\frac{1}{m}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{k}{m}\right)^r t^{2r+1}E^r_{3/2,r/2+2}\left(\frac{-c}{m}t^{3/2}\right). \tag{8.19}$$

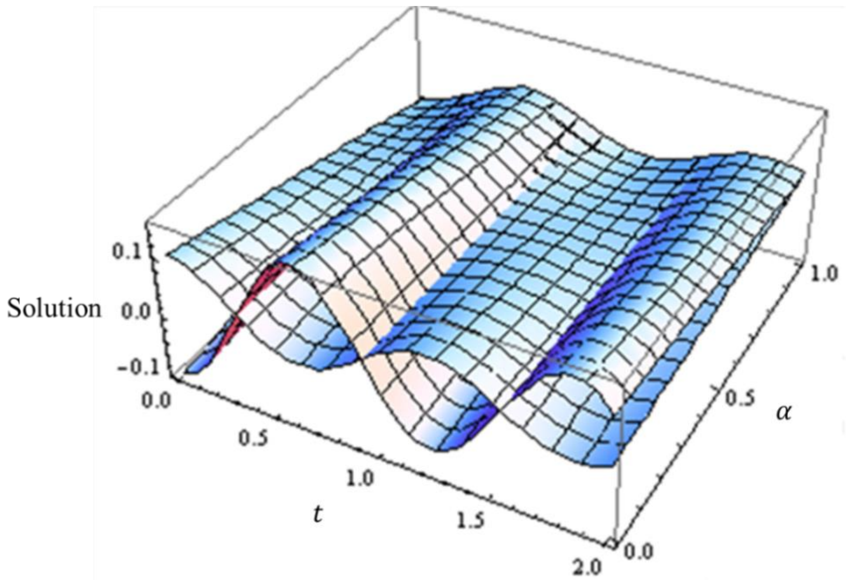
8.3. Numerical Results

For numerical simulations, we use the notations of the parameters $\omega_n = \sqrt{k/m}$, $\eta = c/2m\omega_n^{3/2}$ and value of $m = 1$, where ω_n is the natural frequency and η is the damping ratio.

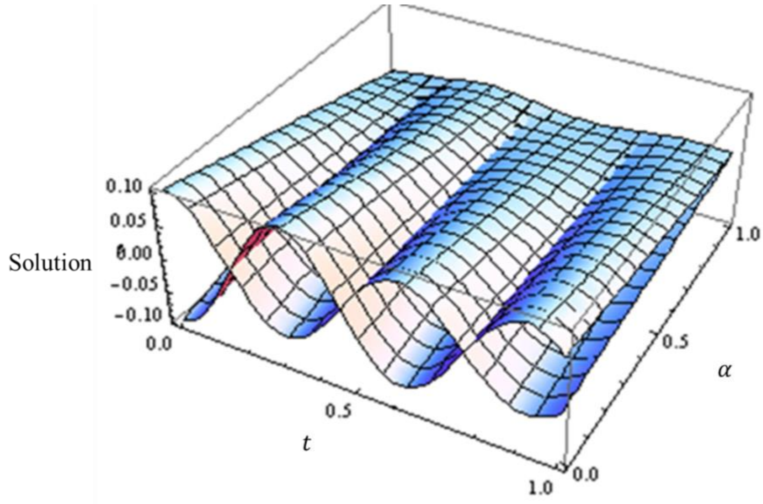
8.3.1. Case studies for uncertain step function response

Eqs. (8.8) to (8.13) provide the desired expressions for the considered loading condition. Fig. (8.1) shows the triangular fuzzy response for Case 1 with natural frequencies $\omega_n = 5$ rad/s (Fig. 8.1 (a)), $\omega_n = 10$ rad/s (Fig. 8.1 (b)) and damping ratio $\eta = 0.05$ for Eqs. (8.8) and (8.9). Similar simulations have been done for Cases 2 and 3. Accordingly, the trapezoidal and Gaussian fuzzy responses are depicted in Figs. 8.2 and 8.3.

In Case 1, the triplet number (a, b, c) defines a triangular membership function, where a and c are the lower and upper bounds at $\alpha = 0$ and b is the nominal (crisp) value at $\alpha = 1$. Hence, for $\alpha = 0$ (Fig. 8.4(a)) and $\alpha = 1$ (Fig. 8.4(b)) along with the crisp solution by Podlunby (1999) for crisp initial condition these are depicted in Fig. (8.4). Similarly, Figs. (8.5) and (8.6) represent the uncertain-but-bounded (interval) solutions for Cases 2 and 3. Here in these cases, $\omega_n = 5$rad/s and $\eta = 0.05$ are considered. Similar interval responses are shown in Figs. 8.7 to 8.9 with $\omega_n = 10$rad/s, and $\eta = 0.05$.

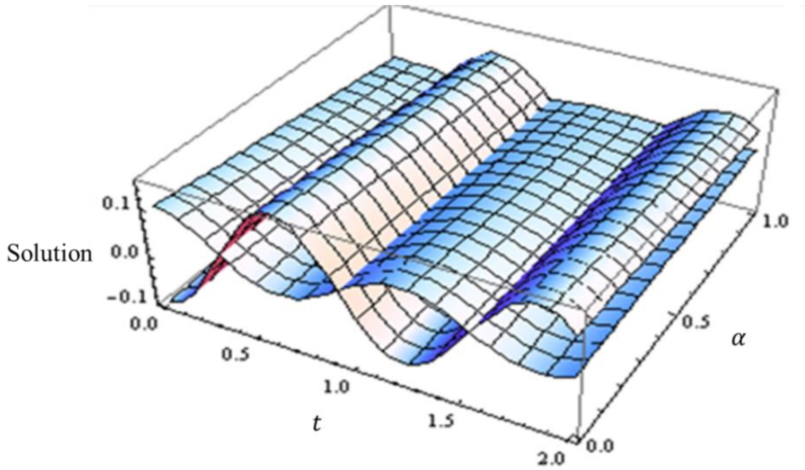


(a)

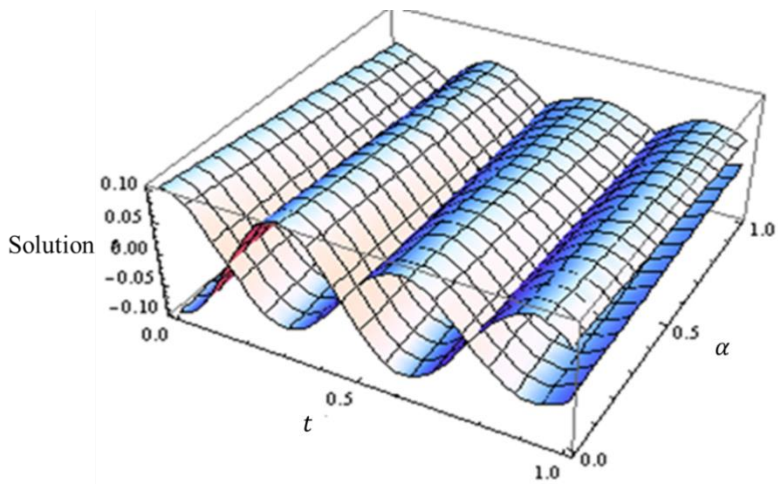


(b)

Fig. 8.1 Triangular fuzzy response subject to unit step load for Case 1 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

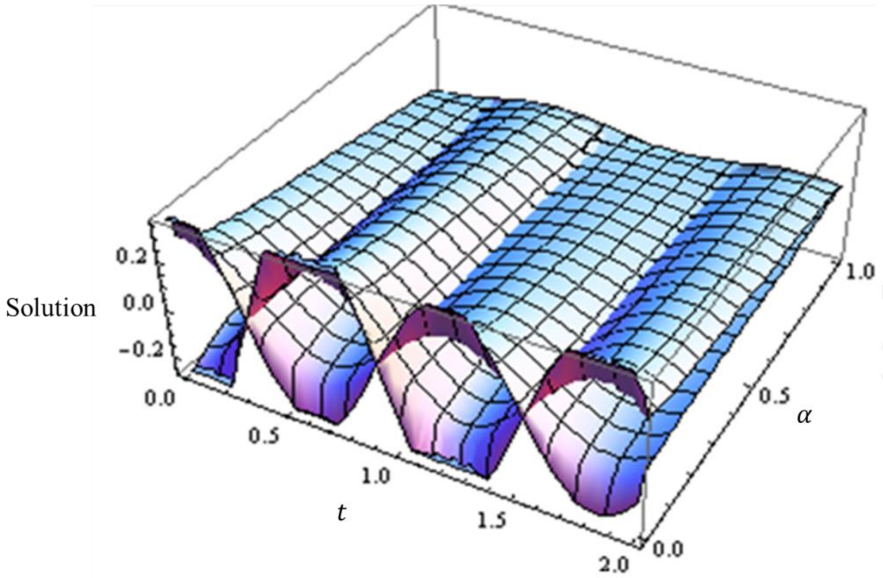


(a)

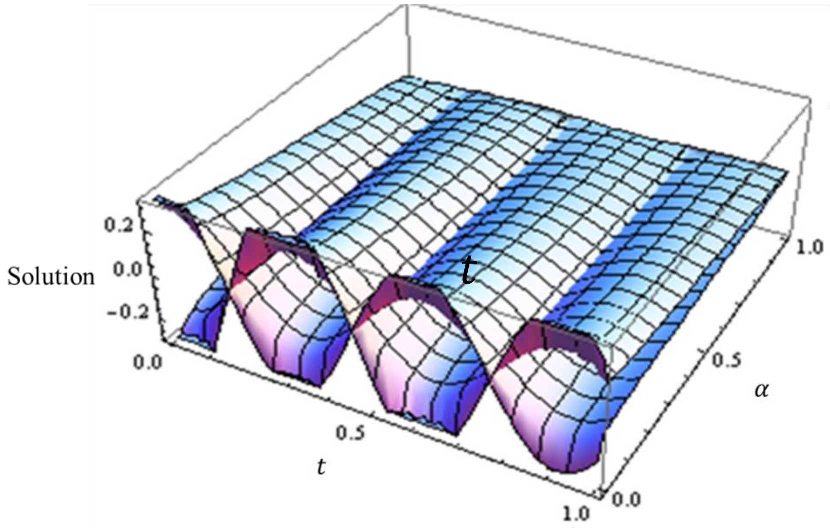


(b)

Fig. 8.2 Trapezoidal fuzzy response subject to unit step load for Case 2 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

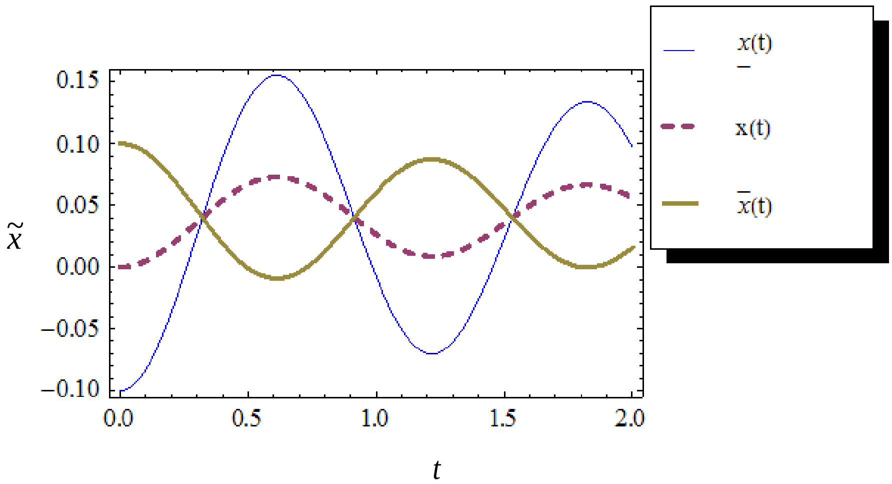


(a)

