ON SUMS AND RECIPROCAL SUM OF GENERALIZED FIBONACCI NUMBERS

A report

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of

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DECLARATION

I hereby declare that the topic " **ON SUM AND RECIPROCAL SUMS OF FIBONACCI NUMBERS** " for completion for my master degree has not been submitted in any other institution or university for the award of any other Degree, Fellowship or any other similar titles.

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CERTIFICATE

This is to certify that the project report entitled **ON SUM AND RECIPROCAL SUMS OF FIBONACCI NUMBERS** submitted by **Bishnu Pada Mandal** to the National Institute of Technology Rourkela for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by him under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

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ABSTRACT

The purpose of this report is to analyze the properties of Fibonacci numbers modulo a Lucas numbers. Any Fibonacci number, except the first two, is the sum of the two immediately preceding Fibonacci numbers and closely related to Fibonacci numbers are Lucas number. Fibonacci numbers are used in the application of computer algorithms. They can be used to compress audio files and generate code. The most recently Fibonacci number have been used to symbolize mathematical relationship in the Davinci code as well as in the TV shows fringe, criminal minds. In this report, some generalized identities of Holliday and Komatsu have been studied results of Liu and Zhao obtained by applying the floor function to the reciprocal of infinite sums of reciprocal generalized Fibonacci numbers and the infinite sums of reciprocal generalized Fibonacci sums have been extended.

Contents

Chapter 1

1 Introduction

The Fibonacci numbers were first mentioned in 1202 in the Liber Abaci, the book was written by Leonardo of Pisa to introduce the Hindu - Arabic numeral system to western Europe.
They can be obtained by the recursive formula

$$F_n = F_{n-1} + F_{n-2} \quad F_0 = 0, F_1 = 1, \quad n \geq 2$$

First derived from the famous " rabbit problem " of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born one pair in a certain population. Let us assume that a pair of rabbit is introduced in to a certain place in the first month of the year. This pair of rabbits take one month to became mature, and every pair of rabbits produces a mixed pair every month, from the second month and all rabbits are immortal.Every pair of rabbits will produce perfectly on schedule.
Suppose that the original pair of rabbits was born on January 1. They take a month to become mature, so there is still only one pair on February 1. On March 1, they are two months old and produce a new mixed pair, so total of two pairs. Continuing on, we find that we will have 3 pairs, in month April, 5 pairs in month May, then 8,13,21,34,...
The Fibonacci sequence is one of the most famous and curious numerical sequence in the mathematics and have been widely studied from both algebraic and combinatorial prospectives.
The Pell sequence $\{P_n\}$ are defined by recurrence. The Pell sequence, also obtained from the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \quad , P_0 = 0, P_1 = 1, n \geq 2$$

is also very important in number theory.
The Diophantine equation $x^2 - dy^2 = 1$, is known as the Pell's equation. Early mathematicians, upon discovering that $\sqrt{2}$ is irrational, realized the link of successive rational

approximations of $\sqrt{2}$ with $x^2 - dy^2 = 1$. The early investigators of Pell's equation were the Indian mathematicians Brahmagupta and Bhaskara. In particular Bhaskara studied Pell's equation for the values $d = 8, 11, 32, 61.$ and 67, Bhaskara found the solution $x = 1776319049, y = 2261590, for\ \mathrm{d} = 61.$

Fermat was also interested in the Pell's equation and worked out some of the basic theories regarding Pell's equation. In general Pell's equation is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a positive non square integer and has a long fascinating history and its applications are wide and Pell's equation always has the trivial solution $(x, y) = (1, 0)$, and has infinite solutions and many problems can be solved using Pell's equation.

