# “NON-LINEAR FORCED VIBRATION ANALYSIS OF AXIALLY FUNCTIONALLY GRADED TAPERED BEAM WITH VARIOUS END CONDITIONS”

**A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF**

**Master of Technology**

**In**

**Machine Design and Analysis**

By

SACHIN SAHU

Roll No: 212ME1280

**Department of Mechanical Engineering**

National Institute of Technology

Rourkela Odisha

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Under Supervision of

Prof. Dr.Anirban Mitra

Department of Mechanical Engineering

National Institute of Technology

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NATIONAL INSTITUTE OF TECHNOLOGY

ROURKELA, ODISHA

CERTIFICATE

This is to certify that the thesis entitled, "**Non-Linear Forced Vibration Analysis Of Axially functionally graded Tapered Beam With Various End Conditions**" by **Sachin sahu** in partial fulfillment of the requirements for the award of Master of Technology Degree in Mechanical Engineering with specialization in "Machine Design & Analysis" at the National Institute of Technology, Rourkela is an genuine work carried out by him under my direction and supervision.

To the best of my knowledge the matter embodied in the thesis has not been submitted to any other University/ Institute for the award of any Degree or Diploma.

Date:

Prof. Anirban Mitra

Dept. of Mechanical Engineering

National Institute of Technology

Rourkela-769008

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SACHIN SAHU
Roll No. 212ME1280
Department of Mechanical Engineering
National Institute of Technology Rourkela
Orissa, India- 769008.

*Dedicated*

*To*

*my Family*

## Nomenclature

T = Total Kinetic energy.

U = Total strain energy.

V = Work potential

$\varepsilon_x^b$ = Strain due to bending of the neutral plane

$\varepsilon_x^s$ = Strain due to stretching of the neutral plane

$\delta$ = Variational operator

ξ = Normalized coordinate in x direction.

*E (ξ)* = Modulus of elasticity of material at any distance *x* in longitudinal axis

A (ξ) = Area of cross-section of beam at any distance *x* in longitudinal axis

*u* = Displacement in axial direction due to stretching of beam.

w = Displacement in transverse direction due to bending of beam.

$\phi_i$ = Denote the sets of orthogonal admissible functions for $w_x$

$\alpha_{i-nw}$ = Denote the sets of orthogonal admissible functions for $u_x$

$\omega$ = excitation frequency

[M] = Mass matrix

[*K*] = Stiffness matrix.

{f} = Load vector

$t_0$ = Root thickness of beam

$E_o$ = Root Elastic Modulus

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## 6. Conclusions and Scope of Future Works

**Abstract-** In the present thesis a study of large amplitude forced vibration of axially functionally graded beams is carried out by considering various boundary conditions and loading condition. The governing differential equation for the system is derived by using Hamilton's principle and the solution to the nonlinear set of equations is generated by implementing a substitution technique. The results of the nonlinear analysis are plotted as characteristics curve i.e. non-dimensional frequency response curve for different classical flexural boundary conditions with different variation of material and geometric parameters. For each boundary condition, variation of material properties (i.e. elastic modulus & density) in the axial direction of the beam is considered with different taper profile of thickness. The variation of material properties taken under consideration are homogenous material, linear variation of both elasticity and density, exponential variation both elasticity and density, linear variation of elasticity with quadratic variation of density. Four different taper profiles are taken into account, namely, uniform, linear, exponential, and parabolic, for thickness variation, while the width of the beam is kept constant.

**Key words-** Axially functionally graded beam, geometric nonlinearity, forced vibration, frequency response, and Hamilton's principle.

# CHAPTER-1

# INTRODUCTION OF FUNCTIONALLY GRADED MATERIAL

## 1.1Introduction

Functionally Graded Materials (FGMs) are one of those exceptional materials where the arrangement of the constituent is varied locally so that certain deviations of the neighborhood material properties are attained. FGMs may be described as, those advanced composite materials in which the volume fractions of two or more materials are attained constantly as a function of position along particular dimension (typically the length and depth) of the structure to bring successfully a required purpose (e.g. mixture of metal and metal or ceramics). It is heterogeneous material, which is defined for those objects with, and/or multiple material objects with clear material zone. By arrangement of material properties in a continuous manner, the effect of inter-laminar stresses developed at the interfaces of the laminated composite due to abrupt change of material properties between neighboring laminas is alleviated. As many members in which very small thickness, i.e. plates and blades used in turbines, nuclear reactor vessels and many other machine parts are perceptive to failure from buckling, large deflections, or too much stresses brought by combine thermo mechanical or thermal loading. Thus, functionally graded materials are especially used in structures subjected to high temperature gradients or where extreme temperature environment are encountered. Primarily they are made from material which are isotropic in nature such as ceramics and metals since role of ceramics provides thermal protection in environments with large thermal gradients (e.g. reactor vessels, semiconductor industry) while metal portion gives strength and acts as structure support. In such conditions the metal provides the strength and toughness while ceramic provides heat and corrosion resistance. Whatever trouble arises in using compound/composite materials those trouble/difficulties can be reduced significantly by using functionally graded material instead of composite materials because functionally graded material changes the material properties from surface to surface or layer to layer. Functionally graded materials are new innovative multifunctional material in which volume fractions of the reinforcement phases deviate without any problem. Furthermore, functionally graded material allows the certain more and numerous properties without any feeble interface. This new idea of materials hinges on mechanics and materials science due to integration of the structural and material importance into the final design of various mechanical and structural components. Moreover, steady change of material properties can be varied according to need in various applications and service conditions.

## 1.2 Origin of Functionally Graded Materials

First functionally graded material concepts introduced in 1984 in the course of a space project plane in japan. There a blending of materials which is used to serve the purpose of a large thermal resistance which is capable to bear a surface temperature of 1727 degree centigrade($^0$C) and a temperature gradient of 727 degree centigrade($^0$C) across a 10 mm section. Recently functionally graded materials concept has become more popular in Europe (Germany). A co-operative research center Transregio (SFB Transregio) is funded since 2006 in order to explore the potential of grading mono-materials, such as aluminum, polypropylene and steel by using thermo mechanically coupled production processes.

## 1.3 Applications of Functionally Graded Materials

Due to increasing demand of present technology, it is need for advanced capability of materials to become an important in engineering field for higher performance systems. FGMs are one of innovative exceptional materials and are being studied for the use in extreme temperature and structural applications. Functionally graded material has a large variety of applications in engineering science practice, which requires materials property to vary as a function of position along particular directions of the material to bring successfully a required purpose. The following applications are noticeable such as,

1) Engineering field (Turbine blade, helicopter blade, beams, bridges, column, shaft, rotor, nuclear reactor, cutting tool, machine parts and engine components; which require corrosion and, wear resistance, mechanical shock etc.)

2) Aerospace field (Aerospace skins, space planes, nuclear reactors,  space structures, nuclear reactors, insulations for cooling structures, Adaptive structures, Rocket engine components, Vibration control etc.)

3) Electronics field (graded band semiconductor, sensor, circuits etc.)

4) Optical field (optical fiber, lens etc.)

5) Biomaterial field (drug delivery system, artificial skin etc.)

6) Commodities (Building materials, Sports good, Car body etc.)

7) Chemical field (Heat Exchanger, Reactor Vessel, Heat Pipe etc.)

8) Defense Field (Nuclear weapons, Missile, High Speed Jet and Rocket, Arms and ammunitions)

9) Energy conversion (Thermo ionic converter, Solar cells, Thermo-electric generator, Fuel cells etc.)

### 1.4 Why FGM than a Composite?

Composite materials and structures are more often used in advanced engineering fields mainly because of their high strength, high stiffness and less weight that is particularly favorable. However, the main downside of composite materials is weakness at the interfaces between adjacent layers and its failure phenomena called de-lamination that may lead to structural failure. Their layer get detached from each other at high temperature and their elasticity is a combine elasticity of parent material found by rule of mixture. Hence, it behaves as an isotropic material. Nevertheless, axially functionally graded material is anisotropic material their property varies along the co-ordinate axis. Furthermore problems in composite which include the difference in coefficient of linear thermal expansion of blending material due to which interface stresses become more serve the and moisture absorption of the fiber and matrix is more in composite than functionally graded material. For anisotropic composition of laminated composite structures frequently, consequences in stress concentrations near material interface which prone to failure in the form adhesive bond separation, matrix cracking and de-lamination. That is why a graded material is preferred than a composite.

## 1.5 Description of Thesis Work

Before proceeding to analysis of any type structure it is very important to study all the research in that field because from this we come to know about past work already done in the research field and comes to know about the present scenario and field where attention to be focused. Therefore, at first in this thesis work, i deal with literature survey in the field of structural element. After studying exhaustive literature review, many studies have been done in field of structure and it is found that functionally graded materials have attracted great attention of researchers because they possess unique properties. Therefore, i start my thesis work dealing with FGMs structure (beam).

Then i started studying the basic information of functionally graded material, and its application. After grasping the basic knowledge of functionally graded material, proceed to nonlinear vibration study to implement analysis of vibration in functionally graded beam. To have a safe and economic design we have to adopt a favourable material and different type of profile of structure to save the excess material to reduce the cost. So in this thesis i adopt four type of beam profile and four type of material to choose optimum material and profile favourable for particular boundary condition and external condition. Then I moved on, how to formulate problem if structures are subject to forced vibration. After i studied energy principle (specifically Hamilton's principle) to formulate the problem given in this thesis and obtained the differential governing equation of problem. Now to solve the governing equation I have to deal with some numerical technique to obtain the solution of governing equation. So I have taken the help of MatLab to solve the governing equation by substitution relaxation technique. Finally generate the result, which is characteristic curve (frequency response curve) of axially functionally graded beam for various profile and material subjected to nonlinear forced vibration with various end conditions.

## 1.6 Layout of Thesis

The context of this thesis is classified into six chapters. Chapter 1 presents an introduction of functionally graded material, and its origin, applications, drawbacks of composite, description of thesis work and the layout of the present work is given in the Chapter 1.

Chapter 2 provides information about research work done in the field of forced vibration, free vibration analysis of homogenous as well as functionally graded materials structure.

Chapter 3 deals with the detail study of free and forced vibration and different type of nonlinearity and it also present the study of different energy methods and variational principles

Chapter 4 discusses the energy principle in formulation of problem. Formulate the governing equation of present problem in thesis

Chapter 5 gives the detail report of validation and result of axially functionally graded tapered beam with different boundary conditions.

Finally, Chapter 6 summarizes conclusions of this project work and scopes for further work are suggested.

# CHAPTER 2

# REVIEW OF RELEVANT LITERATURES

## 2.1 Introduction

In past few year a lot of study have been done in the field of structural elements i.e. rod, bar, beam, strut, column, plate, disk etc. All these structural elements have wide range of applications in engineering. Beams are used in bridges, building structures, columns etc. while rods are used to carry axial load on important components in automobile, aeronautics applications etc. Hence, static and dynamic study of these structural elements is very important to predict their behaviour in different working conditions so that a good design can be accomplished from the point of view of suitable material and prevention of failure. A substantial amount of research work dealing with static and dynamic analysis of bars and beams using different end conditions and applying different methodologies are in existence.

However, in the present scenario due to rapid industrialization and advancement in the fields of aeronautics, construction, defense, space organization, nuclear industry etc., it appears that the isotropic homogeneous materials are unable to meet the need of increasing demands of modern technologies. Thus, new innovative materials with exceptional mechanical features are being fabricated. Functionally graded materials (FGM) are one such class of advanced materials in which engineers and researchers have given great dedication due to their distinctive properties such as good corrosion resistance, thermal resistance, high strength, high toughness, and low density. A useful study of FGM can be found in a book by Y.Miyamoto of functionally graded material and some more fundamentals regarding FGM can be studied in the book written by Suresh and Mortensen. In the following section a few relevant literatures dealing with different types of analysis of uniform and non-uniform FGM beams are reviewed to outline the scope for further studies. It must be stated that an exhaustive review of all the literature available in the corresponding field within the confines of a single chapter is nearly impossible. So, an attempt has been made to review relevant researches to set up the backdrop for the problem taken up in the current thesis work.

## 2.2 Literature Review

Qiusheng and Hong [1996] gave the governing equations for stability and dynamic analysis of a non-uniform taper bar subjected to continuous variable distributed axial loads. Li [2000] investigated the longitudinal vibration analysis of one-step bars subjected to variably distributed stiffness and mass, using the same approach the frequency equation of a multi-step bar with various end conditions was established. Das et al. [2000] dealt with the dynamic analysis of non-uniform bars of variable circular cross sectional area, having different shapes under uniform axial tensile loading and also found out the dynamic analysis by taking applied load to be satisfactorily beyond the yield point of the material. Das et al. [2008] presented the free vibration analysis of prismatic bars of rectangular and circular cross-section under body force loading and also dealt with dynamic behaviour of non-uniform taper bars beyond elastic limit. Again Das et al. [2000] had done an experimental study to find out natural frequency of vibration of tapered bars in elasto-plastic regime and also compare the obtained results from experiment with the results obtained from ANSYS. Cveticanin and uzelac [1999] obtained exact solution and provided four approximate analytical methods for solving the differential equation of longitudinal vibration of a nonlinear rod. Hong et al. [2013] obtained the governing equations for the Functionally Graded axial bars in which variation of material properties are in radial direction and investigated the wave and dynamics characteristics of Functionally Graded axial bars using the spectral element method.

Goel [1976] investigated free transverse vibrations of tapered beams and results were provided for different cases of taper with elastically restrained against rotation. Williams and Banerjee [1985] carried out free vibration analysis of beam and obtained the fundamental frequencies of axially loaded beams with parabolic and linear taper subjected to axial load. Bokaian [1990] conducted a complete study to find out free vibration frequencies and obtained the mode shapes of uniform beams under a constant axial compressive load with various combinations of end condition. Snyder and Wilson [1992] found a method which was presented for calculating the out-of-plane natural frequency of vibration for a horizontally curved thin walled beam continuous over multiple supports. Ochoa [1993] presented an algorithm for the static, stability and vibration analysis of beam and column. According to the procedure, for any type of non-prismatic straight beam various consistent mass and stiffness matrices, damping, and load matrices was obtained without using approximate shape function or breaking up the beam into small sub-element. De Rosa and Auciello [1996] examined the dynamic behavior of tapered beams (both width and thickness are supposed to vary according to a linear law) with rotational and translational displacement at the ends. Abrate [1995] performed a study to determine free vibration frequencies of non-uniform bar and beam with several end conditions.