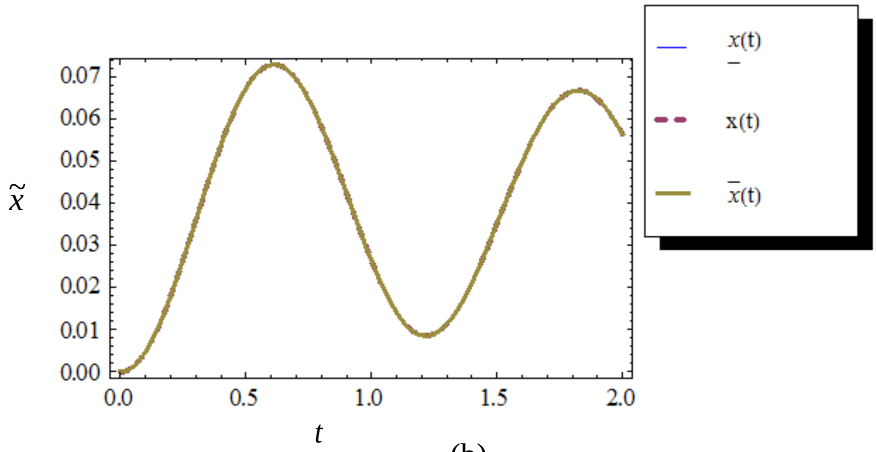


(b)

Fig. 8.3 Gaussian fuzzy response subject to unit step load for Case 3 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

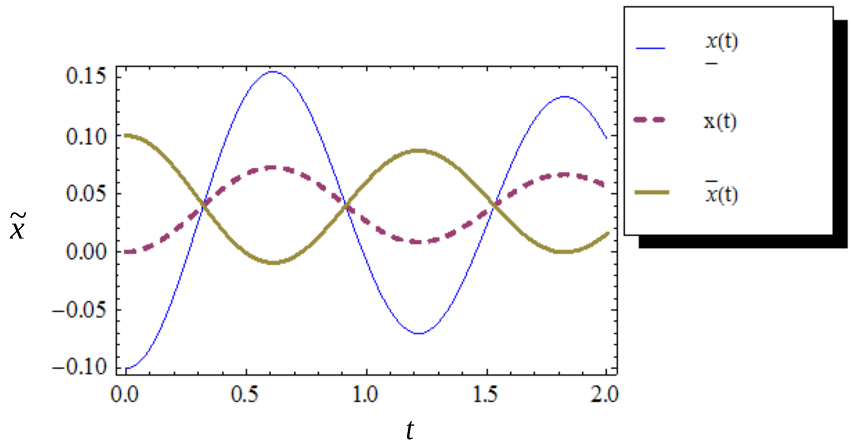


(a)

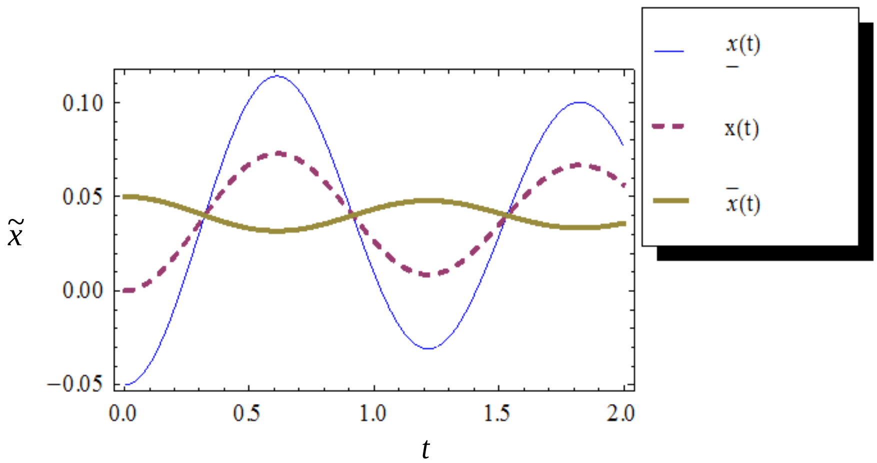


(b)

Fig. 8.4 Uncertain but bounded (interval) response subject to unit step load for Case 1 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 5$ rad/s and damping ratio $\eta = 0.05$

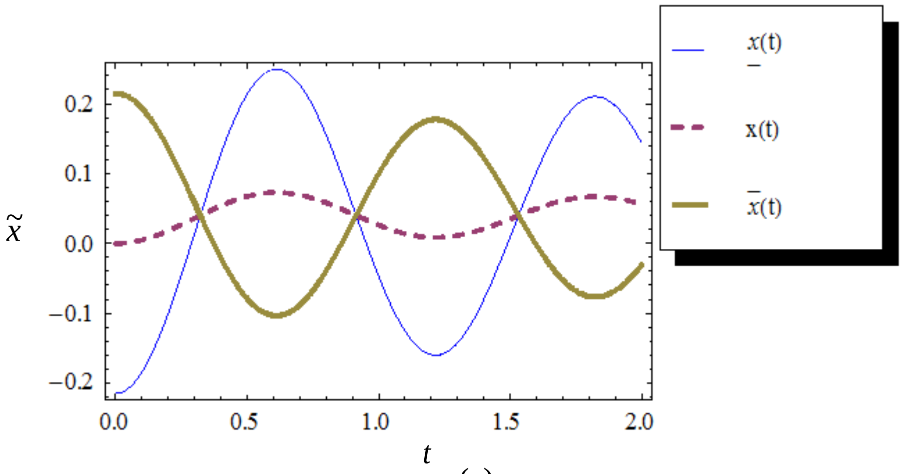


(a)

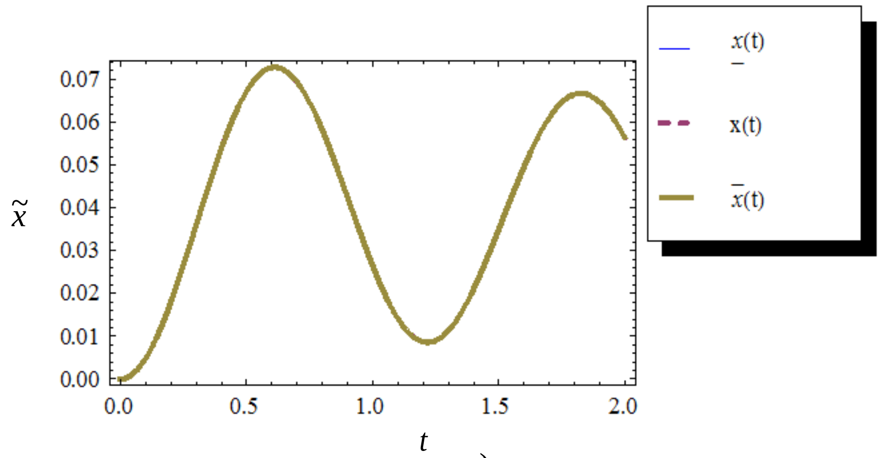


(b)

Fig. 8.5 Uncertain but bounded (interval) response subject to unit step load for Case 2 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 5$ rad/s and damping ratio $\eta = 0.05$

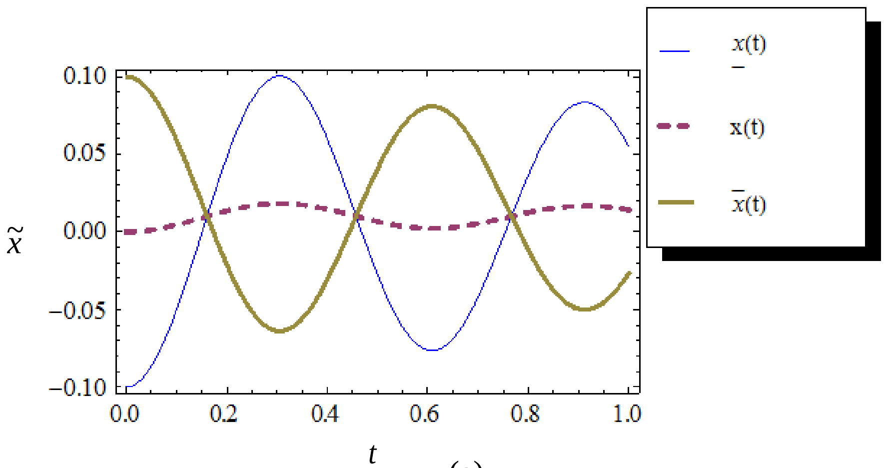


(a)

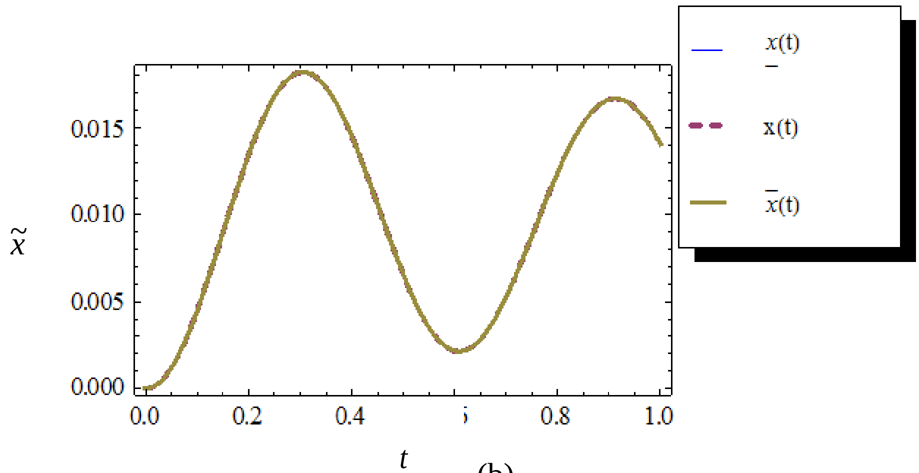


(b)

Fig. 8.6 Uncertain but bounded (interval) response subject to unit step load for Case 3 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 5$ rad/s and damping ratio $\eta = 0.05$

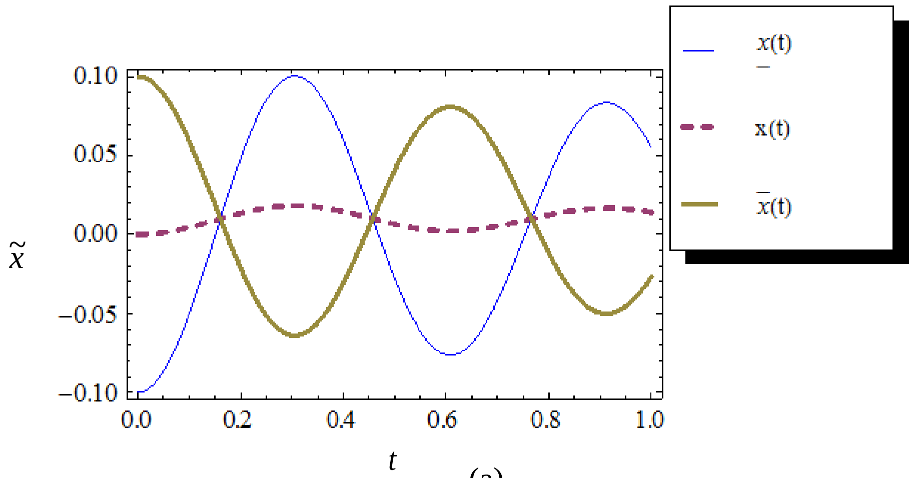


(a)