Chapter 2

2 Preliminaries

2.1 Lucas number

Closely related to the Fibonacci numbers are called the Lucas numbers $1, 3, 4, 7, 11, \ldots.$ Lucas numbers are denoted by L_n and recuresive relation as given below,

$$L_n = L_{n-1} + L_{n-2} \ , L_1 = 1, L_2 = 3, \ n \geq 3.$$

2.2 Binet's formula

Both Fibonacci numbers and Lucas numbers can be defined explicity using Binet's formulas,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$ are the solution of the quadratic equation $x^2 = x + 1$.

2.3 Cassini's identity

Let $\{F_n\}$ be sequence of Fibonacci numbers, defined as

$$F_0 = 1, F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$

Then for $n \geq 1$,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}.$$

2.4 Pell numbers

The Pell numbers can be obtained by the recurrence relation

$$P_n = \begin{cases} 0, & if\ n = 0. \\ 1, & if\ n = 1. \\ 2P_{n-1} + P_{n-2}, & otherwise. \end{cases}$$

The Pell numbers sequence starts from 0 and 1 and then each Pell number is the sum of twice the previous Pell number and the Pell number before that.

2.5 Associated Pell numbers

The Associated Pell numbers can be obtained by the recurrence relation

$$Q_{n+1} = 2Q_n + Q_{n-1}, \quad Q_0 = 1, Q_1 = 1.$$

2.6 Golden ratio

In Mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities.

The golden ratio is also called the golden section or golden mean. It is denoted by ϕ and the approximately value is 1.61803398874989. Two quantities a and b are said to be in the golden ratio ϕ if

$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

Then

$$
\begin{aligned}
\frac{a+b}{a} &= 1+\frac{a}{b} = 1+\frac{1}{\phi} \\
1+\frac{1}{\phi} &= \phi \\
\phi^2 &= \phi + 1 \\
\phi^2 - \phi - 1 &= 0 \\
\phi &= \frac{1+\sqrt{5}}{2} \\
\phi &= 1.61803398874989 \\
\phi &\approx 1.618
\end{aligned}
$$

Chapter 3

3 Properties of Pell and Fibonacci Numbers

3.1 The simplest properties of Pell Numbers

Theorem 3.1.1. Let $a_p = \frac{2}{p+1}\left(\frac{2p}{p+1}\right)^p$. Then $a_p < a_{p+1}$ for $p > 1$

Proof. Note that for all $k > 1$

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \frac{k^2}{k^2-1}$$

Since, also $k^2 > \ k^2 - 1$ for all $k > 1$. Thus we can write for $k > 2$

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \left(\frac{k^2}{k^2-1}\right)^{k-1}$$

Then we may write,

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \left(\frac{k}{2(k-1)} \times \frac{2k}{(k+1)}\right)^{k-1} = \left(\frac{k}{2(k-1)}\right)^{k-1}\left(\frac{2k}{k+1}\right)^{k-1}$$

Therefore we get,

$$\begin{aligned}\frac{1}{k}\left(\frac{2(k-1)}{k}\right)^{k-1} &< \frac{1}{k+1}\left(\frac{2k}{k+1}\right)\left(\frac{2k}{k+1}\right)^{k-1}\\ \frac{2}{k}\left(\frac{2(k-1)}{k}\right)^{k-1} &< \frac{2}{k+1}\left(\frac{2k}{k+1}\right)^{k}\end{aligned}$$

Which gives us for $k > 2$

$$a_{k-1} < a_k$$

Thus, the proof is complete.

Theorem 3.1.2. The characteristic equation of the Pell numbers $x^{p+1} - 2x^p - 1 = 0$ does not have multiple roots for $p > 1$.