Li [2000] investigated free vibration of a different end condition beam by taking flexural displacement of the beam as linear combination of a Fourier series and an auxiliary polynomial function. Zhou and Cheung [2000] obtained the characteristic of free vibration analysis of tapered beams with continuously varying rectangular cross-section by the Rayleigh-Ritz method and also presented the result for three beams of different taper with different truncation factor and boundary condition. Lee [2002] dealt with the free vibrations of three kind of tapered beams of rectangular cross section which is supported by elastic spring at each end and solved the governing equation by employing numerical methods i.e. Runge-Kutta, Regula-Falsi method to find the free vibration frequencies of the system. Bayat [2011] dealt with analysis of non-linear free vibrations of tapered beams with implementation a new procedure of the ancient method called the Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) was used to obtain natural frequencies. Achawakorn and Jearsiripongkul [2012] performed vibration analysis of both uniform and non-uniform beams using the Galerkin's method. The natural frequencies of the Euler-Bernoulli thin beam were determined by approximate method and the results were verified with results obtained from finite element method. Chen [2000] presented forced vibration analysis of non-prismatic beams with variable cross-section. The governing equations of problem were obtained by using Hamilton's principle and formulation of problem utilized the procedure of variational operation. Faruk Firat [2012] investigated forced vibration analysis of curved beams subjected to impulsive loads. The governing equations were derived by Timoshenko beam theory

Yousefi and Rastgoo [2011] put forward a derivation for the natural frequencies of vibration of functionally graded curved beams and obtained the governing equations by the Ritz-method. Murin et al. [2010] obtained the exact solution of bending vibration of functionally graded sandwich beams considering the variation of effective density and effective elasticity. Li et al. [2013] conducted the free vibration analysis of exponentially functionally graded beams with different end condition and obtained their natural frequencies. Chakraborty et al. [2013] established a new beam finite element method to study the thermo elastic behavior of functionally graded beams whose elastic and thermal properties vary along the transverse dimension of the beam and nature of variation of material properties is according to power exponential law. Kang [2009] studied the both small and large deflections of non-linear functionally graded cantilever beams subjected to an end force in which elastic property vary along the depth. Uymaz [2013] conducted both free and forced vibration analysis of functionally graded (FG) nano beams whose material property varies as power law and obtained the solution by Navier method. Zhang Guo-Ce et al. [2011] obtained the steady-state periodic response of an axially moving viscoelastic beam. Xiang [2008] studied free and forced vibration analysis of a functionally graded beam of variable cross-section area under initial stresses produced thermally, based on Timoshenko beam theory.

Akgoz and Civalek [2013] dealt with longitudinal natural frequencies vibration of axially FGM strain gradient microbars by strain gradient elastic method and differential governing equation solution obtained by Rayleigh–Ritz method for two end conditions. Shahba and Rajasekaran [2012] dealt with study of stability and free vibration in transverse direction of functionally axially graded tapered Euler-Bernoulli beam. Dinh Kien [2013] dealt with the dynamic analysis of tapered cantilever beams made of axially FGM considering the effects of non-uniform cross section and material inhomogeneity. Equilibrium equations were formed by finite element technique and governing equation solved by the Newton–Raphson incremental/iterative method.

## 2.3 Problem definition

From the extensive literature review carried out in the previous section it is easy to surmise that dynamic analysis of structural elements is a very popular domain of research. Moreover, due to emergence of functionally graded materials and their applications at the forefront of technology, a lot of research work has been concentrated in the area of analyzing functionally graded (FG) beams. It must be mentioned here that, gradient variation in FG beams may be oriented along the cross-section or axial direction. There is an abundance of research work dealing with FG beams having material property variation along the depth or thickness of the beam. However, in recent times emphasis has shifted to analysis of FG beams with axial material gradation. Literature survey also reveals that maximum studies of past research work are mostly focused on free vibration analysis of axially functionally graded beams and forced vibration studies are an unexplored domain.

A linear forced vibration analysis offers an idea about the unstable regions of operation around the resonance zone in the frequency spectrum. But inclusion of geometric nonlinearity that incorporates large transverse amplitude introduces additional complexities and makes the forced vibration analysis nonlinear in nature. Hence, it is decided to carry out a large amplitude forced vibration analysis of axially functionally graded beams (AFGM) subjected to transverse harmonic excitation. It is also observed from the literature review that most studies consider a single boundary condition or material or geometric parameter. So, a conscious effort is made to study the variations in system response corresponding to changes in material properties, boundary conditions as well as system geometry. The present work is undertaken with the help of Hamilton’s principle in deriving the governing differential equations of the system and nonlinear set of equation is solved by means of an iterative direct substitution method, employing an appropriate relaxation technique. The main advantage of using FGMs instead of traditional materials is that the internal composition of their component materials can be tailored to satisfy the requirements of a particular structure. This work is an important step in being able to properly design mechanical structures using FGMs.

## 2.4 Goal of Present Work

The specific purpose of the present thesis work has been laid down as

- Adapt energy principle (specifically Hamilton's principle) in formulation of geometric nonlinear forced vibration problem.
- Adapt a suitable numerical technique to solve the nonlinear governing equation in MatLab to obtain the solutions.
- Validate the results obtained by present formulation and solution method with previously published results in available literature.
- Compare the frequency response curves corresponding to different axially functionally graded materials for specific boundary condition and specific taper profile.
- Study the forced vibration behavior of axially functionally graded beam with different flexural boundary conditions.

# CHAPTER-3

# VIBRATION, NONLINEARITIES AND ENERGY PRINCIPLE: BASIC CONSIDERATIONS

Vibration may be described as to and fro random motion of a system (for example, particles of an elastic body or medium) about an equilibrium point/ mean position whereby the motion of the system may be periodic or aperiodic. Usually vibration occurs due to elastic force induced by system (commonly known as restoring force) whenever it is disturbed from its mean position. So, vibration is nothing but continuous to and fro motion where there is continuous interchange between potential energy and kinetic energy of the system. Kinetic energy decreases and Potential energy increases when it moves away from equilibrium position and vice-versa. A typical vibratory system usually consists of the following components –

I. Means for storing potential energy
II. Means for storing kinetic energy
III. Means for dissipation of energy

Also, a typical vibratory system may be subjected to the following forces – disturbing force (sometimes known as excitation), restoring force, inertia force and damping force (if the system is non-conservative in nature).

## 3.1 Types of vibration

Vibration of a system can be classified according to different criterion, such as, nature of excitation, whether energy dissipation occurs over a cycle, nature of motion of the particles of the system etc. It is important to remember that external excitation plays a vital role, as the system need to be disturbed from its equilibrium for vibration to occur. External excitation can be categorized into two classes: Instantaneous excitation and Excitations over extended period of time. Depending on the response of the system to these two types of excitations, vibration is classified into -

1) Free vibration: When a system is disturbed at the initial instant and then the excitation is withdrawn with the system being allowed to vibrate to its own accord, it is known as free vibration.
2) Forced Vibration: When a system vibrates under influence of external time varying force, it is called forced vibration and the response of the system is known as forced response.

Vibration can further be classified by presence or absence of an energy dissipating mechanism.

1) Undamped Vibration: The system is conservative and total energy remains constant over a cycle.
2) Damped Vibration: The system is non-conservative and energy is dissipated over a cycle of vibration.

Vibration further can also be classified by the nature of motion of system particles (for a continuous system/medium) with respect to the axis of the system into following categories

1) Longitudinal Vibration (parallel to system axis)
2) Transverse Vibration (perpendicular to system axis)
3) Torsional Vibration (rotation about system axis)

Classification of vibration can also be done according to whether the system response is linear or nonlinear: 1) Linear Vibration and 2) Nonlinear Vibration. Invariably, physical systems and man-made devices are nonlinear in nature and linear models are nothing but mathematical abstractions introduced in order to reduce the complexity of analysis. However, to predict system behaviour with accuracy in certain situations, nonlinearities must be taken into account.

## 3.2 Forced vibration analysis

At the end of the previous chapter, it was decided that the present thesis would deal with forced vibration analysis. So, the current section describes some basic information regarding the topic at hand. Forced vibration occurs when an external repeated force is continuously applied or acts on a body. The force can be a random input, periodic, aperiodic, transient or steady-state input. Again the periodic disturbance may be harmonic or non-harmonic in nature. The external disturbance that acts on the system may be in the form of displacement, velocity or load. For examples, vibration of a machine due to an improper balance, vibration of a system due to uneven distribution of mass, vibration of automobile part due to loose fitting, vibration of reciprocating part due to improper balancing etc. are just some of the practical examples of forced vibration situation.

An interesting phenomenon occurs in forced vibration when the frequency of external vibration becomes equal to the system natural frequency of vibration. At this condition amplitude of vibration of the system rises to infinity and the phenomenon is known as resonance. Consequently the natural frequency of a system also called as resonant frequency. If resonance phenomenon occurs in a system it can lead to an sudden breakdown of the system.

Basically, the main reason behind free and forced vibration analysis is to foretell when the phenomenon of resonance is going to occur and accordingly proceed to reduce or prevent the resonance phenomenon from occurring. Noteworthy decrease in amplitude of vibration can be done by introducing damping, because it can be seen from the frequency response curve of SDoF systems, adding damping can significantly reduce the magnitude of vibration. Also, another option to reduce amplitude of forced vibration is by shifting away from the natural frequency by changing the material or mass or stiffness of the system. If systems natural frequency cannot be altered, then changing the operating frequency range is a feasible solution.

## 3.3 Structural Nonlinearities

It is well established that nonlinearities are inherent in mechanical systems. In structural analysis most commonly encountered types of nonlinearities are -

### 3.3.1. Geometric Nonlinearities

Geometric nonlinearity is attributed to nonlinear strain-displacement and/or nonlinear curvature-displacement relations. The phrase 'large displacement' is exclusively associated with geometric nonlinearity. In such situations deflections (linear or angular) or deformations of the structure/system are large compared to the original dimensions. For example, response of a simple pendulum becomes nonlinear when the angular deflection $(\theta)$ became large. This type of nonlinearity is usually manifested in the potential or strain energy of the system and hence in the stiffness characteristics.

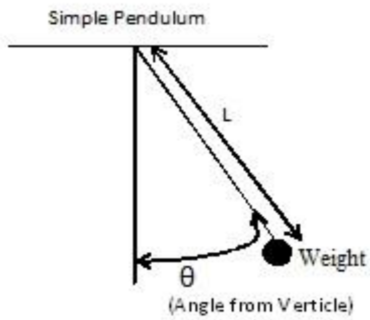



Fig- 3.1 Simple Pendulum

**3.3.2. Material Nonlinearities**

Material nonlinearity arises due to nonlinear stress-strain relationship. Some of models are nonlinear elastic, bilinear elasto-plastic etc. In fact, all materials obey a nonlinear stress-strain (and thus between force-displacement), which must be accounted for when strain variations become large. Material nonlinearities may also include nonlinearities due to material damping.

**3.3.3 Inertia Nonlinearity**

Intertia nonlinearity derives from nonlinear terms containing velocities and/or acceleration in the governing equation of system motion. The kinetic energy of the system is the source of inertia nonlinearity. For example inclusion Coriolis and centripetal acceleration terms make a system nonlinear.

## 3.4 Some basic differences between a linear and nonlinear system

a) The behavior of a nonlinear system is governed by a nonlinear differential equation. Exact solutions do not exist for many nonlinear differential equations.

b) The principle of linear superposition cannot be used to analyze a nonlinear system subjected to a multifrequency excitation. A combination resonance can exist for appropriate combination of excitation frequency

c) Internal resonance can exist in multi-degree-of-freedom and continuous system for appropriate combination of natural frequency.

d) A nonlinear system may have more than one equilibrium point. Equilibrium point may be stable or unstable

e) Steady-state behaviour, if it exists for a nonlinear system, is dependent upon initial conditions.
f) A periodic excitation may lead to a nonperiodic response in a non-linear system.

## 3.5 Energy Methods

Energy principles have a variety of use in structural mechanics which utilize the total potential energy, i.e., potential energy of applied loads and strain energy of a system, to obtain values of an unknown displacement or force at a specific point of the system. These methods are based on conservation of energy i.e. 1$^{st}$ law of thermodynamic. There are various forms of energy methods, for example, Hamilton's principle, principle of minimum total potential energy, principle of virtual work, Galerkin's method, Rayleigh-Ritz method etc. These methods in structural mechanics provide relations between displacements and deformations, and strains and stresses, in the form of energy or work done by internal and external forces acting upon the system and are used to derive the governing equation of systems.
The advantage of these methods is that formulation of the problem is in terms of generalized coordinates. Hence, these methods provide a powerful yet convenient means to formulate the governing differential equations of deformable bodies which may otherwise be intractable to obtain through force and moment balance techniques of Newtonian or vector mechanics. Various energy formulations based on calculus of variations are used as a reliable method of obtaining approximate solutions to practical problems.

### 3.5.1 Principle of Virtual Work

When a body or a system is in equilibrium under the action of a number of forces, there is no displacement and accordingly no work is done. However, if system is imagined to undergo arbitrary but small displacement, some work can be imagined to have been done. The imaginary small displacement given to the system is called virtual displacement and the product of net resultant force and the virtual displacement in the direction of the force is called virtual work. The concept of virtual work is used in solving the problem related to static equilibrium. The principle state that "if a body is in equilibrium, the total virtual work of the force acting on a body is zero for any virtual displacement of the body"

The principle is independent of any constitutive behavior and can be applied to derive the equilibrium equations of continuous deformable solids with elastic or inelastic material Properties. The condition is given by

$$dw = \sum f_x \delta_x + \sum f_y \delta_y + \sum f_z \delta_z = 0$$

$F_x$, $F_y$, $F_z$, are forces acting on x, y, z directions respectively and $\delta_x$, $\delta_y$, $\delta_z$ are virtual displacement in x, y, z direction respectively.

### 3.5.2 Principle of Minimum Total Potential Energy

It is also one of the energy principles which is used to solving problems related to static equilibrium. It can only apply to elastic bodies (linear and nonlinear). It states that for conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremizes the total potential energy and if the extreme condition is a minimum, the equilibrium state is stable. Kinematically admissible displacements are those that satisfy the compatibility and boundary conditions. The total potential energy ($\pi$) of a body is made up of total strain energy stored in the system ($U$) and the work potential ($V$), which implies that, $\pi = U + V$. The strain energy is the elastic energy stored in deformed structure and the work potential $V$, is the negative of the work done by the external forces acting on the structure. Work done by the external forces is simply the forces multiplied by the displacements at the points of application of forces. According to minimum total potential energy principle, the equilibrium condition of the system is obtained by letting $\delta$ ($\pi$) = 0, $\delta$ being the variational operator. Its mathematical form can be given by

$$\delta(\pi) = \delta(U + V) = 0$$

### 3.5.3 Hamilton's Principle

Both the principle of virtual work and minimum total potential energy are limited to Problems of static equilibrium of deformable bodies and are unable to deal with problems of Dynamics on their own. However it is well known that D'Alembert's principle states that a System can be considered to be in equilibrium if inertia forces are taken into account, thereby reducing a dynamic problem into a problem of statics. By the use of D'Alembert's principle along with the principle of virtual work governing equations of the dynamic problem can be derived in a manner similar to the static case. Most widely used principle utilizing the above mentioned scheme is the Hamilton's principle, which is basically a generalization of virtual work principle to dynamics. This principle considers the motion of the entire mechanical system between two finite time instants and is therefore an integral principle. The statement of the Hamilton's principle is: "of all the paths of admissible configuration that the body in motion can take as it goes from configuration 1 at time $\tau_1$ to configuration 2 at time $\tau_2$ , the actual configuration is the path that extremizes time integral of the difference between kinetic and potential energies". Mathematically Hamilton's principle is expressed as,

$$\delta\left(\int_{\tau 1}^{\tau 2} \mathrm{L}\, d\tau\right) = 0$$

where, $L$ is Lagrangian ($L=T-\pi$) and $T$ is the total kinetic energy of the system. It is clear that the principle characterizes the system under consideration by two energy functional, kinetic energy ($T$) and total potential energy ($\pi = U + V$). Consider a single particle, moving in a conservative force field. For such a particle, the kinetic energy $T$ is just a function of velocity of the particle, and the potential energy is just a function of position of the particle. The Lagrangian is thus also a function of the position and the velocity of the particle. In the present work forced vibration analysis of axially functionally graded beams has been formulated through Hamilton's principle.