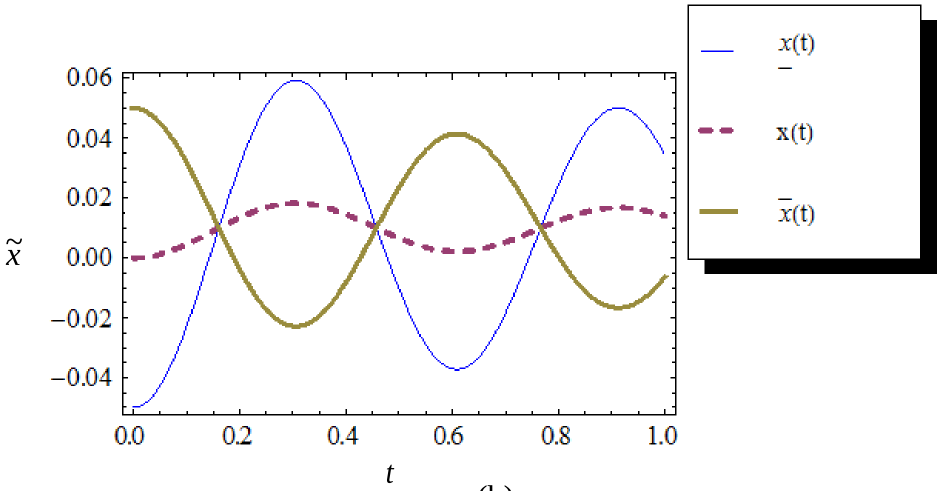


(b)

Fig. 8.7 Uncertain but bounded (interval) response subject to unit step load for Case 1 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

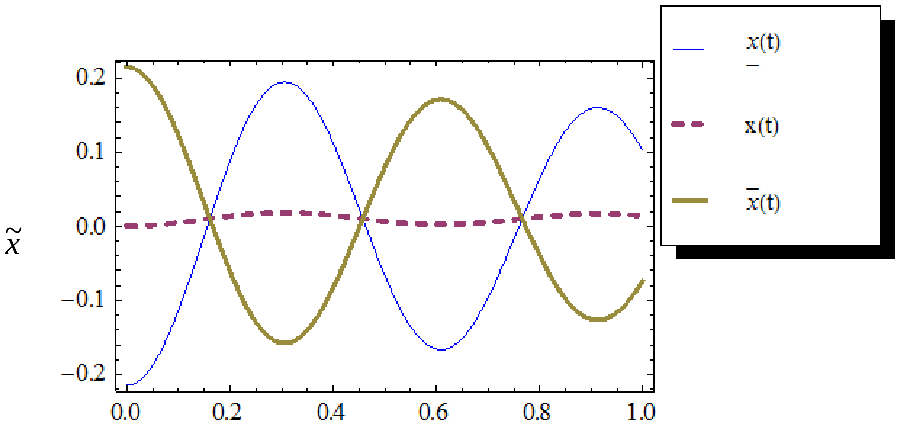


(a)

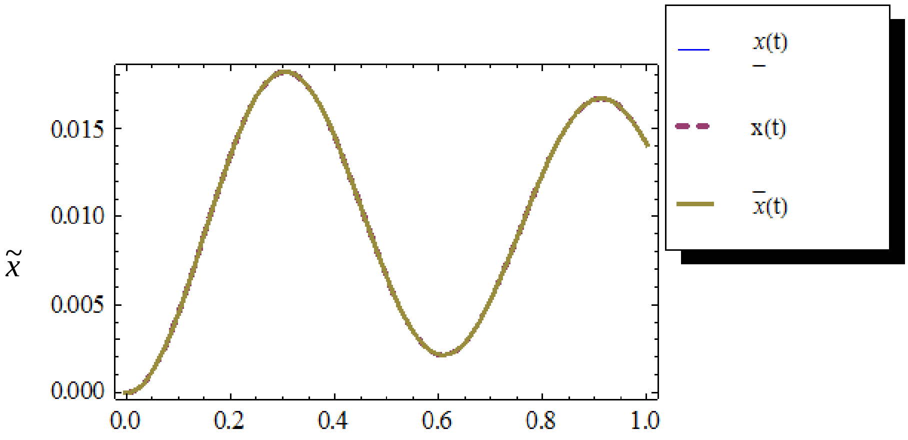


(b)

Fig. 8.8 Uncertain but bounded (interval) response subject to unit step load for Case 2 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$



t (a)



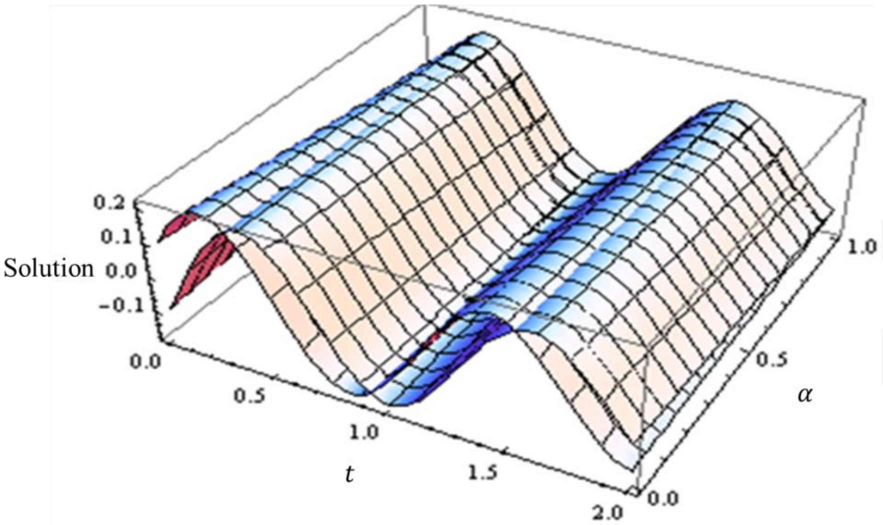
t (b)

Fig. 8.9 Uncertain but bounded (interval) response subject to unit step load for Case 3 when (a) $\alpha = 0$ (b) $\alpha = 1$ with crisp analytical solution (- - -) by Podlunby (1999) where natural frequency $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

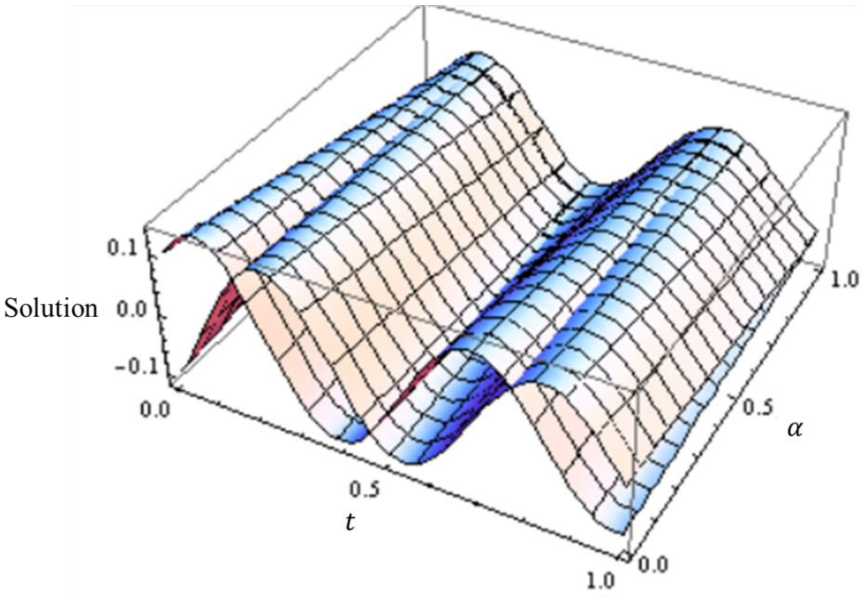
One may see from Figs. 8.1 and 8.3 for Cases 1 and 3 that lower and upper bounds of the fuzzy displacements coincide for $\alpha = 1$, as the fuzzy initial conditions convert to the crisp one (of Chapter 6). Also, it is interesting to note from Figs. 8.4 to 8.9 that for $\alpha = 0$, interval bounds contain the crisp solution. Moreover, interval solution bounds coincide with the crisp solutions for Cases 1 and 3. From Figs. 8.1 and 8.7 it may also be seen that by increasing the value of natural frequency we get less oscillation. This is in general true for all the cases.

8.3.2. Case studies for uncertain impulse function response

Depending on the values of natural frequency ω_n and damping ratio η, different cases have been studied. First, the numerical values of the natural frequency $\omega_n = 5$ rad/s and damping ratio $\eta = 0.05$ are taken. Next, natural frequency $\omega_n = 10$ rad/s with damping ratio $\eta = 0.05$ with unit impulse load is considered for the oscillation. With these parametric values obtained fuzzy displacements are depicted in Figs. 8.10 to 8.12. Also, one can see from Figs. 8.10 and 8.12 for Cases 1 and 3 that lower and upper bounds of the fuzzy displacements coincide for $\alpha = 1$. This is because the fuzzy initial conditions again convert to a crisp one (of Chapter 6).

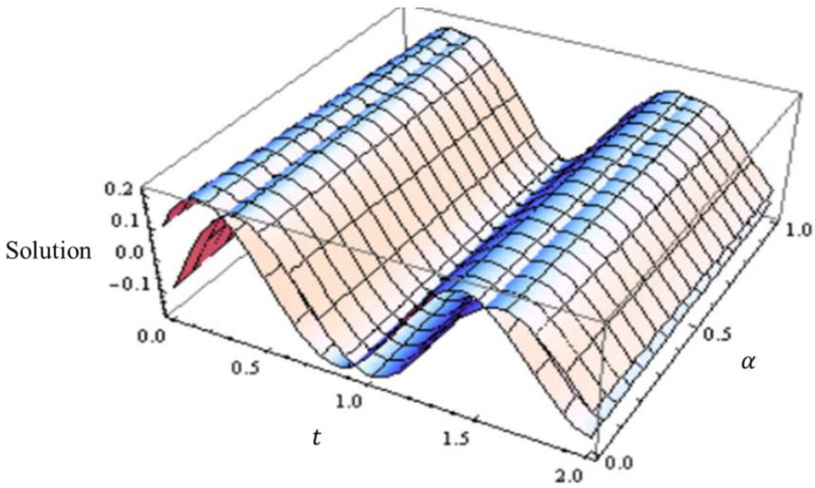


(a)

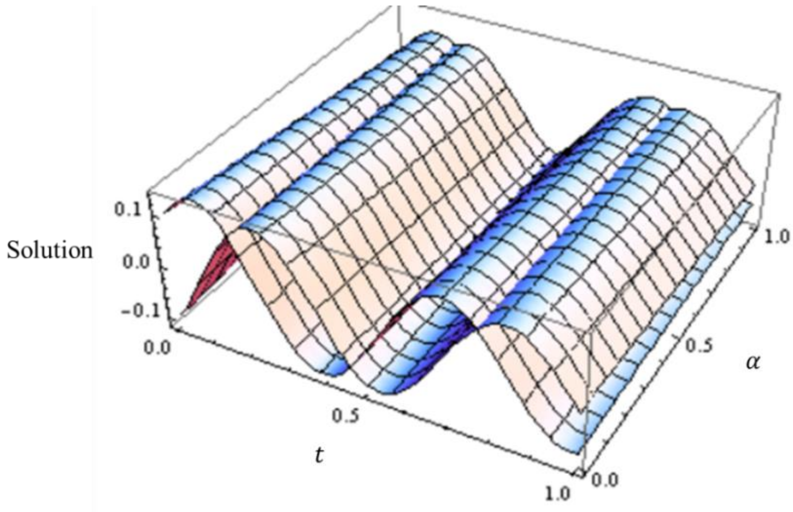


(b)

Fig. 8.10 Triangular fuzzy response subject to unit impulse load for Case 1 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

