Modeling, Identification and Control of Cart-Pole System

A thesis submitted in partial fulfillment

of the requirements for the award of the degree of

Master of Technology

in

Electronics and Communication Engineering

(Specialization: Electronics and Instrumentation)

by

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National Institute of Technology, Rourkela

Odisha- 769008, INDIA

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National Institute of Technology

CERTIFICATE

This is to certify that the thesis titled **"Modeling, Identification and Control of Cart-Pole System"** submitted by **Jyoti Ranjan Pati** bearing roll no. **212EC3378** in partial fulfillment of the requirements for the award of **Master of Technology** in **Electronics and Communication Engineering** with specialization in **"Electronics and Instrumentation Engineering"** during session **2012-2014** at **National Institute of Technology, Rourkela** is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University / Institute for the award of any Degree or Diploma.

Place:

Date:

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Dedicated To My Family, Friends and Teachers

ACKNOWLEDGEMENT

First of all, I would like to express my hearty gratitude towards my guide and philosopher **Dr. Umesh Chandra Pati**, who has been acting as the propelling force for me towards this success. I am sincerely indebted to him for his uninterrupted encouragement, invaluable advice and for motivating me in every aspect of my academic career. His presence and optimistic thoughts have an efficient influence on my present life leading to future. I consider it my good fortune to have an opportunity to work with such a benevolent person.

Next, I want to express my profound respect to Prof. T. K. Dan, Prof. D. P. Acharya, Prof. K. K. Mahapatra, Prof. S. Meher, Prof. A. Swain, Prof. Poonam Singh and Prof. A. K. Sahoo for teaching me and helping me how to learn. They have been the great sources of inspiration to me and I thank them from the bottom of my heart.

I also extend my thanks to all faculty members and staff of the Department of Electronics and Communication Engineering, National Institute of Technology, Rourkela who have encouraged me throughout the course of Master's Degree.

My special thanks to Mr. Subhransu Padhee (PhD Scholar of ECE) and to all my friends and especially to my batch mates for all the thoughtful and mind stimulating discussions we had, which prompted me to think beyond the obvious. I have enjoyed their companionship so much during my stay at NIT, Rourkela.

I am especially indebted to my parents for their love, sacrifice, motivation and support. They are my first teachers after I came to this world and are always with me like friend, philosopher and guide and have set great examples for me how to live, study, and work.

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ABSTRACT

To understand any physical world system, a proper mathematical model is required. With the help of mathematical model, the system can be studied and controlled. There are different ways to develop a mathematical model such as first principle method and system identification method. First principle method is generally used when there is sufficient knowledge of the physical world system but system identification is used when there is no knowledge of the system. System identification is widely used to develop mathematical model of complex, non-linear systems. Cart-pole system is a benchmark problem in control system where the control objective is to balance the inverted pendulum mounted on the cart to a vertical position. This complete system is nonlinear in nature and the mathematical model can't be efficiently calculated using first principle modeling. So system identification method is used to develop the mathematical model of the said system.

This thesis finds out the linearized mathematical model of the said cart-pole system using parametric system identification procedure. Parametric system identification procedure consists of experiment design, model structure selection, parameter estimation and model validation. This thesis also designs linear and non-linear controller for the said system. For linearized model of cart-pole system some of the linear controllers designed are LQR, LQR-Pole-placement-PID, Fuzzy-PID, LQG and H-infinity. For the nonlinear model of cart-pole system, the control techniques discussed are the partial feedback linearization and classical feedback linearization control.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS

FIR---FINITE IMPULSE RESPONSE

ARX-----AUTO REGRESSION EXOGENOUS

ARMAX---AUTO REGRESSION MOVING AVERAGE EXOGENOUS

OE---OUTPUT ERROR

BJ---BOX JENKINS

LQR---LINEAR QUADRATIC REGULATOR

LQG---LINEAR QUADRATIC GAUSSIAN

PID---PROPORTIONAL-INTREGRAL-DERIVATIVE

NN---NEURAL NETWORK

ANN---ARTIFICIAL NEURAL NETWORK

MSE---MEAN SQUARE ERROR

FPE---FINAL PREDICTION ERROR

PRBS---PSEUDO RANDOM BINARY SEQUENCE

CHAPTER1

INTRODUCTION

- Overview
- Literature Review
- Motivation
- Objectives
- Thesis Organization

CHAPTER 1

INTRODUCTION

1.1 OVERVIEW

Inferring models from observation and analyzing their properties is really what the science is all about. Models (Hypothesis, laws of nature, paradigms) have basic feature to link the observations together into some pattern. Modeling is the act of deriving system in mathematical form. In control system modeling of a system is the act of finding out the system's state space equation or system's transfer function. System identification deals with the problems of building mathematical models of dynamical systems based on the observed data from the systems. A system is an object in which variables of different kinds interact and produce observable signals called outputs. A system is also defined as a unit that takes some input, does some operation and produces some output. External signals that can be manipulated by observer are called inputs and others are called disturbances. Dynamic means current output value depends not only on the current external stimuli but also on their earlier values. Process of changing the output according to our requirement is called controlling. Systems designed to control the process outputs are called controllers.

A Cart-Pole or Cart-Inverted pendulum system is a classical problem of nonlinear control. It is a nonlinear, unstable, non-minimum and under actuated system. Though inverted pendulum is a well-studied control problem, but its identification and control is always a topic of interest.

System identification is a procedure where the mathematical model of the system is developed with the help of input, output and disturbance data. From the input and output of inverted pendulum system, the mathematical model is developed. A class of linear and nonlinear models is fitted with the input-output data with the help of a criterion. The model developed is then validated, so the developed model will represent the same data as the real model will represent.

1.2 LITERATURE REVIEW

AARoshdyet.al, presents mathematical modeling of the Inverted Pendulum system from the basic 1st principle and the design of the controller for the modeled system using simple Pole-placement method. It considers two random dominant poles and uses some separation principle to avoid the effect of other system poles. Finally it uses Ackermann's formula to find the feedback gain matrix (K) to place the system poles at desired location [1].

$$K = \begin{bmatrix} 0 & 0 & . & . & . & 0 & 1 \end{bmatrix} P_C^{-1} q(\lambda)$$

Where, P_C is the controllability matrix and

$$q(\lambda) = \left(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\right)\left(s^2 + as + b\right)$$

Where, (a, b) are the two pole locations chosen.

A K Yadav et.al, derives the mathematical model for the Inverted-pendulum by using general physical laws and generates PID and Fuzzy controllers to control the system. It manually adjusts the control parameters of the PID controller. When fuzzy controller is designed it takes the fuzzy controller instead of PID controller [2].

J K Hedrick et.al, represents the ways of transforming the original system models into simpler models to which feedback linearization method can be applied for stabilization. It is completely different from the Jacobian (conventional) linearization method, as it's the complete transformation not assumption. It also explains the lie derivative to find out the Lyapunov equation which helps in finding out the degree of freedom of the system [3].

Russ Tedrake et.al, uses the task space linearization to find collocated and non-collocated partial feedback linearization to design controller for the transformed model and it also solves the swing up control strategy by considering the total energy at the stable and unstable equilibrium points. Here, it uses an extra PD controller for better stabilization [4].

Iraj Hassanzadeh et.al, explains the optimum input output feedback linearization cascade controller that uses the genetic algorithm to design the controller. Due to non-minimum nature only the input output feedback linearization gives internal un-stability, thus it uses a cascade controller that can be a PD controller to stabilize. It adjusts the parameters of the PD controller by genetic algorithm [5].

1.3 MOTIVATION

The motivation for controlling a cart-pole system is immensely due to its wide application in critical balance problems. From general life style to critical control mechanism, the need ofstabilization of inverted pendulum is always there. Like balancing a broom stick on the fingertip to vehicle balance (seg-ways, ice-skating), artificial robot, helicopter, aero-plane, missile and rocket propulsion everywhere this balancing concept is widely adopted. Because of its high non-linearity and un-stability, it has been a bench mark from classical to modern control problem. From past a century intensive research is going on this topic. A white box model for its non-linear part can be found from the physical laws. But to verify its correctness it's always a requirement to identify it by taking it as a black box and a well-known identification method as the reference model. If the output of that black box is nearly comparable to that reference identification model then our generated model is utilizable. Classically based on some assumption the non-linear model was linearized, which in turn was not so effective. So, in modern era the non-linear model is transformed into linear model with some transformation principle without any assumption. Classically controllers were generated based on those assumed linear model but now a days the controllers are designed for the transformed linearized model as well as for the non-linear model directly.

1.4 OBJECTIVES

The objectives of this dissertation are

i. Non-linear modeling of the cart-pole system and then linearizing this non-linear model by both assumption and transformation linearizing methods.
ii. Identification of both the non-linear and linear models of the cart-pole system.
iii. Stabilization of the assumed and transformed linearized models by using different control techniques.
iv. Stabilization of the non-linear model.

1.5 THESIS ORGANIZATION

This thesis is well organized into five chapters including the introductory chapter. The coming four chapters are

Chapter2: Modeling of the Cart-Pole system

The Cart-pole system is mathematically derived from the physical laws of nature using swing up methodology. For modeling parameters of the system must be known. The non-linear equations can be linearized by using some assumption or by using Jacobian methodology.

Chapter 3: Identification of Cart-pole system

This chapter describes identification of the modeled cart-pole system. The linearized model is identified by linear regression methods like ARX, ARMAX, OE and BJ. The non-linear model is identified by ANN. Here, the non-linear model is identified by two main types of network architecture; such as: Elman and feed-forward. The results obtained are compared on the basis of fit %, MSE and FPE for choosing best identification model for the linear model and results are compared on the basis of MSE for choosing better identification method for non-linear model.

Chapter 4: Controller Design

Controlling this type of nonlinear and unstable system is really a challenging task. Hence, so many techniques have been used for control application. This thesis includes LQR, LQR-Pole-placement-PID, Fuzzy-PID, LQG for the stabilization of the linearized model, also H-infinity method has also been used which is also a future work for this thesis. Techniques like Classical Feed-back Linearization and Partial Feed-back Linearization are used for controlling the non-linear model.

Chapter 5: Conclusion

This part represents the overall work and research in brief. It also represents the better identification model and control technique. It also focuses on the future working areas which can be continued for identification and control.

CHAPTER 2

MODELING OF THE CART-POLE SYSTEM

- Cart-Pole system
- Calculation of Equation of Motion

CHAPTER 2

MODELLING OF THE CART-POLE SYSTEM

2.1 Cart-Pole System

Cart-Pole (inverted pendulum) system is one of the classical examples of highly nonlinear, unstable, under-actuated dynamic system. Figure 1 illustrates the schematic diagram of cart-pole system.

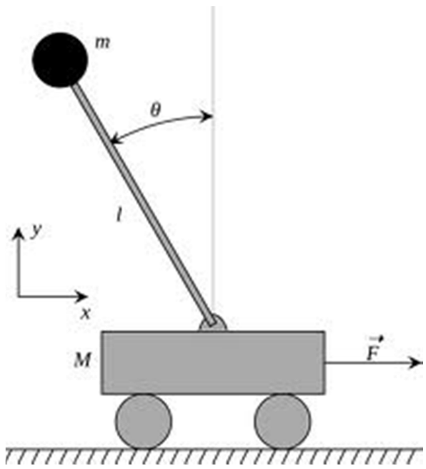


Fig.1: Cart-Pole System

The system consists of a inverted pendulum mounted on the top of a motorized cart. The pendulum is unstable without control and it will simply fall off if the cart is not moved to balance it. The pendulum can be balanced by applying a force to the cart to which the pendulum is attached. A force is given to the cart so that the cart moves in a particular direction. While moving the main objective of the cart is to balance the pole in a specific position. In this particular system, there are two equilibrium points such as the stable equilibrium point (pole position) and the unstable equilibrium point (inverted vertical position). So this system is inherently nonlinear and unstable system due to the existence of multiple equilibrium points.

Here, the cart-pole system has 2 Degree of Freedom (DOF), i.e. linear motion of cart along X-axis and rotational motion of the pendulum on the X-Y plane. Some of the real life applications of cart-pole system are

1. The altitude control of a booster rocket during takeoff
2. Wheel chair like vehicles
3. Seg-ways, Ice-skating
4. Balancing a cart in circular wheel game
5. Helicopter and Aero-plane like aeronautic balancing.

A two dimensional surface of pendulum is considered where the pendulum moves in vertical plane. The control input is the applied force which enables the cart to move and the outputs considered are the horizontal position of the cart and angle of the pendulum made with the vertical line. The pendulum should return to its state of equilibrium (upward vertical position) after being disturbed within stipulated time and it should not move more than the prescribed angle in vertical direction. So, both the cart position and pendulum angle must be monitored.

The known parameters of the laboratory setup cart-pole system are given in the Table I. These values will be used in the future modeling equations.

Table I: Parameters of the Mechanical Setup

Parameter	**Symbol**	**Value**	**Unit**
Mass of cart	M	2	Kg
Mass of pendulum	m	0.2	Kg
Length of pendulum	l	0.5	m
Moment of inertia	I	0.099	Kgm^2
Cart friction Coefficient	b	0	Ns/m
Acceleration due to gravity	g	9.81	m/s^2
Gain of actuator	K	1	-

The control objectives are

1. Settling time should be less than 5 sec
2. Rise time should be less than 0.5 sec

3. Pendulum angle can never be more than 15 degree (0.265 rad) from the vertical
4. Steady state error should be less than 2%
5. Cart should be stabilized with in 10cm step from its desired position.

2.2 Calculation of Equation of Motion

Inverted pendulum has its pivot in space and its centre of mass is above its pivot point. From general physical laws to keep the inverted-pendulum in its equilibrium position the torque acting upon it must be equal to moment of inertia times of angular acceleration.

So, $I\ddot{\theta} = \tau_{net}$ (1)

$$I\ddot{\theta} = -Nl\cos\theta - Pl\sin\theta \quad (2)$$

Where, the N and P are the two reaction forces along vertical (normal reaction) and horizontal.

$$N = m(\ddot{x} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta) \quad (3)$$

and

$$P = m(l\dot{\theta}^2\cos\theta + l\ddot{\theta}\sin\theta + g) \quad (4)$$

$$\begin{bmatrix} \because N = m\ddot{x}_P \\ P = m\ddot{y}_P + mg \\ and \\ x_P = x + l\sin\theta, \\ y_P = -l\cos\theta \end{bmatrix}$$

Putting the values of N and P in Eq. (1), the result obtained is

$$(I + ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta \quad (5)$$

The Lagrange equation [10] is given by

$$L = T - V \quad (6)$$

The system's kinetic energy(T) is given by

$$T = T_C + T_P \quad (7)$$

Where, T_C is the kinetic energy of the cart and is given by

$$T_C = \frac{1}{2} m v_1^{\,2}$$

Where,

v_1 is the velocity of cart and

$$v_1 = \dot{x}$$

$$T_C = \frac{1}{2} M \dot{x}^2$$

And T_P is the kinetic energy of the pendulum and is given by

$$T_P = \frac{1}{2} m v_2^2$$

Where,

v_2 is the velocity of the pendulum and

$$v_2 = \left([\frac{d}{dt}(x - l\sin\theta)]^2 + [\frac{d}{dt} l\cos\theta]^2 \right)^{\frac{1}{2}}$$

$$T_P = \frac{1}{2} m \left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2 \right]$$

So, $T = \frac{1}{2}\left[M\dot{x}^2 + m\left((\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2 \right) \right]$ (8)

The system's potential energy (V) is given by

$$V = V_C + V_P \qquad (9)$$

Where, V_C is the potential energy of the cart and is given by

$$V_C = 0$$

And V_P is the potential energy of the pendulum and is given by

$$V_P = mgl\cos\theta$$

So, $V = mgl\cos\theta$ (10)

Putting Eq. (8) and Eq. (10) in Eq. (6), the Lagrange equation is

$$L=\frac{1}{2}Mv_1{}^2+\frac{1}{2}mv_2{}^2-mgl\cos\theta \tag{11}$$

$$L=\frac{1}{2}(M+m)\dot{x}^2-ml\dot{x}\dot{\theta}\cos\theta+\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta \tag{12}$$

The equation of motion [10] is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=F$$

And $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$ (13)

Putting Eq. (12) in Eq. (13), the result obtained is

$$F=(M+m)\ddot{x}-ml\ddot{\theta}\cos\theta+ml\dot{\theta}^2\sin\theta$$

$$l\ddot{\theta}-g\sin\theta=\ddot{x}\cos\theta \tag{14}$$

From free body diagram along horizontal direction

$$F=M\ddot{x}+N \tag{15}$$

On solving Eq. (5), Eq. (14) and Eq. (15)

$$\ddot{x}=\frac{(I+ml^2)(F+ml\dot{\theta}^2\sin\theta)-gm^2l^2\sin\theta\cos\theta}{I(M+m)+ml^2(M+m\sin^2\theta)} \tag{16}$$

$$\ddot{\theta}=-\frac{ml[F\cos\theta+ml\dot{\theta}^2\sin\theta.\cos\theta-(M+m)g\sin\theta]}{I(M+m)+ml^2(M+m\sin^2\theta)} \tag{17}$$

The standard form of equation for a non-linear system [7] can be given as

$$\frac{dz}{dt}=f(z,u,t) \tag{18}$$

Where, $z=\begin{bmatrix}z_1 & z_2 & z_3 & z_4\end{bmatrix}^T$

$z_1=x$, $z_2=\dot{x}=\dot{z}_1$, $z_3=\theta$ and $z_4=\dot{\theta}=\dot{z}_3$

So,

$$\frac{dz}{dt} = \begin{bmatrix} z_2 \\ \dfrac{(I+ml^2)(F+ml\dot{\theta}^2\sin\theta) - gm^2l^2\sin\theta\cos\theta}{I(M+m)+ml^2(M+m\sin^2\theta)} \\ z_4 \\ -\dfrac{ml[F\cos\theta + ml\dot{\theta}^2\sin\theta.\cos\theta - (M+m)g\sin\theta]}{I(M+m)+ml^2(M+m\sin^2\theta)} \end{bmatrix} \tag{19}$$

These equations are nonlinear in nature. For analysis of linear system these equations need to be linearized. It can be done by two methods as explained below:

2.2.1 By Assumption

For this, let $\theta \cong 0$,

So, $\dot{\theta} = 0$, $\sin\theta \cong \theta$ and $\cos\theta \cong 1$

Hence, the linearized model equations are

$$\ddot{\theta}(lin) = -\frac{mlF + (M+m)g\theta}{I(M+m)+Mml^2} \tag{20}$$

$$\ddot{x}(lin) = \frac{(I+ml^2)F + gm^2l^2\theta}{I(M+m)+Mml^2} \tag{21}$$

The state-space model is given by

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{gm^2l^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{g(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ -\dfrac{ml}{I(M+m)+Mml^2} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{22}$$

By taking Laplace transformation of these two equations and putting the assumed values,

$$\frac{X(s)}{F(S)} = \frac{0.14(s^2 - 7.01)}{0.298s^2(s^2 - 7.2)} \tag{23}$$

On approximation, it becomes

$$\frac{X(s)}{F(S)} = \frac{0.4698}{s^2} \tag{24}$$

$G_{P_1} = \frac{a_1}{s^2}$, It gives cart-position.