# CHAPTER-4

# MATHEMATICAL FORMULATION

## 4.1 Introduction

Beam is a structural element which is generally used to carry transverse load, having its cross sectional dimensions considerably lesser than its length. When a beam is subjected to transverse loading, bending and shear stresses are induced in beam. Beams have a wide variety of application in engineering science such as in construction, civil engineering, mechanical engineering etc. Beside these beams also have applications in automobile industries, aerospace industries, marine engineering, off-shore structures etc. So to have a satisfactory design which fulfills the required purpose, it is essential to carry out thorough investigation of different aspects of beam behaviour.

Beams can be classified into different categories depending on various characteristics, such as, shape of cross-section (rectangular, box, channel, I-section, T-section etc.), geometric profile along the longitudinal dimension (uniform and non-uniform), boundary conditions etc. Beams are associated with end conditions at the two ends and classical end conditions include the following: clamped (C), simply supported (S) and free (F). Combinations of these three give a total of four different classical boundary conditions, namely, clamped-clamped (CC), simply supported (SS), clamped-simply supported (CS) and clamped-free or cantilever (CF), as shown in Fig. 4.1. Apart from that, non-classical boundary conditions, such as, elastically restrained ends, ends resting on elastic foundation etc. also exist. There are also overhang and continuous beams (Fig. 4.1(e) and 4.1(f)) that are used for different purposes. Beams can be subjected to external transverse and/or axial loading in their working environment. However, in the context of the present thesis only transverse loading is considered. Fig. 4.2 shows different types of transverse loading pattern that can be applied on a beam.

As mentioned in the previous paragraph, geometric profile of beams can either be uniform (constant thickness and width along the longitudinal direction) or non-uniform (varying thickness and width along the longitudinal direction). Moreover, in case of non-uniform beams the profile can be linearly taper, exponential taper, parabolic taper etc. Fig. 4.3 shows schematic representations of different taper profiles for thickness (along with an uniform beam).

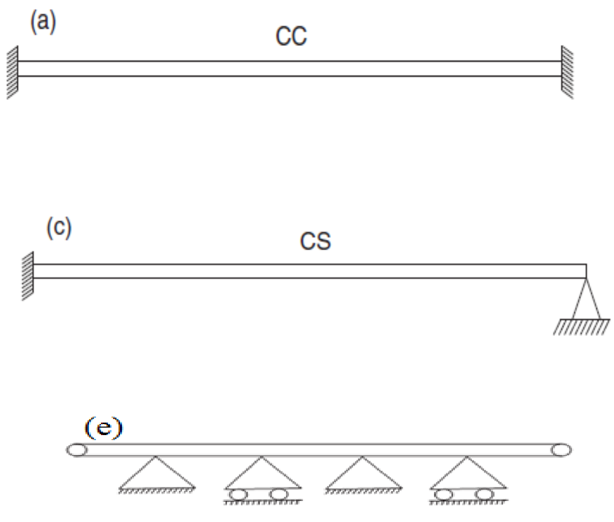



Fig.4.1. Schematic representation of different types of boundary conditions of beams: (a) clamped-clamped (CC), (b) simply supported (SS), (c) clamped-simply supported (CS), (d) clamped-free or cantilever (CF), (e) overhang and (f) continuous.

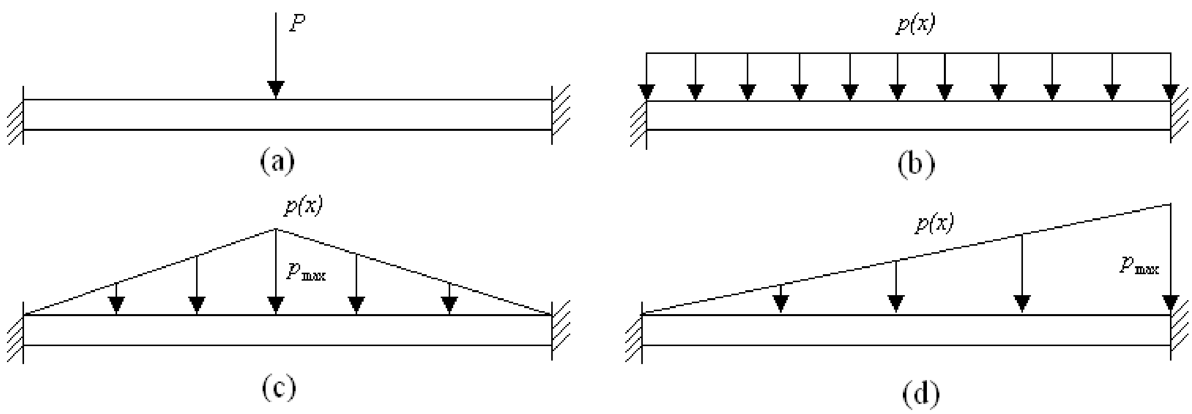



Fig.4.2 Different transverse loading conditions applied on beam: (a) Concentrated load, (b) Uniformly distributed load (UDL), (c) Triangular load and (d) Hat load.

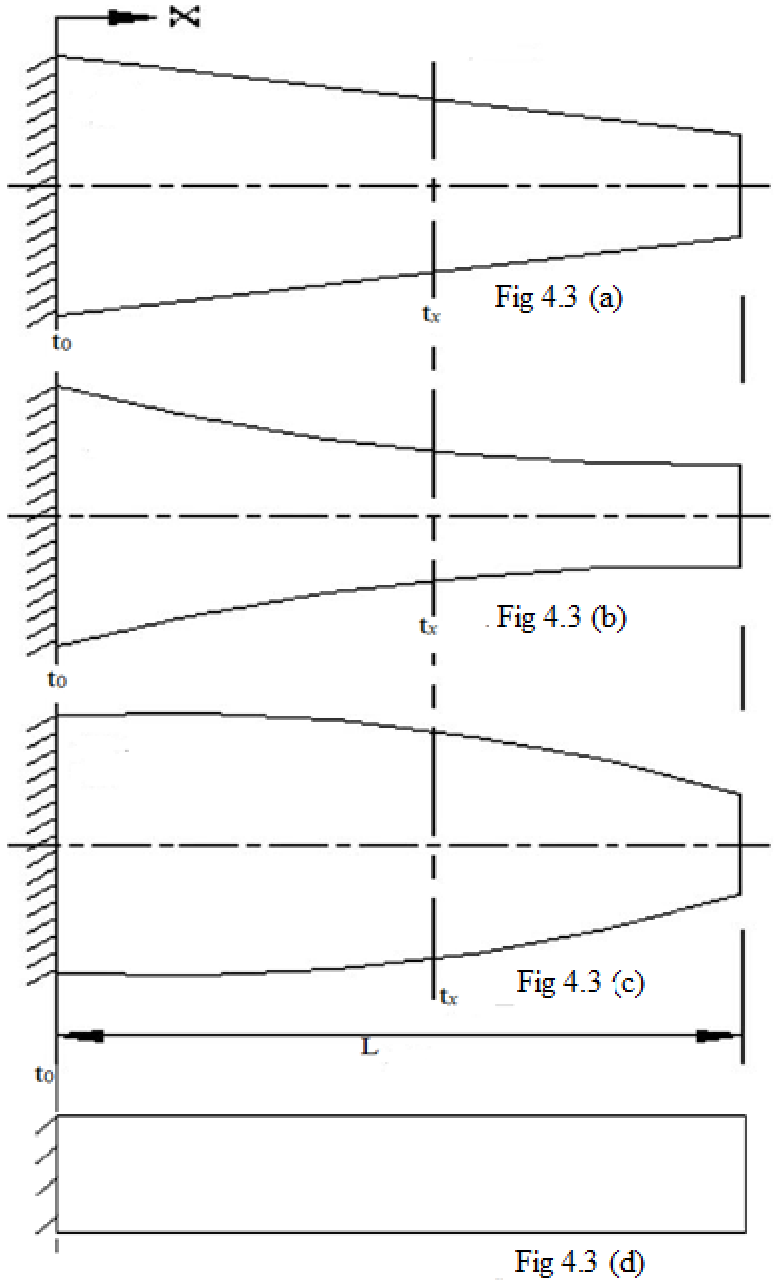



Fig. 4.3. Geometric profile of beam: Cantilever beam with (a) linear taper thickness, (b) exponential taper thickness, (c) parabolic taper thickness and (d) uniform thickness.

## 4.2 Formulation of governing equation

Fig. 4.4 shows an axially functionally graded beam of length $L$ and rectangular cross-section with continuously varying thickness but constant width. However, the formulation can be easily modified to include variation of width along the length. The modulus of elasticity and density are considered to be varying continuously along the length of the beam. The modulus of elasticity, density and thickness at the root of the beam are $E_0$, $\rho_0$ and $t_0$ respectively.

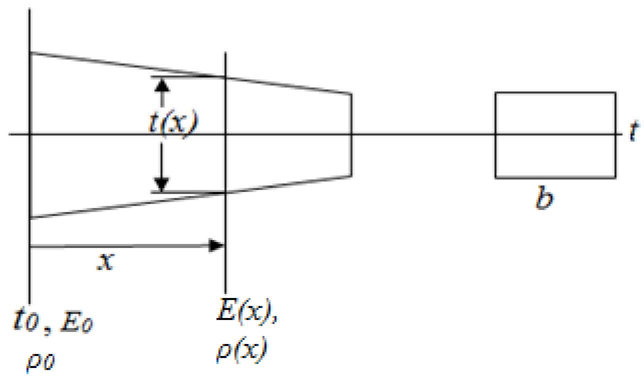



Fig. 4.4. Axially functionally graded linear taper beam.

Large amplitude forced vibration analysis of axially functionally graded (AFG) beams subjected to transverse harmonic excitation is carried out numerically in the present thesis. The problem is tackled in an indirect manner in which the dynamic system is assumed to satisfy the force equilibrium condition at externally applied peak load level. This load level corresponds to the particular excitation amplitude and holds for each frequency. This assumption helps to solve the dynamic problem as an equivalent static problem, where the system response becomes dependent on the excitation frequency and amplitude of the harmonic excitation. In the present study damping in the system is considered negligible and consequently, system response is assumed to have the same frequency as that of the external excitation. Only steady-state dynamic response of AFG beams under transverse harmonic excitation is presented.

The mathematical formulation of the forced vibration study is based on variational form of energy principle and the set of governing equations is derived using Hamilton's principle. In the present thesis, due to consideration of geometric nonlinearity, both stretching and bending of the neutral plane is taken into account. The uniformly distributed transverse load acting on the beam causes transverse displacement *w*, while the displacement *u* is caused by stretching of in-plane. It is to be stated that both *u* and *w* are considered at the neutral plane of the beam. As stated earlier, the governing differential equation of the system is obtained by applying Hamilton's principle, which is given by

$$\delta\left(\int_{\tau 1}^{\tau 2} \mathrm{L}\, \mathrm{d}\tau\right) = 0 \tag{1}$$

In above equation, $L$ is Lagrangian which is given by $L = \{T\text{-}\ (U+V)\}$, which contain of three energy functional i.e. Kinetic Energy T, Strain Energy U, Work Potential V. Whereas, τ denotes time and $\delta$ denotes the variational operator. The Expression of kinetic energy is given by

$$T = \frac{1}{2} b \int_0^l \left\{ \rho(x) t(x) \left(\frac{\partial w}{\partial \tau}\right)^2 + \left(\frac{\partial u}{\partial \tau}\right)^2 \right\} dx \tag{2}$$

The expression of work potential ($V$) for a concentrated load and uniformly distributed load acting simultaneously is given by $V = -Pw - \int_0^L pwdx$. If only uniformly distributed load is acting with no point load then the work potential V of external loading can be expressed as,

$$V = -\int_0^L pwdx \tag{3}$$

The expression of strain energy is given by, $U = \frac{1}{2} \int_{-t/2}^{t/2} \int_0^l (\varepsilon_x^b + \varepsilon_x^s)\, dxdy$ (4)

Where the axial strain of a fiber at a distance y from the neutral plane due to bending action is given by, $\varepsilon_x^b = -y \frac{\partial^2 w}{\partial^2 x}$, while the axial strain due to stretching of the neutral plane is given by $\varepsilon_x^s = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$. It should be noted here that nonlinear strain-displacement relations are used to incorporate the effect of large displacement. Putting these expressions of axial strain due to bending and stretching into equation (4), the expression for strain energy stored in the system is obtained as,

$$U = \frac{1}{2} \int_0^L E(x) I(x) \left(\frac{dw^2}{dx^2}\right)^2 dx + \frac{1}{2} \int_0^L \left[ \left(\frac{du}{dx}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4 + \left(\frac{du}{dx}\right)\left(\frac{dw}{dx}\right)^2 \right] E(x) A(x)\, dx \tag{5}$$

In above equation *E*, *A*, *I* are elastic modulus, cross-section area and moment of inertia of the beam cross section, respectively. In equation (5), following properties of beam about neutral axis are assumed,

$$b\int_{-t/2}^{t/2} dy = \int_{-t/2}^{t/2} A(x) \qquad b\int_{-t/2}^{t/2} y dy = 0 \qquad \text{and} \qquad b\int_{-t/2}^{t/2} y^2 dy = \int_{-t/2}^{t/2} I(x)$$

All the mathematical computations are carried out in normal axial co-ordinate $\xi$ (=$x/L$).

The unknown dynamic displacements $w(x, \tau)$ and $u(x, \tau)$ are assumed to separable in time and space the spatial part of the fields are approximated by finite linear combination of admissible orthogonal functions,

$$w(x,\tau) \cong \sum_{i=1}^{nw} d_i \phi_i(x) \sin(\omega\tau) \tag{6a}$$

$$u(x,\tau) \cong \sum_{i=nw+1}^{nw+nu} d_i \alpha_{i-nw}(x) \sin(\omega\tau) \tag{6b}$$

Where, $\omega$ is the response frequency and $\phi_i$ and $\alpha_{i-nw}$ denote the sets of orthogonal admissible functions for $w$ and $u$, respectively, *nw* and *nu* denote the number of functions for $w$ and $u$, respectively. External harmonic Loading is given by = $p \sin(\omega\tau)$ (7)

*p* is amplitude of external harmonic excitation for uniformly distributed load and $\omega$ is the external excitation frequency.