Proof. Let $f(z) = z^{p+1} - 2z^p - 1$. Suppose that α is a multiple root of $f(z) = 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $f(\alpha) = \alpha^{p+1} - 2\alpha^p - 1$ and $f'(\alpha) = (p+1)\alpha^p - 2p\alpha^{p-1} = 0$. Then

$$f'(\alpha) = \alpha^{p-1}\left((p+1)\alpha - 2p\right) = 0$$

Thus, putting $\alpha = \dfrac{2p}{p+1}$ and hence

$$\begin{aligned} 0 &= -f(\alpha) = -\alpha^{p+1} + 2\alpha^p + 1 = \alpha^p(2-\alpha) + 1 \\ &= \left(\frac{2p}{p+1}\right)^p \left(2 - \frac{2p}{p+1}\right) + 1 \\ &= \frac{2}{p+1}\left(\frac{2p}{p+1}\right)^p + 1 \\ &= ap + 1. \end{aligned}$$

Since By above lemma, $a_2 = \frac{32}{27} > 1$ and $a_p < a_{p+1}$ for $p > 1$, $a_p \neq -1$ which is a contradiction. Therefore the equation $f(z) = 0$, does not have multiple roots.

3.2 Some Properties of Fibonacci Numbers

Theorem 3.2.1. $F_n \equiv 0(mod\ 2)$ if and only if $n \equiv 0(mod\ 3)$.

Proof. We prove by induction that for $k \in \mathbf{N}$ we have

$$F_{3k} \equiv 0(mod\ 2)$$

We have $F_0 = 0 \equiv 0(mod\ 2)$.

We assume that $F_{3k} \equiv 0(mod\ 2)$.

Since, $F_{k+l} = F_l F_{k+1} + F_{l-1} F_k$, we have

$$F_{3(k+1)} = F_{3k+3} = F_3 F_{3k+1} + F_2 F_{3k}$$

Since $F_3 = 2 \equiv 0(mod\ 2)$ and we assumed that $F_{3k} \equiv 0(mod\ 2)$, we have

$$F_{3(k+1)} \equiv 0(mod\ 2)$$

We again prove by induction for $k \in N$

$$F_{3k+1} \equiv 1(mod\ 2)$$

We have $F_1 = 1 \equiv 1(mod\ 2)$.

Let us assume that $F_{3k+1} \equiv 1(mod\ 2)$.

From, $F_{k+l} = F_l F_{k+1} + F_{l-1} F_k$, we have

$$F_{3(k+1)+1} = F_{3k+4} = F_4 F_{3k+1} + F_3 F_{3k}$$

Since $F_{3k} \equiv 0(mod\ 2)$ and $F_4 = 3 \equiv 1(mod\ 2)$, using the assumption $F_{3k+1} \equiv 1(mod\ 2)$, it follows that

$$F_{3(k+1)+1} \equiv 1(mod\ 2)$$

which completes the proof.

Theorem 3.2.2. $F_{5k} \equiv 0(mod\ 5)$ with $k \in \mathbf{N}$

Proof. We again prove this result by induction.

We have $F_0 = 0 \equiv 0(mod\ 5)$

Notice also that $F_5 = 5 \equiv 0(mod\ 5)$

Let assume that $F_{5k} \equiv 0(mod\ 5)$

From, $F_{k+l} = F_l F_{k+1} + F_{l-1} F_k$, we have

$$F_{5(k+1)} = F_{5k+5} = F_5 F_{5k+1} + F_4 F_{5k}$$

Since, $F_{5k} \equiv 0(mod\ 5)$ and we assumed that $F_{5k} \equiv 0(mod\ 5)$, we get

$$F_{5(k+1)} \equiv 0(mod\ 5).$$

Theorem 3.2.3. $F_{k+2} = 1 + \sum_{i=1}^{k} F_i$

Proof. We have

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_2 &= F_0 + F_1 \\ F_3 &= F_1 + F_2 \\ F_4 &= F_2 + F_3 \end{aligned}$$

Continuing the above process, we get

$$F_{k+2} = F_k + F_{k+1}$$

Adding these equalities , we get

$$\begin{aligned} F_{k+2} &= 1+F_1+F_2+...+F_k = 1+\sum_{i=1}^{k} F_i \\ F_{k+2} &= 1+\sum_{i=1}^{k} F_i \end{aligned}$$

3.3 Binet's Formulae for Pell and Fibonacci Numbers

Lemma 3.3.1. The Binet's formula for the Pell sequence is

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

Where $\gamma = 1+\sqrt{2}$, $\delta = 1-\sqrt{2}$.