Similarly, Pendulum angle is

$$\frac{\theta(s)}{F(S)} = \frac{0.3356}{s^2 - 7.2423} \tag{25}$$

$$G_{P_2} = \frac{a_2}{s^2 - b^2}$$

2.2.2 Jacobian Methodology

The linearized form of the Eq. (20) can be of the form [7]

$$\frac{d}{dt}\delta z = J_{=z}\left(z_0, u_0\right)\delta z + J_{=u}\left(z_0, u_0\right)\delta u \tag{26}$$

The reference state is defined as the pendulum stationary and upright with no force.

So, $z_0 = 0$ and $u_0 = 0$.

J stands for Jacobian matrix. The general form of the Jacobian matrix is

$$J_x = \begin{bmatrix} \frac{\partial J_1}{\partial x_1} & . & . & . & \frac{\partial J_1}{\partial x_n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial J_m}{\partial x_1} & . & . & . & \frac{\partial J_m}{\partial x_n} \end{bmatrix}_{m\times n} \tag{27}$$

The elements of the 1st column are all zeros.

The 1st element of the 2nd column is unity and rest elements are all zeros.

The elements of the 3rd column of eq.(22) are

$$J_{=z}\left(z_0,u_0\right) = \left.\frac{\partial f_i}{\partial z_3}\right|_{(z_0,u_0)} \tag{28}$$

1st and 3rd function are not directly related to z_3, these contribution are identically zero.

The second and fourth terms are respectively,

$$\left.\frac{\partial f_2}{\partial z_3}\right|_{(z_0,u_0)} = \frac{-mg}{M} \text{ and } \left.\frac{\partial f_4}{\partial z_3}\right|_{(z_0,u_0)} = \frac{(M+m)g}{Ml}$$

The 3rd element of the 4th column is only zero and rest three elements are zeros.

$$J_{=z}\left(z_0,u_0\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(M+m)g}{Ml} & 0 \end{bmatrix} \tag{29}$$

Deriving the non-linear terms with respect to u, we have

$$J_{=u}\left(z_0,u_0\right)=\begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{Ml}\end{bmatrix} \tag{30}$$

The Jacobian linearization around the upright position is given by

$A_{pendant}= J_{=z}\left(z_0,u_0\right)$ and $B_{pendant}= J_{=z}\left(z_0,u_0\right)$

So,

$$\frac{d}{dt}\delta z=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{\left(M+m\right)g}{Ml} & 0\end{bmatrix}\delta z+\begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{Ml}\end{bmatrix}\delta u \tag{31}$$

There is only simple deviation between two linear modeling of the cart-pole system. Here, the model obtained from the assumed method one will be used for future research work as it takes

$\theta \cong 0$ which is the ideal case.

2.3 Summary

This chapter gives a complete mathematical model of cart-pole system and presents the transfer function and state space model of the same using different techniques.

CHAPTER 3

IDENTIFICATION OF THE CART-POLE SYSTEM

- System Identification
- Identification of Linear Cart-pole system
- Identification of Non-linear Cart-pole system

CHAPTER 3

IDENTIFICATION OF THE CART-POLE SYSTEM

3.1 System Identification

Models that describe the relationship among the system variables in terms of mathematical expressions like difference or differential equations are called mathematical or analytical models, which can be characterized by a no. of adjectives (linear or nonlinear,time continuous or time discrete,deterministic or stochastic,lumped or distributed).Mathematical models can be developed through two ways or routes or by the combination of these two routes.

One route is to split up the system figuratively into subsystems.The properties of these subsystems can be understood by from previous experience. The subsystems are combined mathematically and the model of the whole system is obtained. The route is called modeling and does not need any experimentation of the actual system. The procedure for modeling is application oriented and dependent and basic techniques involve structuring of the process into block diagram with blocks consisting of simple elements. Systems are reconstructed from these basic blocks.The other route to both mathematical and graphical models is directly based on the experiments

Input and output signals from the system are recorded and data analysis is carried out to infer a model. The process of inferring the model is called System Identification. To use the dynamic systems fruitfully they need to be controlled properly. Complex, non-linear systems modeled using the basic first principle modeling approach can't be directly used as the mathematical model for controller design. The mathematical model of such kind of systems is identified by using system identification procedure. This thesis finds out the linearized mathematical model of inverted pendulum system using parametric system identification procedure. Parametric system identification procedure consists of experiment design, model structure selection, parameter estimation and model validation.

3.1.1 System Identification Procedure

Construction of a model involves three entities

a. The data ,
b. A set of candidate models ,and
c. Determination of best model.

THE DATA

User determines which signals to be measured and when to be measured, based on which the input signals is chosen.

SET OF CANDIDATE MODELS

Sometimes careful modeling can lead to generation of model set. The derivation of models to describe the behavior of a system from physical laws(Newton laws, Conservation of mass, Kirchoff's laws etc.) and other well established relations by assumption of ideal components is called White box modeling .In other cases standard linear models are employed without reference to the physical background are called Black box modeling .Combination of these two types of models is called Gray box model.

DETERMINATION OF BEST MODEL

Rules determine candidate models from the data and decide the best model by the assessment of model quality based on the model performance while producing the measured data.

A model can't be accepted as a final and true description of the model, but it can be considered as a better description of certain aspects of the system.

3.1.2 Steps involved in System Identification

Experimental data (input, output and disturbance) and proper knowledge about the system can be used to determine the accurate mathematical model of dynamic systems. This is the primary objective of system identification. A model structure can be assumed for the system from the output process signals. Approximation error criteria withsome axiomatic constraint can be defined based on the model. Figure 2 shows the procedure for system identification.

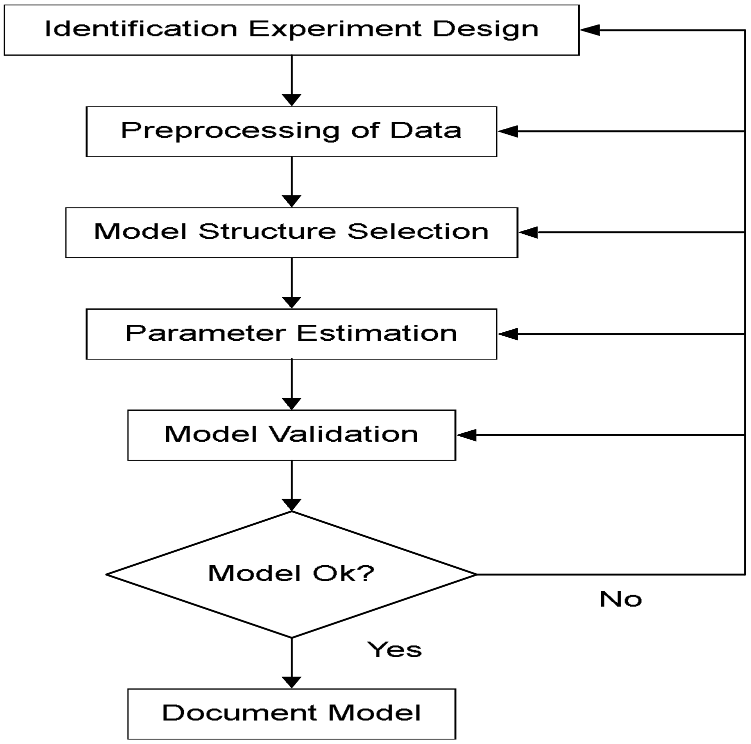


Fig.2: Steps of system identification

Design of Identification Experiment

To get a proper model of any unknown system, input and corresponding output data of the system are required. There are many signals which are considered as input of the system identification experiment, but one of the best input is PRBS, because it is persistently exciting in nature.Maximum length PRBS resembles with white noise and the autocorrelation functionof PRBS is same as the autocorrelation of white noise and it has large energy content and large range of frequencies. A survey of different perturbation signal used in system identification experiment is reported in [8, 9]. The PRBS signal and its auto correlated signal are shown in Figure 3.

Preprocessing of Data

Plant is generally a black box in which different processing operations take place and some unknown output is produced depending upon the input. In the system, some mathematical derivation and calculations are being done like data acquisition. It includes various operations like filtering, scaling, signal conditioning etc. Depending on the input and biases some output data is produced. The data can be anything.

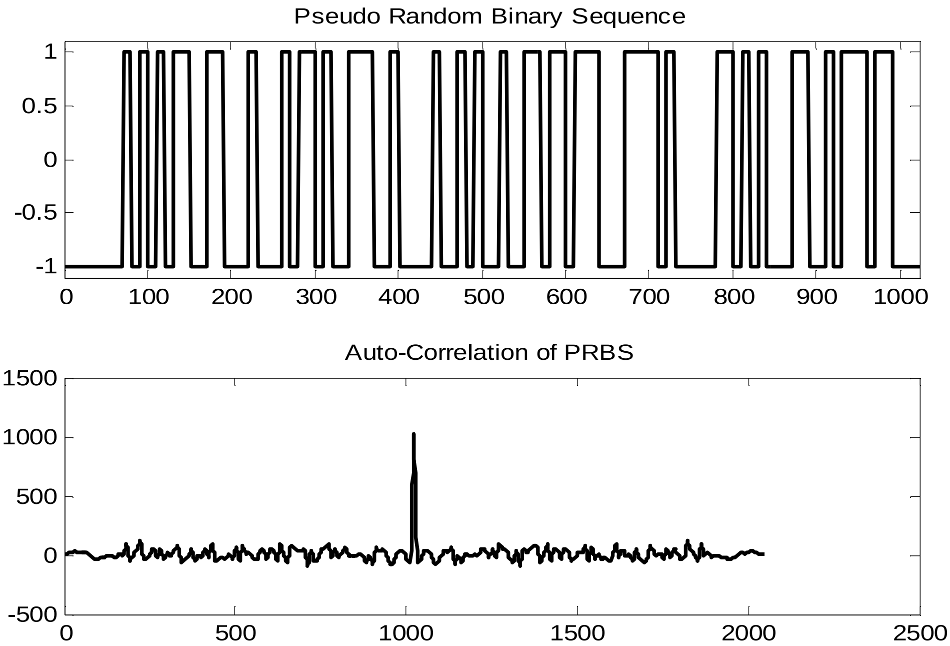


Fig. 3: Pseudo Random Binary Sequence

Model Structure Selection

The resultant output data is collected after PRBS is applied to the system. Pre-processing of signal is carried out to remove noises. In parametric identification, a model structure is selected and the input-output data is fitted in the model structure. Some of the linear parametric model structures [27] are FIR, ARX, ARMAX, OE model and BJ model.

Parameter Estimation

Parameter estimation is carried out to find the parameters of the model structure. Covariance method and least square method are widely used for parameter estimation of time domain data. Parameter estimation method can be online or offline. One of the offline estimation methods is least square method. According to the principle of least square method, the unknown parameter of the model should be chosen in such a way that the sum of squares of the difference between the actual values and computed values multiplied by number of measures the degree of precision is minimum.

Model Validation

Once the model is estimated from the input-output data, the estimated model is validated through direct validation and cross-validation techniques. The quality of the model is evaluated by fit %.

3.2 Identification of Linear Cart-pole system

The linear cart-pole system can be identified by using the linear regression techniques.The linear regression methods used in identification are FIR, ARX, ARMAX, OE and BJ.

In general the input-output relation of a LTI model can be expressed as

$$y(k) = G(q,\theta)u(k) + H(q,\theta)\varepsilon(k) \tag{32}$$

The generalized description of transfer function model can be represented as

$$A(q)y(k) = \frac{B(q)}{F(q)}u(k) + \frac{C(q)}{D(q)}\varepsilon(k) \tag{33}$$

3.2.1 Auto Regressive Exogenous Input (ARX)

ARX model is a simple linear differential equation that describes the relational status of output according to input of the process with the condition that the noise and input /output spectrums have the same characteristic dynamics. The equation is given by

$$y(q) = \frac{B}{A}x(q) + \frac{1}{A}e(q) \tag{34}$$

The input output relation is given by the transfer function block $\frac{B}{A}$.

3.2.2 Auto Regressive Moving Average Exogenous Input (ARMAX)

ARMAX model contains an extra parameter term as compared to ARX model in the error term, so it can be expected that in some case it will give better accuracy as compared to ARX model. The equation is given by

$$y(q) = \frac{B}{A} x(q) + \frac{C}{A} e(q) \tag{35}$$

3.2.3 Output Error Model (OE)

It is different than the above two models as it does not include the noise spectrum coefficient, because here white noise is taken care. The model equation is something different and is given by

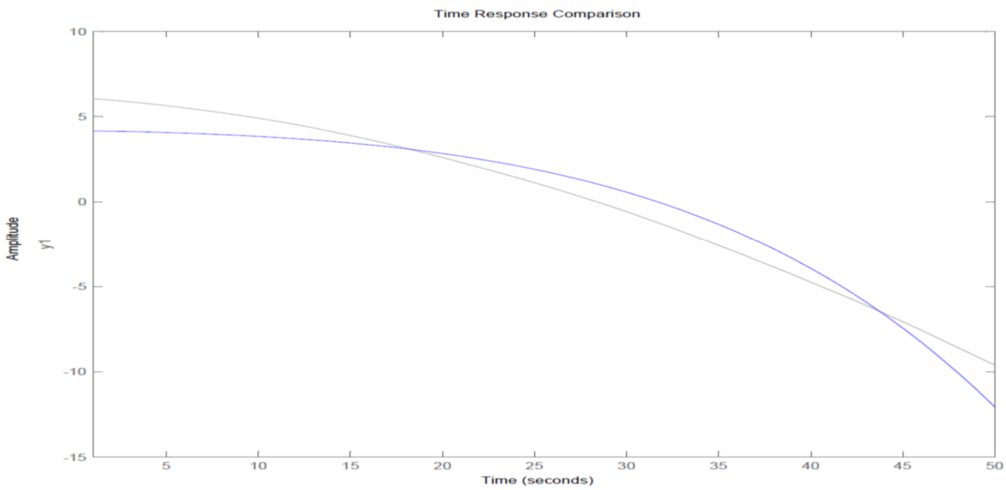
$$y(q) = \frac{B}{F} x(q) + e(q) \tag{36}$$

3.2.4 Box Jenkins Model (BJ)

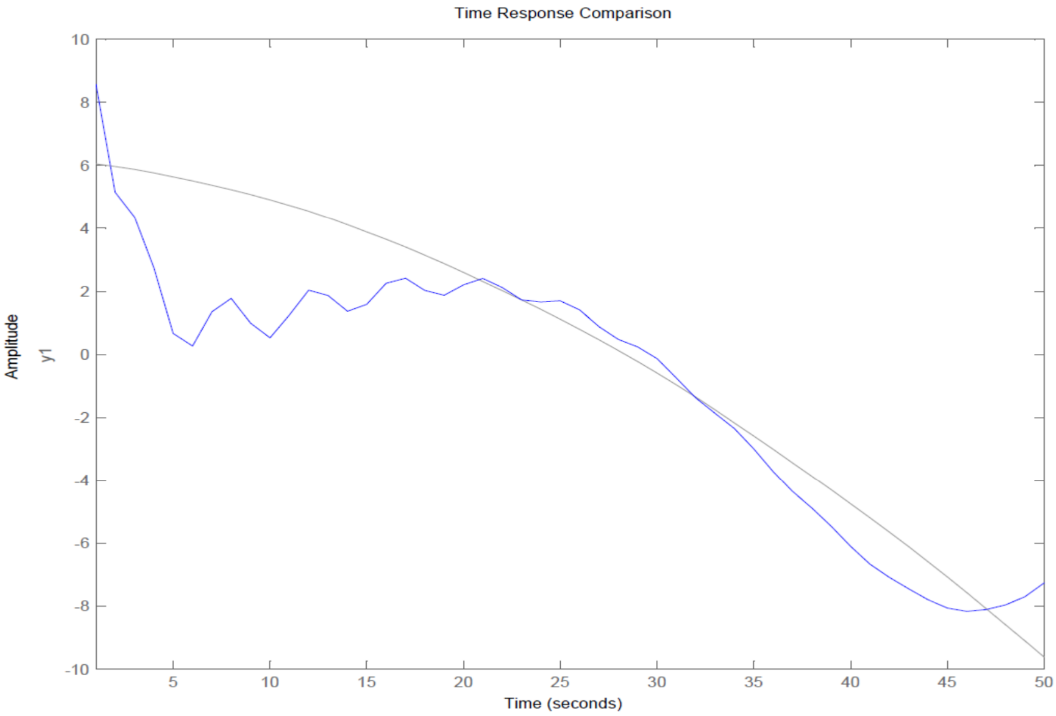
This model has separate transfer functions for the input output spectrum and noise, which is different from the ARX and ARMAX model (in having the same denominator). The equation of this model is

$$y(q) = \frac{B}{F} x(q) + \frac{C}{D} e(q) \tag{37}$$

These methods can be applied either for open loop identification or closed loop identification after the design of controller. Here, these methods are taken for the open loop identification.
The plots of different methods used for identification of pendulum angle are shown in Figure4.