By substituting the relevant energy expressions and using the dynamic displacement fields in equation (6) in equation (1), gives the governing equations of motion for the beam.

$$[K]\{d\} - \omega^2[M]\{d\} = \{f\} \tag{8}$$

[*M*], [*K*] and [*f*] are the mass matrix, stiffness matrix, and load vector, respectively. {*d*} is the vector of unknown coefficients. The following expression of mass matrix, stiffness matrix, and load vector are given below;

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \ [M] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \ \text{and } \{f\} = \begin{Bmatrix} f_{11} \\ f_{12} \end{Bmatrix}$$

The elements of [*K*], [*M*] and {*f*} are:

$$\left[K_{11}\right]=\frac{1}{L^3}\sum_{j=1}^{nw}\sum_{i=1}^{nu}\int_0^1\frac{d^2\phi_i}{d\xi^2}\frac{d^2\phi_j}{d\xi^2}E(\xi)I(\xi)d\xi+\frac{1}{2L^3}\sum_{j=1}^{nw}\sum_{i=1}^{nw}\left(\sum_{i=1}^{nw}d_i\frac{d\phi}{d\xi}\right)^2\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}E(\xi)\,A(\xi)\,d\xi$$

$$+\frac{1}{L^2}\sum_{j=1}^{nw}\sum_{i=1}^{nu}\int_0^1\left(\sum_{i=nw+1}^{nw+nu}d_i\frac{d\alpha_{i-nw}}{d\xi}\right)^2\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}E(\xi)\,A(\xi)\,d\xi$$

$$\left[K_{12}\right]=0,\ \left[k_{21}\right]=\frac{1}{2L^2}\sum_{j=nw+1}^{nw+nu}\ \sum_{i=1}^{nw}\int_0^1\left(\sum_{i=1}^{nw}d_i\frac{d\phi_i}{d\xi}\right)^2\frac{d\phi_i}{d\xi}\frac{d\alpha_{j-nw}}{d\xi}E(\xi)\,A(\xi)\,d\xi,$$

$$\left[k_{22}\right]=\frac{1}{L}\sum_{j=nw+1}^{nw+nu}\ \sum_{i=nw+1}^{nw+nu}\int_0^1\frac{d\alpha_{i-nw}}{d\xi}\frac{d\alpha_{j-nw}}{d\xi}\ E(\xi)\,A(\xi)\,d\xi,$$

$$\left[M_{11}\right]=L\sum_{j=1}^{nw}\ \sum_{i=1}^{nw}\int_0^1\phi_i\phi_j\rho(\xi)A(\xi)d\xi\,,\quad \left[\mathrm{M}_{12}\right]=0,\quad \left[\mathrm{M}_{21}\right]=0,$$

$$\left[M_{22}\right]=L\sum_{j=nw+1}^{nw+nu}\ \sum_{i=1}^{nw+nu}\alpha_{i-nw}\,\alpha_{j-nw}\,\rho(\xi)\,A(\xi)d\xi\,,$$

$$\left\{f_{11}\right\}=L\sum_{j=1}^{nw}\int_0^1 p(\xi)\phi_j d\xi,\ \ \left\{f_{12}\right\}=0,$$

## 4.3 Solution Methodology

As large displacement induced by geometric nonlinearity is incorporated in the formulation, the stiffness matrix $[K(\{d\})]$ becomes a function of the unknown coefficients $\{d\}$. As a result the system governing equations become nonlinear in nature and cannot be solved directly. The governing equation is solved by a direct substitution technique with an appropriate relaxation scheme. It can be said that the matrix $\left[[K]-\omega^2[M]\right]$ is an equivalent stiffness matrix, which represents the dynamic stiffness of the system and for specified amplitude of loading, it depends on the excitation frequency ($\omega$). When the excitation frequency is zero the governing equation becomes $\left[K\right]\{d\}=\{f\}$, which represents a pure static case. The methodology of solution for a particular load step is explained in the form of a flowchart as shown in Fig. 4.5.

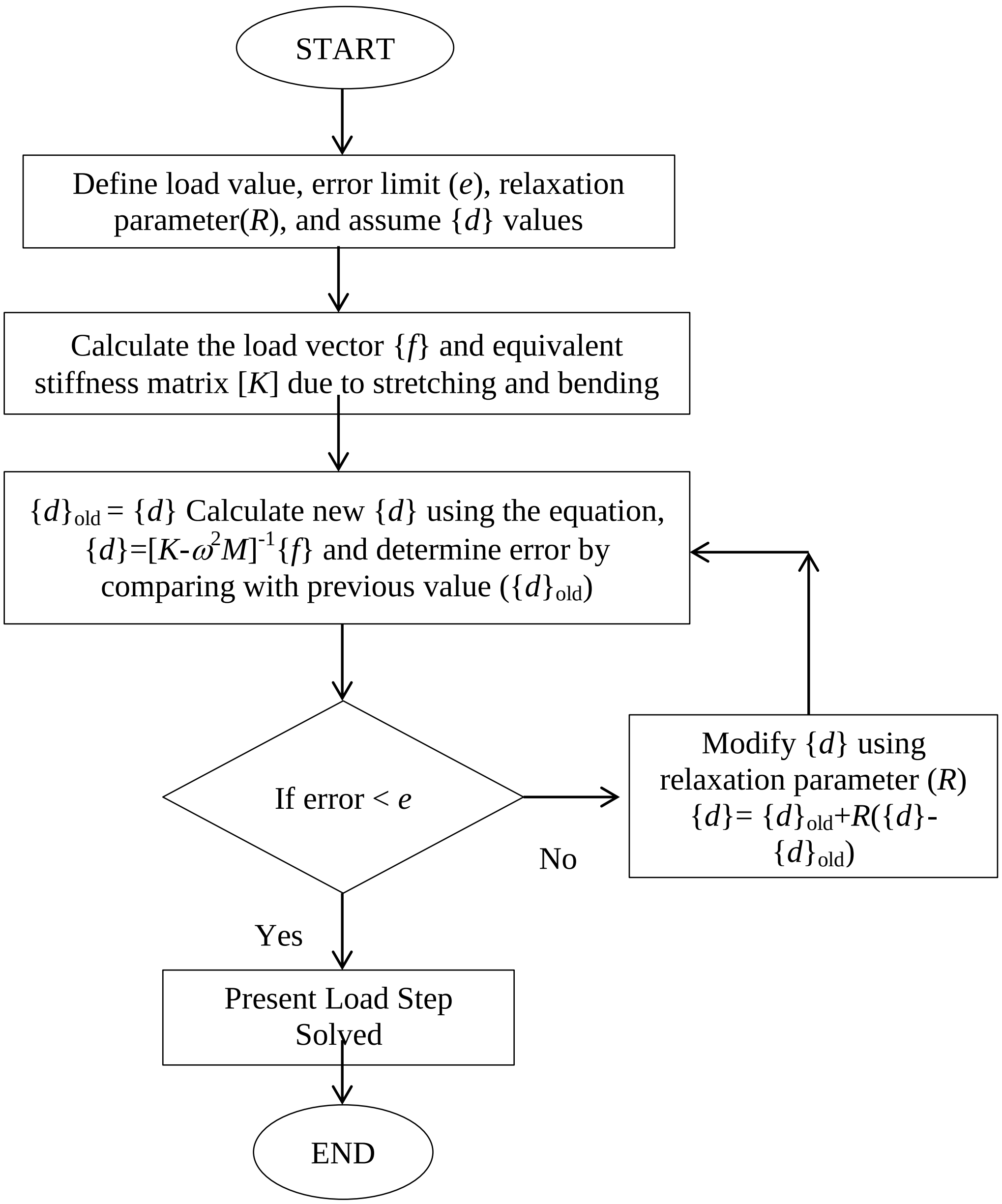


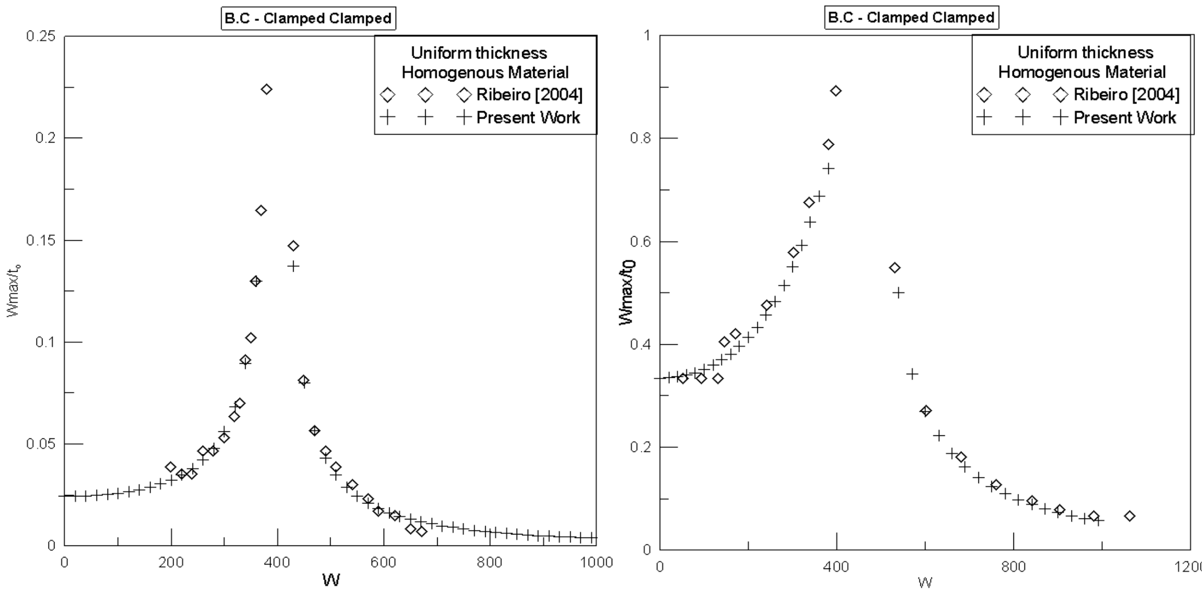

Fig. 4.5 Solution Method Flow chart

# CHAPTER-5
# RESULTS AND DISCUSSION

The present study is conducted with an objective to investigate the large amplitude forced vibration behaviour of axially functionally graded beams under harmonic excitation and also to study the influence of various flexural boundary conditions, material property variations on the frequency response. Four different classical flexural boundary conditions made up of combinations of simply supported (S), clamped (C) and free (F) ends, namely CC, CS, SS and CF, are considered. The results which are non-dimensional frequency response curve are plotted for these boundary conditions with variation of material property (i.e. elastic modulus and density) which vary axially or with longitudinal axis of beam. The effect of variation in taper profile of thickness of the beam on frequency response has been considered as well. However, the width of the beam has been kept constant through the present work.

## 5.1 Validation study

Nonlinear forced vibration for homogenous beam with clamped-clamped end condition subjected to point load at center has already been carried out by Ribeiro [2004] by applying finite element method. In the present thesis results are obtained for nonlinear response of a homogenous beam having clamped-clamped boundary condition with point load at center with a view to compare the results with Ribeiro [2004]. The governing differential equation of motion is obtained by applying Hamilton's principle and the obtained set of nonlinear differential equations is solved through method of substitution with successive relaxation scheme in MatLab. The results are generated for a 0.406m long beam having square (0.02 m × 0.002 m) cross section with following material properties: elastic modulus ($E$) = 71.72 GPa and density ($\rho$) = 2800 kg/m$^3$. It should be mentioned here that the material properties are considered as constant for the validation study. The frequency response curve is presented in a non-dimensional plane, in which the ordinate represents normalized maximum displacement and abscissa represents excitation frequency. The validation plots are shown in Figures 6(a) and 6(b) for excitation amplitudes $P$ = 0.134 N and 2 N (point load), respectively. From the validation plots it is evident that the present formulation and solution methodology yield satisfactory results and can be utilized to conduct geometric nonlinear forced responses of axially functionally graded tapered beam under harmonic excitation.




(a) (b)

Fig 5.1. Comparison of frequency response of a homogeneous clamped uniform beam under concentrated harmonic excitation at the mid-span of the beam. (a) $P$ = 0.134 N and (b) $P$ = 2.0 N.

## 5.2 Large amplitude frequency response

In the present thesis large amplitude forced vibration analysis of axially functionally graded beam is investigated. The governing differential equation of the problem is obtained by employing Hamilton's principle and with the help of MATLAB the obtained nonlinear governing differential equation is solved by substitution technique with successive relaxation scheme. The results are generated for axially functionally graded tapered beams with different combinations of boundary conditions, taper profile and material models. The geometric parameter values are $L$ = 0.406 m, $b$ = 0.02 m and $t$ = 0.002 m. The non-dimensional frequency response curves are plotted for uniformly distributed harmonic excitation having intensity of excitation amplitude of 2 N/m. In the non-dimensional frequency response curve $y$-axis signifies the normalized maximum amplitude of vibration ($w_{max}/t_0$) and $x$-axis signifies normalized forcing frequency ($\omega/\omega_n$). The normalization of amplitude is done by root thickness of beam and normalization of the forcing frequency is done by the first natural frequency or fundamental frequency ($\omega_n$).

Table 5.1 Geometric properties of beam

| Length (in m) | Width (in m) | Thickness ( in m) |
|---|---|---|
| 0.406 | 0.02 | 0.002 |

Table 5.2 Material properties of beam

| Elastic modulus (GPa) | Density (kg/m$^3$) | Poissons ratio |
|---|---|---|
| 71.72 | 2800 | 0.33 |

Table – 5.3 Various Taper Profiles

| | |
|---|---|
| Uniform Thickness | $t\ (\xi) = t_0$ |
| Linear Tapered Thickness | $t\ (\xi) = t_o\ (1\text{-}b\ \xi)$ |
| Exponential Taper Thickness | $t\ (\xi) = t_0 \exp\ (-b\ \xi)$ |
| Parabolic Taper Thickness | $t\ (\xi) = t_0\ (1\text{-}b\ \xi^2)$ |

where, $t_0$ is root thickness and $t(\xi)$ is the thickness at a distance $\xi$ form root. Taper ratio, which is a constant parameter, is designated by $b$ = 0.2, $\xi$ is the normalized coordinate in $x$ direction.

