MULTIPLIERS BETWEEN ORLICZ SEQUENCE SPACE

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By

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Certificate

This is to certify that the project work embodied in the dissertation **Multipliers between Orlicz Sequence Space** which is being submitted by **Sunima Naik, Roll No. 412MA 2115**, has been carried out under my supervision.

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(Sunima naik)

ABSTRACT

Let M, N be Orlicz functions and let $D(l_M, l_N)$ be the space of all diagonal operators (that is multipliers) acting between the Orlicz sequence spaces l_M and l_N. We prove that the space of multipliers $D(l_M, l_N)$ coincides with (and is isommorphic to) the Orlicz sequence space $l_{M_N^*}$, where M_N^* is the Orlicz function defined by $M_N^*(\lambda) = \sup\{N(\lambda x) - M(x), x \in (0,1)\}$.

Key words and phrases. Compact multiplier, Linear operator, Multipliers, Orlicz function, Orlicz sequence space, $\triangle_2$-condition.

Contents

CHAPTER 1

1 Historical Glimpses and Motivation

1.1 Orlicz and Modular Sequence Spaces

Orlicz spaces have their origin in the Banach space researches of 1920. Indeed, after the development of Lebesgue theory of integration and inspired by the functions t^p in the definitions of the spaces l^p and L^p, Orlicz spaces were rst proposed by Z. W. Birnbaum and W. Orlicz in [1] and later developed by Orlicz himself in [12], [13]. The study and applications of this theory was picked up again in Poland, USSR and Japan after the war years. Around the year 1950, H. Nakano [11] studied Orlicz spaces with the name 'modulared spaces'. However, the theory became popular for researches in the western countries after the publication of the book Linear Analysis by A. C. Zaanen. This possibly resulted in the translation of the monograph of M. A. Krasnoselskii and Ya.B. Rutickii on Convex Functions and Orlicz Spaces by Leo F. Boron from Russian to English, and after the appearance of the English version of this book in 1961, the theory has been eectively used in many branches of Mathematics and Statistics, e.g, dierential and integral equations, harmonic analysis, probability etc.

Prior to the researches of W. Orlicz, it was W. H. Young [18] who, motivated by the functions $u^p(u \geq 0)$ and $v^p(v \geq 0)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 < p, q < \infty$, introduced a function $v = \varphi(u)$ for $u \geq 0$ such that φ is continuous and strictly increasing with $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. If $u = \psi(v)$ is inverse of φ, he defined

$$\Phi(a) = \int_0^a \varphi(u)du \quad , \quad \Psi(b) = \int_0^b \psi(v)dv$$

for $a, b \geq 0$. These functions are known as Youngs functions in the literature, and besides being convex, satisfy the Youngs inequality

$$ab \leq \Phi(a) + \Psi(b)$$

for $a, b \geq 0$. Young introduced the classes Y_Φ and Y_Ψ consisting of measurable functions f for which $\int \Phi(|f(x)|)dx < \infty$ and $\int \Psi(|f(x)|)dx < \infty$, respectively. These spaces failed to form the vector spaces. However, if Φ satisfies $\triangle_2$condition in the sense that there exists a constant $C > 0$ such that $\Phi(2u) \leq C\Phi(u)$ holds for all $u \geq 0$,Y_Φ becomes a vector space. In the process of norming the spaces Y_Φ, Y_Ψ, Orlicz considered the class L_Φ of all measurable functions f satisfying

$$|f|_\Phi = \sup\left\{\int |fg|dx : \int \Psi(|g|)dx \leq 1\right\} < \infty$$

and proved that $(L_\Phi, \|.\|_\Phi)$ is a normed linear space. In general, $Y_\Phi \subset L_\Phi$, however, if ϕ satisfies $\triangle$ condition defined as above, $Y_\Phi = L_\Phi$, cf.[12], [13]

Orlicz sequence spaces which are of the particular type of Orlicz spaces, attracted the attention of mathematicians with a certain specic purpose of solving problems in Banach space theory. Though the spacesl_M and h_M were respectively introduced by Orlicz [12] and Y. Gribanov [3], a clear and detailed exposition of results on Orlicz sequence spaces was given by K. Lindberg [7,8] who got interested in finding the solution of the problem, Does a Banach space have a complemented subspace isomorphic to either l^p, $1 \leq p < \infty$, or c_0? He succeeded in getting the affirmative answer to this problem for certain type of Orlicz sequence spaces. Lindberg also proved that h_M is a closed subspace of l_M for which the sequence e^n, forms a symmetric basis. He also considered the $\triangle_2$-condition for small x and proved the equivalence of this condition with the equality of the spaces h_M and l_M.

In 1971 J. Lindendenstruss and L. Tzafriri [9], besides proving that every Orlicz sequence space contains a subspace isomorphic to l_P for some $p \geq 1$, showed that there are Orlicz sequence spaces with a unique (upto equivalence) symmetric basis. In their subsequent work [9], they investigated the structure of those subspaces of an Orlicz sequence space which are themselves Orlicz sequence spaces. They also considered an example of a reexive Orlicz sequence space which does not contain any $l_p,\ 1 < p < \infty$,as a complemented subspace. In [10] Lindendenstruss and Tzafriri considered the problem of identifying those ps for which l_p is isomorphic to a subspace of a given Orlicz sequence space l_M. Indeed, they proved that if an Orlicz function M satisfies $\triangle_2$ condition at 0, then is l_p isomorphic to a subspace of l_M iff $p \in [\alpha_M, \beta_M]$, where $\alpha_M = \sup\left\{p : \sup_{0<x,t\leq 1} \dfrac{M(ts)}{M(t)x^p} < \infty\right\}$ and $\beta_M = \inf\left\{p : \sup_{0<x,t\leq 1} \dfrac{M(ts)}{M(t)x^p} > 0\right\}$.