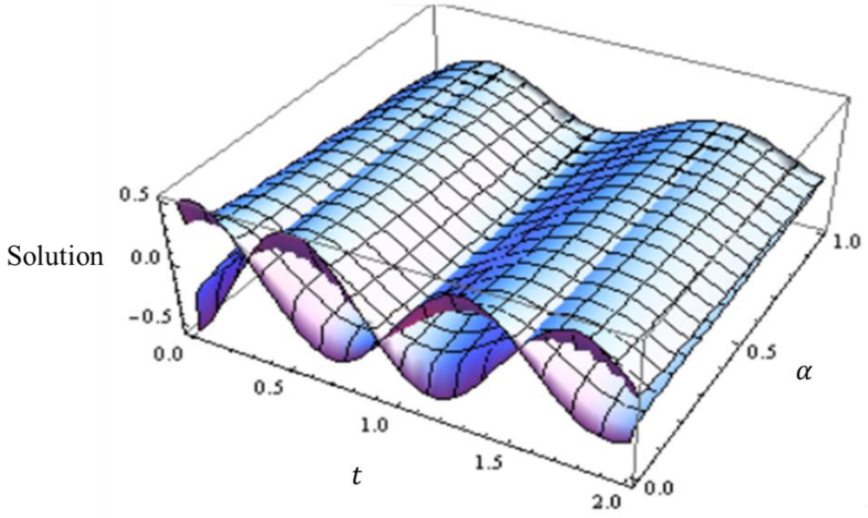


(a)

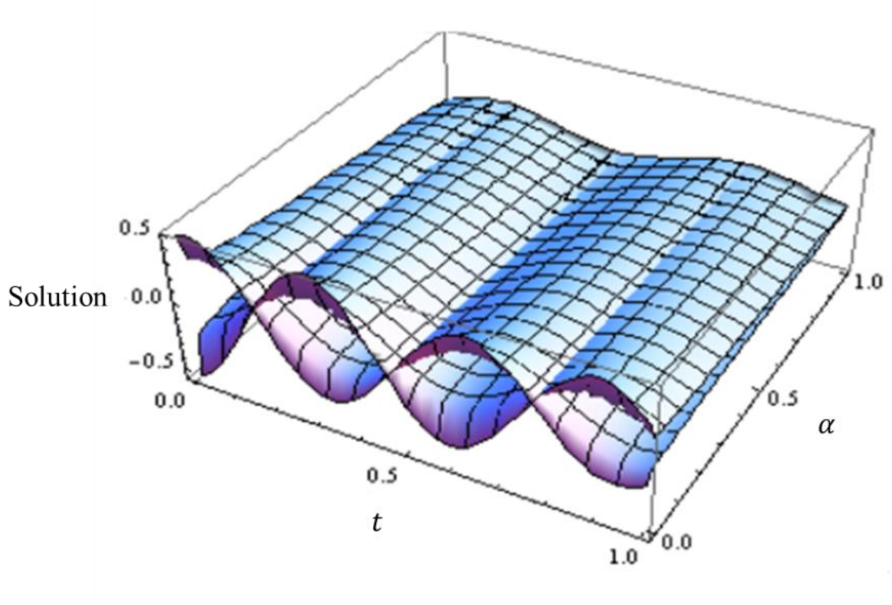


(b)

Fig. 8.11 Trapezoidal fuzzy response subject to unit impulse load for Case 2 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$



(a)



(b)

Fig. 8.12 Gaussian fuzzy response subject to unit impulse load for Case 3 with natural frequency (a) $\omega_n = 5$ rad/s (b) $\omega_n = 10$ rad/s and damping ratio $\eta = 0.05$

Homotopy perturbation method with fuzzy based approach has successfully been applied to obtain the uncertain solution of an imprecisely defined fractionally damped spring-mass mechanical system subject to a unit step and impulse load, where the fractional derivative is considered of order 1/2. Also these type of systems may be solved by other well known semi analytical methods such as Varitational Iteration Method (VIM) and Adomain Decomposition Method (ADM) etc.

Chapter 9

Uncertain Fractionally Damped Continuous System

The content of this chapter has been communicated for publication.

Chapter 9

Uncertain Fractionally Damped Continuous System

In this chapter, fuzzy fractionally damped continuous system viz. beam has been studied using the double parametric form of fuzzy numbers subject to unit step and impulse loads. Triangular fuzzy numbers are used to represent the initial conditions. Using alpha cut, corresponding beam equation is first converted to an interval based equation. Next, it has been transformed to crisp form by applying double parametric form of fuzzy numbers. Finally, Homotopy Perturbation Method (HPM) has been used for obtaining the fuzzy response.

9.1. Fuzzy Fractionally Damped Viscoelastic Beam

Let us consider a fuzzy linear differential equation which describes the dynamics of the above system with damping as an arbitrary fractional derivative of order λ

$$\rho A\frac{\partial^2\tilde{v}}{\partial t^2}+c\frac{\partial^\lambda\tilde{v}}{\partial t^\lambda}+EI\frac{\partial^4\tilde{v}}{\partial x^4}=F(x,t). \tag{9.1}$$

Eq. (9.1) may be written as

$$\frac{\partial^2\tilde{v}}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda\tilde{v}}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4\tilde{v}}{\partial x^4}=\frac{F(x,t)}{\rho A}, \tag{9.2}$$

where ρ, A, c, E and I are the mass density, cross sectional area, damping coefficients per unit length, Young's modulus of elasticity and moment of inertia of the beam. $F(x,t)$ is the externally applied force and $\tilde{v}(x,t)$ is the transverse fuzzy displacement. $\frac{\partial^\lambda}{\partial t^\lambda}$ is the fractional derivative of order $\lambda\in(0,1)$ of the fuzzy displacement function $\tilde{v}(x,t)$. Initial conditions are considered as fuzzy viz. $\tilde{v}(0)=\tilde{v}'(0)=(-0.1,0,0.1)$.

As per the single parametric form, we may write Eq. (9.2) as

$$\left[\frac{\partial^2\underline{v}(x,t;\alpha)}{\partial t^2},\frac{\partial^2\overline{v}(x,t;\alpha)}{\partial t^2}\right]+\frac{c}{\rho A}\left[\frac{\partial^\lambda\underline{v}(x,t;\alpha)}{\partial t^\lambda},\frac{\partial^\lambda\overline{v}(x,t;\alpha)}{\partial t^\lambda}\right]$$
$$+\frac{EI}{\rho A}\left[\frac{\partial^4\underline{v}(x,t;\alpha)}{\partial x^4},\frac{\partial^4\overline{v}(x,t;\alpha)}{\partial x^4}\right]=\frac{F(x,t)}{\rho A} \tag{9.3}$$

subject to fuzzy initial conditions

$[\underline{v}(x,0;\alpha),\bar{v}(x,0;\alpha)]=[\underline{v}'(x,0;\alpha),\bar{v}'(x,0;\alpha)]=[0.1\alpha-0.1,0.1-0.1\alpha]$ where, $\alpha\in[0,1]$.

Next, using the double parametric form (as discussed in Definition 2.7), Eq. (9.3) can be expressed as

$$\left\{\beta\left(\frac{\partial^2\bar{v}(x,t;\alpha)}{\partial t^2}-\frac{\partial^2\underline{v}(x,t;\alpha)}{\partial t^2}\right)+\frac{\partial^2\underline{v}(x,t;\alpha)}{\partial t^2}\right\}$$
$$+\frac{c}{\rho A}\left\{\beta\left(\frac{\partial^\lambda\bar{v}(x,t;\alpha)}{\partial t^\lambda}-\frac{\partial^\lambda\underline{v}(x,t;\alpha)}{\partial t^\lambda}\right)+\frac{\partial^\lambda\underline{v}(x,t;\alpha)}{\partial t^\lambda}\right\} \tag{9.4}$$
$$+\frac{EI}{\rho A}\left\{\beta\left(\frac{\partial^4\bar{v}(x,t;\alpha)}{\partial x^4}-\frac{\partial^4\underline{v}(x,t;\alpha)}{\partial x^4}\right)+\frac{\partial^4\underline{v}(x,t;\alpha)}{\partial x^4}\right\}=\frac{F(x,t)}{\rho A}$$

subject to the initial conditions

$\{\beta(\bar{v}(x,0;\alpha)-\underline{v}(x,0;\alpha))+\underline{v}(x,0;\alpha)\}==\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}$ and

$\{\beta(\bar{v}'(x,0;\alpha)-\underline{v}'(x,0;\alpha))+\underline{v}'(x,0;\alpha)\}=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}$, where, $\beta\in[0,1]$.

Let us now denote

$$\left\{\beta\left(\frac{\partial^2\bar{v}(x,t;\alpha)}{\partial t^2}-\frac{\partial^2\underline{v}(x,t;\alpha)}{\partial t^2}\right)+\frac{\partial^2\underline{v}(x,t;\alpha)}{\partial t^2}\right\}=\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2},$$

$$\left\{\beta\left(\frac{\partial^\lambda\bar{v}(x,t;\alpha)}{\partial t^\lambda}-\frac{\partial^\lambda\underline{v}(x,t;\alpha)}{\partial t^\lambda}\right)+\frac{\partial^\lambda\underline{v}(x,t;\alpha)}{\partial t^\lambda}\right\}=\frac{\partial^\lambda v(x,t;\alpha,\beta)}{\partial t^\lambda},$$

$$\left\{\beta\left(\frac{\partial^4\bar{v}(x,t;\alpha)}{\partial x^4}-\frac{\partial^4\underline{v}(x,t;\alpha)}{\partial x^4}\right)+\frac{\partial^4\underline{v}(x,t;\alpha)}{\partial x^4}\right\}=\frac{\partial^4 v(x,t;\alpha,\beta)}{\partial x^4},$$

$\{\beta(\bar{v}(x,0;\alpha)-\underline{v}(0;\alpha))+\underline{v}(x,0;\alpha)\}=v(x,0;\alpha,\beta)$ and

$\{\beta(\bar{v}'(x,0;\alpha)-\underline{v}'(x,0;\alpha))+\underline{v}'(x,0;\alpha)\}=v'(x,0;\alpha,\beta)$.

Substituting these values in Eq. (9.4) we get

$$\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\lambda v(x,t;\alpha,\beta)}{\partial t^\lambda}+\frac{EI}{\rho A}\frac{\partial^4 v(x,t;\alpha,\beta)}{\partial x^4}=\frac{F(x,t)}{\rho A}, \tag{9.5}$$

with initial conditions $v(x,0;\alpha,\beta)=v'(x,0;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}$.

Now, Eq. (9.5) has been solved using HPM. According to HPM, we may construct a simple homotopy for an embedding parameter $p\in[0,1]$, as follows

$$(1-p)\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2} + p\left[\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\lambda v(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v(x,t;\alpha,\beta)}{\partial x^4} - \frac{F(x,t)}{\rho A}\right] = 0, \quad (9.6)$$

or

$$\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2} + p\left[\frac{c}{\rho A}\frac{\partial^\lambda v(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v(x,t;\alpha,\beta)}{\partial x^4} - \frac{F(x,t)}{\rho A}\right] = 0. \quad (9.7)$$

For $p=0$, Eqs. (9.6) and (9.7) become a linear equation i.e. $\frac{\partial^2 v(x,t;\alpha,\beta)}{\partial t^2} = 0$, which is easy to solve. For $p=1$, Eqs. (9.6) and (9.7) turns out to be the same as the original Eq. (9.5).

We assume the solution of Eq. (9.6) or (9.7) as a power series expansion in p as

$$v(x,t;\alpha,\beta) = v_0(x,t;\alpha,\beta) + pv_1(x,t;\alpha,\beta) + p^2 v_2(x,t;\alpha,\beta) + p^3 v_3(x,t;\alpha,\beta) + \cdots, \quad (9.8)$$

where, $v_i(x,t;\alpha,\beta)$ for $i = 0,1,2,3,\cdots$ are functions yet to be determined.