Table – 5.4 Different types of variation of different axially functionally graded material.

| | | | |
|---|---|---|---|
| Homogenous Elasticity | Material-1 | $E\ (\xi) = E_0$ | $\rho\ (\xi) = \rho_0$ |
| Linear Elasticity | Material-2 | $E\ (\xi) = E_0\ (1\text{-}c\xi)$ | $\rho\ (\xi) = \rho_0\ (1\text{-}c\xi)$ |
| Exponential Elasticity | Material-3 | $E\ (\xi) = E_0 \exp\ (-c\xi)$ | $\rho\ (\xi) = \rho_0 \exp\ (-c\xi)$ |
| Linear Elasticity & Quadratic density | Material-4 | $E\ (\xi) = E_0\ (1+ \xi)$ | $\rho\ (\xi) = \rho_0\ (1+ \xi + \xi^2)$ |

where $E_0$ is root elastic modulus and $E(\xi)$ is elastic modulus at a distance $\xi$ form root. The material variation parameter is c = 0.5.

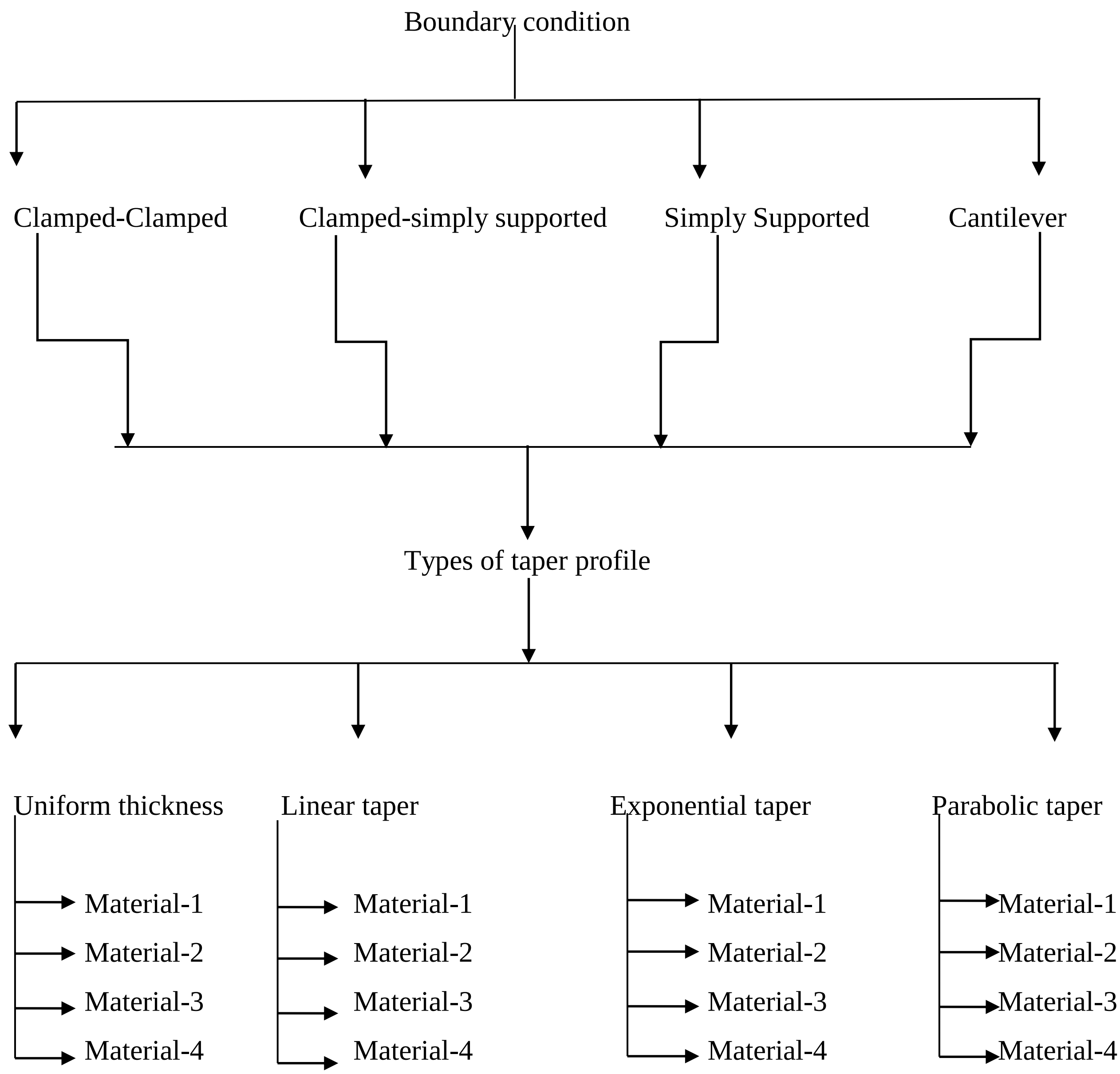



Fig.5.2 Flowchart for result generation

The non-dimensional frequency response curves are generated for four types of boundary conditions, Clamped-Clamped, Clamped-Simply supported, Simply supported-Simply supported and Clamped-free. For each of the boundary conditions appropriate start functions are selected and Gram-Schmidt orthogonalisation principle is implemented through MatLab to generate the higher order functions. For each type of boundary condition, three different types of AFG material models, as well as homogeneous material, are considered (Table 5.4). Three different types of taper profiles (namely, linear taper, parabolic taper and exponential taper) for the thickness of the beam are also taken into consideration (Table 5.3). For comparison purposes, results for uniform thickness beams are also provided. Figure 5.2 shows the sequence of generated results.

### 5.2.1 First natural frequency of AFG beams

Fundamental linear frequency of AFG beams for various boundary conditions, material models and taper profiles are necessary for normalization of excitation frequency. It should be mentioned here that these results are not generated as part of the present thesis work as it requires a separate large amplitude free vibration analysis of the system, which is presented in the paper by Kumar and Mitra [2013]. However, for ready reference the results for fundamental linear frequency are furnished.

Table 5.5 - Fundamental frequency of clamped-clamped boundary condition (in rad/sec)

| | Material-1 | Material-2 | Material-3 | Material-4 |
|---|---|---|---|---|
| Uniform thickness | 395.1254 | 391.7314 | 395.7343 | 361.5347 |
| Linear Taper thickness | 354.5911 | 353.5176 | 356.609 | 321.7029 |
| Exponential Taper thickness | 354.5911 | 431.2346 | 436.2091 | 403.0575 |
| Parabolic Taper thickness | 360.9258 | 359.7733 | 362.9087 | 327.5011 |

Table 5.6 - Fundamental frequency of clamped-simply supported boundary condition (in rad/sec)

| | Material-1 | Material-2 | Material-3 | Material-4 |
|---|---|---|---|---|
| Uniform thickness | 272.787 | 283.8499 | 282.2727 | 234.4454 |
| Linear Taper thickness | 252.2686 | 262.7375 | 261.2755 | 216.2482 |
| Exponential Taper thickness | 293.1321 | 304.9224 | 303.1916 | 252.3754 |
| Parabolic Taper thickness | 259.4856 | 269.7295 | 268.3694 | 222.9689 |

Table 5.7 - Fundamental frequency of simply supported boundary condition (in rad/sec)

| | Material-1 | Material-2 | Material-3 | Material-4 |
|---|---|---|---|---|
| Uniform thickness | 174.954 | 174.166 | 174.5244 | 160.0456 |
| Linear Taper thickness | 144.1899 | 155.1686 | 155.7421 | 144.1899 |
| Exponential Taper thickness | 192.8407 | 192.9979 | 193.1252 | 175.5793 |
| Parabolic Taper thickness | 164.3821 | 162.6455 | 163.2666 | 151.1218 |

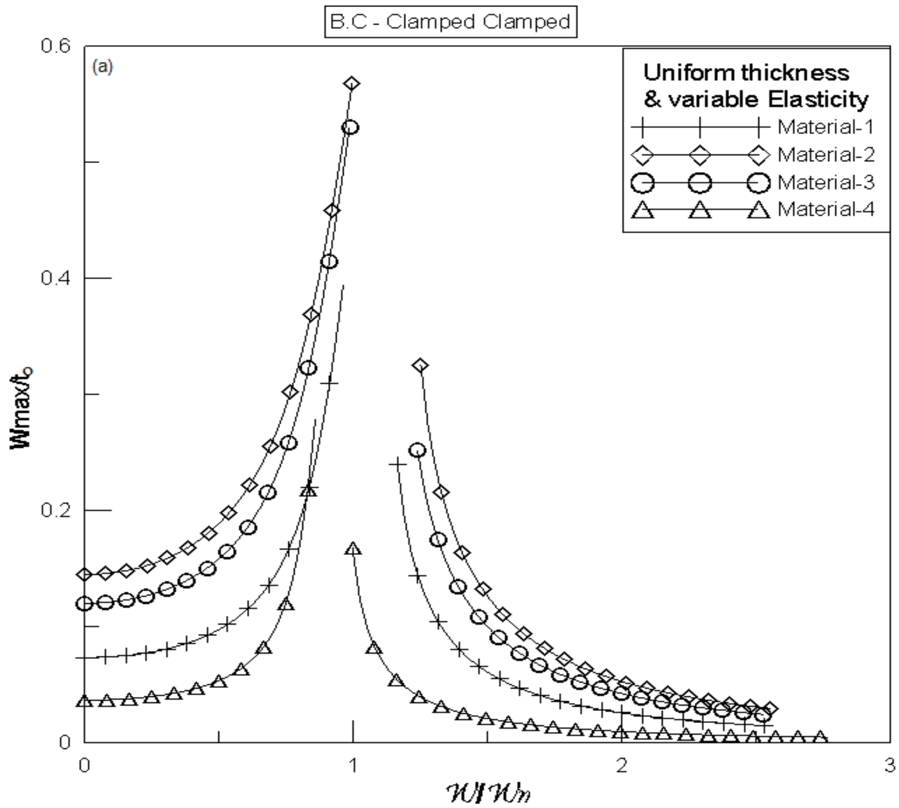
Table 5.8 - Fundamental frequency of clamped free boundary condition (in rad/sec)

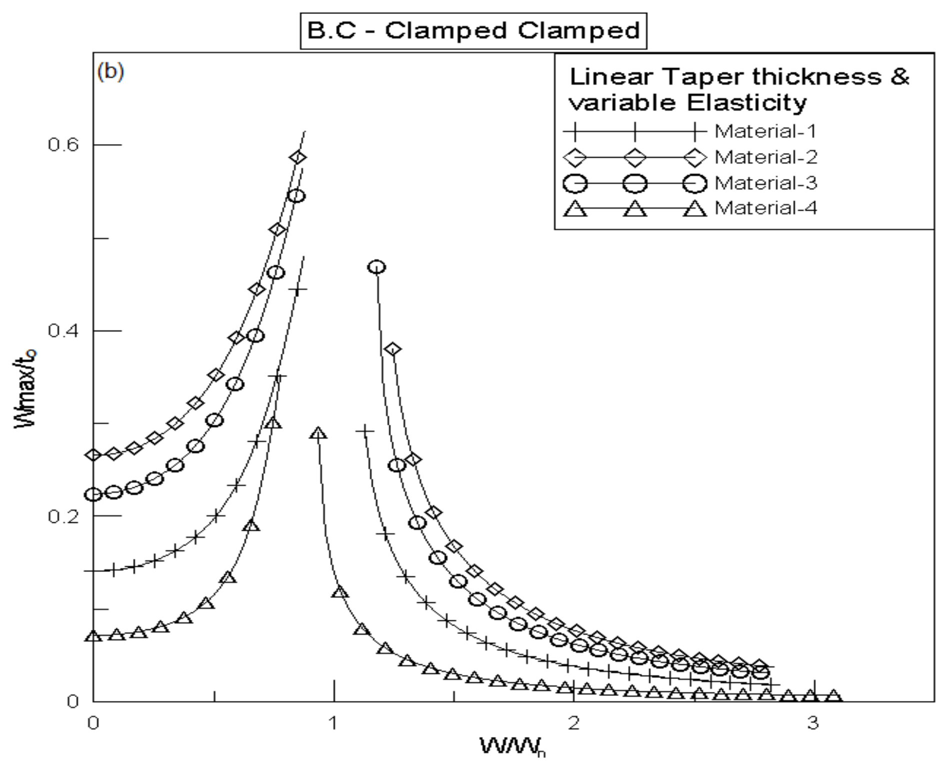
| | Material-1 | Material-2 | Material-3 | Material-4 |
|---|---|---|---|---|
| Uniform thickness | 62.319 | 76.4748 | 72.4808 | 42.9945 |
| Linear Taper thickness | 63.9652 | 78.128 | 74.102 | 44.4114 |
| Exponential Taper thickness | 60.7919 | 74.9079 | 70.9509 | 41.7059 |
| Parabolic Taper thickness | 65.7232 | 80.3188 | 76.2126 | 45.5845 |

### 5.2.2 Effect of Boundary Conditions, Material Model and Taper Profile on Large Amplitude Frequency Response

In the present section new benchmark results for large amplitude frequency response is presented for various combinations of material model, boundary condition and taper profile. As mentioned earlier, the results are plotted in non-dimensional frequency-amplitude plane. Figure 5.3 shows the nonlinear frequency response corresponding to a clamped-clamped (CC) axially functionally graded taper beam under harmonic excitation. The figure contains four different plots for four different taper profiles as mentioned in Table 5.3. Each of these plots consists of four nonlinear frequency response curves corresponding to four material models described in Table 5.4. Similarly, results for other boundary conditions (CS, SS and CF) are presented in an identical manner through Figures 5.4 – 5.6. It must be pointed out that the response curves in the present scenario correspond to primary resonance condition (external excitation frequency = fundamental frequency of the system). It is clearly evident from the figures that for all the different cases the response consists of two distinct branches. In one branch the response amplitude continuously increases with increase in excitation frequency, while in the other branch it decreases (at least initially) with increasing excitation frequency. Each response curve is characterized by a discontinuous zone in the frequency spectrum, where the present solution methodology could not obtain converged results. This discontinuous region occurs near the fundamental frequency of the system and it can be surmised that it is due to the resonance state of the system. It can be concluded that a more efficient and robust solution technique needs to be employed in order to obtain converged results near the resonance state.

The figures provide a preliminary indication that the response curves tilt slightly towards the right, which means the free vibration frequency of the system increases with increase in vibration amplitude. The scenario is known as hardening type nonlinearity. The static deflection of the system can also be determined from the location where the response curve intersects the vertical axis, which signifies zero excitation frequency.