(a)


Time Response Comparison
Amplitude
y1
Time (seconds)
10
8
6
4
2
0
-2
-4
-6
-8
-10
5
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20
25
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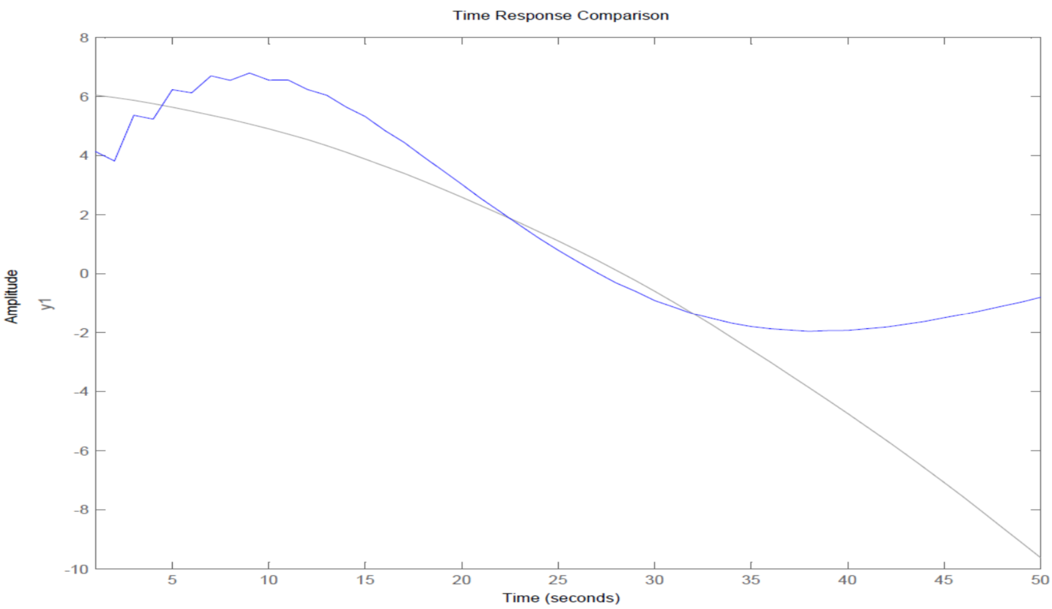
50

(b)


Time Response Comparison
Amplitude
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Time (seconds)
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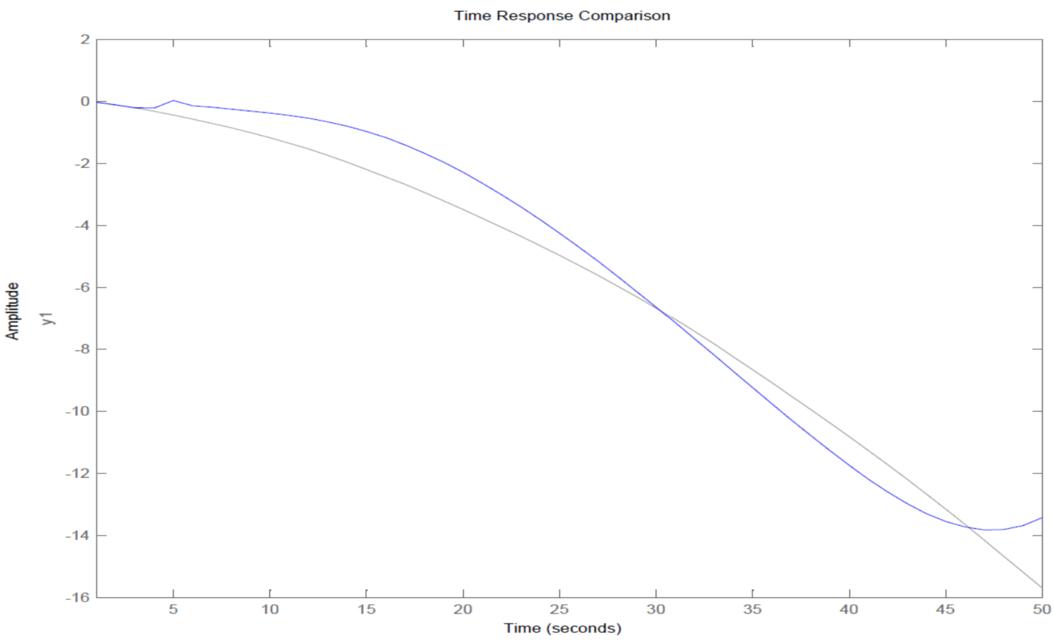
(c)



(d)

Fig 4: Identification plots of pendulum angle of linear cart-pole system

(a) ARX (b) ARMAX (c) OE and (d) BJ

These methods are compared on the basis of their Fit % , MSE and FPE which is shown in TableII.

TableII: Comparison of different regression methods

Type	**Parameters**	**Transfer function**	**MSE**	**FPE**	**FIT %**
ARX	[2 1 4]	$\frac{-0.001367s}{s^2-2.066s+1.068}$	0.00092	0.0019	76.70
ARMAX	[5 3 2 1]	$\frac{-0.0001814s^4+0.0007501s^3+0.0007681s^2}{s^5-2.922s^4+3.841s^3-3.472s^2+2.086s-0.5221}$	1.058e-05	1.614e-05	57.65
OE	[3 3 1]	$\frac{-0.005015s^2+0.005939s-0.007802}{s^3-1.118s^2-0.5799s+0.7206}$	0.001363	0.0015	37.09
BJ	[5 4 3 3 1]	$\frac{-0.001087s^4-0.0009s^3+0.0002s^2-0.0007s-0.001}{s^5-1.82s^4+0.869s^3+0.032s^2}$	5.224e-06	7.775e-06	82.06

From the results it can be concluded that BJ is the best identification among all and in between ARX and ARMAX there is little bit of confusion. ARX has better fit % but the MSE and FPE are larger. So, according to requirement ARX or ARMAX can be chosen.

3.3 Identification of Non-linear Cart-pole system

NARX and NARMAX non-linear model structures can be used in the identification of real world dynamic problems, but it makes the computation complex and time consuming. To overcome this problem an alternative like neural network (NN) can be applied to handle nonlinearity. It imitates the functionality of human brain and applies the power of human brain to a wide variety of applications like signal processing, pattern recognition, system identification and control. When it consists of neurons, named as artificial neural network (ANN).

The advantages of the ANN are

1. The main advantage of an ANN is that it can be trained to perform any desired function by adjusting the weights (of the connections) between two layers or elements. They can be trained as then transferred into some process where adaptive learning takes place.

2. ANN has memory as the weights in the neurons. They can be adjusted according to the requirement.

3. ANN consists of parallel elements operating simultaneously, which increases the speed of calculation as compared to the sequential processing.

The disadvantages of the ANN are

1. ANN operates as black boxes as the rule of operation are completely unknown. So, it is not possible to convert the neural structure into known structures like ARMAX.

2. Comparatively more time is required to train the neural network.

3.3.1 Neural Network Structures

Mainly there are three types of ANN structures, such as: single layer feed-forward network, multi-layer feed-forward network and recurrent networks. The most common type of single layer feed-forward network is the Perceptron. This dissertation describes the use of multilayer feed-forward network for identification of the non-linear cart-pole system.

A block diagram representing the basic operational sequence of a multilayer feed-forward parallel process network is shown in Figure 5. The 1st layer is called as the input layer, the final layer is called as the output layer and the middle one is called as the hidden layer.

Inputs to the perceptron which computes the output, which are individually weighted and summed to an activation function F needed to introduce nonlinearities into the network. It is the reason for which multi-layer networks are powerful in representing nonlinear functions.

3.4 Types of Learning

Supervised, reinforced and un-supervised learning methods are the three main types of neural network learning methods. The output of NN is compared with target and the error signal generated,is used to update the weights in supervised learning. Reinforced learning is supervisedtype but no target is there, the algorithm is given a grade of the ANN performance. Input data is used to update the weights in unsupervised learning. The ANN can be used to cluster different input patterns into different classes.The supervised learning methodology is used which is shown in the Figure 5.

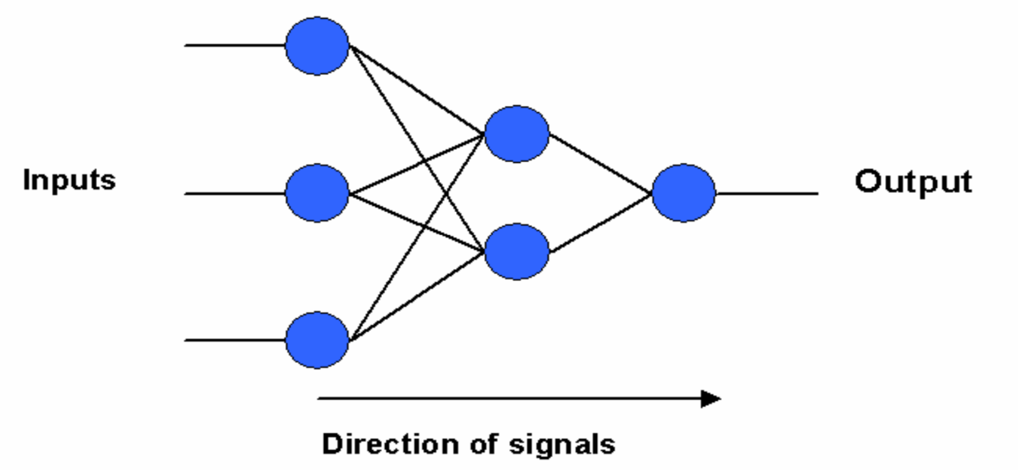


Fig. 5: Multi-layer feed-forward network

The output of the perceptron is

$$y(k) = f(w^T(k).x(k)) \tag{38}$$

Back propagation algorithm is used to update the weights. The error, difference between the desired output and the actual output, can be formulated as

$$e(k) = d(k) - y(k) \tag{39}$$

The back propagation algorithm used for updating the weights is given by

$$w(k+1) = w(k) + \mu e(k).x(k) \tag{40}$$

Where, μ is the learning rate which determines to what extent the newly acquired information will override the old information. The range of learning rate is $0 \le \mu \le 1$. If $\mu = 0$ the agent won't learn anything and if $\mu = 1$ the agent will consider only the most recent information.

The process and the ANN both receive the same input and the outputs of these two are compared to produce the error signal which updates the weights. The block diagram representing this, is shown in Figure 6.

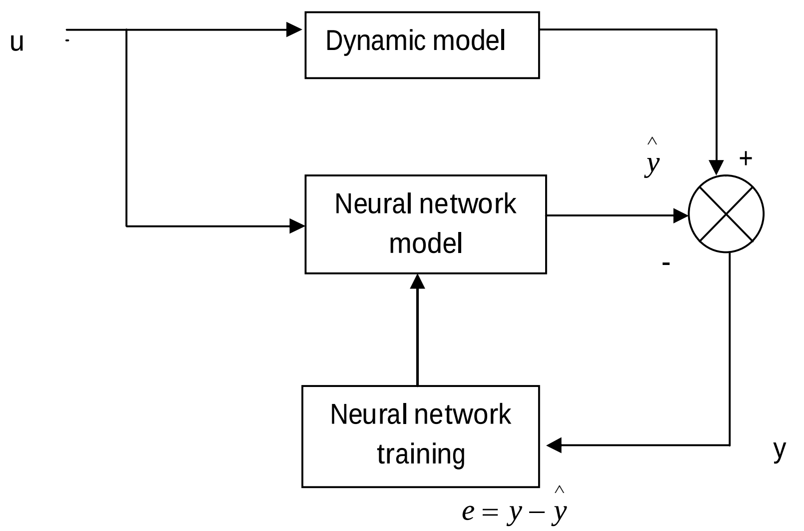


Fig. 6: Block diagram of Supervised learning.

Here the system is identified by using the Elman and Feed-forward neural networks. Here, closed loop identification is used instead of open loop identification. A control is derived for the non-linear plant by general inversion technique [10, 11]. The control law is given by

$$u = \frac{f_2}{h_2}\left(h_1 + k_1(\theta - \theta_d) + k_2\dot{\theta} + c_1(x - x_d) + c_2\dot{x}\right) - f_1 \tag{41}$$

Where,

$$f_1 = m\left(l\sin\theta.\dot{\theta}^2 - \frac{3}{8} g\sin 2\theta \right)$$

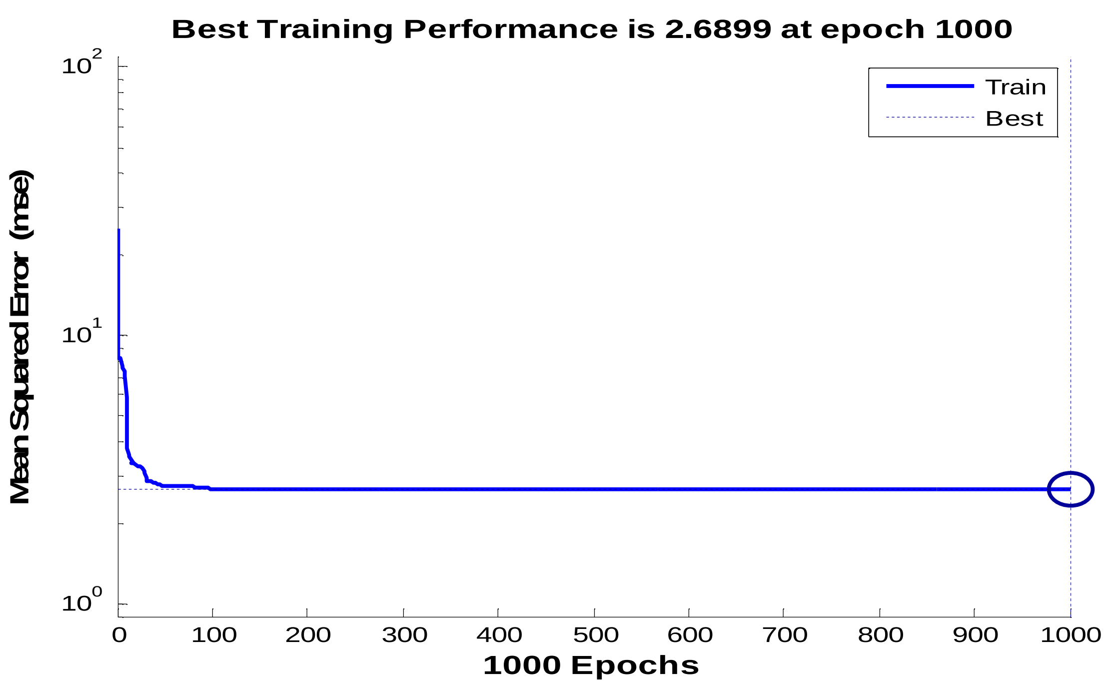
$$f_2 = M + m\left(1 - \frac{3}{4}\cos^2\theta \right)$$

$$h_1 = \frac{3}{4l} g\sin\theta \text{ and } h_2 = \frac{3}{4l}\cos\theta$$

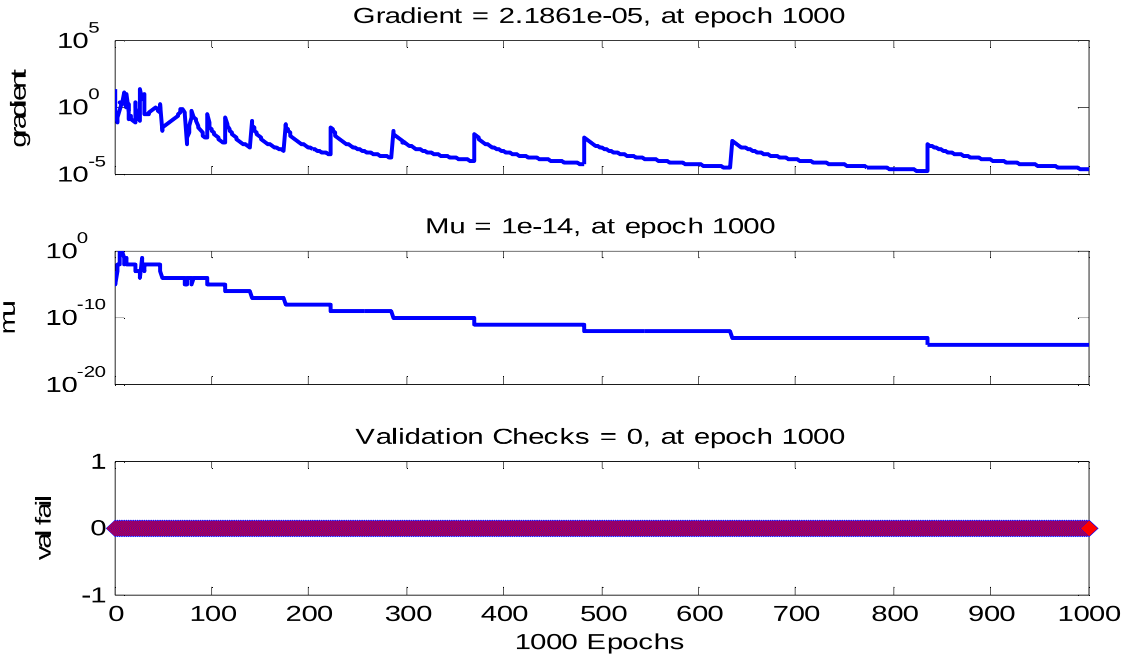
$\theta_d = 0 = x_d$ and k_1=25, k_2=10 , c_1=1 and c_2=2.6.

These values are obtained from the paper [10]. The training is done by the Levenberg- Marquardt methodology. For both the networks the learning rate is 0.001 and the transfer function is tan-sigmoid function. The identified outputs are compared on the basis of MSE.

The performance and training set of aElman network generated for a random input with four hidden layers are shown in Figure 6. The identification of angle is shown in figure 8.