Substituting Eq. (9.8) into Eq. (9.6) or (9.7) and equating the terms with the identical powers of p, we have

$$p^0 : \frac{\partial^2 v_0(x,t;\alpha,\beta)}{\partial t^2} = 0, \quad (9.9)$$

$$p^1 : \frac{\partial^2 v_1(x,t;\alpha,\beta)}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\lambda v_0(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v_0(x,t;\alpha,\beta)}{\partial x^4} - \frac{F(x,t)}{\rho A} = 0, \quad (9.10)$$

$$p^2 : \frac{\partial^2 v_2(x,t;\alpha,\beta)}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\lambda v_1(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v_1(x,t;\alpha,\beta)}{\partial x^4} = 0, \quad (9.11)$$

$$p^3 : \frac{\partial^2 v_3(x,t;\alpha,\beta)}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\lambda v_2(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v_2(x,t;\alpha,\beta)}{\partial x^4} = 0, \quad (9.12)$$

$$p^4 : \frac{\partial^2 v_4(x,t;\alpha,\beta)}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\lambda v_3(x,t;\alpha,\beta)}{\partial t^\lambda} + \frac{EI}{\rho A}\frac{\partial^4 v_3(x,t;\alpha,\beta)}{\partial x^4} = 0, \quad (9.13)$$

and so on.

Choosing initial approximation $v(x,0;\alpha,\beta)=v'(x,0;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}$ and applying the inverse operator L_{tt}^{-1} (which is the inverse of the operator $L_{tt}=\dfrac{\partial^2}{\partial t^2}$) on both sides of each Eqs. (9.9) to (9.13), one may obtain the following equations

$$v_0(x,t;\alpha,\beta)=v'(x,0;\alpha,\beta)t+v(x,0;\alpha,\beta), \tag{9.14}$$

$$v_1(x,t;\alpha,\beta)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^{\lambda}v_0(x,t;\alpha,\beta)}{\partial t^{\lambda}}-\frac{EI}{\rho A}\frac{\partial^4 v_0(x,t;\alpha,\beta)}{\partial x^4}+\frac{F(x,t)}{\rho A}\right), \tag{9.15}$$

$$v_2(x,t;\alpha,\beta)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^{\lambda}v_1(x,t;\alpha,\beta)}{\partial t^{\lambda}}-\frac{EI}{\rho A}\frac{\partial^4 v_1(x,t;\alpha,\beta)}{\partial x^4}\right), \tag{9.16}$$

$$v_3(x,t;\alpha,\beta)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^{\lambda}v_2(x,t;\alpha,\beta)}{\partial t^{\lambda}}-\frac{EI}{\rho A}\frac{\partial^4 v_2(x,t;\alpha,\beta)}{\partial x^4}\right), \tag{9.17}$$

$$v_4(x,t;\alpha,\beta)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^{\lambda}v_3(x,t;\alpha,\beta)}{\partial t^{\lambda}}-\frac{EI}{\rho A}\frac{\partial^4 v_3(x,t;\alpha,\beta)}{\partial x^4}\right), \tag{9.18}$$

and so on.

Substituting these terms in Eq. (9.8) with, $p\to 1$ one may get the approximate solution of Eq. (9.5) as follows.

$$v(x,t;\alpha,\beta)=v_0(x,t;\alpha,\beta)+v_1(x,t;\alpha,\beta)+v_2(x,t;\alpha,\beta)+v_3(x,t;\alpha,\beta)+\cdots$$

To obtain the lower and upper bound of the solution in single parametric form, we may substitue $\beta=0$ and 1 respectively. These may be represented as $v(x,t;\alpha,0)=\underline{v}(x,t;\alpha)$ and $v(x,t;\alpha,1)=\overline{v}(x,t;\alpha)$.

9.2. Uncertain Response Analysis

Let us consider the external applied force $F(x,t)$ as

$$F(x,t)=f(x)g(t),$$

where $f(x)$ is a specified space dependent deterministic function, and $g(t)$ is a time dependent process. In the following paragraph, we will examine the fuzzy response of the dynamic system (9.5) subject to unit step and impulse loading conditions.

9.2.1. Unit step function response

We will now consider the response of the fuzzy fractionally damped beam subject to a unit step load of the form $g(t) = Bu(t)$ where $u(t)$ is the heaviside function and B is a constant. By using HPM we have

$$v_0(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}(1+t), \tag{9.19}$$

$$v_1(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(-\frac{c}{\rho A}\frac{t^{3-\lambda}}{\Gamma(4-\lambda)}\right)+\frac{fBt^2}{2\rho A}, \tag{9.20}$$

$$v_2(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(\frac{c^2}{\rho^2A^2}\frac{t^{5-2\lambda}}{\Gamma(6-2\lambda)}\right)-\frac{fBc}{\rho^2A^2}\frac{t^{4-\lambda}}{\Gamma(5-\lambda)}$$
$$-\frac{EIBf^4}{\rho^2A^2}\frac{t^4}{\Gamma(5)}, \tag{9.21}$$

$$v_3(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(-\frac{c^3}{\rho^3A^3}\frac{t^{7-3\lambda}}{\Gamma(8-3\lambda)}\right)+\frac{fBc^2}{\rho^3A^3}\frac{t^{6-2\lambda}}{\Gamma(7-2\lambda)}$$
$$+\frac{2EIBcf^4}{\rho^3A^3}\frac{t^{6-\lambda}}{\Gamma(7-\lambda)}+\frac{E^2I^2Bf^8}{\rho^3A^3}\frac{t^6}{\Gamma(7)}, \tag{9.22}$$

$$v_4(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(-\frac{c^4}{\rho^4A^4}\frac{t^{9-4\lambda}}{\Gamma(10-4\lambda)}\right)-\frac{fBc^3}{\rho^4A^4}\frac{t^{8-3\lambda}}{\Gamma(9-3\lambda)}$$
$$-\frac{3EIBc^2f^4}{\rho^4A^4}\frac{t^{8-2\lambda}}{\Gamma(9-2\lambda)}-\frac{3E^2I^2Bcf^8}{\rho^4A^4}\frac{t^{8-\lambda}}{\Gamma(9-\lambda)}-\frac{3E^3I^3Bf^{12}}{\rho^4A^4}\frac{t^8}{\Gamma(9)}, \tag{9.23}$$

and so on, where $f^{(i)} = \dfrac{\partial^i f}{\partial x^i}$. The solution can be written in general form as

$$v(x,t;\alpha,\beta) = \{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left\{1+\sum_{k=0}^{\infty}\frac{t^{(2-\lambda)k+1}}{\Gamma((2-\lambda)k+2)}\right\}$$
$$+\frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+r)!\,t^{(2-\lambda)j}}{j!\,\Gamma((2-\lambda)j+2r+3)}. \tag{9.24}$$

As discussed above, the solution bound in single parametric form may be obtained by putting $\beta = 0$ and 1. This may be represented as

$$v(x,t;\alpha,0) = \underline{v}(x,t,\alpha) \text{ and } v(x,t,\alpha,1) = \overline{v}(x,t,\alpha)$$

where

$$\underline{v}(x,t;\alpha,0)=\frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+r)!t^{(2-\lambda)j}}{j!\Gamma((2-\lambda)j+2r+3)} \quad (9.25)$$

and

$$\bar{v}(x,t;\alpha,1)=\frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2(r+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+r)!t^{(2-\lambda)j}}{j!\Gamma((2-\lambda)j+2r+3)}. \quad (9.26)$$

9.2.2. Unit impulse function response

In this section, we study response of the beam subject to unit impulse load of the form $g(t)=\delta(t)$ where $\delta(t)$ is the unit impulse function. Using HPM in this case again, we have

$$v_0(x,t;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}(1+t), \quad (9.27)$$

$$v_1(x,t;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(-\frac{c}{\rho A}\frac{t^{3-\lambda}}{\Gamma(4-\lambda)}\right)+\frac{fBt}{\rho A}, \quad (9.28)$$

$$v_2(x,t;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(\frac{c^2}{\rho^2A^2}\frac{t^{5-2\lambda}}{\Gamma(6-2\lambda)}\right)-\frac{fBc}{\rho^2A^2}\frac{t^{3-\lambda}}{\Gamma(4-\lambda)}$$
$$-\frac{EIBf^4}{\rho^2A^2}\frac{t^3}{\Gamma(4)}, \quad (9.29)$$

$$v_3(x,t;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left(-\frac{c^3}{\rho^3A^3}\frac{t^{7-3\lambda}}{\Gamma(8-3\lambda)}\right)+\frac{fBc^2}{\rho^3A^3}\frac{t^{5-2\lambda}}{\Gamma(6-2\lambda)}$$
$$+\frac{2EIBcf^4}{\rho^3A^3}\frac{t^{5-\lambda}}{\Gamma(6-\lambda)}+\frac{E^2I^2Bf^8}{\rho^3A^3}\frac{t^5}{\Gamma(6)}, \quad (9.30)$$

and so on, where $f^{(i)}=\frac{\partial^i f}{\partial x^i}$. Hence, the solution can be written in general form as

$$v(x,t;\alpha,\beta)=\{\beta(0.2-0.2\alpha)+(0.1\alpha-0.1)\}\left\{1+\sum_{k=0}^{\infty}\frac{t^{(2-\lambda)k+1}}{\Gamma((2-\lambda)k+2)}\right\}$$
$$+\frac{1}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+r)!t^{(2-\lambda)j}}{j!\Gamma((2-\lambda)j+2r+2)}. \quad (9.31)$$

Again to obtain the solution bounds in single parametric form, we may put $\beta = 0$ and 1 to get the lower and upper bounds of the solution respectively as

$$v(x,t;\alpha,0) = \underline{v}(x,t,\alpha) \text{ and } v(x,t,\alpha,1) = \bar{v}(x,t,\alpha)$$

where

$$\underline{v}(x,t;\alpha,0) = \frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j \frac{(j+r)!\,t^{(2-\lambda)j}}{j!\,\Gamma((2-\lambda)j+2r+2)}, \quad (9.32)$$

and

$$\bar{v}(x,t;\alpha,1) = \frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)} t^{2r+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j \frac{(j+r)!\,t^{(2-\lambda)j}}{j!\,\Gamma((2-\lambda)j+2r+2)}. \quad (9.33)$$

9.3. Numerical Results

Numerical results corresponding to the discussed loads have been considered in this section. Eqs. (9.24) and (9.31) provide the desired expressions for the considered loading condition. We have assumed a simply supported beam, hence one may have

$$f(x) = \sin\left(\frac{\pi x}{L}\right).$$

For numerical simulations, let us denote c/m and $EI/\rho A$ respectively as $2\eta\omega^{3/2}$ and ω^2 where, ω is the natural frequency and η is the damping ratio. The values of the parameters are taken as $B = 1, \rho A = 1, L = \pi$, $x = 1/2$ and $m = 1$.

9.3.1. Case studies for fuzzy unit step response

Depending on the natural frequency ω, damping ratio η and arbitrary order fractional derivative λ subject to unit step load, two different cases have been considered as follows. In the first case, the numerical values of the parameters are taken as $\omega = 5$ rad/s, $\eta = 0.5$ and $\lambda = 0.2$. In the second case, $\omega = 10$ rad/s, $\eta = 0.05$ and $\lambda = 0.5$ have been considered. For first and second cases, obtained fuzzy responses with respect to time are depicted in Figs. 9.1 and 9.2 respectively.