Almost at the same time in 1973, J. Y. T. Woo [16] generalized the concept of Orlicz sequence spaces to modular sequence spaces which envelope the modular sequence spaces considered earlier by H. Nakano [11] as particular case; indeed take $M_n(x) = x^{p_n}$ for some $p \in [1, \infty)$ in definition of Modular sequence spaces from Section 1.3. He generalized some of the results of Lindendenstruss and Tzafriri proved by them in [9] to modular sequence spaces; for instance, they proved that every modular sequence space contains l^p for some $p \in [1, \infty)$. Indeed, in order to carry out their investigations on modular sequence spaces for results valid for Orlicz sequence spaces, Woo introduced the notion of almost equality for two sequences $\{M_n\}$ and $\{N_n\}$ of Orlicz functions ($\{M_n\}$ and $\{N_n\}$ are said to be almost equal if there exists $a_n > 0$ for all $n \in N$ such that $M_n(x) = N_n(x)$ for all $x > a_n$ and $\sum_n M_n(a_n) < \infty$) and the uniform $\triangle_2$-condition (cf. p-13 for denition). He also investigated the duals of modular sequence spaces and characterized their reflexivity. This study was further continued by him in [17].

In 1977, N. J. Kalton [5], studied Orlicz sequence spaces without the condition of convexity in the denition of an Orlicz function. He observed that in case of locally bounded spaces, some results of Lindendenstruss and Tzafriri hold with identical proofs (as pointed out by them in [10], p-369); but it was not so with the non-locally bounded case. He succeeded in nding many interesting features distinguishing the two theories.

CHAPTER 2

2 Preliminaries

2.1 Definitions

Definition 3.1.1. Let M be function with

$$M : [0, \infty) \to [0, \infty)$$

is continuous, non-negative, convex, even function such that $M(0) = 0$ and for some $a \neq 0, M(a) \neq 0$.

Then $M(x)$ is called an Orlicz function.

Definition 3.1.2. (Orlicz Sequence Space) l_M is a Banach space of real sequence, $\{x_i\}_{i=1}^{\infty}$ such that for some $r > 0$, $\sum_i M\left(\frac{x_i}{r}\right) < \infty$ that is,

$$l_M = \left\{ \{x(i)\} : \sum_{i=1}^{\infty} M\left(\frac{x_i}{r}\right) < \infty, \text{ for some } r > 0 \right\}$$

.

with the norm,

$$\|\{x_i\}_{i=1}^{\infty}\|_M = \inf\left\{ r > 0 : \sum_i M\left(\frac{x_i}{r}\right) \leq 1 \right\}$$

.

The space l_M with norm $\|.\|_M$ is called Orlicz sequence space.

Representation of $M(x)$: An Orlicz function $M(x)$ has the representation,

$$M(x) = \int_0^{|x|} p(t)dt$$

where $p(t)$ is the right derivative of $M(x)$.

If $p(t)$ is a right continuous, non decreasing, non negative function defined on the non negative reals. Then $\int_0^{|x|} p(t)dt$ call the representation of $M(x)$.

According to behaviour of $p(t)$: there will be three distinct properties of p

(i) There is an $a > 0$ such that $p(0) = a$.

(ii) There is an $x_0 > 0$ such that for $0 \leqslant x \leqslant x_0, \;\; p(x) = 0$.

(iii) $p(0) = 0$ and for $x > 0, \;\; p(x) > 0$.

First two generate l_1 and l_∞ respectively.

The third is called M- function.

Definition 3.1.3. (Complementary M- function): Let $M(x)$ be an M- function with representation $M(x) = \int_0^{|x|} p(t)dt$

Define,

$$q(s) = \sup_{p(t) \leqslant s} t$$

Then $q(s)$ is a right-continuous, non-decreasing function defined on the non-negative reals such that

$$q(0) = 0 \text{ and } q(s) > 0 \text{ for } s > 0$$

$$N(x) = \int_0^{|x|} tq(s)ds$$

is an M-function and it is called the M-function complementary to $M(x)$.

$M(x)$ is also complementary of $N(x)$. Hence $M(x)$ and $N(x)$ are complementary M-function.

Example: Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $M(x) = \frac{1}{p}x^p$ and $N(x) = \frac{1}{q}x^q$ are complementary M-function.

Definition 3.1.4. (The sequence space $\bar{l_M}$)

$$\bar{l_M} = \left\{ \{x(i)\}_{i=1}^{\infty} : \||\{x_i\}_{i=1}^{\infty}\||_M = \sup_{\sum N(y_i) \leq 1} \left(\sum x_i y_i \right) < \infty \right\}$$

The space $\bar{l_M}$ with norm $\|.\|_M$ is a Banach space over the reals.

h_M**- space** :-

Let $M(x)$ be an Orlicz function and

$$h_M = \left\{ \{x_i\}_{i=1}^{\infty} : \sum_i M\left(\frac{x_i}{r}\right) < \infty, \text{for all } \ r > 0 \right\}$$

with

$$\|\{x_i\}_{i=1}^{\infty}\|_M = \inf \left\{ r > 0; \sum_i M\left(\frac{x_i}{r}\right) \leq 1 \right\}$$

h_M is a subset of l_M.