B.C - Clamped Clamped
(a)
Uniform thickness & variable Elasticity
Material-1
Material-2
Material-3
Material-4
0.6
0.4
0.2
0
0
1
2
3
Wmax/to
W/Wn


B.C - Clamped Clamped
(b)
Linear Taper thickness & variable Elasticity
Material-1
Material-2
Material-3
Material-4
0.6
0.4
0.2
0
0
1
2
3
Wmax/to
W/Wn

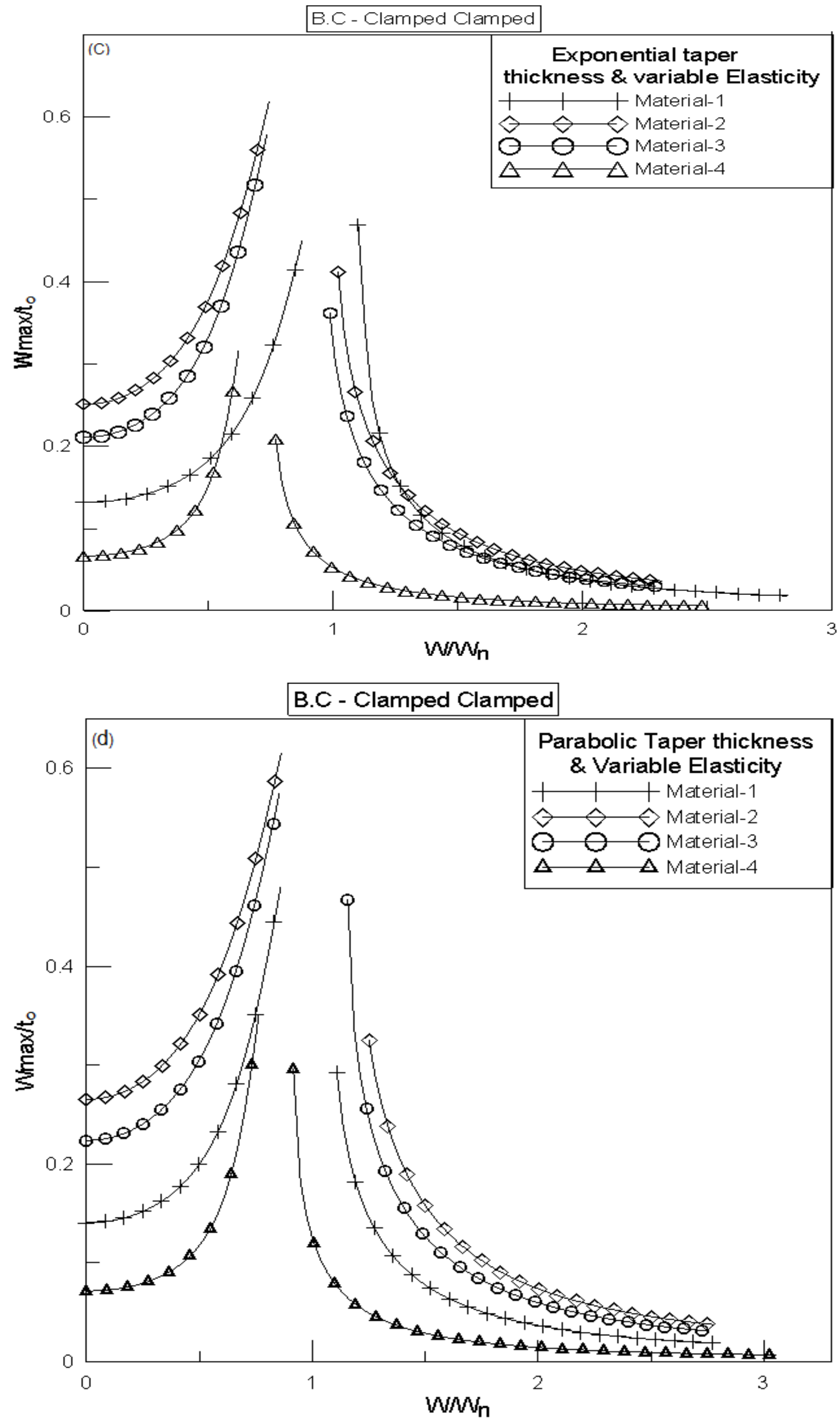
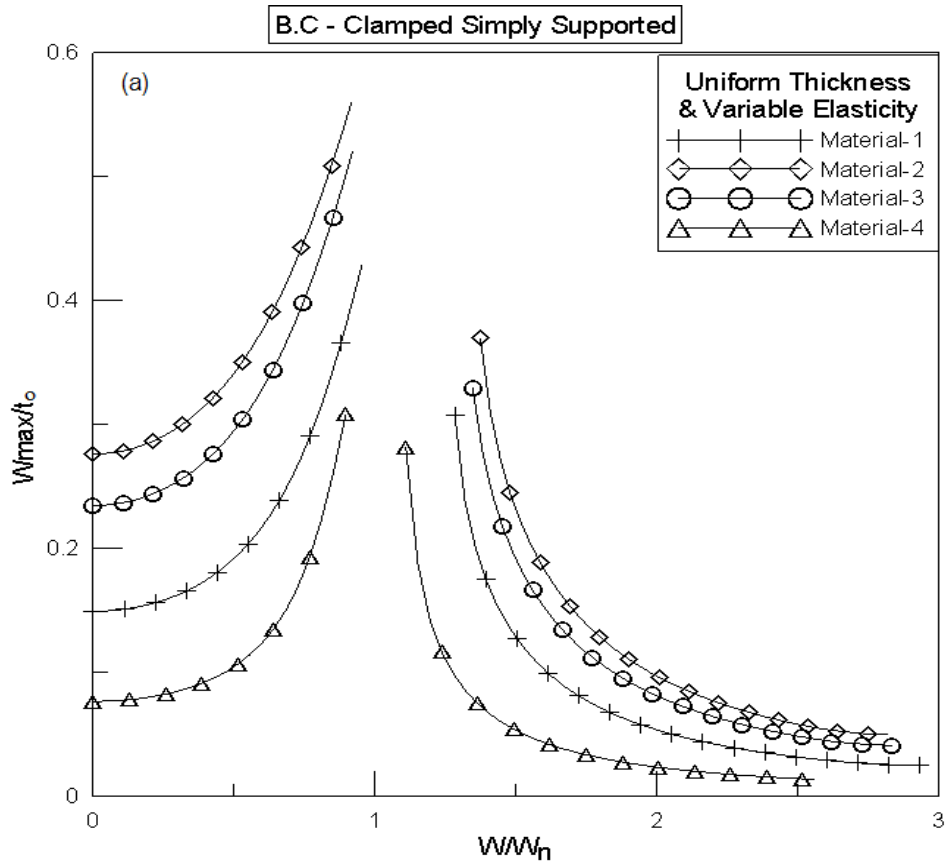
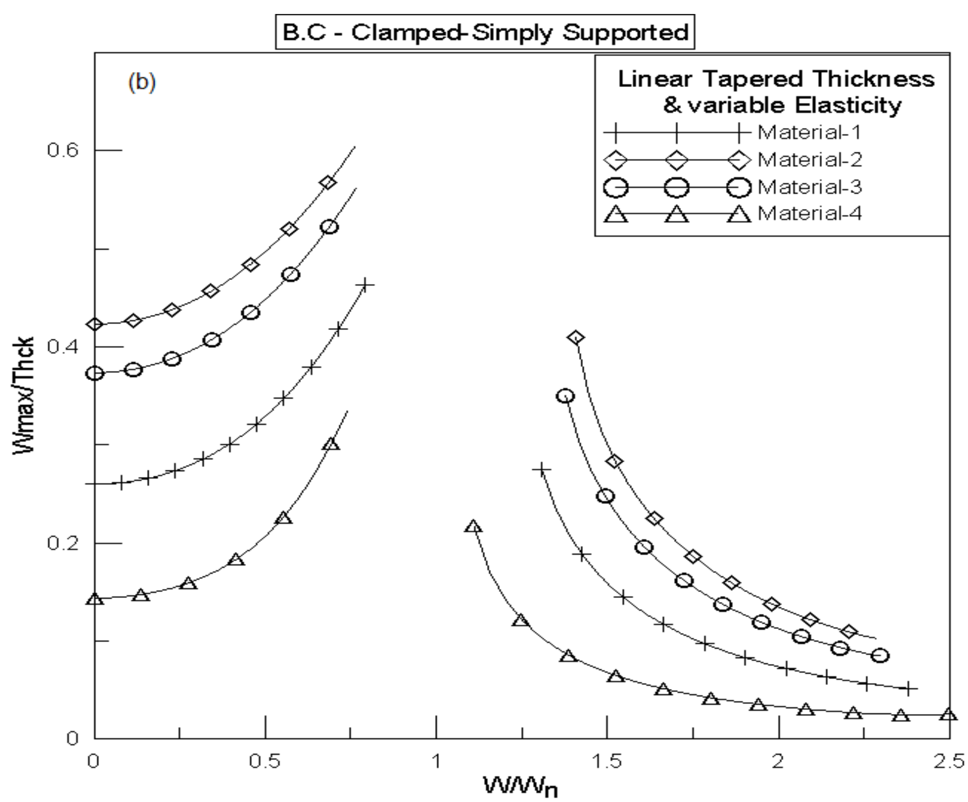
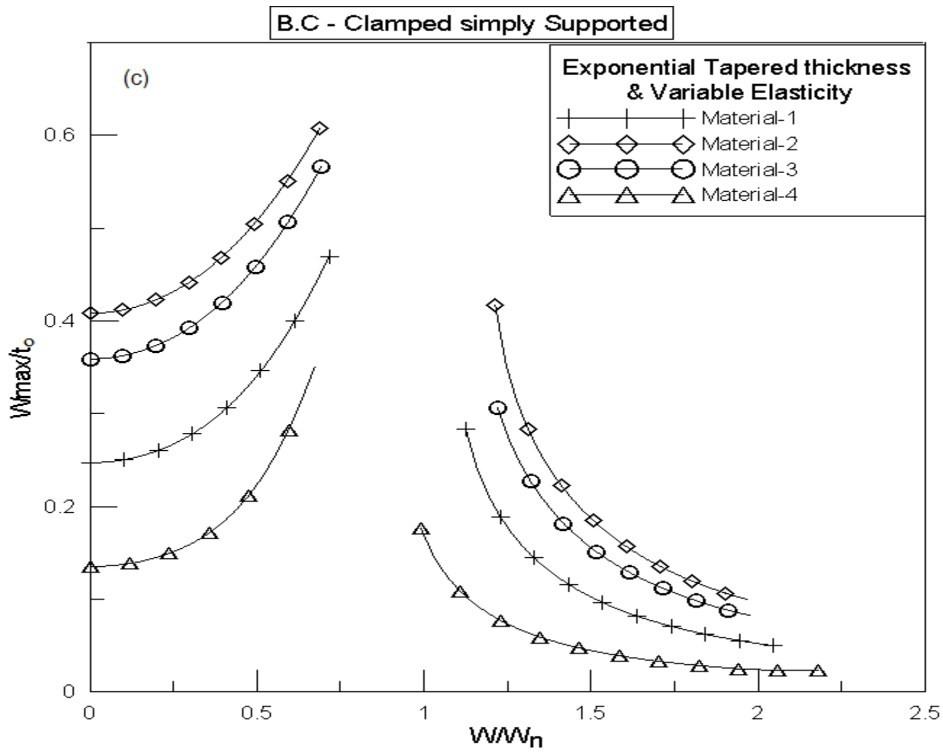



Fig. 5.3 Non dimensional frequency response curve of clamped-clamped beam under uniform distributed excitation of amplitude 2 N/m (a) uniform thickness (b) linear Taper thickness (c) exponential Taper thickness (d) parabolic taper thickness


B.C - Clamped Simply Supported
(a)
Uniform Thickness & Variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/to
W/Wn
0
0.2
0.4
0.6
1
2
3


B.C - Clamped-Simply Supported
(b)
Linear Tapered Thickness & variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/Thck
W/Wn
0
0.2
0.4
0.6
0.5
1
1.5
2
2.5