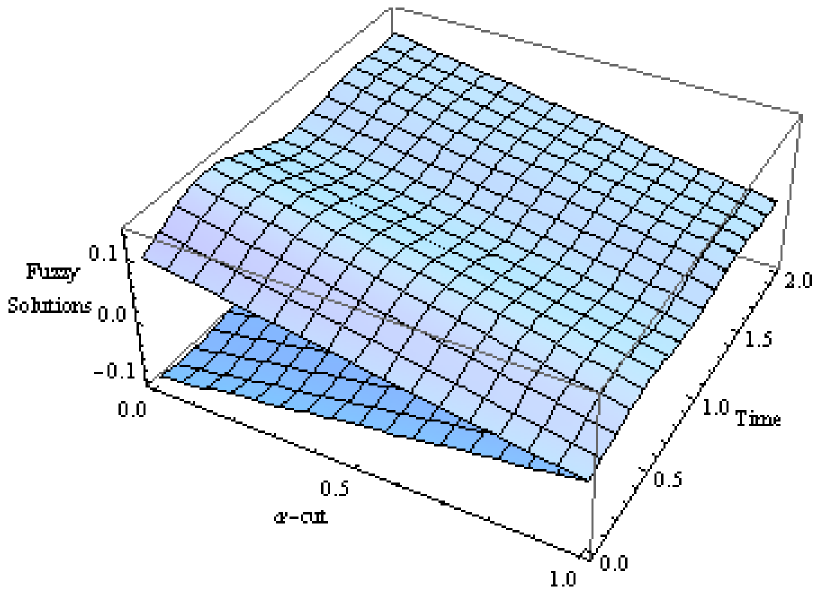


Fig. 9.1 Fuzzy unit step response for $\omega = 5$ rad/s , $\eta = 0.5$ and $\lambda = 0.2$

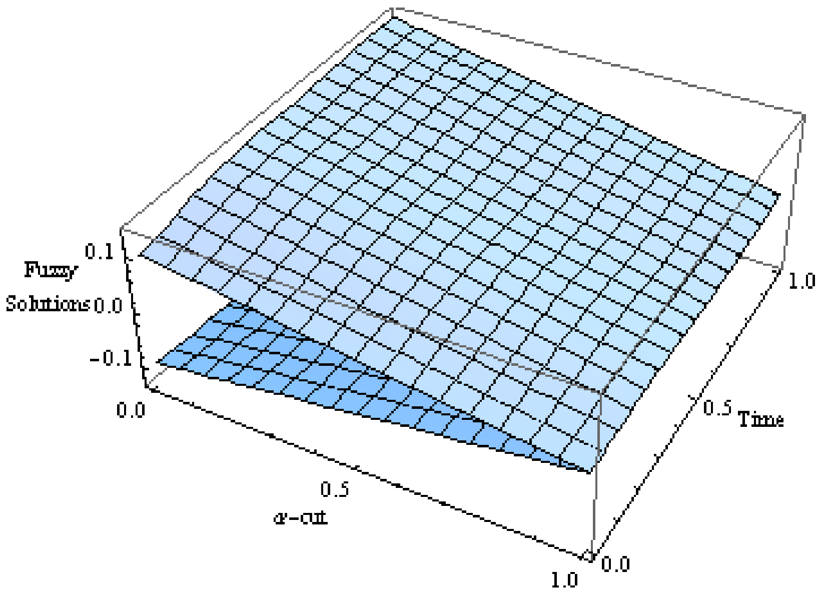
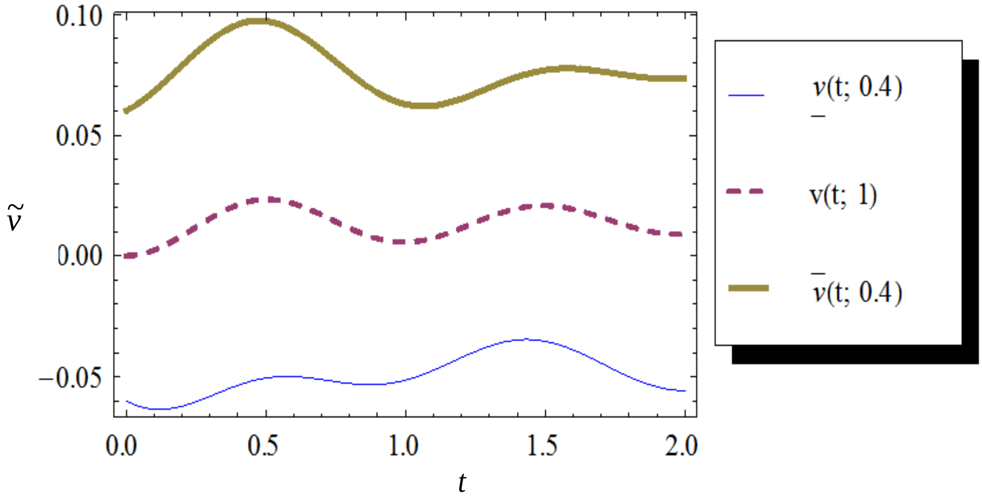


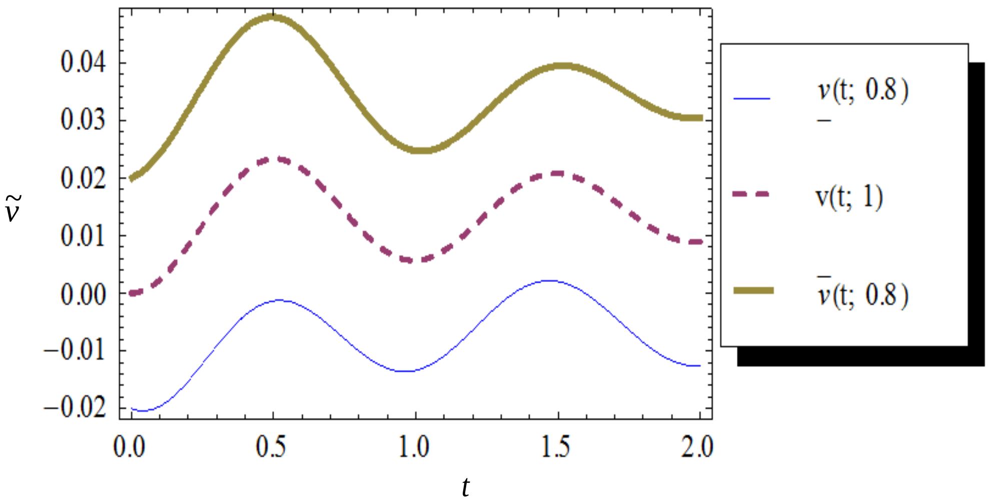
Fig. 9.2 Fuzzy unit step response for $\omega = 10$ rad/s , $\eta = 0.05$ and $\lambda = 0.5$

Figs. 9.3 and 9.4 give the effects of interval unit step responses for the particular membership α . For $\alpha = 1$, the lower and upper bounds of the solution coincide with each other and are denoted as $\underline{v}(t;1) = \overline{v}(t;1) = v(t;1)$ which is actually the crisp solution given in Chapter 7. Fig. 9.3 represents the interval solution for $\alpha = 0.4$ and 0.8 with $\alpha = 1$ for the first case. Similarly, Fig. 9.4 cites the results for the second case with $\alpha = 1$.

We now vary the fractional order derivative with the same parametric values as considered for Fig. 9.3. As such, Figs. 9.5 and 9.6 present the interval unit step responses for $\lambda = 0.5$ and 0.8 respectively.



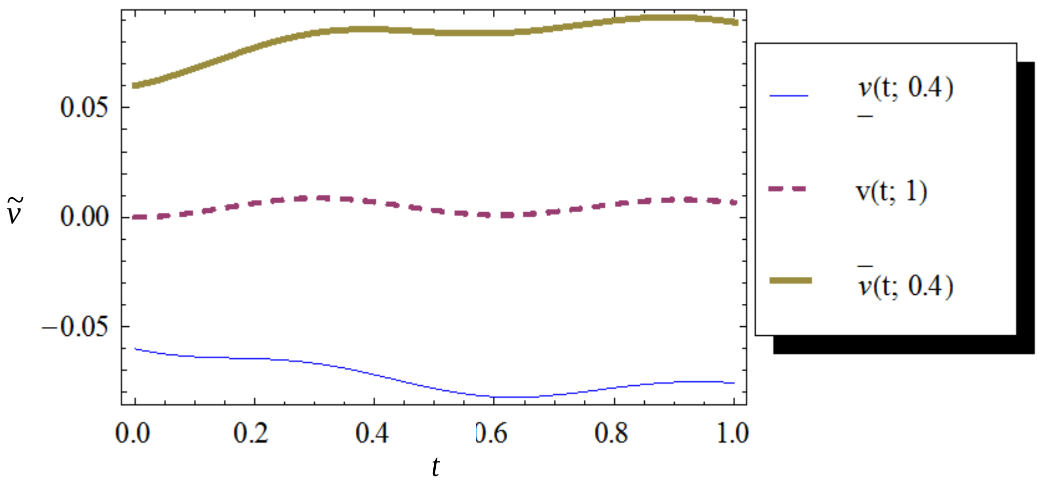
(a)



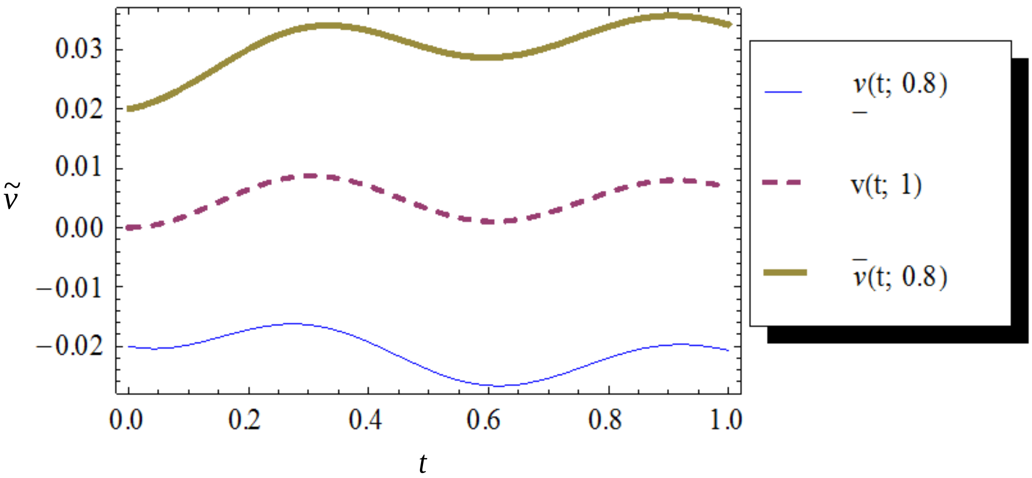
(b)

Fig. 9.3 Interval unit step response for (a) $\alpha = 0.4$, (b) $\alpha = 0.8$ with $\omega = 5$ rad/s, $\eta = 0.5$

$\lambda = 0.2$ and $\alpha = 1$

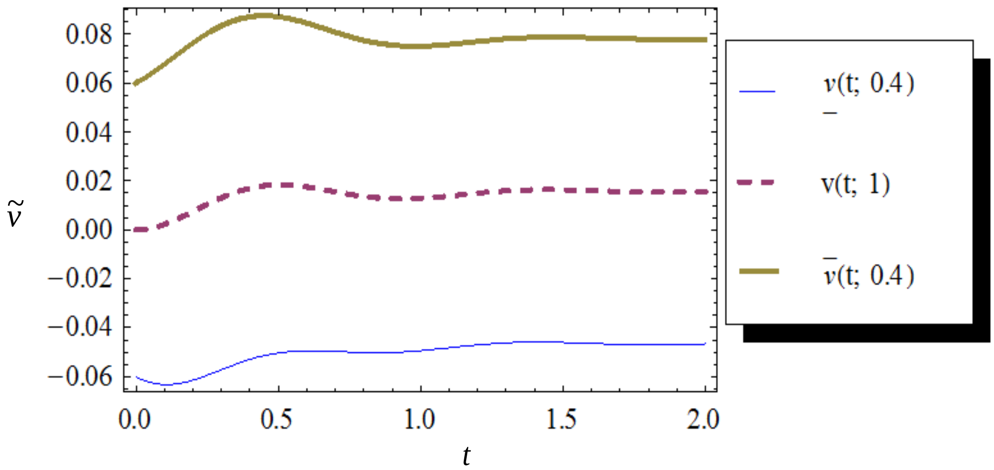


(a)