Unit Vector Space:-

The set $\{e_i\}_{i=1}^{\infty}$ where $e_i = \{0, 0, ..., 0, 1, 0, ...\}$ (1 in the ith position) is the set of unit vector for the Orlicz space l_M. The unit vectors form a symmetric basis for c_0 (space of sequence convergences to zero with supremum norm) and $l_p, 1 \leqslant p < \infty$. This basis is called the unit vector basis.

Definition 3.1.5.: Let $M_1(x)$ and $M_2(x)$ are equivalent if there are positive constants A, B, a, b and x_0 , such that for all $0 \leqslant x \leqslant x_0$,

$$AM_2(ax) \leqslant M_1(x) \leqslant BM_2(bx).$$

2.2 Basic Results of Orlicz sequence spaces.

Proposition 3.2.1. *Let $M(x)$ be an M-function and let $\{x_i\}_{i=1}^{\infty}$ be a real sequence. Then $\{x_i\}_{i=1}^{\infty}$ is in l_M iff $\{x_i\}_{i=1}^{\infty}$ is in $\bar{l}_M$. And,*

$$\|\{x_i\}_{i=1}^{\infty}\|_M \leq |||\{x_i\}_{i=1}^{\infty}|||_M \leq 2\|\{x_i\}_{i=1}^{\infty}\|_M.$$

Proposition 3.2.2. *Let $M(x)$ and $N(x)$ be complementary M-functions. Let $M(x) = \int_o^{|x|} p(t)dt$ be the representation of $M(x)$. Then,*

(a) *For all $x, y \geq 0,\ \ xy \leq M(x) + N(x)$.*

(b) *For all $x \geq 0,\ \ xp(x) = M(x) + N(x)$.*

(c) *For all $\{x_i\}_{i=1}^{\infty}$ in l_M,*

$$\sum_i x_i y_i \leq |||\{x_i\}_{i=1}^{\infty}|||_M \qquad if \ \ \sum N(y_i) \leq 1,$$

and

$$\sum_i x_i y_i \leq |||\{x_i\}_{i=1}^{\infty}|||_M \cdot \sum N(y_i) \qquad if \ \ \sum N(y_i) \leq 1.$$

Proposition 3.2.3. *An M-function $M(x)$ satisfies the $\triangle_2$-condition for small x if for all $\mathbf{Q} > 0$ there are $K > 0$ and $x_0 > 0$ such that $M(\mathbf{Q}x) \leq KM(x)$ for all $0 \leq x \leq x_0$. Using convexity of $M(x)$, it follows that the above is equivalent to the existence of K and x_0 such that $M(\mathbf{2}x) \leq KM(x)$ for all $0 \leq x \leq x_0$.*

Proposition 3.2.4. *Let $M(x)$ be an M-function with representation $M(x) = \int_0^{|x|} p(t)dt$. $M(x)$ satisfies the $\triangle_2$-condition for small x iff there are $K > 0$ and $x_0 > 0$ such that $1 \leq \dfrac{xp(x)}{M(x)} \leq K$ for all $0 < x < x_0$.*

Proposition 3.2.5. *Let $M(x)$ be an M-function. The following are equivalent*:

(a) *$M(x)$ satisfies the $\triangle_2$-condition for small x.*

(b) $l_M = h_M$.

(c) *l_M is separable.*

(d) *l_M has a symmetric basis.*

Proposition 3.2.6. *Let $M_1(x)$ and $M_2(x)$ be Orlicz functions. If $M_1(x)$ is equivalent to $M_2(x)$, then l_{M_1} and l_{M_2} are isomorphic. If l_{M_1} and l_{M_2} are separable, then $M_1(x)$ is equivalent to $M_2(x)$ iff the unit vectors bases of l_{M_1} and l_{M_2} are equivalent.*

Proposition 3.2.7. *Let $M(x) = \int_0^{|x|} p(t)dt$ be an M-function which satisfies the $\triangle_2$-condition for small x. Then there is an M-function $M_1(x)$ which is equivalent to $M(x)$ such that $M_1(x)$ has the representation $M_1(x) = \int_0^{|x|} p_1(t)dt$, where $p_1(t)$ is a continuous, strictly increasing function with $p_1(0) = 0$.*

CHAPTER 3

3 Multipliers between Orlicz Sequence Space

Let $M(t)$, $t \geq 0$ be a Orlicz function i.e,

$$M : [0, \infty) \to [0, \infty)$$

is convex, non-decreasing (constant and increasing) such that $M(0) = 0$ and $M(t) \to \infty$ as $t \to \infty$.

Orlicz sequence space

$$l_M = \left\{ \{x_i\}_{i=1}^{\infty} : \sum_i M\left(\frac{|x_i|}{\alpha}\right) < \infty;\ \text{for some } \alpha > 0 \right\}$$

is a linear space.

Equipped with norm

$$\|x\|_M = \inf\left\{ \rho : \sum_i M\left(\frac{|x_i|}{\alpha}\right) \leqslant 1 \right\}$$

is Banach space, that is, it is complete(Cauchy sequence is convergences).

To show : l_M **is complete**.