Lemma 3.3.2. Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, so that α and β are roots of the equation $x^2 = x+1$. Then $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, for all $n \geq\ 1$.

Proof. When $n = 1$, $F_1 = 1$, Which is true. Let us suppose that it is true for $n = 1, 2, 3, ...n$. Then $F_{k-1} = \dfrac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}}$ and $F_k = \dfrac{\alpha^k - \beta^k}{\sqrt{5}}$. Adding these two equations, we get

$F_k + F_{k-1} = \dfrac{\alpha^k}{\sqrt{5}}(1+\alpha^{-1}) + \dfrac{\beta^k}{\sqrt{5}}(1+\beta^{-1})$. Then $F_{k+1} = \dfrac{\alpha^{(k+1)} + \beta^{(k+1)}}{\sqrt{5}}$.

Chapter 4

4 Sums of Reciprocal Fibonacci Numbers

4.1 Introduction

Let a, b be two positive integers and c non negative integers. The generalized Fibonacci numbers $F_n(c;a,b)$ are defined by the following relation

$$G_0 = c, \quad G_1 = 1 \quad and \quad G_{n+1} = aG_n + bG_{n-1}, \quad (n \geq 1)$$

Where $F_n(0;1,1) = F_n$, are the Fibonacci numbers.
$F_n(2;1,1) = L_n$, are the Lucas numbers.
$F_n(0;2,1) = P_n$, are the Pell numbers.

4.2 Some results related to the Reciprocals of generalized Fibonacci numbers

Theorem 4.2.1. Let a,b be positive integers and c non negative integers. Then for $n \geq 1$, we have

(1) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n b^{n-1}(1 - ac - bc^2)$

(2) $\sum_{i=0}^{n} F_i = \frac{1}{a+b-1}(F_{n+1} + bF_n + ac - c - 1)$

Theorem 4.2.2. Let $F_n = V_n(c;1,1)$ for $c \geq 1$, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} \right)^{-1} \right\rfloor = F_n - 1 \quad (n \geq n_0),$$

Proof. By using the above lemma 4.2.1.

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n(1 - c - c^2)$$

and

$$\sum_{i=0}^{n} F_n = F_{n+2} - 1$$

Suppose $c \geq 1$, we have

$$\begin{aligned}
\frac{1}{F_{n-1}} - \frac{1}{F_{n+1}-1} - \frac{1}{\sum_{i=0}^{n} F_i} &= \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}-1} - \frac{1}{F_{n+2}-1} \\
&= \frac{F_{n+1}}{(F_n - 1)(F_{n+2}-1)} - \frac{1}{(F_{n+1}-1)} \\
&= \frac{2F_n - 1 + (-1)^{n+1}(c^2+c-1)}{(F_n - 1)(F_{n+1}-1)(F_{n+2}-1)}
\end{aligned}$$

Since F_n is monotonic increasing with n, we can take n so large that $2F_n \geq (-1)^n(c^2+c)$ for a fixed c. Hence, the numerator of the right-hand side of the above identity is positive if $n \geq N_1$ for some positive integer N_1, So we get

$$\begin{aligned}
\frac{1}{F_{n-1}} &\geq \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{F_{n+1}-1} \\
&\geq \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{\sum_{i=0}^{n+1} F_i} + \frac{1}{F_{n+2}-1} \\
&\geq \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{\sum_{i=0}^{n+1} F_i} + \frac{1}{\sum_{i=0}^{n+2} F_i} + \ldots
\end{aligned}$$