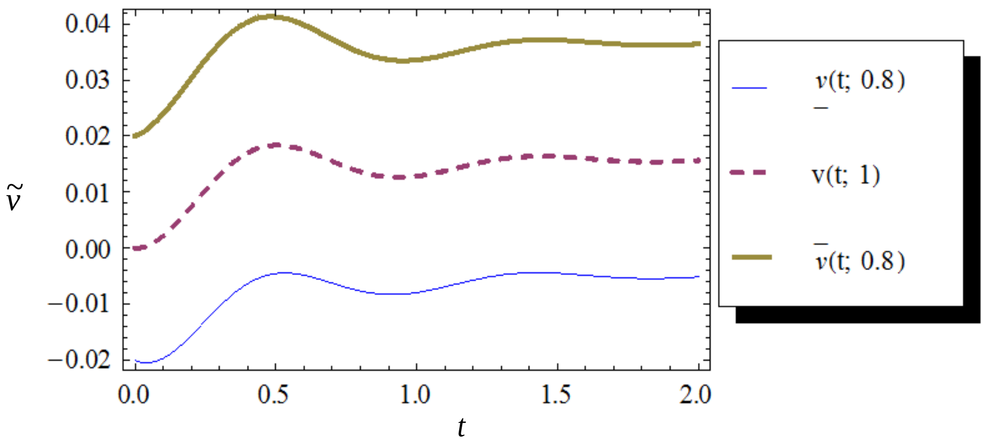


(b)

Fig. 9.4 Interval unit step response for (a) $\alpha = 0.4$, (b) $\alpha = 0.8$ with $\omega = 10$ rad/s, $\eta = 0.05$, $\lambda = 0.5$ and $\alpha = 1$



(a)



(b)

Fig. 9.5 Interval unit step response for (a) $\alpha = 0.4$, (b) $\alpha = 0.8$ with $\omega = 5$ rad/s, $\eta = 0.5$ $\lambda = 0.5$ and $\alpha = 1$

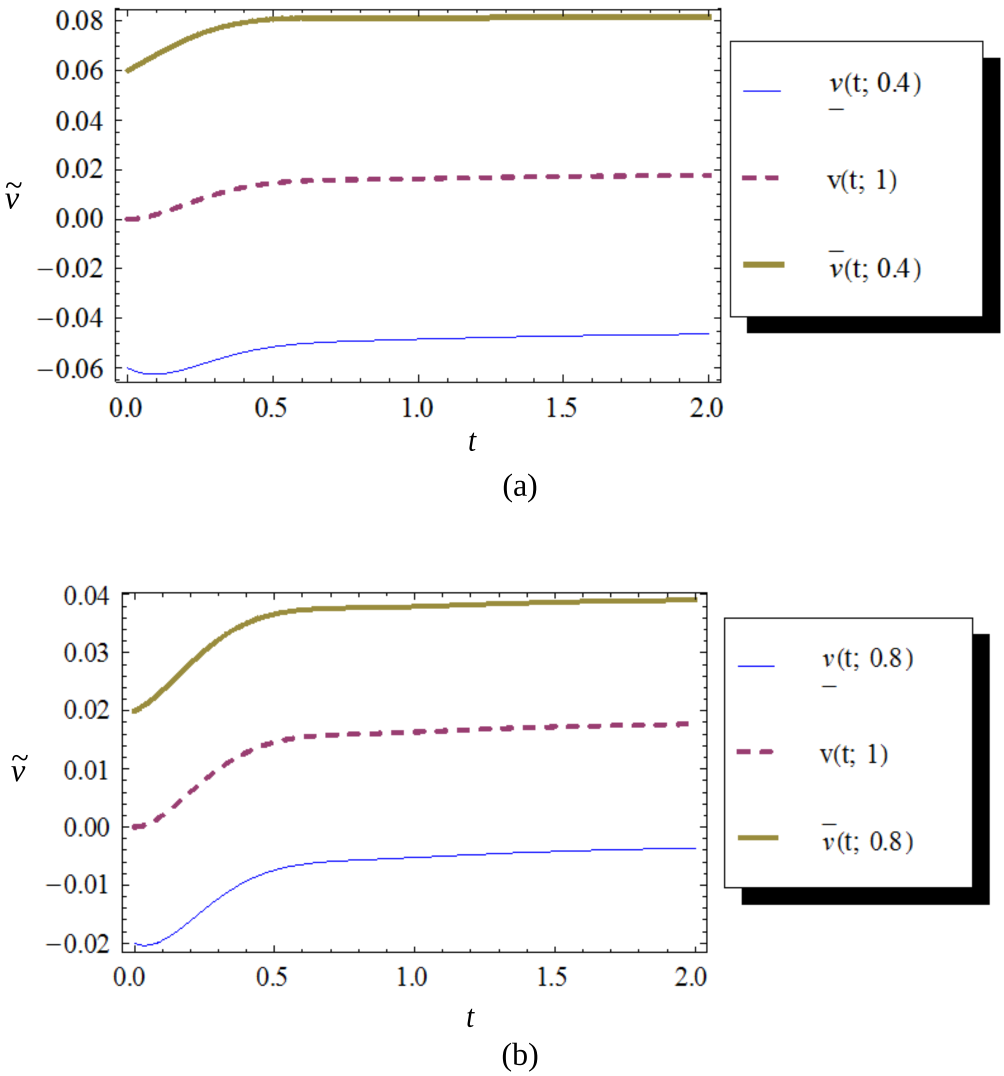


Fig. 9.6 Interval unit step response for (a) $\alpha = 0.4$, (b) $\alpha = 0.8$ with $\omega = 5$ rad/s , $\eta = 0.5$, $\lambda = 0.8$ and $\alpha = 1$

From Figs. 9.4 to 9.6 it can be seen that the uncertain width of the solution gradually decreases by increasing the membership value α. One may also observe from Figs. 9.3, 9.5 and 9.6 that the oscillation of the uncertain bounds of the unit step response gradually decreases by increasing the order of the fractional derivative.

9.3.2. Case studies for fuzzy unit impulse response

Depending on the system parameters viz. natural frequency ω, damping ratio η and arbitrary order fractional derivative λ, four different cases have been considered as follows subjected to unit impulse load.

Case 1 : $\omega = 5$ rad/s , $\eta = 0.5$ and $\lambda = 0.2$

Case 2 : $\omega = 10$ rad/s , $\eta = 0.5$ and $\lambda = 0.5$

Case 3: $\omega = 5$ rad/s , $\eta = 0.05$ and $\lambda = 0.8$

Case 4 : $\omega = 10$ rad/s , $\eta = 0.05$ and $\lambda = 0.2$.

Accordingly, for all the cases from first to four, obtained fuzzy unit impulse responses are shown in Figs. 9.7 to 9.10 respectively. Similar interpretations may be drawn as mentioned in the problem of fuzzy step responses about change of parametric values and the corresponding results.

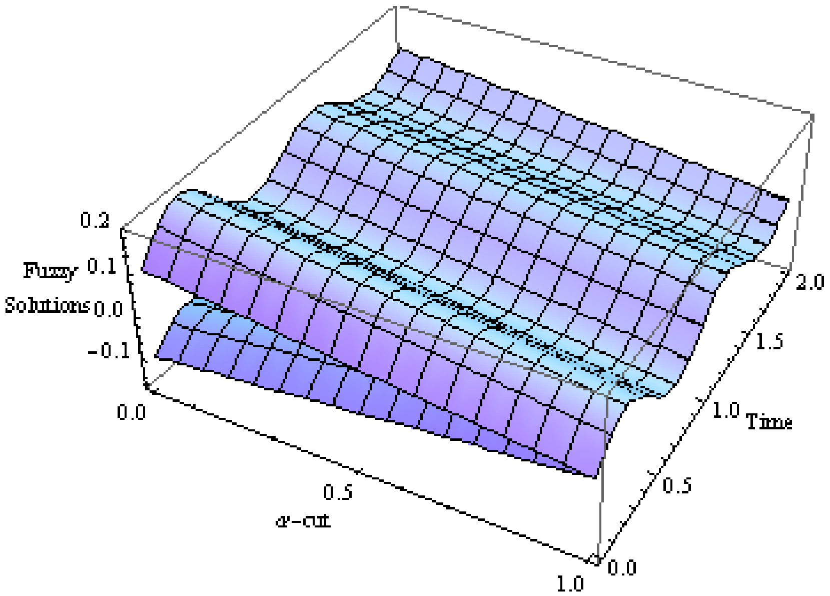


Fig. 9.7 Fuzzy unit impulse response for $\omega = 5$ rad/s , $\eta = 0.5$ and $\lambda = 0.2$ (Case 1)

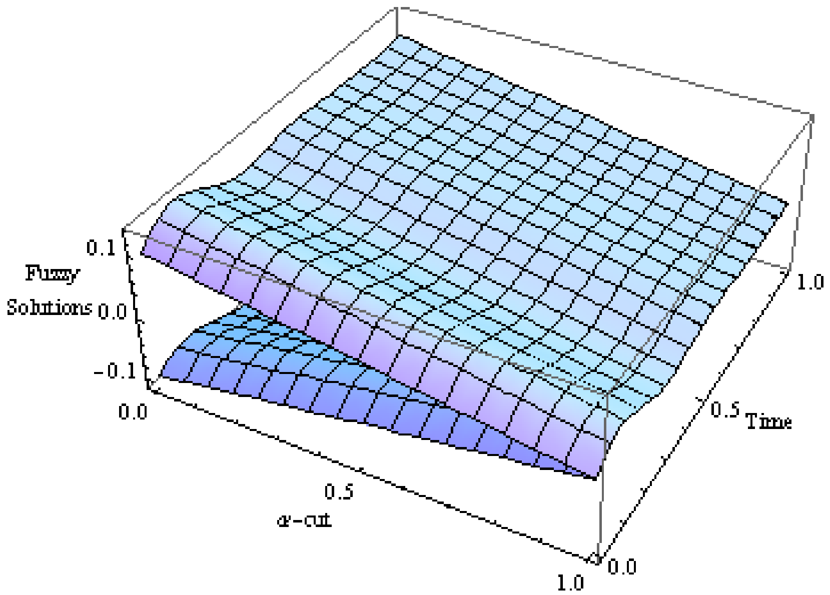


Fig. 9.8 Fuzzy unit impulse response for $\omega = 10$ rad/s , $\eta = 0.5$ and $\lambda = 0.5$ (Case 2)

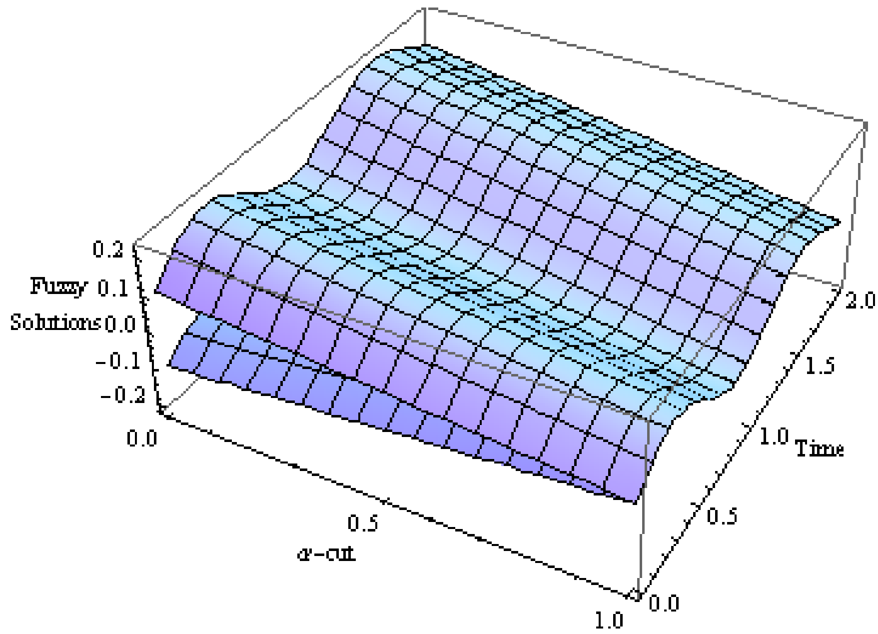


Fig. 9.9 Fuzzy unit impulse response for $\omega = 5$ rad/s , $\eta = 0.05$ and $\lambda = 0.8$ (Case 3)