Let (x_n) be any Cauchy sequence in the space l_M where $x_m = (\xi_1^m, \xi_2^m, ...,)$. Then for every $\epsilon > 0$, there is an N such that for all $m, n > N$,

$$\begin{aligned} d(x_m, x_n) \quad &= |x_m - x_n| < \epsilon \\ &= \sum_i \left| M\left(\frac{|\xi_i^m|}{\alpha}\right) - M\left(\frac{|\xi_i^n|}{\alpha}\right) \right| < \epsilon,\ \textit{for some } \alpha > 0 \\ &= \sum_i \left| M\left(\frac{|\xi_i^m|}{\alpha} - \frac{|\xi_i^n|}{\alpha}\right) \right| < \epsilon. \end{aligned}$$

It follows that for every $i = 1, 2, ...$ we have

$$\left| M\left(\frac{|\xi_i^m|}{\alpha}\right) - M\left(\frac{|\xi_i^n|}{\alpha}\right) \right| < \epsilon$$

we choose a fixed i, we see that $(\xi_i^1, \xi_i^2, ...)$ is a Cauchy sequence of numbers. It convergences since R and C are complete.

say $\xi_i^m \to \xi_i$ as $m \to \infty$, using this limit.

We define

$$x = (\xi_1, \xi_2, ...)$$

To show : $x \in l_M$ and $x_m \to x$.

we have for all $m, n > N$.

$$\sum_{i=1}^{k} \left| M\left(\frac{|\xi_i^m|}{\alpha}\right) - M\left(\frac{|\xi_i^n|}{\alpha}\right) \right| < \epsilon \qquad (k = 1, 2, ...)$$

letting $n \to \infty$, we obtain for $m > N$.

$$\sum_{i=1}^{\infty} \left| M\left(\frac{|\xi_i^m|}{\alpha}\right) - M\left(\frac{|\xi_i^n|}{\alpha}\right) \right| < \epsilon \tag{1}$$

This show that $x_m - x = \left(M\left(\frac{|\xi_i^m|}{\alpha}\right) - M\left(\frac{|\xi_i|}{\beta}\right)\right) \in l_M$

since $x_m \in l_M$.

Hence by triangle inequality,

$$\begin{aligned}
|\xi_i| &= |(\xi_i - \xi_i^m) + (\xi_i^m)| \\
&\leqslant |\xi_i - \xi_i^m| + |\xi_i^m| \\
&\Rightarrow \sum_i M\left(\frac{|\xi_i|}{\alpha}\right) \leqslant \sum_i M\left(\frac{|\xi_i - \xi_i^m|}{\alpha}\right) + \sum_i M\left(\frac{|\xi_i|}{\alpha}\right) \\
&< 2\epsilon = \epsilon.
\end{aligned}$$

This inequality holds for every i,

This implies that $x = (\xi_i) \in l_M$ and from (1) we obtain $d(x_m, x) < \epsilon$. This show that $x_m \longrightarrow x$.

Since (x_m) was arbitrary Cauchy sequence, So l_M is complete.

Definition ($\triangle_2$-condition): An Orlicz function $M(t)$ is said to satisfy the $\triangle_2$-condition if

$$M(2t) \leq CM(t), \; t \in (0, 1), \quad \text{for some constant} \;\; C > 0.$$

Proposition 4.1.:

Let M be an Olicz function, then the subspace

$$h_M = \left\{ x = (x_i) : \sum M\left(\frac{|x_i|}{\rho}\right) < \infty,\ \forall\, \rho > 0 \right\}$$

is a closed subspace of l_M and the vectors $(e_n)_1^\infty$ (where $e_n = (e_{ni}), e_{ni} = 0$ if $i \neq n$, $e_{nn} = 1$) form a basis in it.

Proof.

Let $X_n \in h_M$ where $h_M = \left\{ x = (x_i^n) : \sum_i M\left(\frac{|x_i|}{r}\right) < \infty,\ \text{ for all } r > 0 \right\}$ and $X_n = \{x_i^n\}_{i=1}^\infty$ and

$$\|X_n\|_M = \inf\left\{ r > 0 : \sum_i M\left(\frac{|x_i^n|}{r}\right) \leqslant 1 \right\}$$

Let X_n converges to X_0 in the norm $\|.\|_M$.

To show: If $X_0 = \{x_i^{(0)}\}_{i=1}^\infty$, then $\sum_i M\left(\dfrac{x_i^{(0)}}{r}\right) < \infty$, for all $r > 0$.

But since $\frac{1}{r}X_n \longrightarrow \frac{1}{r}X_0$, for each $r > 0$ and let $\frac{1}{r}X_n \in h_M$.

It is only necessary to show : $\sum_i M\left(x_i^{(0)}\right) < \infty$.

For each n, let $X_n = \{x_i^{(n)}\}_{i=1}^\infty$. Choosing N such that for $n \geqslant N$,

$$\|X_n - X_0\|_M \leqslant \frac{1}{2}$$

it follow that,

$$\begin{aligned}
&\|X_n - X_0\|_M \leqslant \frac{1}{2} \\
\Rightarrow\ & 2\|X_n - X_0\|_M \leqslant 1 \\
\Rightarrow\ & 2\|x_i^{(n)} - x_i^{(0)}\| \leqslant \frac{\|x_i^{(n)} - x_i^{(0)}\|}{\|X_n - X_0\|_M} \leqslant 1,\ \ \forall n \geqslant N \\
\Rightarrow\ & M\left(2\|x_i^{(n)} - x_i^{(0)}\|\right) \leqslant M\left(\frac{\|x_i^{(n)} - x_i^{(0)}\|}{\|X_n - X_0\|_M}\right) \leqslant 1,\ \ \forall n \geqslant N \\
\Rightarrow\ & \sum_i M\left(2\|x_i^{(n)} - x_i^{(0)}\|\right) \leqslant \sum_i M\left(\frac{\|x_i^{(n)} - x_i^{(0)}\|}{\|X_n - X_0\|_M}\right) \leqslant 1,\ \ \forall n \geqslant N
\end{aligned}$$