Thus,

$$\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} \leq \frac{1}{F_n - 1} \quad (n \geq N_1) \tag{1}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{\sum_{i=0}^{n} F_i} - \frac{1}{F_n} + \frac{1}{F_{n+1}} &= \frac{1}{F_{n+2}-1} - \frac{1}{F_n} + \frac{1}{F_{n+1}} \\
&= \frac{1 - F_{n+1}}{F_n(F_{n+2}-1)} + \frac{1}{F_{n+1}} \\
&= \frac{F_{n-1} + (-1)^n(c^2+c-1)}{F_nF_{n+1}(F_{n+2}-1)}
\end{aligned}$$

Similarly, we can take n so large that $F_{n-1} + (-1)^n(c^2+c-1) > 0$ for a fixed c. Hence, the numerator of the right-hand side of the above identity is positive if $n \geq N_2$ for some positive integer N_2, we get

$$\begin{aligned}
\frac{1}{F_n} &< \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{F_{n+1}} \\
&< \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{\sum_{i=0}^{n+1} F_i} + \frac{1}{F_{n+2}} \\
&< \frac{1}{\sum_{i=0}^{n} F_i} + \frac{1}{\sum_{i=0}^{n+1} F_i} + \frac{1}{\sum_{i=0}^{n+2} F_i} + \ldots
\end{aligned}$$

Thus,

$$\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} > \frac{1}{F_n} \quad (n \geq N_2) \tag{2}$$

Combining the two inequalities (1) and (2), we get

$$\frac{1}{F_n} < \sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} \leq \frac{1}{F_n - 1}$$

Where $n \geq n_0 = \max\{N_1, N_2\}$, which completes the proof.

Theorem 4.2.3. Let $a \geq b \geq 1$ and $F_n = V_n(0; a, b)$, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} \right)^{-1} \right\rfloor = F_n - 1 \quad (n \geq N_a),$$

Where $N_a = 3$, for $a = 1$ and $N_a = 2$, for $a \geq 2$

Proof. The case $a = 1$, has already been proved, it is sufficient to show the case $a \geq 2$. Defining $S_n = \sum_{i=0}^{n} F_i$, we have

$$\begin{aligned} \frac{1}{S_n} - \frac{1}{F_n} + \frac{1}{F_{n+1}} &= \frac{1}{F_{n+1}} - \frac{S_{n-1}}{F_n S_n} \\ &= \frac{F_n S_n - F_{n+1} S_{n-1}}{F_n F_{n+1} S_n} \\ &= \frac{F_{n+1} - F_n + (-1)^{n+1}}{a F_n F_{n+1} S_n} \\ &> 0 \end{aligned}$$

and for $n \geq 2$

$$\begin{aligned} \frac{1}{F_n - 1} - \frac{1}{F_{n+1} - 1} - \frac{1}{S_n} &= \frac{S_{n-1} + 1}{(F_n - 1) S_n} - \frac{1}{F_{n+1} - 1} \\ &= \frac{F_{n+1} S_{n-1} - F_n S_n + a S_n}{(F_n - 1)(F_{n+1} - 1) S_n} \\ &= \frac{2F_n + (-1)^n - 1}{a (F_n - 1)(F_{n+1} - 1) S_n} \\ &> 0 \end{aligned}$$

Then, we get

$$\frac{1}{F_n} < \sum_{k=n}^{\infty} \frac{1}{S_k} \leq \frac{1}{F_{n-1}} \quad (n \geq 2)$$

which completes the proof.

Chapter 5

5 Fibonacci Numbers Modulo a Lucas Number

5.1 Introduction

Let $m_1, m_2, ...$ be positive integers, then

$$k := \sup_{x \in (0,1)} \min_i \|xm_i\|.$$

Here, for $x \in \mathbf{R}$, $\|x\|$ is the distance to the nearest integer. Observe that if we have a finite number of integers $m_1, m_2, ...m_n$ and

$$k_1 := \max_{m=m_j+m_l} \frac{1}{m} \min_i |km_i|_m$$

Where $|x|_m$ denotes the absolute value of the absolutely least remainder of x mod m.