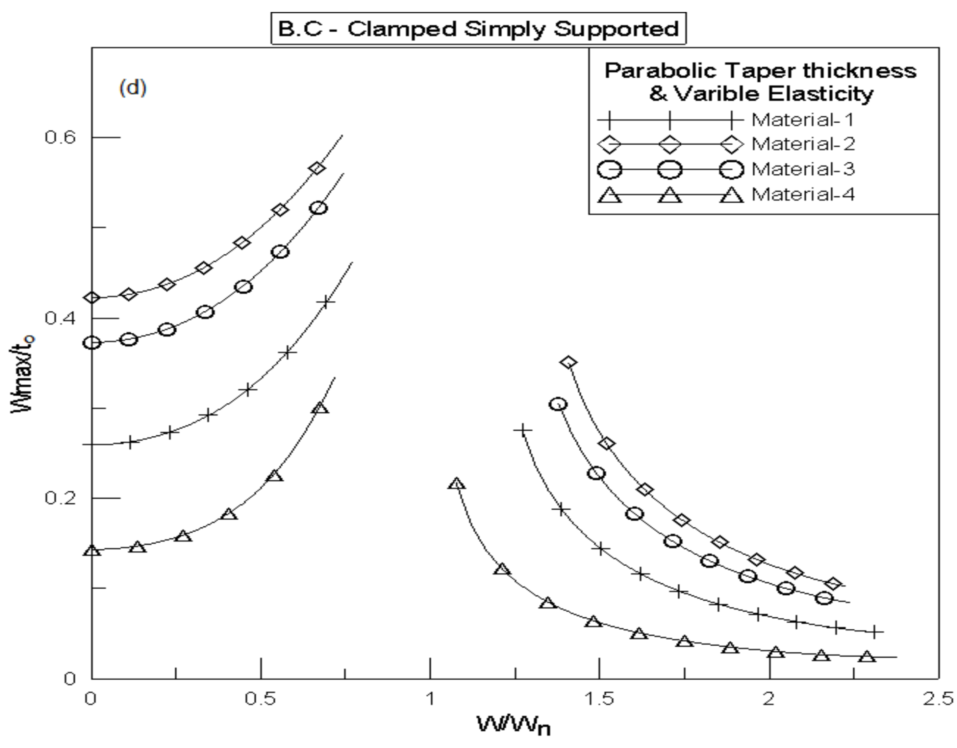
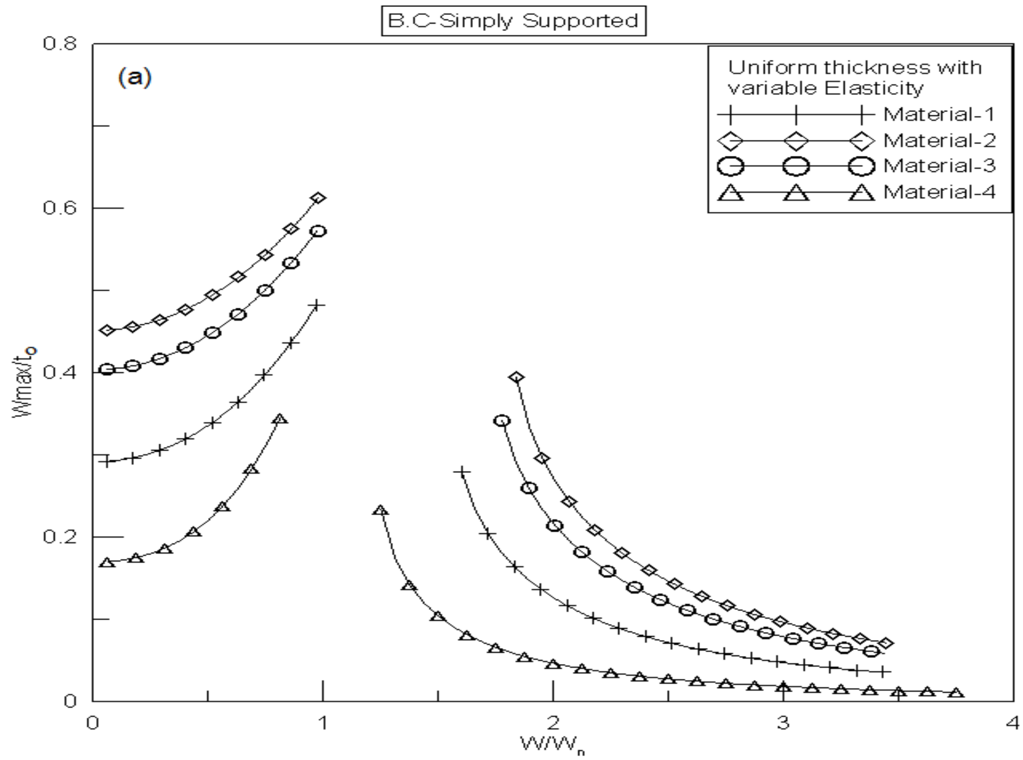
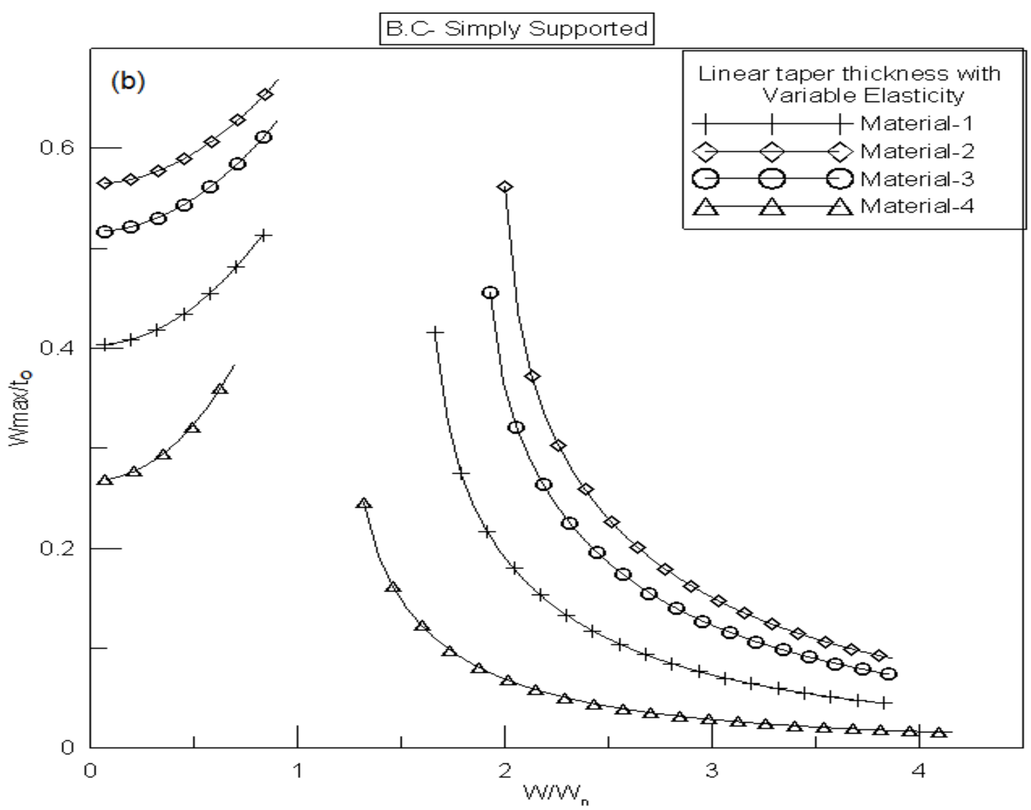




Fig. 5.4 Non dimensional frequency response curve of clamped-simply supported beam under uniform distributed load of 2N/m (a) uniform thickness (b) linear Taper thickness (c) exponential Taper thickness (d) parabolic taper thickness


B.C-Simply Supported
(a)
Uniform thickness with variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/$t_o$
W/$W_n$
0
0.2
0.4
0.6
0.8
1
2
3
4


B.C- Simply Supported
(b)
Linear taper thickness with Variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/$t_o$
W/$W_n$
0
0.2
0.4
0.6
1
2
3
4

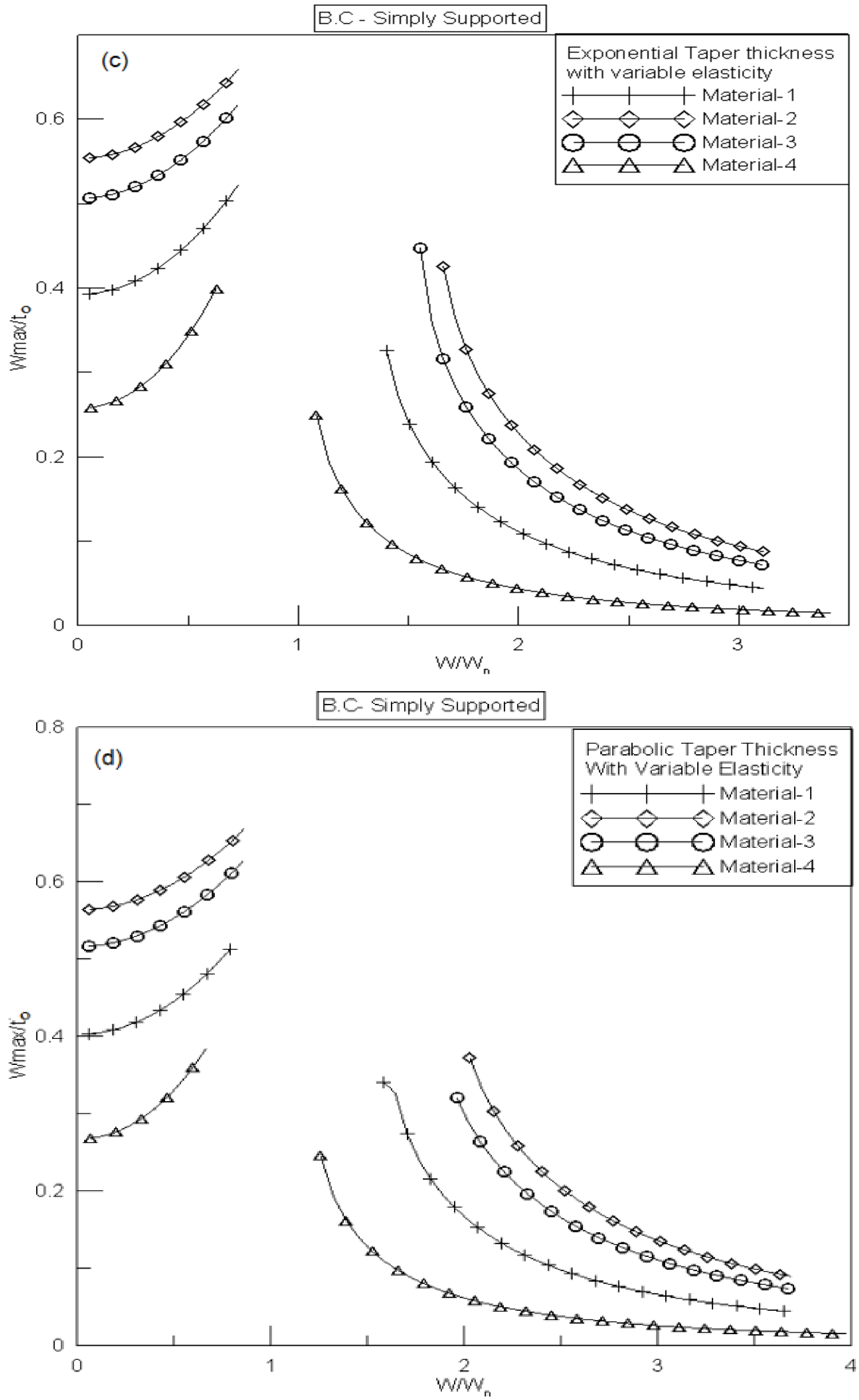
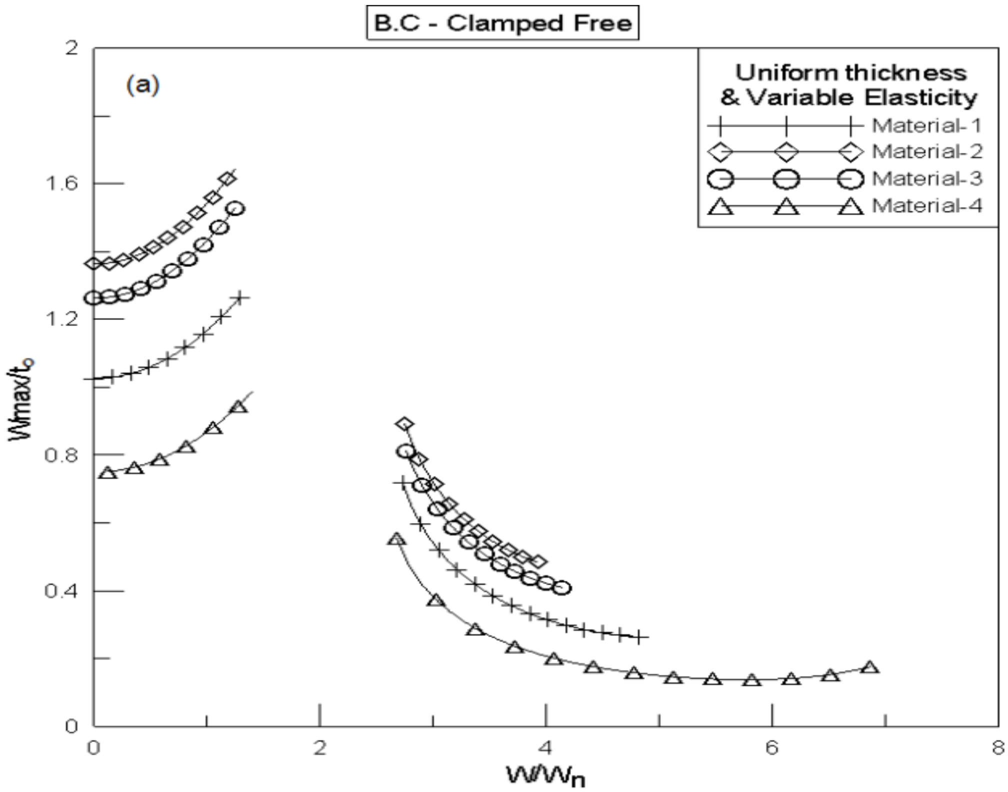
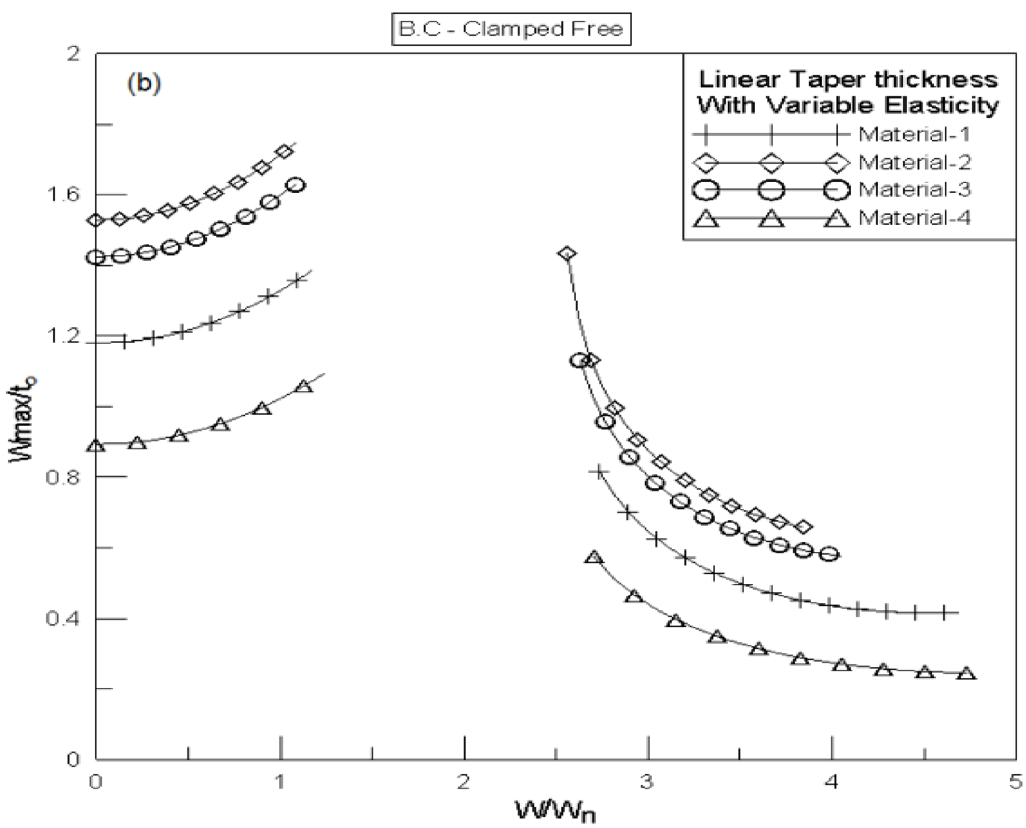
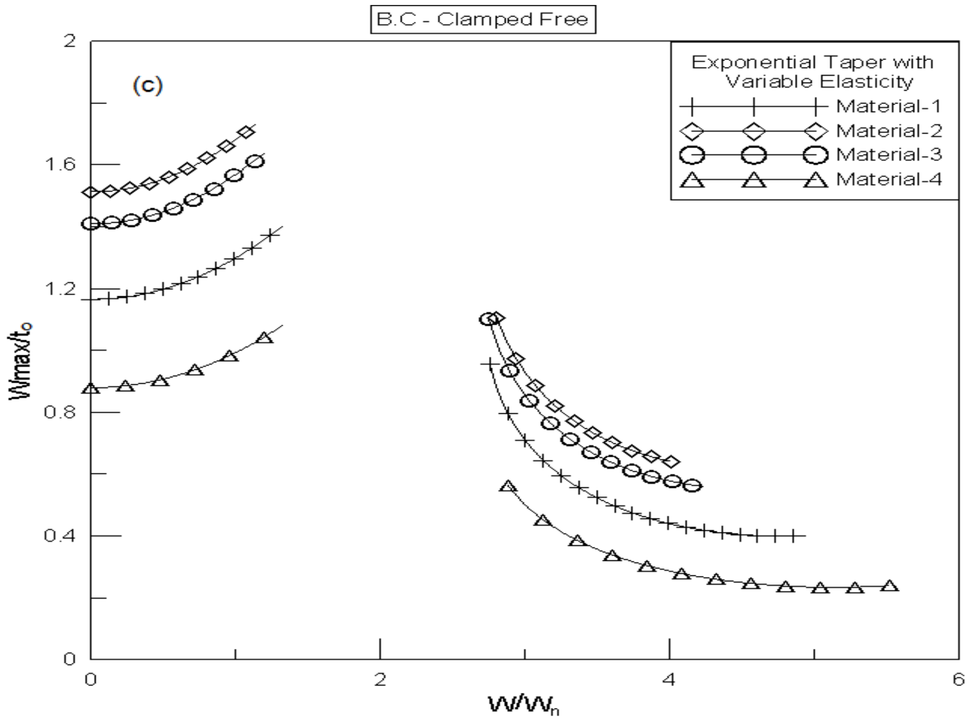