Therefore convexity of $M(x)$,

$$\begin{aligned}
\sum_i M(x_i^{(0)}) &= \sum_i M\left(\frac{2x_i^{(n)} - 2x_i^{(n)} + 2x_i^{(0)}}{2}\right) \\
&= \sum_i M\left(\frac{2x_i^{(n)} - 2(x_i^{(n)} - x_i^{(0)})}{2}\right) \\
&\leqslant \frac{1}{2}\sum_i M\left(2\left(x_i^{(n)}\right)\right) + \frac{1}{2}\sum_i M\left(2|x_i^{(n)} - x_i^{(0)}|\right) \\
&< \infty.
\end{aligned}$$

Hence h_M is closed in l_M. Let $\{e_i\}_{i=1}^{\infty}$ be the set of unit vector. If for each $\{x_i\}_{i=1}^{\infty}$ in h_M, $\sum_i x_i e_i$ convergence in the norm $\|.\|_M$ to $\{x_i\}_{i=1}^{\infty}$ then $\{e_i\}_{i=1}^{\infty}$ is basis for h_M. Given $\{x_i\}_{i=1}^{\infty}$ in h_M and $0 < \epsilon < 1$, choose an integer N such that $\sum_{i=1}^{\infty} M\left(\frac{x_i}{\epsilon}\right) \leqslant 1$, This can be done since $\{x_i\}_{i=1}^{\infty}$ is in h_M. Now for $n \geqslant N$,

$$\begin{aligned}
\|\{x_i\}_{i=1}^{\infty} - \sum_{i=1}^{n} x_i e_i\| &= \inf\left\{r > 0 : \sum_{i=n+1}^{\infty} M\left(\frac{x_i}{r}\right) \leqslant 1\right\} \\
&\leqslant \inf\left\{r > 0 : \sum_{i=N}^{\infty} M\left(\frac{x_i}{r}\right) \leqslant 1\right\} \\
&\leqslant \epsilon
\end{aligned}$$

Hence $\{e_i\}_{i=1}^{\infty}$ forms a basis for h_M. //.

Definition:

Let $M(t)$ and $N(t)$ be two Orlicz functions. A sequence of scalars $\lambda = (\lambda_i)$ is called a multiplier between the Orlicz space l_M and l_N if for each

$$x = (x_i) \in l_M, \text{ we have}$$
$$\lambda x := (\lambda_i x_i) \in l_N.$$

Each Multiplier λ defines a continuous diagonal operator

$$T_\lambda : l_M \longrightarrow l_N$$

Therefore we identify multipliers with diagonal operators and denoted by $D(l_M, l_N)$ the space of all multipliers between l_M and l_N. Regarded with the usual operator norm it is a Banach space.

Consider the function

$$M_N^*(s) = \max(0, \sup_{t\in[0,1]}\{N(st) - M(t)\}), s \geqslant 0 \tag{2}$$

by the definition, we have

$$\begin{aligned} &M_N^*(s) = \sup_{t\in[0,1]}(N(st) - M(t)) \\ &\Rightarrow M_N^*(s) \geqslant (N(st) - M(t)) \\ &\Rightarrow N(st) \leqslant M_N^*(s) + M(t) \end{aligned} \tag{3}$$

which generalizes the classical Young inequality.

Definition (Equivalent):

Two Orlicz function $M(t)$ and $\bar{M}(t)$ are equivalent, if

$$\exists\ c > 0,\ t_0 > 0 \quad : \quad c^{-1}M(c^{-1}(t)) \leqslant \bar{M}(t) \leqslant cM(ct), \quad \forall\ t \in [0, t_o]$$

Equivalent Orlicz function generate one and the same Orlicz sequence space and define equivalent norm on it.

Proposition 4.2.: *If S is an Orlicz function then*

$$\|(y_i)\|_S > 1 \Rightarrow \|(y_i)\|_S \leqslant \sum_i S(|y_i|)$$

Proof:

Since S is a convex function and $S(0) = 0$.

We have for every $\beta > 1$ that

$$
\begin{aligned}
S(\beta^{-1}(t)) \quad &= S(\beta^{-1}t + (1-\beta^{-1}).0) \\
&\leqslant \beta^{-1}S(t) + (1-\beta^{-1})S(0) \\
&= \beta^{-1}(S(t)) + (1-\beta^{-1}).0 \\
&= \beta^{-1}S(t)
\end{aligned}
$$

we know,

$$\|(y_i)\|_S = \inf\left\{\beta > 0 : \sum_{i=1}^{\infty} S\left(\frac{|y_i|}{\beta}\right) \leqslant 1\right\}$$

for every β such that,

$$
\begin{aligned}
& 1 < \beta < \|(y_i)\|_S \\
\Rightarrow\ & 1 < \sum_{i=1}^{\infty}\left(\frac{S|y_i|}{\beta}\right) \\
& \leqslant \frac{1}{\beta}\sum_{i=1}^{\infty}(S(y_i)) \\
\Rightarrow\ & \beta \leqslant \sum_{i=1}^{\infty}(S(y_i))
\end{aligned}
$$

So, letting $\beta \longrightarrow \|(y_i)\|_S$, we obtain

$$\|(y_i)\|_S \leqslant \sum_{i=1}^{\infty}(S|y_i|).$$

Proposition 4.3.: *If S is an Orlicz function then*

$$\|(y_i)\|_S < 1 \Rightarrow \sum_i S(|y_i|) \leqslant \|(y_i)\|_S$$

Proof:

Since S is a convex function and $S(0) = 0$.