(a)



(b)

Fig. 7: Elman Network : (a) Performance and (b) Training set

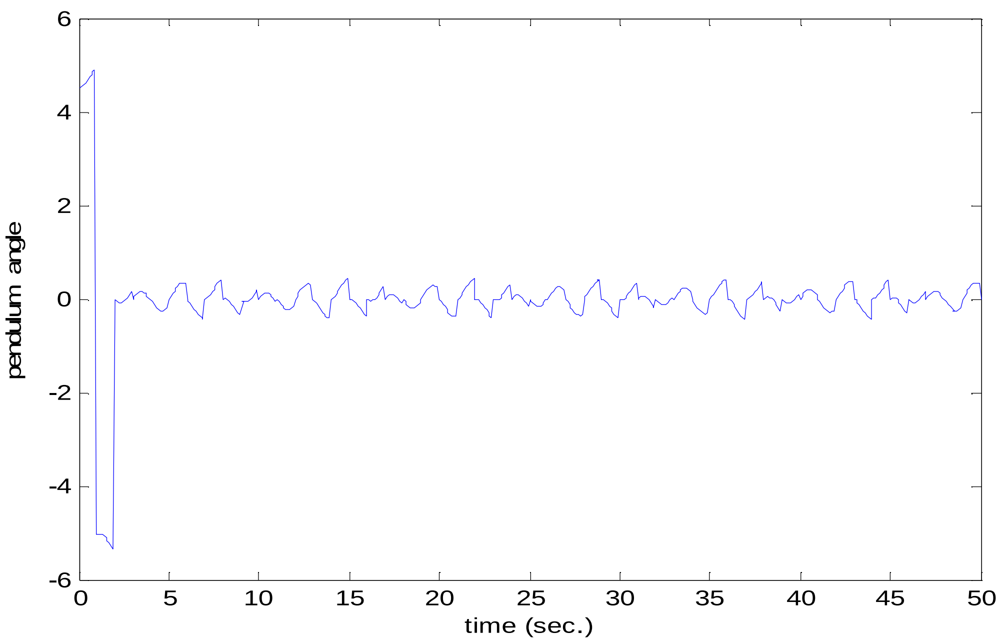
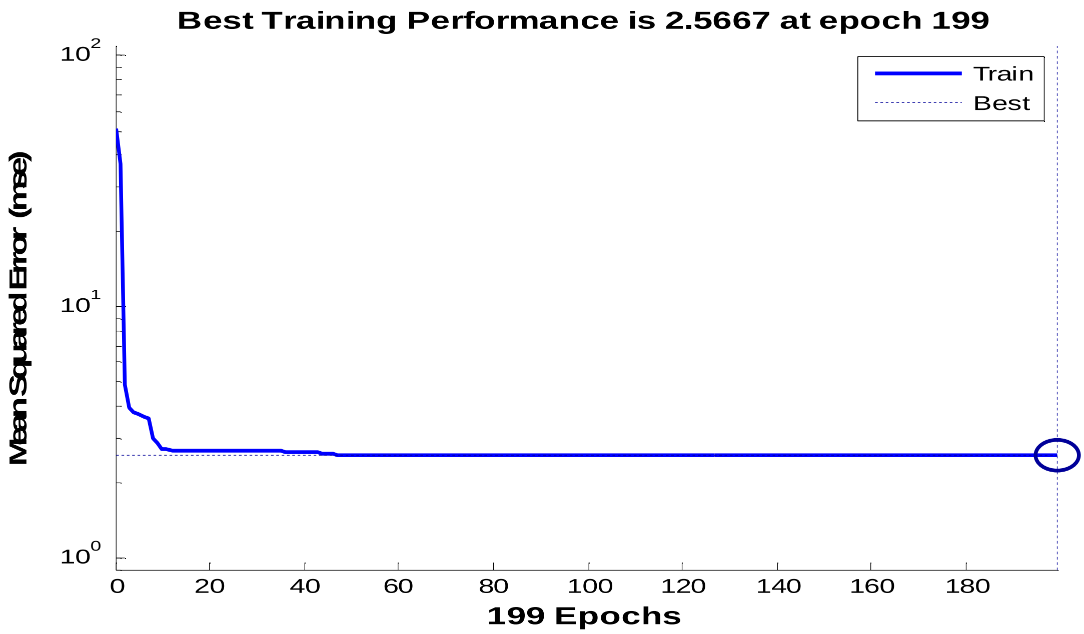
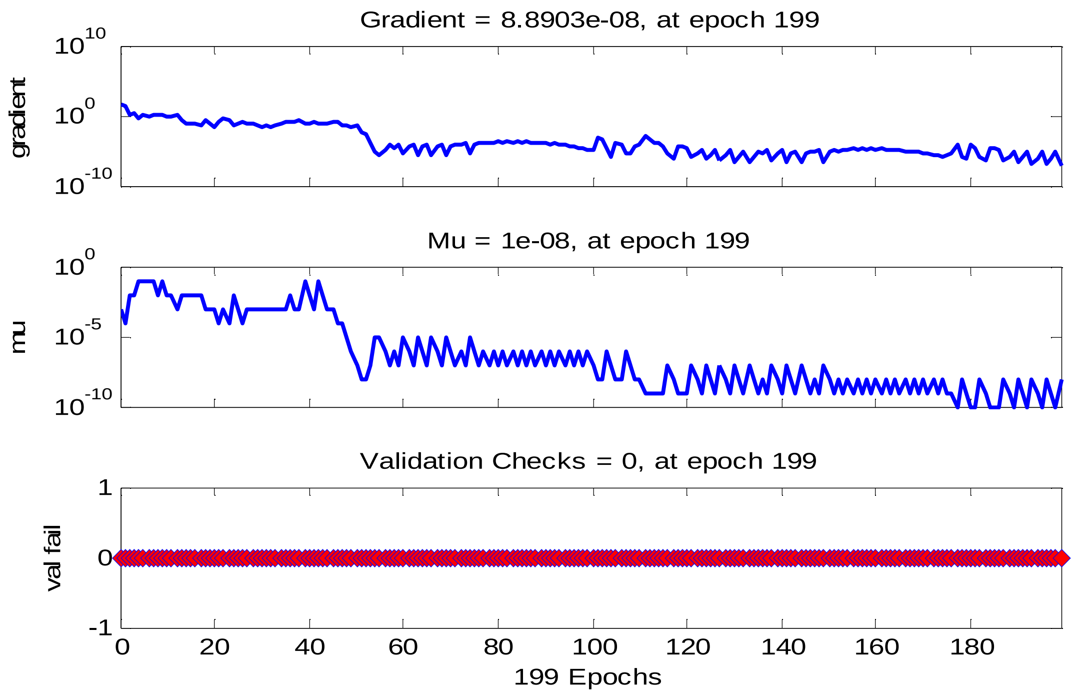


Fig.8: The identification of the angle of Elman network

The performance, training set and regression are shown in the figure 9 and the identified angle plot is shown in figure 10.



(a)



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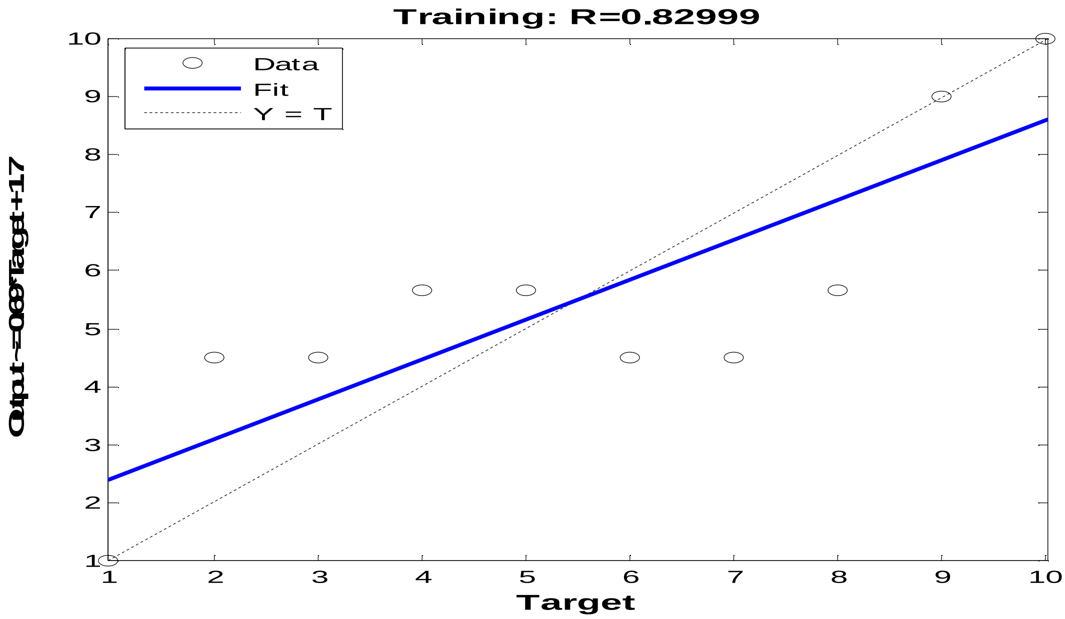


(c)

Fig. 9 : Feed-forward network : (a) Performance (b) Training Set and (c) Regression.

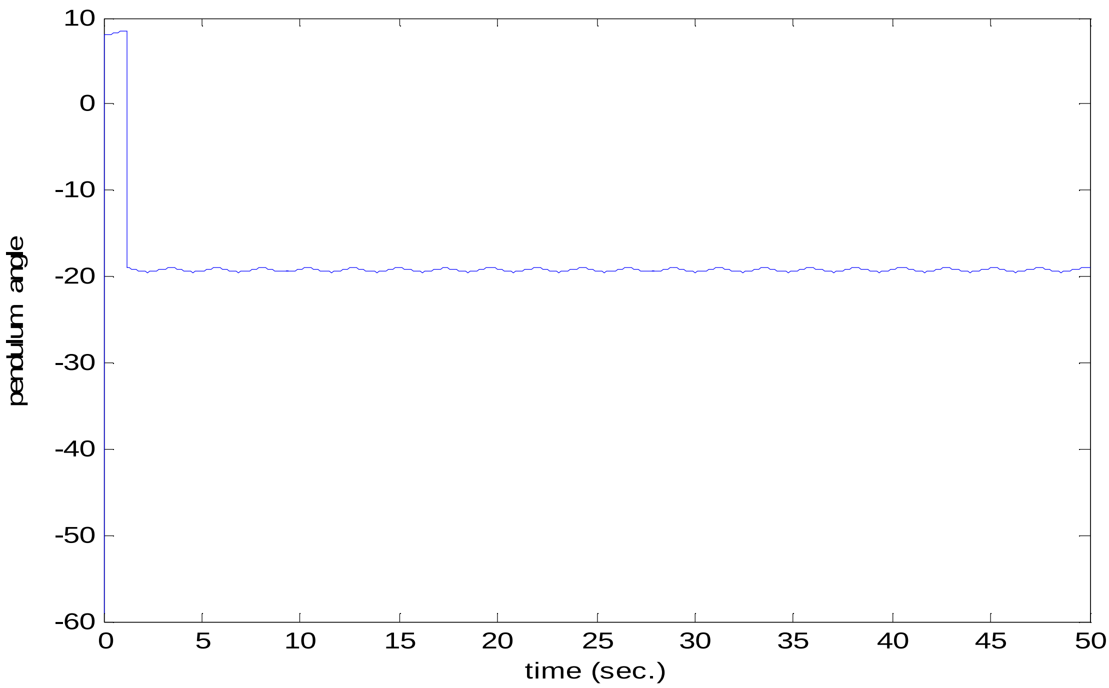


Fig. 10: Identified pendulum angle of feed-forward network.

The MSEs obtained for the identified pendulum angle are 1.1878 for Elman NN and 363.1531 for feed-forward NN. So, after identification it is proved that the Elman NN is better to use for identification.

3.5 Summary

This chapter gives an introduction of system identification and performs linear and nonlinear system identification of cart-pendulum system. Black box identification is vital for controller design as the black box identification gives the accurate mathematical model of the system. For complex systems, black box identification is the only way to develop mathematical model.

CHAPTER 4

CONTROLLER DESIGN FOR THE CART-POLE SYSTEM

- Control of Linear Cart-pole system
- Control of Non-linear Cart-pole system

CHAPTER 4

CONTROLLER DESIGN

4.1 Control ofLinear Cart-pole system:

For controlling the obtained linear model five control mechanisms are used. These five mechanisms are explained below.

4.1.1 Linear Quadratic Regulator method

LQR method [12] uses a state space approach to analyze a system. It provides a way to compute the state feedback control gain matrix which stabilizes the system. The system for LQR methodology is given by

$$\dot{x} = Ax + Bu$$

$$y = Cx \tag{42}$$

The objective is to find out a control function u which willminimize the performance index (cost function) and is given by the equation

$$J = \int_0^\infty (x^* Q x + u^* R u) dt \tag{43}$$

Where, Q = positive semi-definite matrix

R = positive definite matrix.

Both Q and R determine the relative importance of error. The minimum value of the cost function is achieved if x and u are zero. This type of problem is called regular problem.

The above equation can be simplified while finding a controller for optimization purpose, which is based on the positive definite solution to the equation given below

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{44}$$

This is called as reduced matrix Riccati equation, Where, P is a positive definite Hermitian or real symmetric matrix. If the system is stable, then there exists one positive definite matrix P to satisfy the equation. The positive definite solution of this algebraic Riccati equation (ARE) gives an asymptotically stable system.

For the controller to exist, the system should be controllable and observable[13]. For this system

$$T = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \tag{45}$$

T should be controllable and it is controllable if T is a full rank matrix i.e. $rank(T) = 4$.

And

$$S = \begin{bmatrix} C' \\ A'C' \\ (A^2)'C' \\ (A^3)'C' \end{bmatrix} \tag{46}$$

S should be observable and it is, if S is a full rank matrix i.e. $rank(S) = 4$.

For the Riccati Eq. (44)

$$Q = C^T C \tag{47}$$

and R is a m×m square matrix, where m is the no. of columns in the matrix B.

For the cart-pole system B has one column so, R is a single element and is taken as 1.

If the C in S is not able to satisfy the observability condition then C_q another matrix is found that that can satisfy the observability condition.

Then the Q matrix is given by

$$Q = C_q^{\,T} C_q \tag{48}$$

The feedback control law is given by

$$u = -Kx \tag{49}$$

Where, K is the gain of the feedback control matrix and is given by

$$K = R^{-1} B^T P \tag{50}$$

P is the solution of Riccati Eq. (44).

For the stability the closed loop structure of the system is determined and is given by

$$G_C = A - BK \quad (51)$$

If the Eigen values of G_C are on the left half of the imaginary axis, then the system is proved to be stabilized.

The said system satisfies the condition of controllability and observability and the unique solution to the ARE is a positive definite matrix. So, the design of LQR controller to stabilize the system is possible and done by MATLAB coding. The solution to the continuous time ARE is given by

$$P = 10^4 \begin{bmatrix} 0.0002 & 0.0003 & 0.0056 & 0.0007 \\ 0.0003 & 0.0005 & 0.0122 & 0.0014 \\ 0.0056 & 0.0122 & 1.3390 & 0.1573 \\ 0.0007 & 0.0014 & 0.1573 & 0.0185 \end{bmatrix}$$

The closed loop Eigen values are

L =-0.4856 + 0.4856i

-0.4856 - 0.4856i

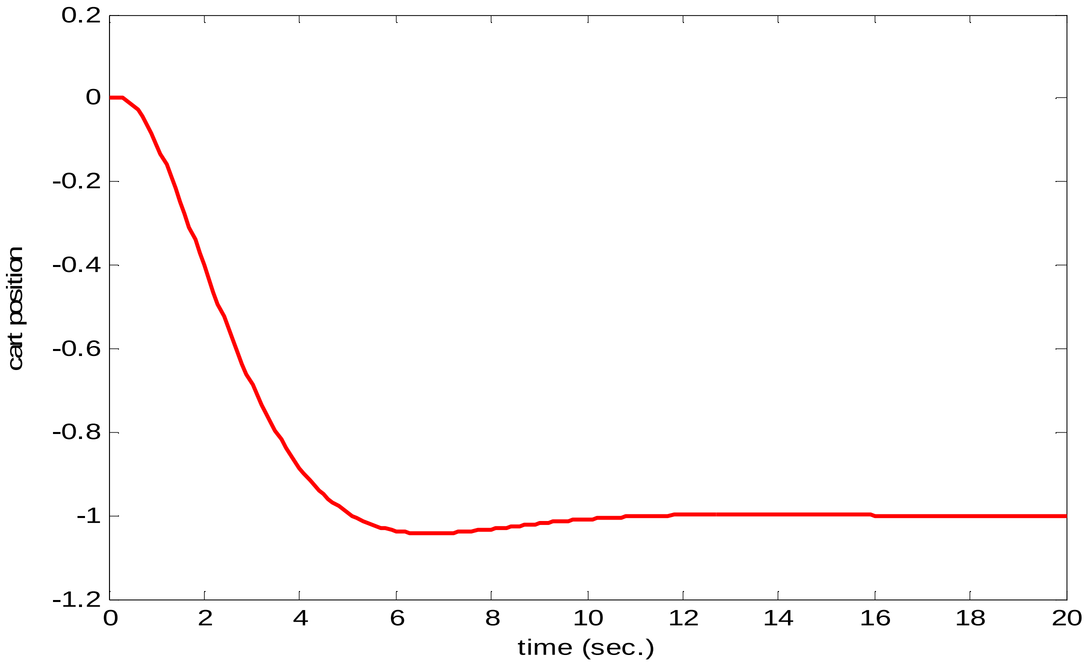
-8.5100 + 0.0200i

-8.5100 - 0.0200i

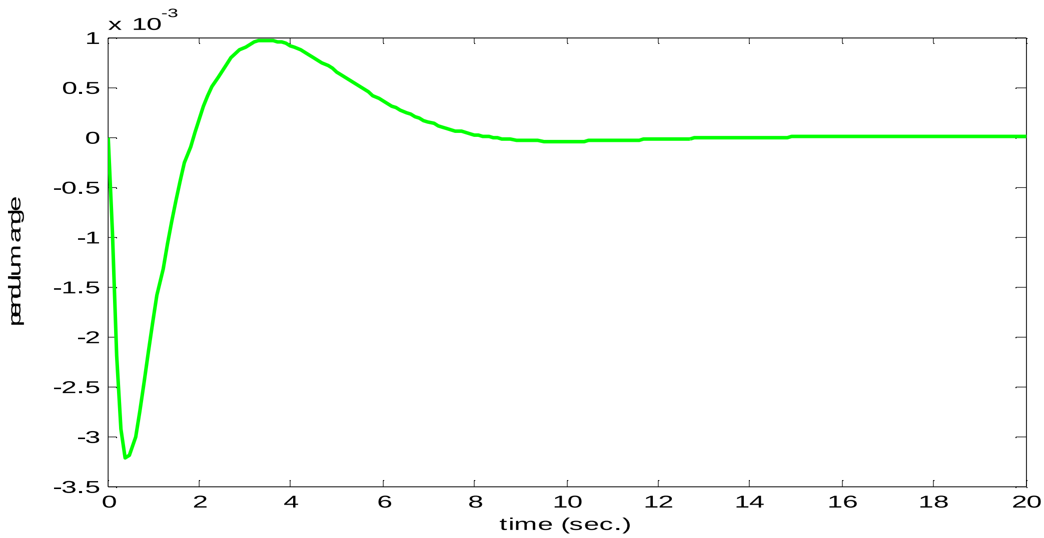
The closed loop feedback gain matrix is

k=[-1.0000 -2.2945 -477.3847 -56.0869]

All the closed loop Eigen values are situated on the left hand side of the imaginary axis. So, the designed controller asymptotically stabilizes the cart-pole system. The plots obtained for the cart position and pendulum angle are shown in the figure 11.