Fig. 5.5 Non dimensional frequency response curve of simply supported beam under uniform distributed load of 2N/m (a) uniform thickness (b) linear Taper thickness (c) exponential Taper thickness (d) parabolic taper thickness


B.C - Clamped Free
(a)
Uniform thickness
& Variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/to
W/Wn
0
0.4
0.8
1.2
1.6
2
0
2
4
6
8


B.C - Clamped Free
(b)
Linear Taper thickness
With Variable Elasticity
Material-1
Material-2
Material-3
Material-4
Wmax/to
W/Wn
0
0.4
0.8
1.2
1.6
2
0
1
2
3
4
5



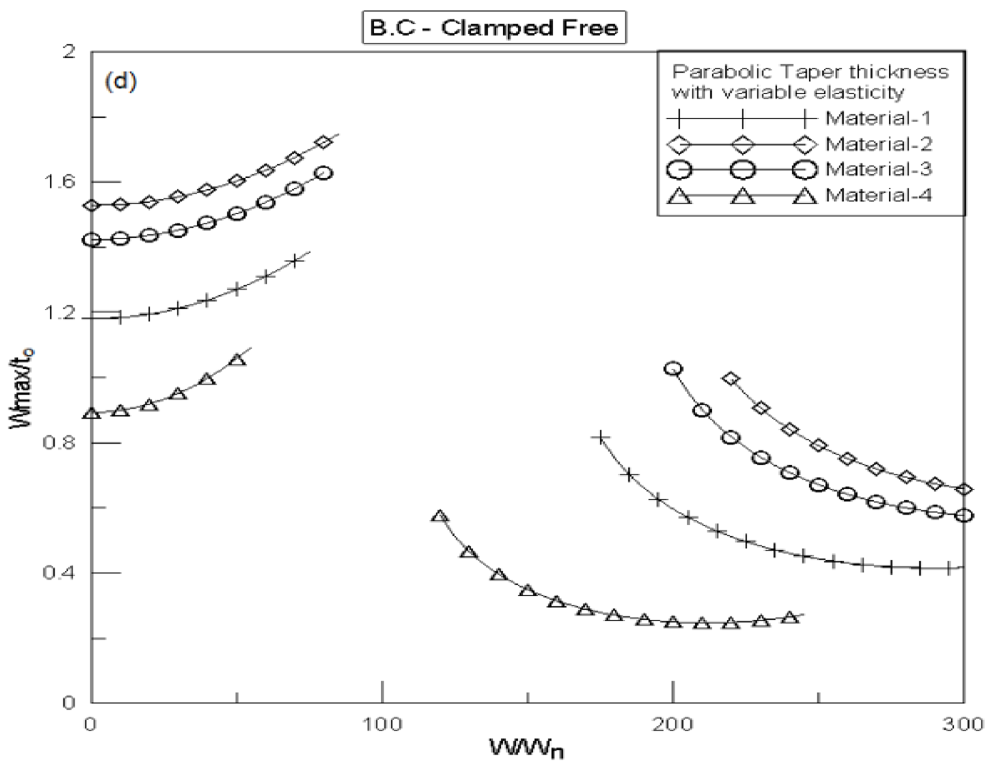




Fig. 5.6 Non dimensional frequency response curve of Clamped-free beam under uniform distributed load of 2N (a) uniform thickness (b) linear Taper thickness (c) exponential Taper thickness (d) parabolic taper thickness

### 5.2.3 Frequency Response for Higher Modes

The frequency response curves in Figures 5.3 - 5.6 are presented only in the neighborhood of the first mode of vibration and correspond to primary resonance condition. However, if the excitation frequency is increased further, in certain cases the response amplitude starts to increase again. Figures 5.7 – 5.10 clearly indicate that as the excitation frequency nears a subsequent higher natural frequency of the system, increase in response amplitude is observed.

These figures present the nonlinear frequency response curves for clamped-simply supported (CS) AFG beams under harmonic excitation for various taper profiles. It is clear from these figures that in the vicinity of a higher vibration frequency, the vibration amplitude increases.

If the excitation frequency is further increased a separate branch, where the response decreases with increase in forcing frequency, is obtained.

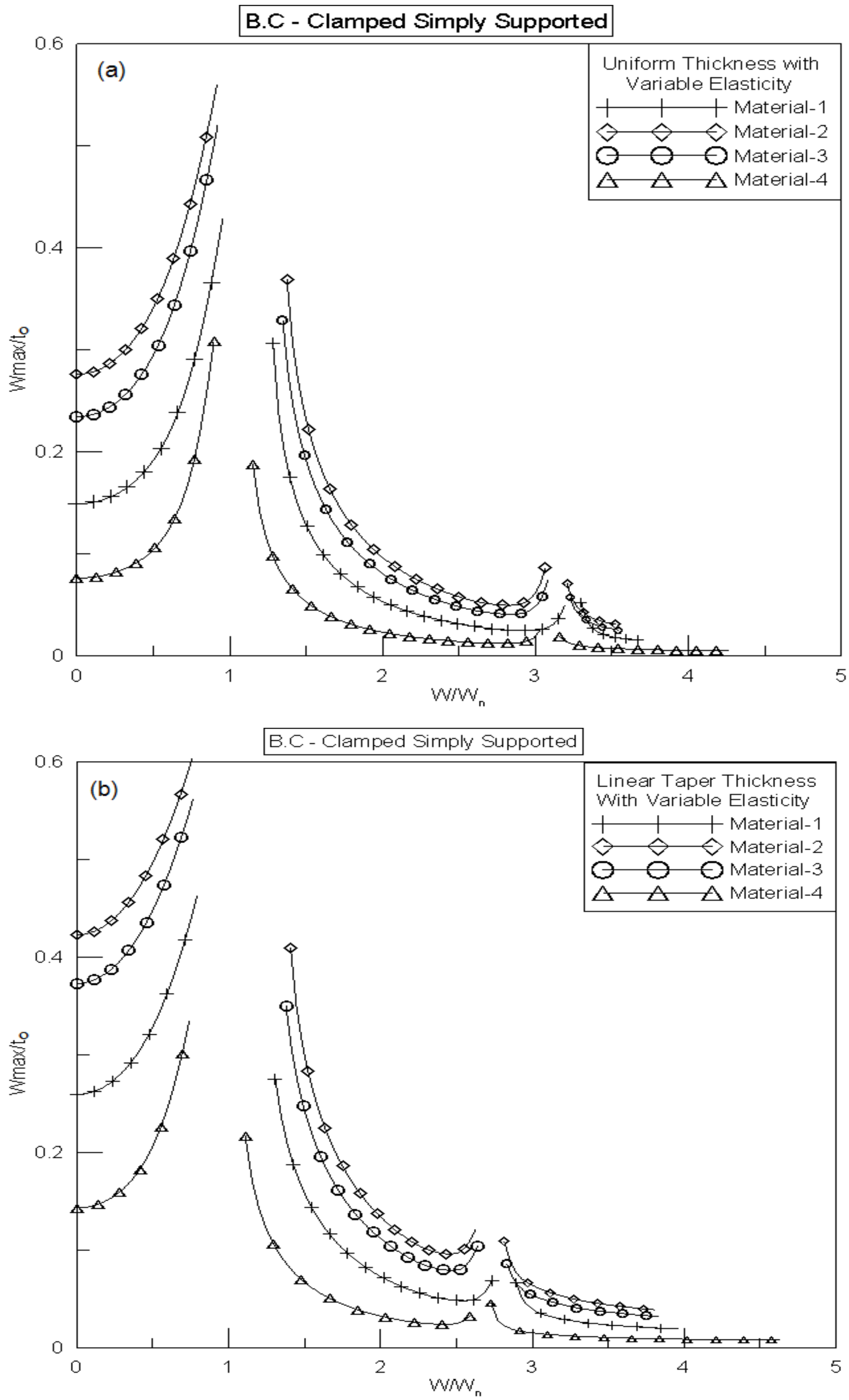
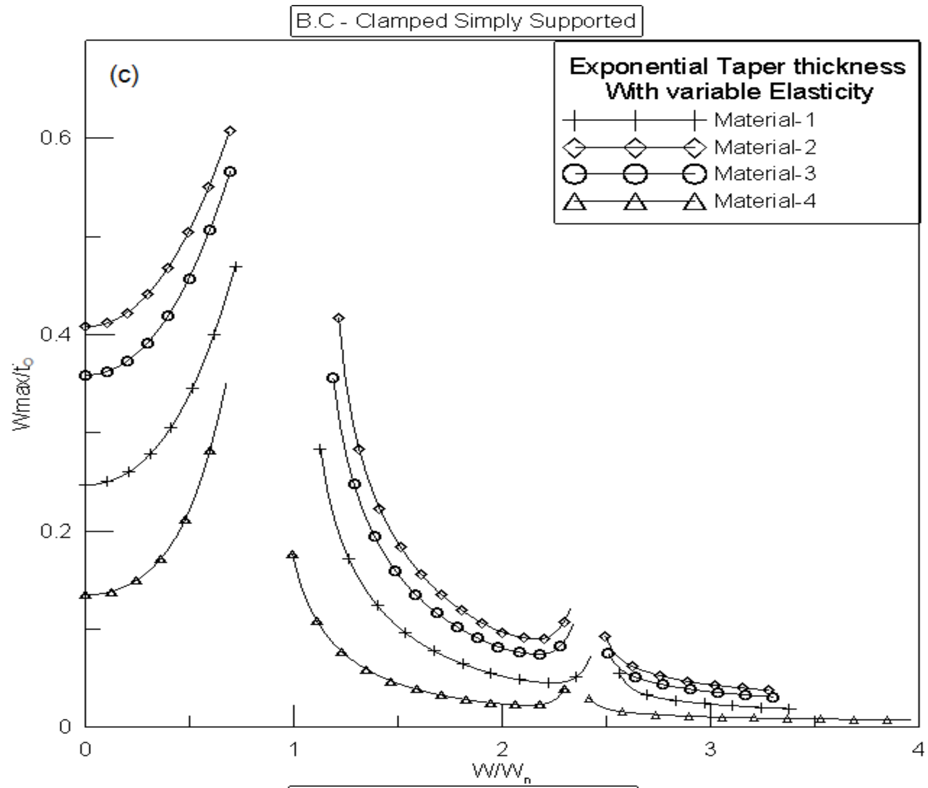



Fig 5.7 Non dimensional frequency response curve with double resonance zone of clamped-simply supported beam under uniform distributed load of 2N/m (a) uniform thickness (b) Linear taper thickness



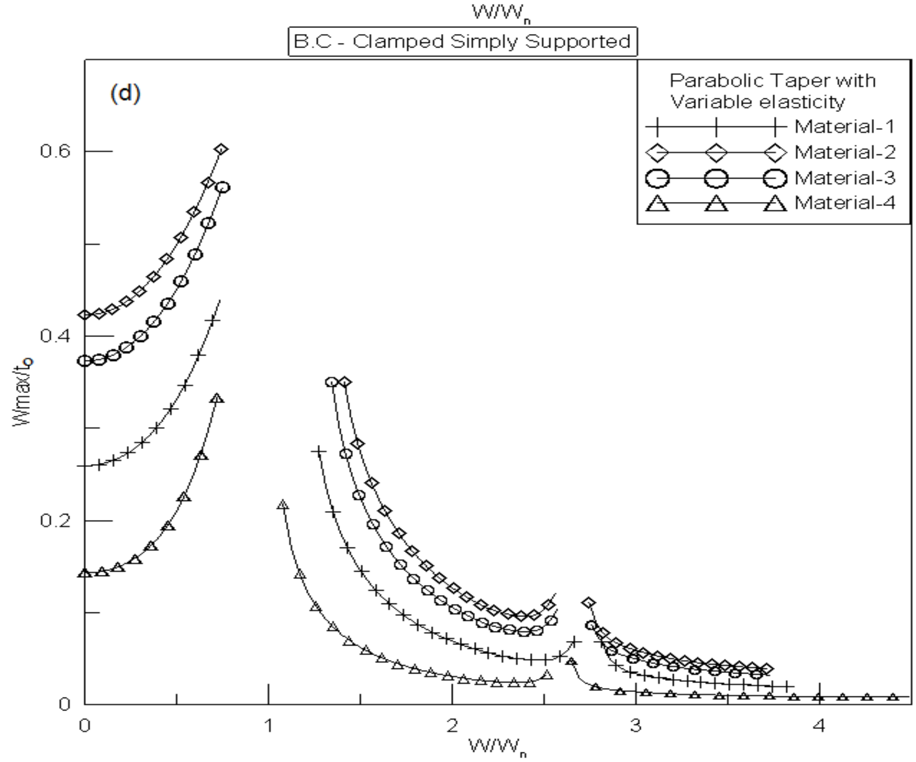




Fig 5.8 Non dimensional frequency response curve with double resonance zone of clamped-simply supported beam under uniform distributed load of 2N/m (c) Exponential taper thickness (d) parabolic taper thickness

# Chapter-6

# Conclusion

## 6.1 Conclusion

In this thesis, forced vibrations analysis of axially functionally graded beams without damping is presented. The fundamental frequencies of system are taken from literature review. By the use of Hamilton's principle governing differential equations of motion is derived and governing equation is solved by substitution technique with successive relaxation scheme in MatLab. The results are presented in the non-dimensional characteristics curve (frequency-response plot). The validation of the obtained result is done by the prior published paper. The frequency response curves for various combinations of material, taperedness of thickness and boundary conditions are obtained. Finally, it can be concluded that the present work can be used for vibration analysis of the axially functionally graded beam for different end conditions and different loading pattern

## 6.2 Scope of Future work

- In present problem, damping is not taken into account in formulation of problem. Always there is chance of damping. By including damping in present problem we got solution of problem close to the reality.
- In this thesis, the problem is formulated in elastic region; it can be carried up to post elastic region.

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