We have for every $\alpha \in (0,1)$ that

$$
\begin{aligned}
S(\alpha(t)) \quad &= S(\alpha t + (1-\alpha).0)\\
&\leqslant \alpha S(t) + (1-\alpha)S(0)\\
&= \alpha(S(t)) + (1-\alpha).0\\
&= \alpha S(t).
\end{aligned}
$$

we know,

$$\|(y_i)\|_S = \inf\left\{\alpha > 0 : \sum_{i=1}^{\infty} S\left(\frac{|y_i|}{\alpha}\right) \leqslant 1\right\}$$

There for every α, such that

$$\|(y_i)\|_S < \alpha < 1$$

Given

$$
\begin{aligned}
&\|(y_i)\|_S < \alpha\\
\Rightarrow &\sum_{i=1}^{\infty} S\left(\frac{|y_i|}{\alpha}\right) \leqslant 1\\
\Rightarrow &\sum_{i=1}^{\infty} S\left(\frac{\alpha|y_i|}{\alpha}\right) \leqslant \alpha\\
\Rightarrow &\sum_{i=1}^{\infty} S\left(\frac{\alpha|y_i|}{\alpha}\right) \leqslant \alpha \sum_{i=1}^{\infty} S\left(\frac{|y_i|}{\alpha}\right) \leqslant \alpha\\
\Rightarrow &\sum_{i=1}^{\infty} S(|y_i|) \leqslant \alpha
\end{aligned}
$$

So, letting $\alpha \longrightarrow \|y_i\|_S$ we obtain,

$$\sum_i S(|y_i|) \leqslant \|(y_i)\|_S.$$

Proposition 4.4.:

If $\lambda \in l_{M_N^*}$ then it is a multiplier from l_M into l_N. Moreover, the following generalization of the Holder inequality hold:

$$\|\lambda x\|_N \leqslant 2\|\lambda\|_{M_N^*}\|x\|_M, \qquad \forall\ \lambda \in l_{M_N^*}, \quad \forall\ x \in l_M.$$

proof:

Fix $\lambda = (\lambda_i) \in l_{M_N^*}$ and $x = (x_i) \in l_M$, and let

$$\rho > \|\lambda\|_{M_N^*}, \quad r > \|x\|_M.$$

Consider the sequences $\tilde{\lambda} = \dfrac{\lambda}{\rho}$ and $\tilde{x} = \dfrac{x}{r}$.

For $\tilde{\lambda}_i > 0$ and $\tilde{x}_i > 0$.

Consider the function

$$\begin{aligned}
&M_N^*(\tilde{\lambda}_i) = \max\Big(0, \sup_{\tilde{x}\in[0,t]}\{N(\tilde{\lambda}_i\tilde{x}_i) - M(\tilde{x}_i)\}\Big)\\
&\Rightarrow M_N^*(\tilde{\lambda}_i) = \sup_{\tilde{x}\in[0,t]}\Big(\{N(\tilde{\lambda}_i\tilde{x}_i) - M(\tilde{x}_i)\}\Big)\\
&\Rightarrow M_N^*(\tilde{\lambda}_i) \geqslant N(\tilde{\lambda}_i\tilde{x}_i) - M(\tilde{x}_i)\\
&\Rightarrow N(\tilde{\lambda}_i\tilde{x}_i) \leqslant M(\tilde{x}_i) + M_N^*(\tilde{\lambda}_i)
\end{aligned}$$

Take summation in the both side

$$\Rightarrow \sum_i N(|\tilde{\lambda}_i\tilde{x}_i|) \leqslant \sum_i M(|\tilde{x}_i|) + \sum_i M_N^*(|\tilde{\lambda}_i|) \tag{4}$$

Given

$$\begin{aligned}
&r > \|x\|_M\\
&\Rightarrow \sum_i M\left(\frac{|x_i|}{r}\right) \leqslant 1\\
&\Rightarrow \sum_i M\left(|\tilde{x}_i|\right) \leqslant 1
\end{aligned}$$

and

$$\begin{aligned}
&\rho > \|\lambda\|_{M_N^*}\\
&\Rightarrow \sum_i M_N^*\left(\frac{|\lambda_i|}{\rho}\right) \leqslant 1\\
&\Rightarrow \sum_i M_N^*(|\tilde{\lambda}_i|) \leqslant 1
\end{aligned}$$

From (4),

$$\begin{aligned}
\Rightarrow \sum_i N(|\tilde{\lambda}_i\tilde{x}_i|) \quad &\leqslant \sum_i M(|\tilde{x}_i|) + \sum_i M_N^*(|\tilde{\lambda}_i|)\\
&\leqslant 1 + 1\\
&\leqslant 2 \qquad (5)
\end{aligned}$$

From, Proposition (4.2), we have

$$\|(y_i)\|_S > 1 \Rightarrow \|(y_i)\|_S \leqslant \sum_i S(|y_i|) \tag{6}$$

From (5) and (6),

$$\begin{aligned} \|\tilde{\lambda}_i \tilde{x}_i\|_N &\leqslant \sum_i N\left(|\tilde{\lambda}_i \tilde{x}_i|\right) \leqslant 2 \\ \Rightarrow \|\tilde{\lambda}_i \tilde{x}_i\|_N &\leqslant 2 \\ \Rightarrow \|\tilde{\lambda}\tilde{x}\|_N &\leqslant 2 \\ \Rightarrow \|\frac{\lambda}{\rho}\frac{x}{r}\|_N &\leqslant 2 \\ \Rightarrow \|\lambda x\|_N &\leqslant 2\rho r \end{aligned}$$

Letting $\rho \longrightarrow \|\lambda\|_{M_N^*}$ and $r \longrightarrow \|x\|_M$, we obtain the following

$$\Rightarrow \|\lambda x\|_N \leqslant 2\|\lambda\|_{M_N^*}\|x\|_M$$

and $\lambda x \in l_N$ and λ is multiplier between l_M into l_N, that is, $\lambda \in D(l_M, l_N)$.