(a)



(b)

Fig. 11: LQR outputs: (a) Cart position (b) Pendulum angle.

4.1.2 LQR-Pole-placement-PID Controller

Two outputs, cart-displacement and pole-angle are produced, so it is better to control them by two individual PID controllers to have perfect stabilization [14]. The transfer function of the two PID controllers are given as

$$G_{C1} = K_{P1} + \frac{K_{I1}}{s} + K_{D1}$$

and

$$G_{C2} = K_{P2} + \frac{K_{I2}}{s} + K_{D2} \tag{52}$$

The characteristic equation of the controlled linear system is given by

$$1 + G_{P1}G_{C1} + G_{P2}G_{C2} = 0 \tag{53}$$

So, the resultant equation is

$$s^5 + s^4(a_2K_{D2} + a_1K_{D1}) + s^3(a_2K_{P2} + a_1K_{P1} - b^2) + s^2(a_2K_{I2} + a_1K_{I1} + a_1K_{D1}b^2) - sa_1K_{P1}b^2 - a_1K_{I1}b^2 = 0 \tag{54}$$

Generally, Q is a diagonal matrix. Element at row1 and column1 weights to the position of cart.Element at row2 and column2 weights to the velocity of cart.Element at row3 and column3 weights to the pendulum angle and element at row4 and column4 weights to the angular velocity of the pendulum.The Q matrix can be taken as

$$\begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix} \tag{55}$$

Here, the velocity of cart and angular velocity of pole are not considered, so, q_3 and q_4 are zeros.

So,

$$q1 = \frac{1}{t_s\,(x_{\max})},\; q4 = \frac{1}{t_s(\angle\theta_{\max})^2} \tag{56}$$

Where, t_s is the desired settling time, $x_{\max}$ is the constraint on the cart position and

$\angle\theta_{\max}$ is the constraint on $\angle\theta$.

The maximum values of t_s, x_{max} and $\angle\theta_{\max}$ are given in TableI.

The R matrix can be taken as

$$\rho\begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \tag{57}$$

Where, ρ is a constant and $\rho > 0$ and the selection of ρ can be arbitrary.

$$r_i = \frac{1}{(u_{i\max})^2} \tag{58}$$

Where, $u = Fx_{\max}$

F is the applied force,

Or $F = -\begin{bmatrix} \rho^{-\frac{1}{2}} & \sqrt{2} & \rho^{-\frac{1}{4}} \end{bmatrix}$

But, here R is constant and R = 100, as it should have rows equal to the columns in B.

Pole-Placement Method

This method is somewhat similar to that of root locus method. But the basic difference is that root locus includes only dominant closed loop poles while the pole placement method includes all the closed loop poles at desired locations.

The necessary and sufficient condition for pole placement method is that the system should be completely state controllable, such that the closed loop poles can be placed at any desired locations by state feedback through an appropriate state feedback gain matrix.

Here, the poles are obtained by LQR method by the formula $\left[K \quad S \quad \lambda\right] = lqr(A \quad B \quad Q \quad R)$

Where, λ is the position of poles.

K is the state feedback unit gain matrix. It can be determined by the Eq. (50)

In this K has four values, so λ also has four values.

According to Eq. (54), there are five poles. So, fifth pole is assumed to be six times the real part of the dominant pole. After getting the poles the characteristic equation obtained is

$$(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)(s-\lambda_5) = 0 \qquad (59)$$

Comparing Eq. (54) & Eq. (59) and assuming one of the six control parameters of the two PID controllers, the rest five parameters can be found out. These control parameters can also be varied by varying the Q matrix and R.

LQR-Pole Placement PID

It has been proved earlier that the pole-placement method is controllable.

Here, I have taken two combinations of the Q and R matrices by considering the formulae given in Eq. (55). For first set the values are

$$Q = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R=1.

The closed loop poles are at

l =-1.0265 + 1.0266i

-1.0265 - 1.0266i

-8.5101 + 0.0341i

-8.5101 - 0.0341i

As it has been explained above by changing the two matrices Q and R the control parameters can be changed, so the second set is obtained by multiplying a factor 20 to the Q matrix and taking R=100.

$$\text{So, } Q = \begin{bmatrix} 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R=100

Here the closed loop poles are

e =-0.7258 + 0.6448i

-0.7258 - 0.6448i

-8.5101 + 0.0153i

-8.5101 - 0.0153i

On comparing the pole location for two sets, the dominant pole in set 2 is more nearer to the origin. So, the speed of the response is more and as it is on the left side of imaginary axis also stable. So, for controlling the system second set is taken. Hence, the fifth pole is -4.3548.

Now the work is to choose one of the six control parameters. Comparing Eq. (54) and Eq. (59), putting all the closed loop poles and assuming, $K_{D1} = -5$, all other controller parameters obtained are

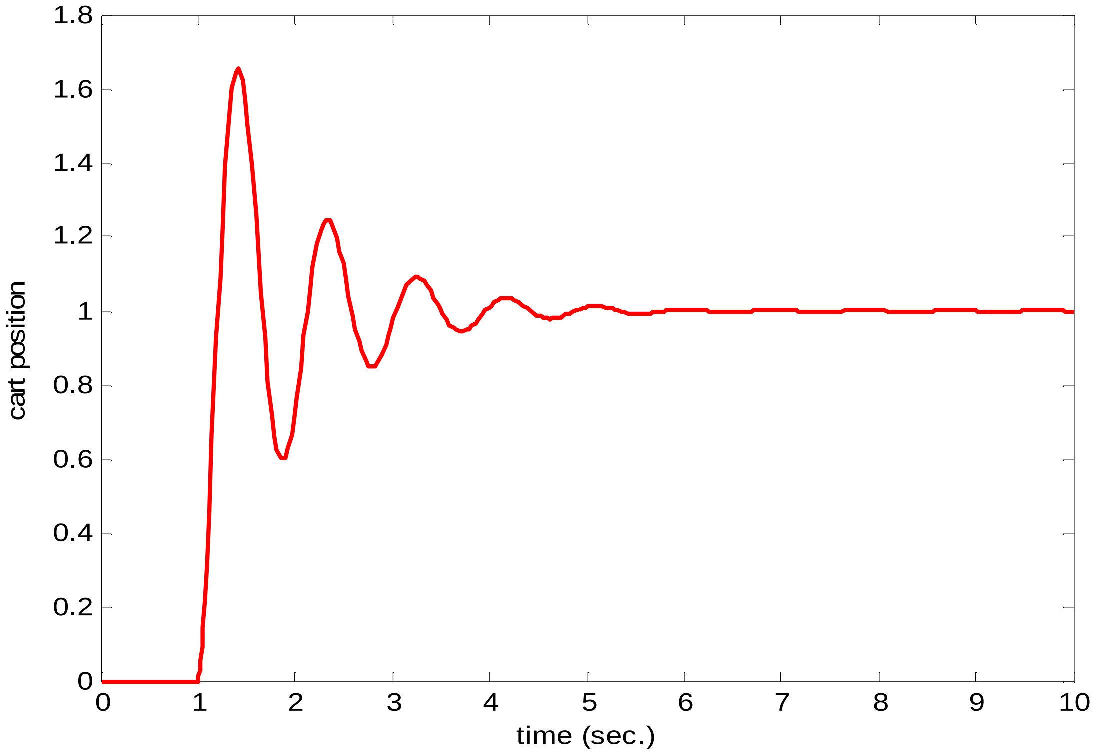
$$K_{I1} = -87.8591$$
$$K_{P1} = -175.1688$$

$$K_{D2} = 74.9735$$
$$K_{I2} = 1705.3235$$
$$K_{P2} = 798.2723$$

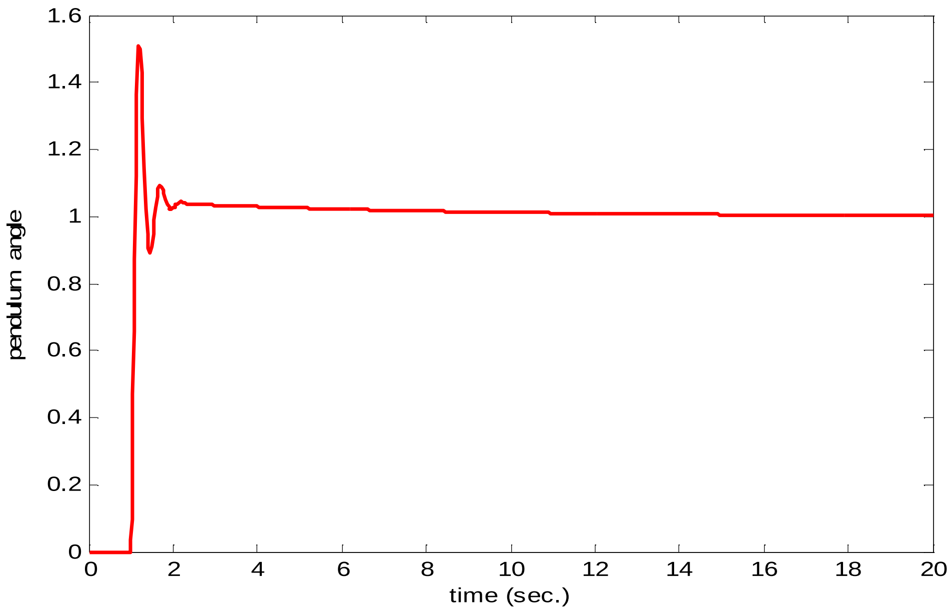
By simulation the plots obtained for the step input are shown in figure 12.

But if $K_{D1} = 5$ is assumed, then on the basis of the rest controller parameters, the system can't be stabilized hence the value of $K_{D1} = -5$ is taken for finding out the controller parameters.

As the position curve is going after the required settling time, i.e. 5 sec. a little bit manual adjustment can be done to have the perfect curve. And the curves obtained after a little manual adjustment are shown in the figure 12.1for the reference input as 0.1.

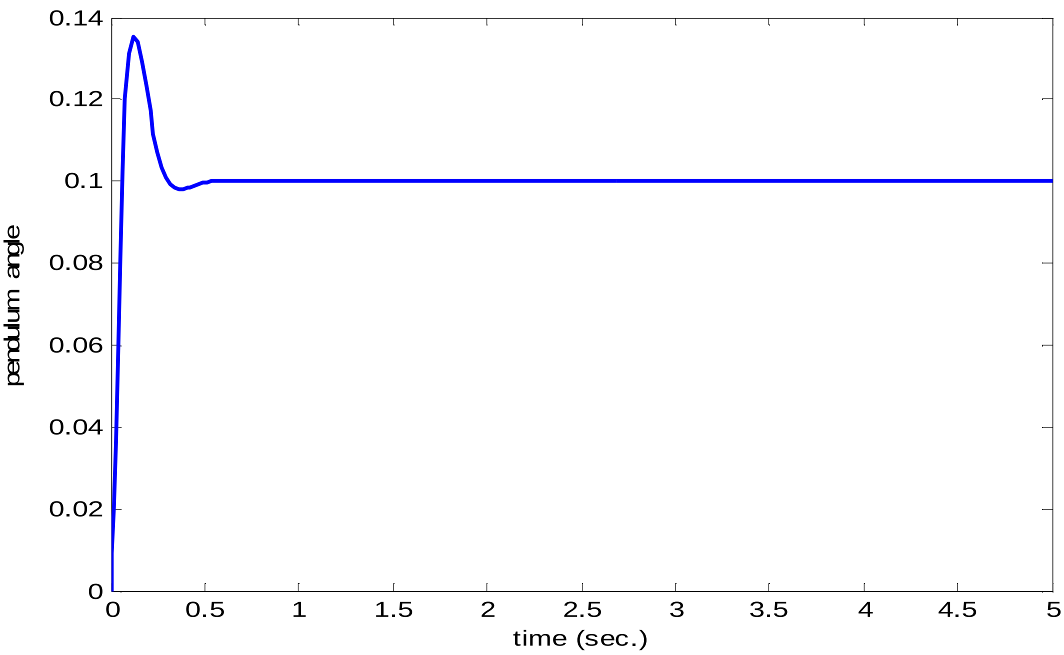


(a)

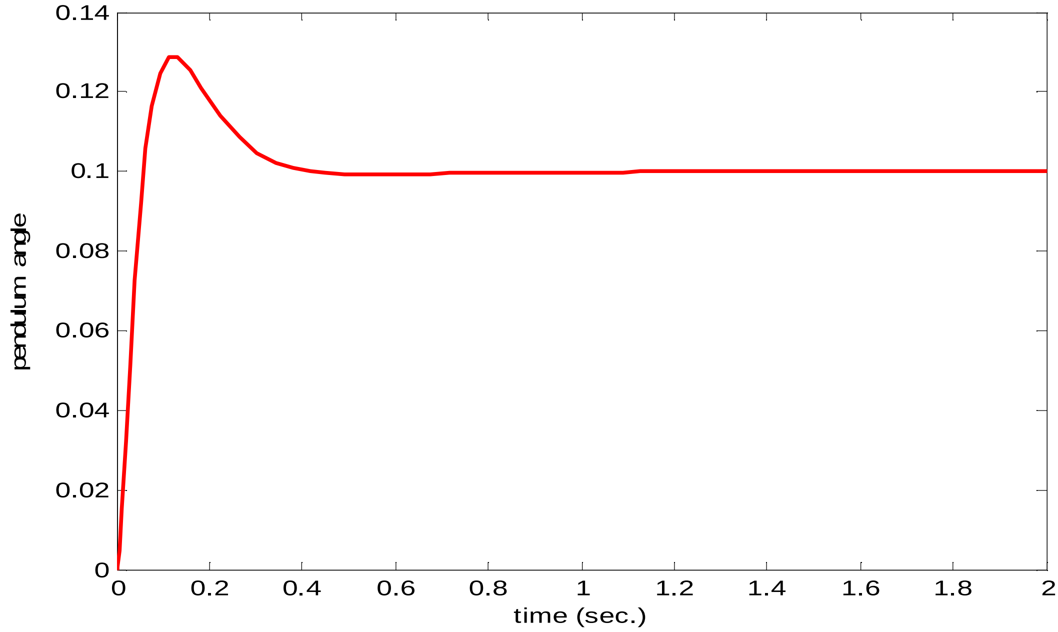


(b)

Fig. 12: Output of LQR-Pole-placement-PID: (a) Cart-position (b) Pendulum-angle



(a)



(b)

Fig. 12.1: Output after little variation in parameters: (a) Cart-position (b) Pendulum-angle

4.1.3 Fuzzy-PID Controller Design

Conventional controller design requires a precise mathematical model of the system to be controlled. But it is difficult to get accurate mathematical model of nonlinear system like inverted pendulum system. To counter such kind of issues, fuzzy logic is used as it offers a way to develop controller using logic and human intuition [15, 16].

Though fuzzy control does not need the mathematical equations but it needs the proper expert knowledge about the system. It provides vagueness or imprecision of the input-output description of the system by using the fuzzy sets. Fuzzy logic is employed to a set of fuzzy rules i.e. the controlling knowledge of a human about the plant. In this text the Mamdani type FIS fuzzy system is used for the controller design. For better result a PD+fuzzy controller or PID+fuzzy controller can be used. Here, two fuzzy logic controllers (FLCs) are used. The inputs to the FLCs are the errors in the cart position, cart velocity and errors in pendulum angle and pendulum angular velocity. The output of the FLCs is the force. The general block diagram of fuzzy inference system used for fuzzy controller development using PD controller is shown in

Figure 13. This generalized inference system can be extended to the system with the use of PID controller. But the fuzzy controller can take maximum of two inputs so another input, output of Integration controller can be used as shown in Figure 14.

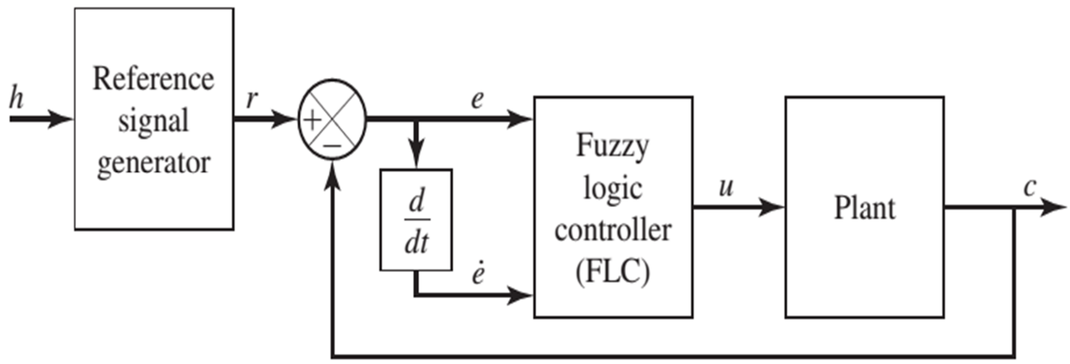


Fig. 13: Block diagram of PD+Fuzzy Controller system.