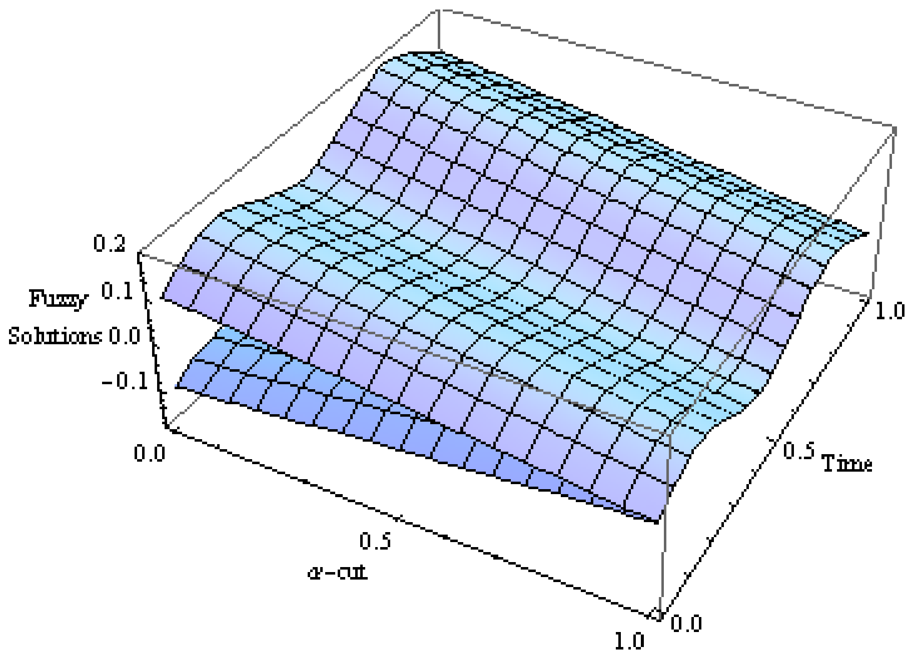


Fig. 9.10 Fuzzy unit impulse response for $\omega = 10$ rad/s , $\eta = 0.05$ and $\lambda = 0.2$ (Case 4)

It is a gigantic task to include here all the results with respect to various parameters involved. For both the problems for $\alpha = 1$, fuzzy initial conditions convert into crisp initial conditions. It is interesting to note that for both the responses (unit step and impulse), lower and upper bounds of the fuzzy solutions are the same for $\alpha = 1$ which are the results obtained in Chapter 7.

Homotopy perturbation method has successfully been applied to obtain the uncertain dynamic responses of fuzzy fractionally damped simply supported beam using double parametric form of fuzzy numbers. Double parametric form of fuzzy numbers converts the corresponding differential equation in crisp form, which is found to be efficient and straight forward to solve.

Chapter 10

Conclusions and Future Directions

Chapter 10

Conclusions and Future Directions

Based on the present work carried out for uncertain static and dynamic analysis of imprecisely defined structures, conclusions are drawn along with the recommendation for future work. In this investigation, fuzzy and interval techniques are used to handle the uncertainties in the geometry, material and load parameters in the static and dynamic problems of structures. The main purpose of this study is to develop computationally efficient methods to solve the above uncertain problems.

In the following paragraphs, conclusions are drawn with respect to the various proposed methods and the application problems mentioned in the previous chapters.

10.1. Conclusions

- Two new methods for fuzzy complex system of linear equations and five methods for fuzzy real system of linear equations with related theorems have been proposed here. Among these methods, Method 2 for fuzzy complex system of linear equations and Method 3 for fuzzy real system of linear equations was found to be more efficient and straight forward. In these two methods, fuzzy addition and subtraction concepts have been incorporated to find the solution. The solution process consists of three phases. First the system is solved by adding the lower and upper bounds of the unknown variable vector and right hand side vector. Then the system is solved for fuzzy subtraction using the related theorem discussed in Chapter 3. Finally fuzzy addition and subtraction solutions are used to get the solution of the original system. Example problems are solved to show the efficacy of the methods.

- Two methods based on single and double parametric form of fuzzy numbers have been developed to get the non-negative solution of fully fuzzy system of linear equations in Chapter 3 as well. Using single parametric form, the $n \times n$ fully fuzzy system of linear equations has been converted to $2n \times 2n$ crisp system of linear equations. On the other hand, double parametric form of fuzzy number converts the $n \times n$ fully fuzzy system of linear equations to a crisp system of same order.

Literature review reveals for non-negative solution that the original fuzzy system are usually converted to a number of crisp systems of linear equations depending upon the type of fuzzy numbers involved in it. Compared to all the existing and single parametric based methods, proposed double parametric form of fuzzy number based method was found to be easy, straight forward and computationally efficient (reduces the computational cost) as it does not change the order of the original system.

- In addition to the above, a linear programming problem approach has also been proposed to find the generalised solution of Fully Fuzzy System of Linear Equations (FFSLE) in Chapter 3. In the considered FFSLE, there is no restriction on the sign of the elements of coefficient matrix as opposed to the need for the other method. The proposed method has also been applicable when the elements of the fuzzy unknown vector are both non-negative and non-positive. In this procedure, first we have to determine the sign of the elements of the solution vector by a proposed Theorem 3.19 of Chapter 3. Thus we may predict the fuzzy solution vector as non-negative, non-positive or both non-negative and non-positive elements. Finally, one may convert the FFSLE to a Linear Programming Problem (LPP) to have the solution following the discussions in Section 3.2.1.3 of Chapter 3.

- Applicability of the developed methods for fuzzy and fully fuzzy system of linear equations has been examined with various type of imprecisely defined structural problems for static analysis. In this respect, bar, beam, truss and rectangular sheet structures have been considered. From the results obtained by the present methods (Chapter 4), we may observe that uncertain spread of the static responses gradually increases by increasing the spread of the uncertain applied force(s). Material and geometric properties have also been considered as uncertain in some of the structural problems. Further it is revealed that the sensitivity of the response changes from node to node depending upon the uncertainties in the material and/or geometric properties. Thus it is important to have a detailed knowledge of the structural response in order to get a complete understanding of the uncertain structural behaviour.

- With regards for dynamical (vibration) problems, an algorithm has been developed by extending the method of Chen et al. (1995) for computing fuzzy eigenvalues and eigenvectors. Spring mass, multistorey shear building and beam structures have been considered for the uncertain dynamic analysis. Uncertainties have been considered in the material and geometric properties. Obtained results are compared in special cases to have validation and efficacy of the proposed algorithm.

- Next, the identification procedure of the uncertain stiffness parameters of multistorey frame structure has been investigated. Bounds of the identified uncertain stiffness are obtained by using a proposed fuzzy based iteration algorithm in Chapter 5. The algorithm systematically modifies and identifies the uncertain structural parameters, viz. the column stiffness for a frame structure. It uses the prior known estimates of uncertain parameters and corresponding vibration characteristics and then the algorithm estimates the bounds of present parameters utilizing the known uncertain dynamic data from some experiments. Numerical procedure is tested by incorporating two sets of data. It may be noted that the accuracy of the results depends upon many factors viz. on the uncertain bound of the experimental data, initial design values of the parameters, the fuzzy computation, norm as defined etc. The present investigation may be a first of its kind to handle the identification procedure for uncertain data.

- To have the completeness for uncertain dynamic problems, responses of fractionally damped discrete and continuous system have been investigated here first with crisp parameters. In this regard, a single degree of freedom spring-mass mechanical system (discrete) with fractional damping of order 1/2 and a viscoelastic beam (continuous) with fractional damping of arbitrary order have been considered. Homotopy Perturbation Method (HPM) has been used to compute the dynamic response of the system, subject to unit step and impulse loads. For spring mass system we may refer to Chapter 6 to note that increasing the value of the damping ratios, the oscillation of the displacements gradually decreases. Similarly we may refer to Chapter 7 for the beam, that increase in the order of the fractional derivative decreases the oscillation. In other words, the beam suffers more oscillations for smaller value of the order of fractional derivative. Similar observations may be made by keeping the order of the fractional derivative constant and varying the damping ratios and we may observe that increasing the value of the damping ratios decrease the oscillations.

- Lastly, uncertain dynamic responses of arbitrary order discrete and continuous systems have been investigated in Chapters 8 and 9 with fuzzy initial conditions. To study this, semi analytical method such as HPM and double parametric form of fuzzy numbers have been used. Uncertain dynamic response of the above problems subject to unit step and impulse loads have been obtained. For discrete system, usual fuzzy based approach is used whereas for continuous system the proposed double parametric based method has been incorporated. The double parametric based method has been found to be efficient as it converts the original fuzzy differential equation to a crisp uncoupled differential equation.

In view of the above, this thesis develops various methods to analyse system of fuzzy or interval system of linear equations for static problems and fuzzy or interval eigenvalue problems as well as semi analytical methods with double parametric form of fuzzy numbers for dynamic problems. These methods have been thoroughly investigated by solving different structural problems. It may however be noted that there are few limitations on these methods which actually will open a new vista for future research and are discussed in the following section.

10.2. Future Directions

Although exhaustive investigations are done related to the titled problems, we may not claim that the proposed methods are most general and full proof for solving any type of fuzzy system of linear equations, fuzzy eigenvalue problems and fuzzy differential equations. As such, there may still be some gaps which are the future direction of research and are incorporated below.

- Developed methodologies can be extended to fully fuzzy differential equations where the coefficients, variables and initial/boundary conditions may all be considered as fuzzy.

- Hybrid type of fuzzy numbers may be introduced in the fuzziness viz. by taking combinations of different fuzzy numbers in the coefficients, variables and initial/boundary conditions.

- Another hybrid type of uncertainties such as combinations of fuzziness and randomness may also be considered. Although this require intelligent methods to handle both fuzzy and random uncertainties in the computation.

- Theoretical concepts regarding the methods for the solution of fuzzy and fully fuzzy system of linear equations and inequations may be investigated in greater details.

- Similarly, theoretical concepts regarding the numerical methods for the solution of eigenvalue problems may also be investigated in greater details.

- Numerical methods along with existence and uniqueness etc. may also be studied for the solution of fuzzy and fully fuzzy system of linear equations.

- Fuzzy and interval computations are themselves complex to handle (as mentioned in various chapters). Accordingly, new concepts about these computations should be developed.

- Present analysis of uncertain static and dynamic problems may further be extended to higher dimensional structural problems.

- In reference to the uncertain parameter identification problem, improvements may also be done by proposing other iterative and system identification methods. Moreover, the problem may also be extended for nonlinear structures.

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2. Behera, D., Chakraverty, S. (2014) Vibration analysis of imprecisely defined multistory shear structure, Applied Soft Computing (to be submitted) (Elsevier).

(c) Journals (Communicated)

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1. Behera, D., Chakraverty, S. (2013) Parameter identification of multi-storey frame structure from uncertain dynamic data, Eleventh International Conference on Recent Advances in Structural Dynamics (RASD-2013), 1^{st}-3^{rd} July, University of Pisa, Italy;
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