5.2 Main Results

Let $M = \{F_2, F_3, ...F_t\}$. Let $n \geq 1$ be an integer such that $4k+2 \leq t \leq 4k+5$ and $m = F_{2k+2} + F_{2k+4} = L_{2k+3}$.

In this section, we use the following identities.

1. Cassini's identity: $F_{k+1}F_{k-1} - F_k^2 = (-1)^{k+1}$.
2. $F_k^2 + F_{k+1}^2 = F_{2k+1}$
3. d'Ocagne's identity: $F_mF_{k+1} - F_{m+1}F_k = (-1)^k F_{m-k}$
4. $F_{2k} = \sum_{i=0}^{k-1} F_{2i+1}$
5. $F_{2k+1} - 1 = \sum_{i=1}^{k} F_{2i}$

5.3 Properties of Fibonacci Numbers Modulo m

Lemma 5.3.1.

(a) $F_2F_{2k+2} \equiv F_{4k+5}F_{2k+2} \pmod m$

(b) $F_2F_{2k+2} \equiv -F_{4k+4}F_{2k+2} \pmod m$

(c) $F_3F_{2k+2} \equiv F_{4k+3}F_{2k+2} \pmod m$

(d) $F_4F_{2k+2} \equiv -F_{4k+2}F_{2k+2} \pmod m$

Proof. (a) We have

$$\begin{aligned} mF_{2k+2} &= (F_{2k+2} + F_{2k+4})F_{2k+2} \\ &= F_{2k+2}^2 + F_{2k+4}F_{2k+2} \\ &= F_{2k+2}^2 + (F_{2k+3} + F_{2k+2})(F_{2k+3} - F_{2k+1}) \\ &= F_{2k+2}^2 + F_{2k+3}^2 + F_{2k+3}F_{2k+2} - F_{2k+3}F_{2k+1} - F_{2k+2}F_{2k+1} \\ &= F_{2k+2}^2 + F_{2k+3}^2 + (F_{2k+2} + F_{2k+1})F_{2k+2}^2 - F_{2k+3}F_{2k+1} - F_{2k+2}F_{2k+1} \\ &= F_{2k+2}^2 + F_{2k+3}^2 + F_{2k+2}^2 - F_{2k+3})F_{2k+1} \\ &= F_{2k+2}^2 + F_{2k+3}^2 + (-1)^{2k+1} \qquad [Cassini's\ identity] \\ &= F_{4k+5} - 1 \end{aligned}$$

Hence, we get

$$F_{4k+5}F_{2k+2} = (1 + mF_{2k+2})F_{2k+2} \equiv F_{2k+2} = F_2F_{2k+2} \pmod m$$

$$\text{i.e } F_2F_{2k+2} \equiv F_{4k+5}F_{2k+2} \pmod m$$

Proof. (b) By using (a)

$$\begin{aligned} F_2F_{2k+2} &\equiv F_{4k+5}F_{2k+2} \pmod m \\ &= (F_{4k+6} - F_{4k+4})F_{2k+2} \\ &= F_{4k+6}F_{2k+2} - F_{4k+4}F_{2k+2} \\ &= (L_{2k+3}F_{2k+3})F_{2k+2} - F_{4k+4}F_{2k+2} \qquad [Since\ L_kF_k = F_{2k}] \\ &\equiv -F_{4k+4}F_{2k+2} \ (modm) \end{aligned}$$

i.e $F_2F_{2k+2} \equiv -F_{4k+4}F_{2k+2} \pmod{m}$

Proof. (c)