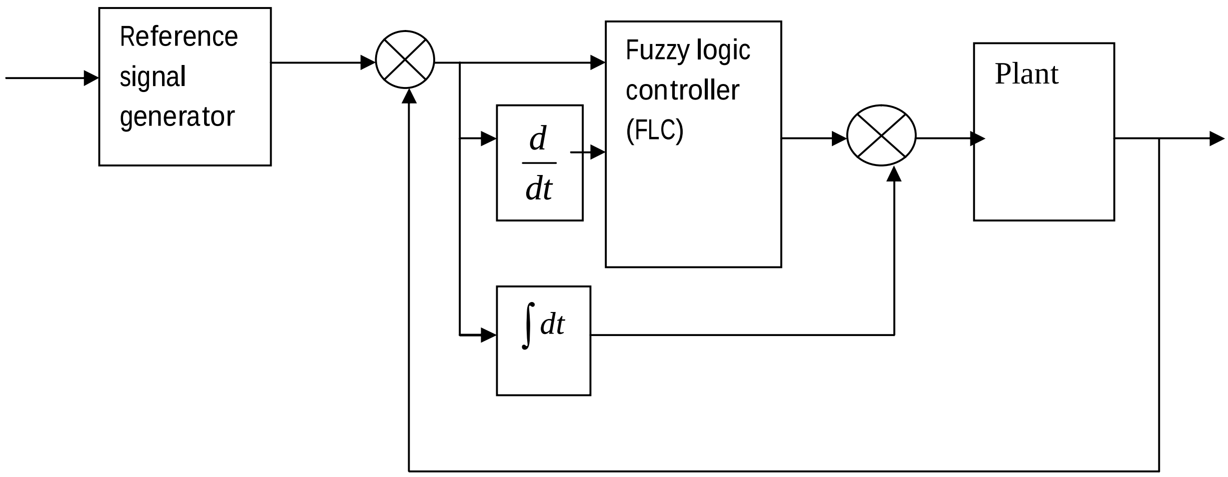
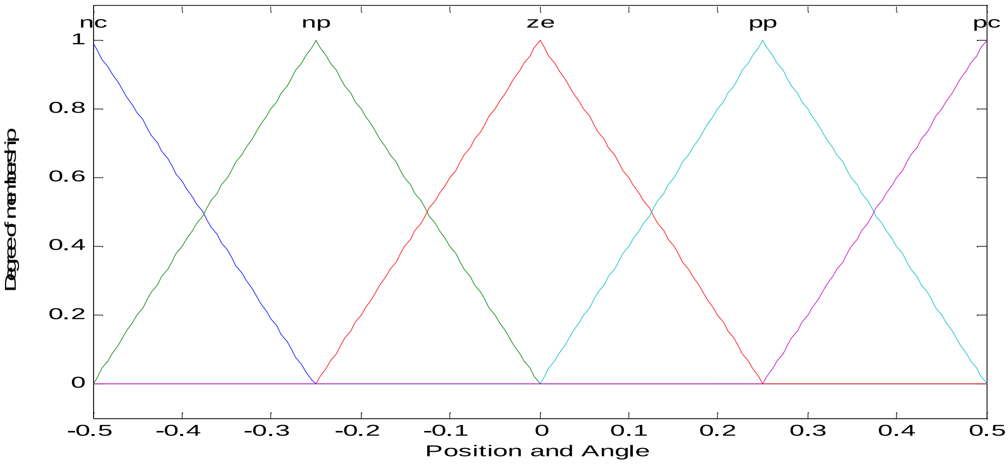


Fig. 14: Block diagram of PID+Fuzzy Controller system.

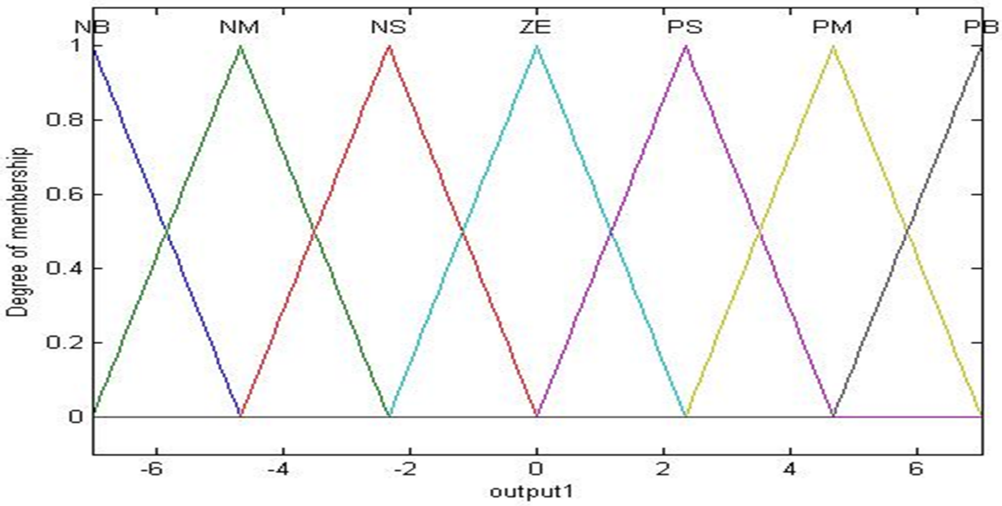
Membership Function and Fuzzy Rule Base

The membership function of fuzzy a set Z on the universe of discourse X is defined as

μ_Z : X →[0,1] where each element of X is mapped to a value between 0 and 1. This value is called membership value or degree of membership and it represents the grade of each element in X to the fuzzy set Z. Membership function represents a fuzzy set graphically. It represents the degree of truth. Degree of truth is the probable value of approaching a particular point. All the inputs are supposed to have five membership functions. So, a set of twenty five fuzzy rules are prepared for individual FLC. The output is considered to have seven membershipfunctions.



(a)



(b)

Fig. 15: Membership function: (a) Membership function of inputs;

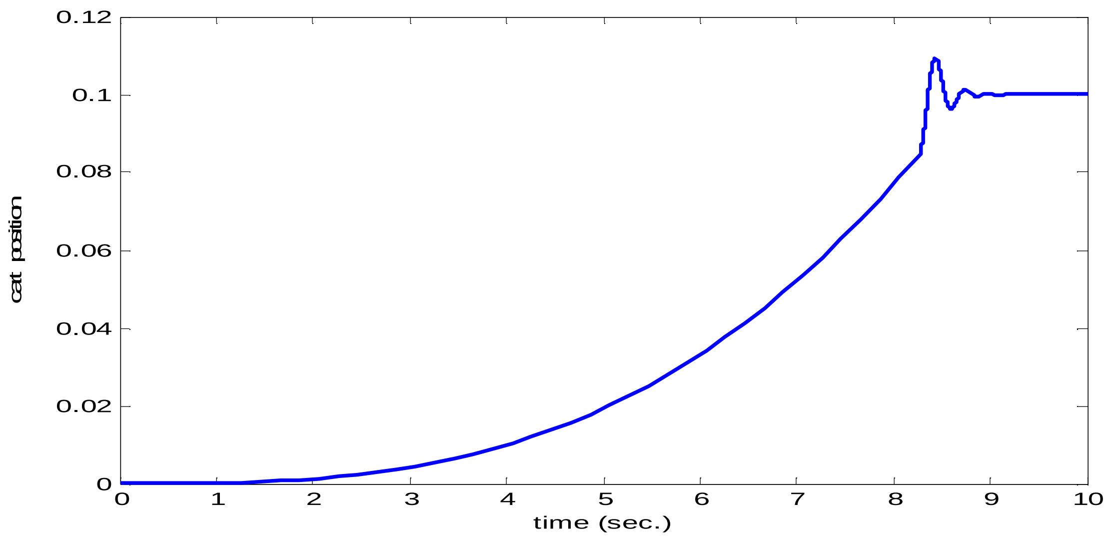
(b) Membership function of output.

This method of getting the fuzzy rule base is completely IF-THEN type linguistic representation. In this thesis the rule used is IF A and B, THEN C. A and B are the premises and C is called the consequence. The membership function for the inputs and outputs of the fuzzy controller are shown in Figure 15 and the rule base for the controller to be designed is given in Table III.

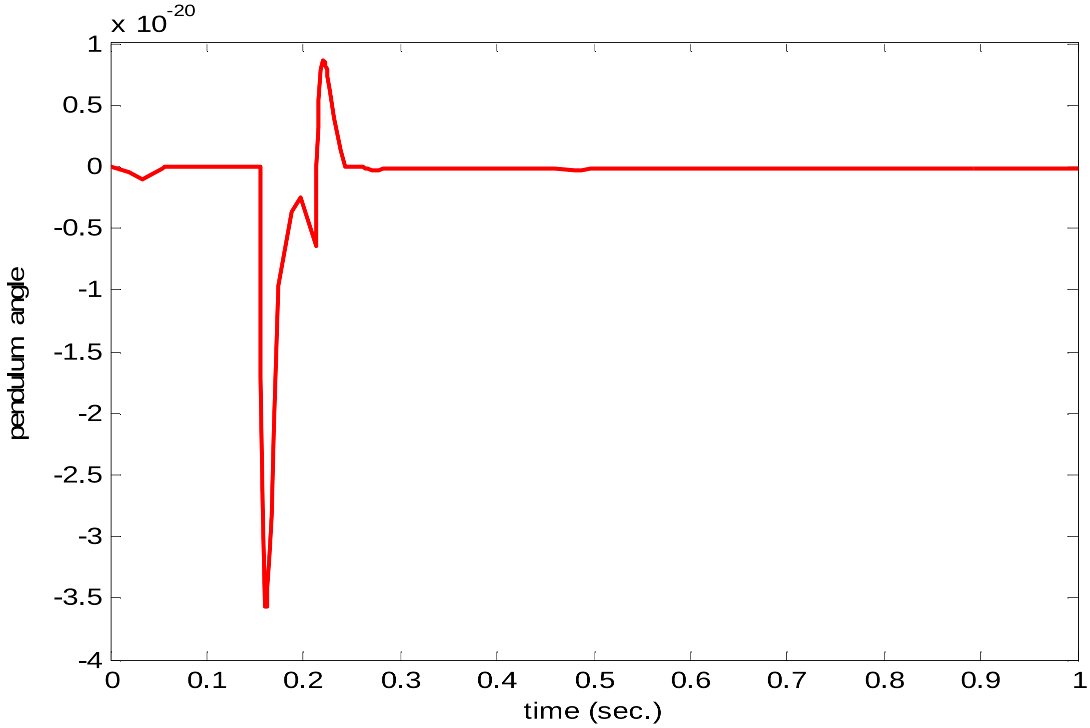
Table III: Fuzzy rule base

Angular/Linear Velocity Error	Pendulum Angle/Cart Position				
	Angle Error/Position Error				
	NB	NS	ZE	PS	PB
NB	NB	NB	NM	NM	ZE
NS	NB	NM	NS	ZE	PS
ZE	NM	NS	ZE	PS	PM
PS	NS	ZE	PS	PB	PB
PB	ZE	PM	PM	PB	PB

The parameters of the PD or PID controllers here can be adjusted manually. The output obtained is shown in figure 16.



(a)



(b)

Fig. 16: Output of Fuzzy-PID controller: (a) Cart position (b) Pendulum angle.

4.1.4 Linear Quadratic Gaussian

LQG control is one of the fundamental control problem which can be applied to both linear time in-variant and time variant systems. Its application to the linear time invariant system includes the design of feedback controller [12, 17]. These feedback type of controllers are designed by the technique same as LQR.

LQG is nothing but a compensator, which is the combination of a linear quadratic estimator (LQE) and a linear quadratic regulator (LQR). The LQE is nothing but a Kalman filter also known as observer. These two can be designed and computed independently.

Kalman filter is feasible estimation approach. It can provide a relative accurate result by fusing multiple sensory measurements. It can minimize mean square error of a series of noisy measurements. Kalman filter provides an optimal recursive data processing algorithm. The algorithm works on the way in which the estimated value of the present state is determined from the estimated value of the previous state and from the present measured data instead of all the entire measured data. It estimates the state of a discrete time controlled process governed by linear stochastic equation.

The LQG controller is a dynamic system like the system which it controls. So they both have same dimensions. LQG does not guarantee automatic robustness, it is checked for the closed loop system after its design. So it guarantees the closed loop stability.

$$\dot{x} = Ax + Bu + wx$$

$$y = Cx + v \tag{60}$$

Where, w represents random noise disturbance input(process noise) and *v* is the random measurement(sensor) noise.

The controller Riccati equation is same as Eq. (44).

The controller is somewhat different than that of LQR and is given by

$$u = -\hat{x}K(t) \tag{61}$$

Where,

$$K = R^{-1}B'P$$

Where,

$\hat{x}$ is the estimate of x.

The Riccati equation for the observer or filter is

$$A\Sigma + \Sigma A' + Q_0 - \Sigma C' R_0^{-1} C\Sigma = 0 \tag{62}$$

Where, $\sum$ is the intensity of the estimation error.

The observer or filter or the regulator is represented as

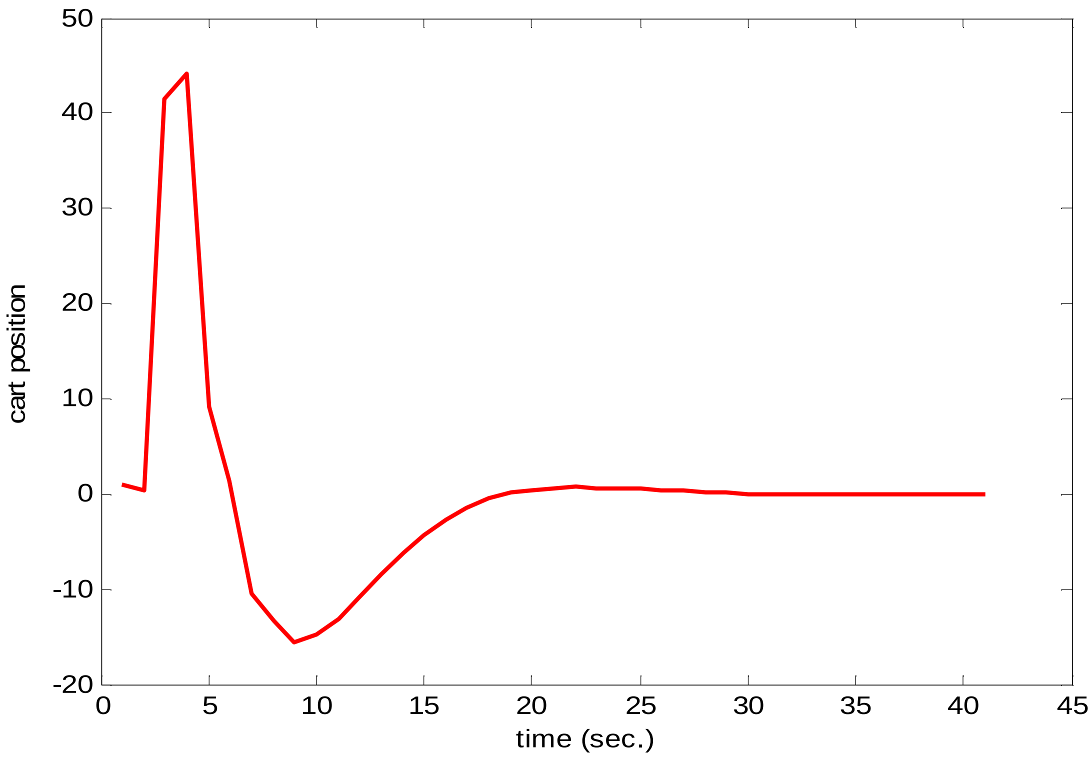
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{63}$$

Where,

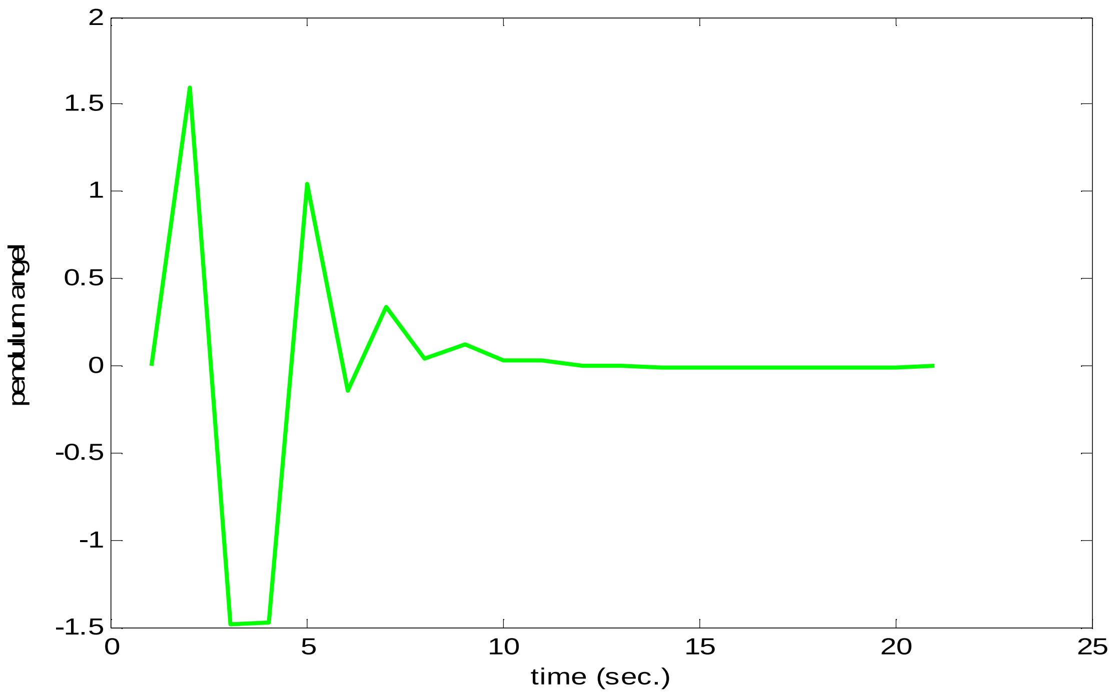
$L = \Sigma C' R_0^{-1}$ is the observer gain.