Hence, $l_{M_N^*} \subset D(l_M, l_N)$. //.

Proposition 4.5.:

Show that $D(l_M, l_N) \subset l_{M_N^*}$, That is, if λ is a multiplier from l_M into l_N then $\lambda \subset l_{M_N^*}$.

Proof:

Consider, in the space of multiplier $D(l_M, l_N)$, the operator norm

$$\|\lambda\|_0 = \sup\{\|\lambda x\|_N : \|x\|_M = 1\} \tag{7}$$

We may assume without loss of generality that $M(1) = 1$ and $N(1) = 1$

Then we have, $\forall\ i$

$$\|e_i\|_M = \inf\left\{\rho : \sum_i M\left(\frac{|e_i|}{\rho}\right) \leqslant 1\right\}$$

and

$$\|e_i\|_N = \inf\left\{r : \sum_i M\left(\frac{|e_i|}{r}\right) \leqslant 1\right\}$$

But we assume $M(1)=1$ and $N(1)=1$ So, $\|e_i\|_M$ and $\|e_i\|_N=1$

Let fix a multiplier,

$$\lambda=(\lambda_i)\in D(l_M,l_N) \tag{8}$$

$\lambda=(\lambda_i)\in D(l_M,l_N)$ such that $\|\lambda\|_0=\dfrac{1}{2}$

Then,

$$|\lambda_i|=\lambda\|e_i\|_N=\|\lambda e_i\|_N\leqslant\|\lambda\|_0\|e_i\|_M\|=\frac{1}{2}\|e_i\|_M=\frac{1}{2}$$

$$\Rightarrow|\lambda_i|\leqslant\frac{1}{2}$$

Since M and N are Orlicz function they are continuous.

Thus for every $i=1,2,3,\ldots\exists$ an $x_i\in[0,1]$ such that

$$M_N^*|\lambda_i|=N(|\lambda_i|x_i)-M(x_i)$$

that is,

$$N(|\lambda_i|x_i)=M(x_i)+M_N^*(|\lambda_i|) \tag{9}$$

consider the sequence $(x_i)_{i=1}^{\infty}$. Since by our assumption $\|\lambda\|_0=\dfrac{1}{2}$.

we have from Proposition (4.3), $(\|(y_i)\|_S<1\Rightarrow\sum_i S(|y_i|)\leqslant\|(y_i)\|_S)\quad\forall\ i,$

$$\begin{aligned}N(|\lambda_i|x_i)\quad&\leqslant\||\lambda_i|x_i\|_N\\&=\|\lambda_i x_i e_i\|_N\\&\leqslant\frac{1}{2}\|x_i e_i\|_M\\&\leqslant\frac{1}{2}\end{aligned}$$

Therefore

$$M(x_i)=N(|\lambda_i|x_i)-M_N^*(|\lambda_i|)<\frac{1}{2}\ ,\qquad i=1,2,\ldots$$

We shall prove by induction that

$$\sum_{i=1}^{n}M(x_i)\leqslant\frac{1}{2}$$

It is shown that the statement is true for $n = 1$. Consider the sequences

$$\xi^{(n)} = \sum_{i=1}^{n} x_i e_i, \;\; n = 1, 2, ...$$

Assume that the claim is true for some n. Then

$$\sum_{1}^{n+1} M(x_i) = \sum_{1}^{n} M(x_i) + M(x_{n+1}) \leq \frac{1}{2} + \frac{1}{2} \leq 1$$

So, $\|\xi^{n+1}\|_M \leq 1$.

Therefore, we obtain From equation (9) and Proposition (4.3) ,

$$\begin{aligned}\sum_{i=1}^{n+1} M(x_i) \leq \sum_{i=1}^{n+1} N(|\lambda_i| x_i) \quad & \leq \|\lambda_i x_i e_i\|_N \\ & \leq \|\lambda \xi^{n+1}\|_N \\ & \leq \|\lambda\| \|\xi^{n+1}\|_M \\ & \leq \frac{1}{2}\end{aligned}$$

Which prove the claim.

Since,

$$\sum_{1}^{n} M(x_i) < \frac{1}{2}$$

for every n, we have

$$\sum_{1}^{\infty} M(x_i) \leq \frac{1}{2}$$

Thus $x \in l_M$ and $\|x\|_M < 1$.

Now from equation (9), it follows

$$\sum_{i=1}^{\infty} M_N^*(\lambda_i) \leq \sum_{i=1}^{\infty} N(|\lambda_i| x_i) \leq \|\lambda x\|_N \leq \frac{1}{2}\|x\|_M \leq \frac{1}{2}$$

Hence,

$$\lambda \in l_{M_N^*} \; and \; \|\lambda\|_{M_N^*} \leq 1 \tag{10}$$

From equation (8) and (10), we get

$$D(l_M, l_N) \subset l_{M_N^*}. \qquad //.$$

Proposition 4.6.:

Prove that: $\|\mu\|_{M_N^*} \leq 2\|\mu\|_0, \quad \forall \ \mu \in D(l_M, l_N)$.