$$\begin{aligned} F_3F_{2k+2} &= (F_2+F_1)F_{2k+2} \\ &= F_2F_{2k+2} + F_1F_{2k+2} \\ &= F_{4k+5}F_{2k+2} + F_2F_{2k+2} \quad [By\ using\ (a)\ and\ F_1 = F_2] \\ &= F_{4k+5}F_{2k+2} - F_{4k+4}F_{2k+2} \quad [By\ using\ (b)] \\ &\equiv (F_{4k+5} - F_{4k+4})F_{2k+2} \\ &\equiv F_{4k+3}F_{2k+2} \pmod{m} \end{aligned}$$

i.e $F_3F_{2k+2} \equiv F_{4k+3}F_{2k+2} \pmod{m}$

Proof. (d)

$$\begin{aligned} F_4F_{2k+2} &= (F_3+F_2)F_{2k+2} \\ &= F_3F_{2k+2} + F_2F_{2k+2} \\ &\equiv (F_{4k+3} - F_{4k+4})F_{2k+2} \quad [By\ using\ (b)\ and\ (c)\,] \\ &\equiv -F_{4k+2}F_{2k+2} \ (modm) \end{aligned}$$

i.e $F_4F_{2k+2} \equiv -F_{4k+2}F_{2k+2} \pmod{m}$

Lemma 5.3.2. $F_{p+5}F_{2k+2} = (F_{p+4}+F_{p+3})F_{2k+2} \equiv \epsilon F_{4k+1-p}F_{2k+2} \pmod{m}$ for each $0 \le p \le 2k-2$, where

$$\epsilon = \begin{cases} +1, if\ p\ is\ even. \\ -1, if\ p\ is\ odd. \end{cases} \tag{3}$$

Proof. Using the recurrence relation $F_{K+1} = F_k + F_{k-1}$ for $k \ge 1$ and Lemma 4.3.1, the proof follows by induction on p.

Lemma 5.3.3.

(a) $F_2F_{2k+2} \equiv F_{2k+2} \pmod{m}$.

(b) $F_3F_{2k+2} \equiv -F_{2k+3} \pmod{m}$.

(c) $F_4F_{2k+2} \equiv -F_{2k+1} \pmod{m}$.

(d) $F_5F_{2k+2} \equiv 2F_{2k+2} - F_{2k+1} \pmod{m}$.

Proof. (a)

$$\begin{aligned} F_2F_{2k+2} &= 1F_{2k+2} \\ &\equiv F_{2k+2} \ (mod\ m) \end{aligned}$$

i.e $F_2F_{2k+2} \equiv F_{2k+2} \ (mod\ m)$.

Proof. (b)

$$\begin{aligned} F_3F_{2k+2} &= 2F_{2k+2} \ \ (As\ F_3 = 2) \\ &\equiv -(F_{2k+1} + F_{2k+2}) \\ &= -F_{2k+3} \ (mod\ m) \end{aligned}$$

i.e $F_3F_{2k+2} \equiv -F_{2k+3} \ (mod\ m)$.

Proof. (c)

$$\begin{aligned} F_4F_{2k+2} &= (F_2 + F_3)F_{2k+2} \\ &= F_2F_{2k+2} + F_3F_{2k+2} \\ &\equiv (F_{2k+2} - F_{2k+3}) \quad [F_2 = 1\ and\ using\ (b)] \\ &\equiv -F_{2k+1} \ (mod\ m) \end{aligned}$$

i.e $F_4F_{2k+2} \equiv -F_{2k+1} \ (mod\ m)$.

Proof. (d)

$$\begin{aligned} F_5F_{2k+2} &= 5F_{2k+2} \quad (AsF_5 = 5) \\ &= (F_3 + F_4)F_{2k+2} \quad [using\ (b)\ and\ (c)] \\ &= F_3F_{2k+2} + F_4F_{2k+2} \\ &\equiv 2F_{2k+2} - F_{2k+1} \ (mod\ m) \end{aligned}$$

i.e $F_5F_{2k+2} \equiv 2F_{2k+2} - F_{2k+1} \ \ (mod\ m)$.

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