Q_0 and R_0 represent the intensity of the process and sensor noise parameters and these are assumed by the user. Solution of the Kalman filter exists if and if Q_0 and R_0 are positive semi definite and positive definite matrices respectively.

LQG results in an asymptotically stable closed loop system. The controller minimizes the average cost function of LQR resulting in an optimal solution. The output is shown in figure 17.



(a)



(b)

Fig. 17: Output of LQG: (a) Cart position (b) Pendulum angle

4.1.5 H-Infinity

All aforesaid methods are time domain methods for design of controller. But H-infinity is a frequency domain optimization method for the design of robust controller. The main objective of h-infinity is to choose a stable and proper transfer function from a space of transfer functions. It is done by comparing the H_∞norm of all the transfer functions. The H_∞ norm of a transfer function is given below,

The H_∞ norm [18] for a stable transfer matrix G(s) is given as

$$\|G\|_\infty = \sup_w \sigma_{\max} |G(jw)| \tag{64}$$

$\sigma_{\max}$ is the maximum singular value.

The mathematical computation is quite complicated, so it is done on graphical basis by comparing the peak in the Bode magnitude plot of all the transfer functions.

The non-linear system [19] can be represented as

$$\dot{x} = F(x, w, u)$$

$$z = Z(x, u)$$

$$y = Y(x, w) \qquad (65)$$

The plant can be represented [12] as

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{11} w + D_{12} u$$

$$y = C_2 x + D_{21} w + D_{22} u \qquad (66)$$

The packed matrix notation is given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad (67)$$

But, here according to non-linear representation the matrices D_{11} and D_{22} are zero.

The basic block diagram for the H_∞ controller system is given in Fig.18

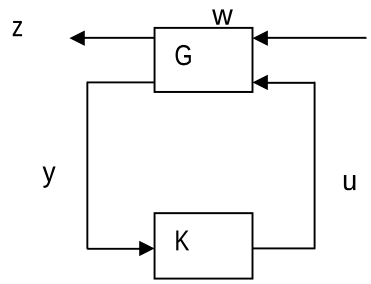


Fig. 18: H_∞ - Block diagram

Where, G is the generalized plant and K is the controller. G contains the cart-pole system and all the weighting functions.

w contains all the external inputs along with disturbances, noises and commands. z is the error signal, y is the measured variable and u is the control input.

The H_∞ state estimator contains a more term than the LQG estimator [12][18-20], which is given by

$$\dot{\hat{x}} = A\hat{x} + B_2 u + B_1 \hat{w} + Z_\infty K_e (y - \hat{y}) \qquad (68)$$

Where,

$\hat{w} = \gamma^{-2} B_1^T X_\infty \hat{x}$ and

$\hat{y} = C_2 \hat{x} + \gamma^{-2} D_{21} B_1^T X_\infty \hat{x}$

γ is called as the trade of parameter.

The closed loop system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -B_2 K_c \\ Z_\infty K_e C_2 & A - B_2 K_c + \gamma^{-2} B_1 B_1^T X_\infty - Z_\infty K_e (C_2 + \gamma^{-2} D_{21} B_1^T X_\infty) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 \\ Z_\infty K_e D_{21} \end{bmatrix} w$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} C_1 & -D_{12} K_c \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ D_{12} \end{bmatrix} w \qquad (69)$$

Where,

$$Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty \right)^{-1}$$

$$X_\infty = Ric \begin{bmatrix} A - B_2 \tilde{D}_{12} D_{12}{}^T C_1 & \gamma^{-2} B_1 B_1{}^T - B_2 \tilde{D}_{12} B_2{}^T \\ -\tilde{C}_1{}^T \tilde{C}_1 & -\left(A - B_2 \tilde{D}_{12} D_{12}{}^T C_1 \right)^T \end{bmatrix}$$

$$Y_\infty = Ric \begin{bmatrix} \left(A - B_1 D_{21}{}^T \tilde{D}_{21} C_2 \right)^T & \gamma^{-2} C_1{}^T C_1 - C_2{}^T \tilde{D}_{21} C_2 \\ -\tilde{B}_1 \tilde{B}_1^T & -\left(A - B_1 D_{21}{}^T \tilde{D}_{21} C_2 \right) \end{bmatrix}$$

These are in the Hamiltonian matrix format. The Riccati equation of the Hamiltonian equation will be

$$A^T X + XA - X[B_1, B_2] \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} X + C^T C = 0 \qquad (70)$$

The two terms X_∞ and Y_∞ are the solutions to the controller and the estimator Riccati equations respectively. For the existence of a stabilizing compensator, X_∞ and Y_∞ should have positive semi-definite solution and

$$\rho\left(X_\infty Y_\infty\right) < \gamma^2 \qquad (71)$$

Where,

$\rho\left(X_\infty Y_\infty\right)$ represents the spectral radius, the maximum Eigen values of ($X_\infty Y_\infty$).

Where,

$$\tilde{C}_1 = \left(I - D_{12}\tilde{D}_{12}D_{12}{}^T\right)C_1,$$

$$\tilde{B}_1 = B_1\left(I - D_{21}{}^T\tilde{D}_{21}D_{21}\right),$$

$$\tilde{D}_{12} = \left(D_{12}{}^T D_{12}\right)^{-1},$$

$$\tilde{D}_{21} = \left(D_{21}D_{21}{}^T\right)^{-1},$$

$$K_c = \tilde{D}_{12}\left(B_2{}^T X_\infty + D_{12}{}^T C_1\right),$$

And

$$K_e = \left(Y_\infty C_2{}^T + B_1 D_{21}{}^T\right)\tilde{D}_{21}.$$

K_e corresponds to the L in the LQG.

The estimator gain is given by $Z_\infty K_e$.

H_∞ OUTPUT

The output of controller and estimator Riccati equations arefound to be positive definite matrices and have values:

$$X_\infty = 10^3\begin{bmatrix} 0.0016 & 0.0014 & 0.0281 & 0.0033 \\ 0.0014 & 0.0020 & 0.0430 & 0.0051 \\ 0.0281 & 0.0430 & 3.3683 & 0.3956 \\ 0.0033 & 0.0051 & 0.3956 & 0.0465 \end{bmatrix}$$

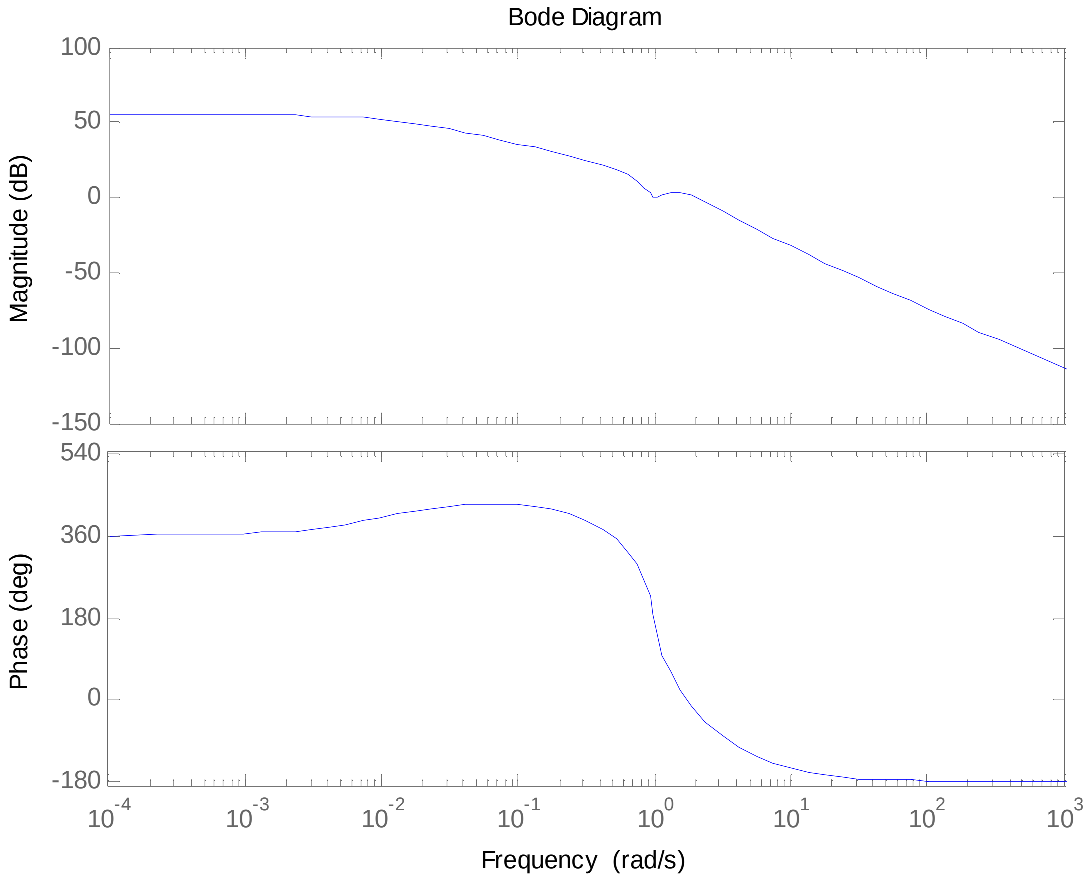
And

$$Y_{\infty} = 10^{3}\begin{bmatrix} 0.0014 & 0.0004 & 0.0001 & 0.0006 \\ 0.0004 & 0.0019 & 0.0006 & 0.0059 \\ 0.0001 & 0.0006 & 0.0161 & 0.0059 \\ 0.0006 & 0.0059 & 0.1448 & 1.3094 \end{bmatrix}$$

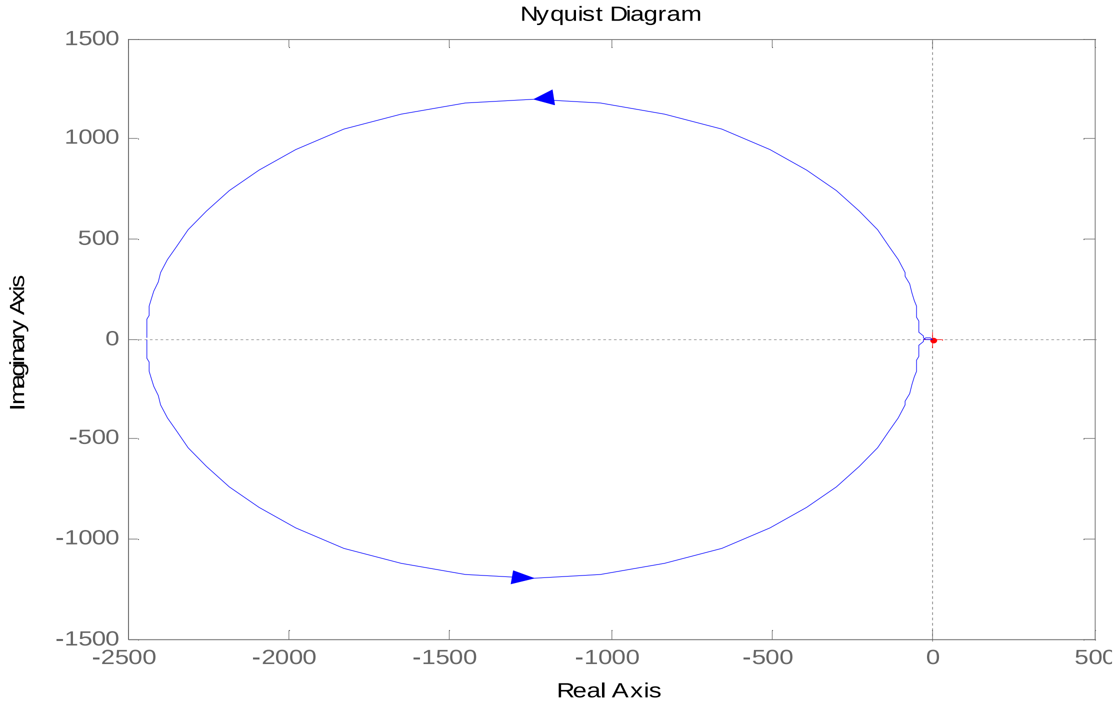
and

$$\rho\left(X_{\infty}Y_{\infty}\right) \cong \gamma^{2} = 2.2973*10^{5}$$

So, there exists the stabilizing compensator. The bode plot and nyquistplot for the compensator are shown in the figure no. 19.



(a)



(b)

Fig. 19: Compensator outputs: (a) Bode plot (b) Nyquist plot

4.2 Control of Non-linear Cart-pole System

Feedback linearization [20-25] is a technique used for controlling non-linear systems. This technique involves the linearization of non-linear system into an equivalent linear system by transformation through the change of variables and using suitable control input. It is better as compared to conventional (Jacobian) linearization, because it's not an assumption but variable transformation. Feedback linearization applied to non-linear system having the form as follows

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x) \qquad (72)$$

Where, $x \in \mathbb{R}^n$ is the state vector, u is the vector of inputs and y is the vector of outputs.

The goal is to develop a control input of the form

$$u = a(x) + b(x)v \qquad (73)$$

Where, v is the new input.

The transformation used here must be diffeo-morphism, i.e. it is not only invertible (bijective) but also both the transformed model and inverse are smooth, only because to ensure that the transformed model is the equivalent of the original one. Lie derivative helps to understand the structure of a system.

To produce transformed model some techniques are used: one of them is the Lie derivative [26]. It is nothing but the time derivative of the output state vector.

$$\dot{y} = \frac{dh(x)}{dt} = \frac{dh(x)}{dx}\dot{x}$$

$$= \frac{dh(x)}{dx} f(x) + \frac{dh(x)}{dx} g(x)u \tag{74}$$

According to Lie algebra the lie derivative is given by

$\frac{dh(x)}{dx} f(x) = L_f h(x)$ and

$$\frac{dh(x)}{dx} g(x) = L_g h(x) \tag{75}$$

If $L_g h(x) \neq 0$ for some $x = x_0$, then combiningEq.(74) and Eq. (75), the input transformation is found to be

$$u = \frac{1}{L_g h}\left(-L_f h + v\right) \tag{76}$$

Where, $v = \dot{y}$.

If $L_g h(x) = 0$ for all *x*, then the second derivative is evaluated.

The $\dot{y}$ can be differentiated to have

$$\ddot{y} = L_f{}^2 h(x) + L_g L_f h(x)u \tag{77}$$

If $L_g L_f h(x)u = 0$ for all *x* then it is differentiated again until some integer n, so that

$L_g L_f{}^{n-1} h(x)u \neq 0$ of the general equation

$$y^n = L_f{}^n h(x) + L_g L_f{}^{n-1} h(x) u \quad (78)$$

$L_g L_f{}^{n-1} h(x) u \neq 0$ indicates the relative degree of the system is n.

Hence, the relative degree of the system is determined by the lie derivative.

Now continuing the equation the input transformation is obtained, given by

$$u = \frac{1}{L_g L_f{}^{n-1} h}\left(-L_f{}^n h + v\right) \quad (79)$$

Here, $v = y^n$.

From Eq. (19), Eq. (72) and Eq. (79)

$$f(x) = \begin{bmatrix} \dot{x} \\ \dfrac{ml\left(m l g \sin\theta.\cos\theta + (I + ml^2)\dot{\theta}^2 \sin\theta\right)}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta} \\ \dot{\theta} \\ -\dfrac{ml\left((M+m) g \sin\theta + ml\dot{\theta}^2 \sin\theta.\cos\theta\right)}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta} \end{bmatrix}$$

And

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{(I+ml^2)}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta} \\ 0 \\ -\dfrac{ml\cos\theta}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta} \end{bmatrix} \quad (80)$$

The second lie derivative of y is given by

$$L_f{}^2 h(x) = \frac{ml\left(m l g \sin\theta.\cos\theta + (I + ml^2)\dot{\theta}^2 \sin\theta\right)}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta}$$

And

$$L_g L_f h(x) = \frac{I + ml^2}{(I+ml^2)(M+m) - m^2 l^2 \cos^2\theta} \quad (81)$$

For the cart-pole system output is the column vector two elements: cart-position (x) and pendulum angle (θ).

$$h = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{82}$$

Let, $y = x$

On applying lie derivative

$\dot{y} = \dot{x}$

So, $L_g h(x) = 0$

$\dot{y} = \dot{x} = f(x) + g(x)u$

The second lie derivative of y is given by

$$L_f{}^2 h(x) = \frac{ml\left(m\lg\sin\theta.\cos\theta + (I + ml^2)\dot{\theta}^2\sin\theta\right)}{(I + ml^2)(M + m) - m^2l^2\cos^2\theta}$$

And

$$L_g L_f h(x) = \frac{I + ml^2}{(I + ml^2)(M + m) - m^2l^2\cos^2\theta} \tag{83}$$

If

$y = \theta$

$\dot{y} = \dot{\theta}$

And

$L_g h(x) = 0$

The second lie derivative of y is given by

$$L_f{}^2 h(x) = -\frac{ml\left((M + m)\mathrm{g}\sin\theta + ml\dot{\theta}^2\sin\theta.\cos\theta\right)}{(I + ml^2)(M + m) - m^2l^2\cos^2\theta}$$

And

$$L_g L_f h(x) = -\frac{ml\cos\theta}{(I+ml^2)(M+m)-m^2l^2\cos^2\theta} \tag{84}$$

So,

$$\ddot{y} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_4 \end{bmatrix} + \begin{bmatrix} g_2 \\ g_4 \end{bmatrix} u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$u = \begin{bmatrix} g_2 \\ g_4 \end{bmatrix}^{-1} \begin{bmatrix} v_1 - f_2 \\ v_2 - f_4 \end{bmatrix} \tag{85}$$

From this the degree of freedom is found to be two.