Proof:

Suppose $\mu \in D(l_M, l_N)$ in an arbitrary multiplier. Consider the sequences $\lambda = \dfrac{\mu}{\rho}$, where $\rho = 2\|\mu\|_0$. We see in Proposition (4.5), if $\mu \in D(l_M, l_N)$ be arbitrary . Then $\mu \in l_{M_N^*}$.

Given $\mu \in D(l_M, l_N)$ and $\rho = 2\|\mu\|_0$. So, $\rho \in D(l_M, l_N)$

Then we have , $\lambda \in l_{M_N^*}$ and $\|\lambda\|_{M_N^*} = \|\dfrac{\mu}{\rho}\|_{M_N^*} \leq 1$,

Hence, $\mu \in l_{M_N^*}$ and

$$\|\mu\|_{M_N^*} \leq 2\|\mu\|_0. \qquad //.$$

Proposition 4.7.:

Prove that: $\|\mu\|_0 \leq 2\|\mu\|_{M_N^*}, \ \forall \ \mu \in M_N^*$.

Proof:

Suppose $\mu \in l_{M_N^*}$, Consider the sequence $\lambda = \dfrac{\mu}{\rho}$ Where $\rho = 2\|\mu\|_{M_N^*}$, So, $\rho \in l_{M_N^*}$.

Then $\lambda \in l_{M_N^*}$. We known that from Proposition (4.4), $l_{M_N^*} \subset D(l_M, l_N)$

So, $\lambda \in D(l_M, l_N)$,

$$\begin{aligned} \|\lambda\|_0 \quad &= \left\|\frac{\mu}{\rho}\right\|_0 \leq 1 \\ &\Rightarrow \left\|\frac{\mu}{2\|\mu\|_{M_N^*}}\right\|_0 \leq 1 \\ &\Rightarrow \frac{1}{2\|\mu\|_{M_N^*}}\|\mu\|_0 \leq 1 \\ &\Rightarrow \|\mu\|_0 \leq 2\|\mu\|_{M_N^*} \end{aligned}$$

Theorem-1:

For every pair of Orlicz function M, N the sequence space $D(l_M, l_N)$ and $l_{M_N^*}$ coincide as sets, and moreover, they are isomorphic as Banach spaces.

Proof:

From Proposition(4.4) and Proposition (4.5), We have

$$l_{M_N^*} \subset D(l_M, l_N)$$

and

$$D(l_M, l_N) \subset l_{M_N^*}$$

$$\Rightarrow D(l_M, l_N) = l_{M_N^*}$$

From Proposition (4.6) and Proposition (4.7), we have

$$\|\mu\|_{M_N^*} \leq 2\|\mu\|_0$$

and

$$\|\mu\|_0 \leq 2\|\mu\|_{M_N^*}$$

$$\Rightarrow \|\mu\|_{M_N^*} \leq 2\|\mu\|_0$$
$$\Rightarrow \frac{1}{2}\|\mu\|_{M_N^*} \leq \|\mu\|_0 \leq 2\|\mu\|_{M_N^*}$$

This show that $D(l_M, l_N)$ and $l_{M_N^*}$ are isomorphic as Banach spaces.

Definition (Degenerate):

An Orlicz function S is called *Degenerate*, if $S(t) = 0$ for some $t > 0$; Then the corresponding Orlicz sequence space l_S coincides with l_∞.

Remark-1

$D(l_M, l_N) = l_\infty$ *if and only if the orlicz function* M_N^* is degenerate.

Example: It is well known that for $p, q \geq 1$.

$$D(l_p, l_q) = \begin{cases} l_r, \quad \dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p}, & \text{if } p > q \\ l_\infty, & \text{if } p \leq q, \end{cases}$$

Proof:

Consider $M(t) = \dfrac{t^p}{p}$ and $N(t) = \dfrac{t^q}{q}$. If $P > q$, then it is easy to see that for each fixed $s \in (0,1)$ the expression $N(st) - M(t) = \dfrac{(st)^q}{q} - \dfrac{t^p}{p}$ attains its maximum for $t \in [0,1]$ at $t = s^{\frac{q}{(p-q)}}$.Thus for $s \in [0,1]$

$$M_N^*(s) = \left(\frac{1}{q} - \frac{1}{p}\right) s^{\frac{pq}{(p-q)}} = \frac{s^r}{r}$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, hence $D(l_q, l_p) = l_r$. In this case $p \leq q$, if $s^q \leq \frac{q}{p}$, then

$$N(st) - M(t) = \frac{(st)^q}{q} - \frac{t^p}{p} \leq o, \quad t \in [0,1]$$

Therefore $M_N^*(s) = 0$ for $s \leq \left(\frac{q}{p}\right)^{\frac{1}{q}}$, that is M_N^* is a degenerate Orlicz function, hence $D(l_p, l_q) = l_\infty$ //.

Remark-2

Let $D_c(l_M, l_N)$ be the space of all compact multipliers between the spaces l_M and l_N. It is easy to see by Proposition-4.1. that each multiplier from the subspace $h_{M_N^*}$ is compact (as limit of finitely supported multipliers), thus

$$h_{M_N^*} \subset D_c(l_M, l_N).$$

In particular, if the function M_N satisfies the $\triangle_2$-condition near zero,then each multiplier between the spaces l_M and l_N is compact, that is

$$D(l_M, l_N) = D_c(l_M, l_N).$$

Question. It is true that every compact multiplier between the spaces l_M and l_N is a limit of finitely supported multipliers ?

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