4.2.1 Classical Feedback Linearization

The general structure for the fictitious output [23] can be of the form

1. $\eta = x + f(\theta, x)$
2. $\eta = x + f(\theta)$ (86)

The 1st one validates if the system is completely linearizable (full state linearizable) and if not then 2nd option is used.

The dynamic equation of the pendulum using the collocated partial feedback linearization is given by

$$\ddot{\theta} = a\sin\theta - b\cos\theta . u \tag{87}$$

Where,

$a = \frac{g}{l}$, $b = \frac{1}{l}$ and $u = \ddot{x}$, u is called as the new control input.

The said cart-pole system is not full state linearizable, so according to the pseudo code Eq.(86) and Eq. (87), the second derivative will be

$$\ddot{\eta}_1 = u(1 - f'(\theta)b\cos\theta) + f'(\theta)a\sin\theta + f''(\theta).\dot{\theta}^2$$

$$= v \tag{88}$$

v is the external controller.

The input feedback linearization control is given by

$$u = \frac{-v + f'(\theta)a\sin\theta + f''(\theta).\dot{\theta}^2}{f'(\theta)b\cos\theta - 1} \tag{89}$$

For the zero dynamics the control input obtained is given by

$$u_0 = \frac{f'(\theta)a\sin\theta + f''(\theta).\dot{\theta}^2}{f'(\theta)b\cos\theta - 1} \tag{90}$$

To stabilize the zero dynamics, the function $f(\theta)$ needs to be chosen. For the above zero dynamics $f(\theta) = k_1 \sin\theta$. So,

$$\eta_1 = x + k_1 \sin\theta \tag{91}$$

The energy of the zero dynamics is

$$E_0 = \frac{1}{2}(k_1 b\cos^2\theta - 1).\dot{\theta}^2 + a(1 - \cos\theta) \tag{92}$$

Asymptotically stability at the origin v can't be proved, so an extra term (u_1) to the zero dynamics control input is added to have stability.

u_1 is taken in such a way that $\dot{E}_0 \le 0$. The u_1 is taken as

$$u_1 \square k_2 \frac{\dot{\theta}\cos\theta}{k_1 b\cos^2\theta - 1} \tag{93}$$

The new output (η_2), obtained from the zero dynamics is given by

$$\ddot{\eta}_2 = \ddot{\eta}_1 + k_2\dot{\theta}\cos\theta$$

$$\dot{\eta}_2 = \dot{\eta}_1 + k_2 \sin\theta \tag{94}$$

Where, the control gains k_1 and k_2 are given as

$k_1 > 1/b > 0$ and $k_2 > 0$.

So, the control input for the zero dynamics is given by

$$u_0 + u_1 = \frac{k_1 \sin\theta(a\cos\theta - \dot{\theta}^2)}{k_1 b\cos^2\theta - 1} + k_2 \frac{\dot{\theta}\cos\theta}{k_1 b\cos^2\theta - 1} \quad (95)$$

Now to make the zero dynamics of the system asymptotically stable the external controller is taken as

$$v = -k_3\dot{\eta}_2 \quad (96)$$

Where, $\dot{\eta}_2 = \dot{x} + k_1\dot{\theta}\cos\theta + k_2\sin\theta$.

Now the Lyapunov function for the input control can be determined as

$$V = E_0 + \frac{1}{2}k_3\dot{\eta}_2^2 \quad (97)$$

From this Lyapunov function the external controller for the system can be derived, which is given by

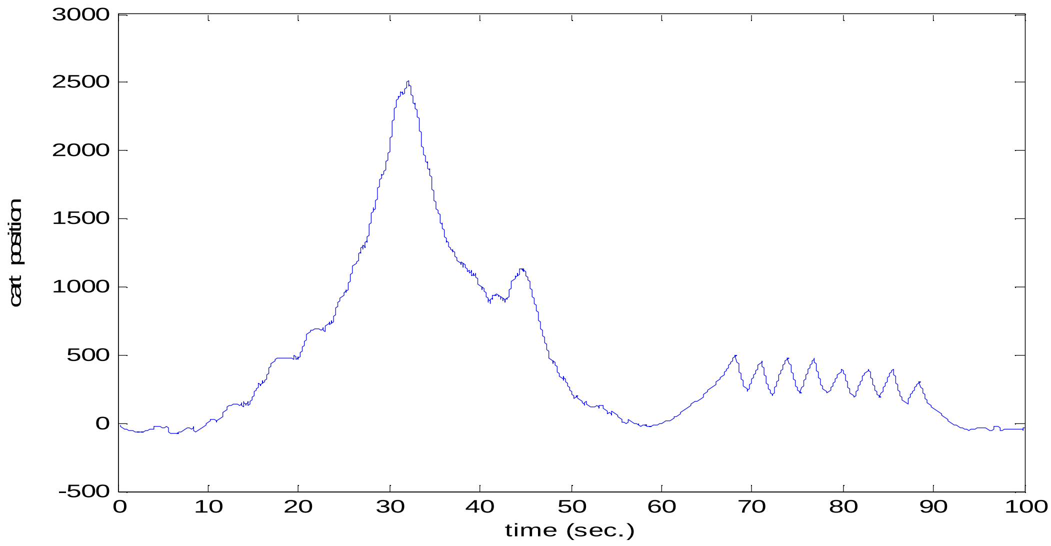
$$v = -k_4(b\dot{\theta}\cos\theta + k_3\dot{\eta}_2) \quad (98)$$

Where, $k_4 > 0$

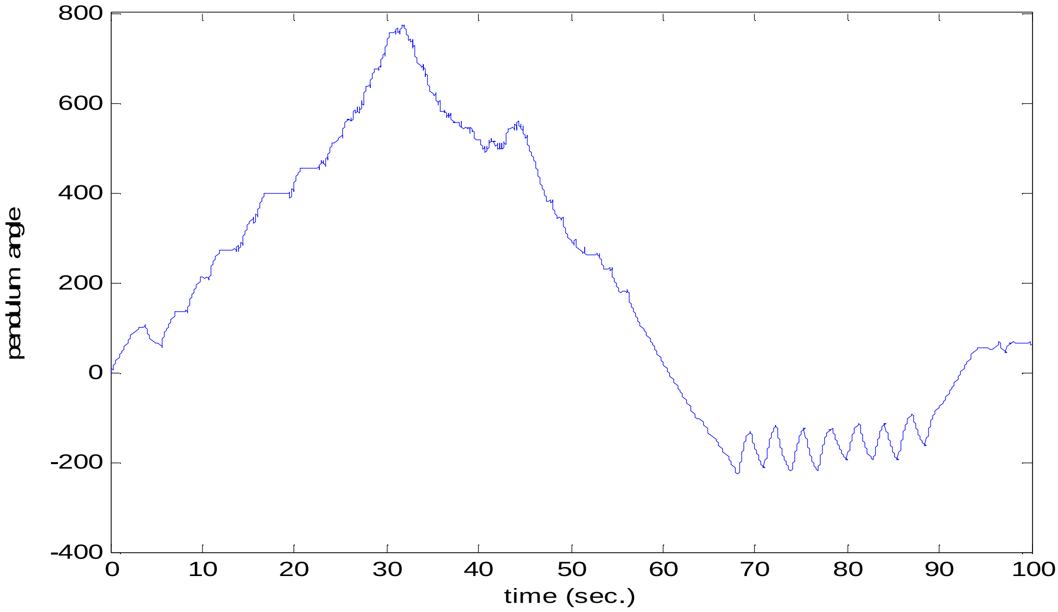
Now the input feedback linearization control obtained for the system is given by

$$u = \frac{-v + k_1\sin\theta(a\cos\theta - \dot{\theta}^2) + k_2\dot{\theta}\cos\theta}{k_1 b\cos^2\theta - 1} \quad (99)$$

The stabilized system output using the classical feedback linearization is shown in figure 21.



(a)



(b)

Fig. 20: Results of classical feedback linearization: (a) Cart position (b) Pendulum angle

4.2.2 Partial (Input-Output) Feed-back Linearization

The compact form of Euler-Lagrange equation [22, 24, 25] for system with two degree of freedom is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + h(p) = Q_q \qquad (100)$$

Where, $M(q)$ is a symmetric square matrix and is given by

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad Q_q = \begin{bmatrix} u \\ 0 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \; h(p) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

(101)

So,

$$C(q,\dot{q}) = \begin{bmatrix} 0 & ml\cos\theta \\ 0 & 0 \end{bmatrix}$$

$$h(p) = \begin{bmatrix} 0 \\ -m\lg\sin\theta \end{bmatrix}$$

$$M(q) = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & I+ml^2 \end{bmatrix} \qquad (102)$$

The dynamic system with degree of freedom two can be expanded as

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_1 + C_1 + h_1 = 0 \qquad (103)$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_1 + C_2 + h_2 = u \qquad (104)$$

It is divided into two types: Collocated and Non-collocated Linearization.

Collocated Linearization

Let us consider the first equation

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_1 + C_1 + h_1 = 0$$

So,

$$\ddot{q}_1 = -M_{11}^{-1}\left(M_{12}\ddot{q}_2 + C_1 + h_1\right)$$

Putting $\ddot{q}_1$ in the eq. (104)

$$\bar{M}_{22}\ddot{q}_2 + \bar{C}_2 + \bar{h}_2 = u \qquad (105)$$

Where,

$$\bar{M}_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}$$

$$\bar{C}_2 = C_2 - M_{21}M_{11}^{-1}C_1$$

$$\bar{h}_2 = h_2 - M_{21}M_{11}^{-1}h_1$$

The feedback linearization controller for the collocated mode is given as

$$u = \bar{M}_{22}v_2 + \bar{C}_2 + \bar{h}_2 \qquad (106)$$

Where,

$v_2 = \ddot{q}_2$.

v_2 behaves like an external control input and is given by

$$v_2 = \ddot{q}_{2d} + k_d(\dot{q}_{2d} - \dot{q}_2) + k_p(q_{2d} - q_2) \qquad (107)$$

$\ddot{q}_{2d}$, $\dot{q}_{2d}$ and q_{2d} are the desired trajectories.

According to the swing up technique

$q_{2d} = \frac{\pi}{2}$, so $\dot{q}_{2d} = 0 = \ddot{q}_{2d}$.

Thus,

$$v_2 = k_p(q_{2d} - q_1) - k_d\dot{q}_2 \qquad (108)$$

Non-collocated Linearization

$$\ddot{q}_{22} = -M_{12}^{+}[M_{11}\ddot{q}_1 + C_1 + h_1] \qquad (109)$$

M_{12}^{+} is the right Moore-Penrose Pseudo-Inverse which is given by

$$M_{12}^{+} = M_{12}^{T}\left(M_{12}M_{12}^{T}\right)^{-1}$$

Putting $\ddot{q}_{22}$ in the eq. (104)

$$\bar{M}_{21}\ddot{q}_1 + \bar{C}_2 + \bar{h}_2 = u \quad (110)$$

Where,

$$\bar{M}_{21} = M_{21} - M_{22}M^{+}{}_{12}M_{11}$$

$$\bar{C}_2 = C_2 - M_{22}M_{12}{}^{+}C_1$$

$$\bar{h}_2 = h_2 - M_{22}M_{12}{}^{+}h_1$$

For

$v_2$in terms of pseudo inverse $M_{12}{}^{+}$ can be defined as

$$v_2 = -M_{12}{}^{+}[M_{11}v_1 + C_1 + h_1] \quad (111)$$

Where,

$$v_1 = \ddot{q}_1$$

v_1 is called as the additional control input and is given by

$$v_1 = \ddot{q}_{1d} + k_d(\dot{q}_{1d} - \dot{q}_1) + k_p(q_{1d} - q_1) \quad (112)$$

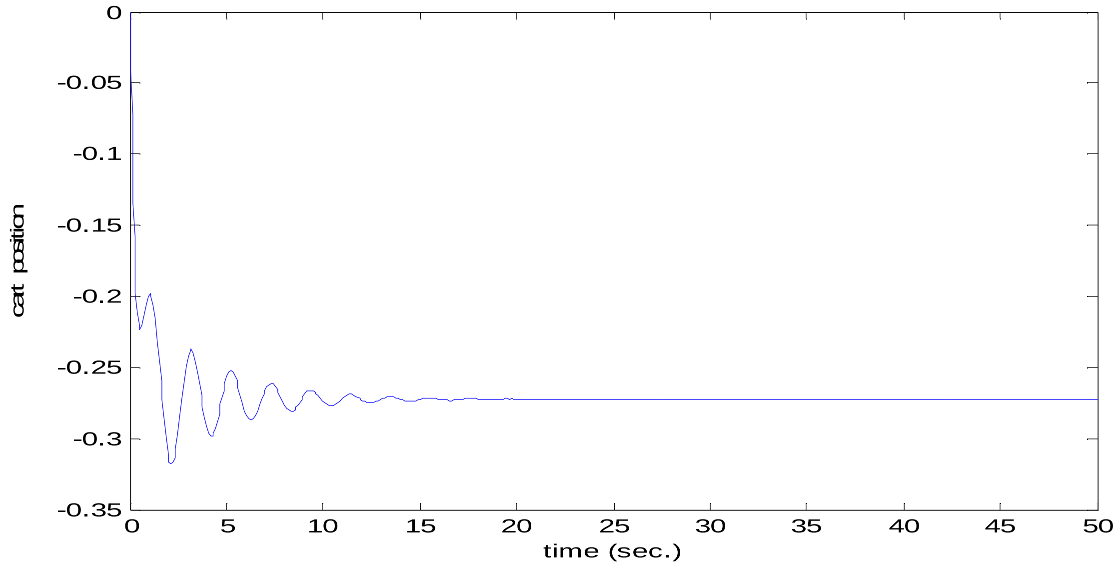
By swing up control technique q_{1d} has some finite value, so

$$\dot{q}_{1d} = 0 = \ddot{q}_{1d}.$$

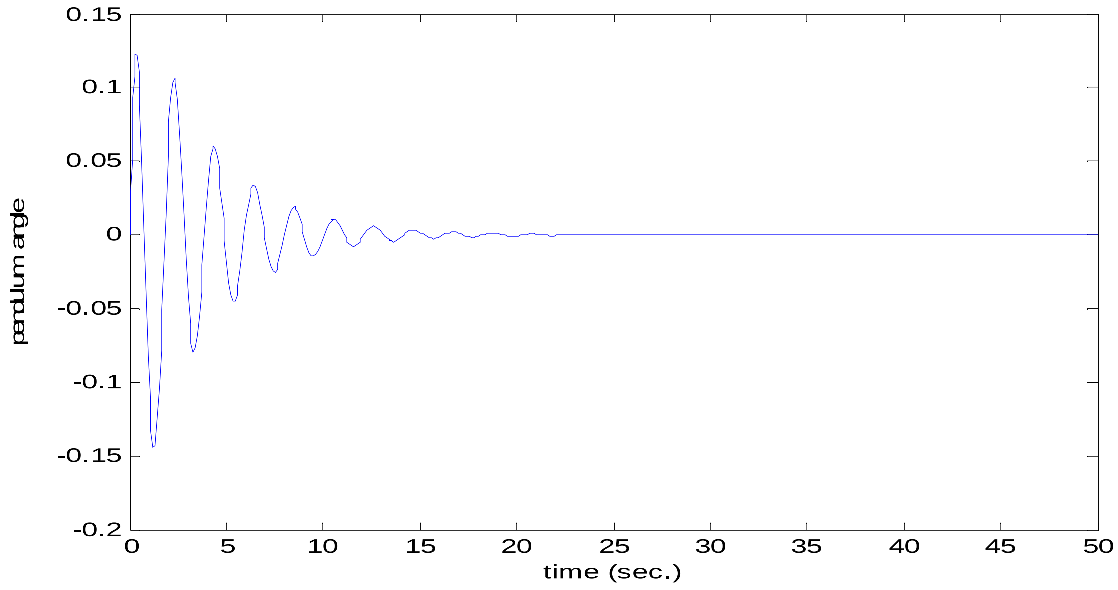
Hence, the additional control input is given by

$$v_1 = k_p(q_{1d} - q_1) - k_d\dot{q}_1 \quad (113)$$

The output using the partial feedback linearization is shown in figure 19.



(a)



(b)

Fig. 21: Result of Partial feedback linearization: (a) Position (b) Angle

4.3 Summary

This chapter develops different controllers (linear and nonlinear) for cart-pole system.

CHAPTER 5

CONCLUSION

- Conclusion
- Future Work

CHAPTER 5

CONCLUSION

5.1 Conclusion

This thesis studies different methods of system identification of a physical system and develops suitable controllers. The classical example of control system is “cart-pendulum system” which is considered as the physical system and different linear and non-linear system identification techniques are used to identify the dynamics of the system. The identified dynamics of the system is further validated with the actual mathematical model obtained from the first principle model.

Several parametric model identification structure such as, ARX, ARMAX, OE and BJ are evaluated to identify the linear mathematical model of the system. Considering the nonlinear dynamics of the system, neural network topology is used for identification.

By comparing MSE, FPE and Fit %, BJ is found to be the best identification method for the linear model. Comparing MSE the Elman network is proved to be better than the Feed-forward network for identification of the non-linear system.

After identification different controllers are designed to achieve the desired control objective. Some of the linear controllers which are designed are LQR, LQR-Pole-placement-PID, LQG and H-infinity techniques. Some intelligent controllers are also developed such as fuzzy controller and fuzzy-PID controller.

Some of the nonlinear controllers, designed to meet the control requirements, are classical feedback linearization and partial feedback linearization control law.

5.2 Future Work

1. The control laws discussed above can be implemented in an FPGA unit.
2. Different neural and evolutionary control techniques can be implemented for better performance.
3. Instead of analog domain discrete domain can be used and digital controllers can be designed to reduce time and increase speed of operation.

DISSEMINATION OF THE RESEARCH WORK

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