PHASE-ONLY IMAGE SYNTHESIS USING FAST GENERALIZED FOURIER FAMILY TRANSFORM (FGFT)

K RAM PRABHAKAR

DEPARTMENT OF ELECTRICAL ENGINEERING
NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA
ROURKELA, INDIA

May 2014

Phase-Only Image Synthesis Using Fast Generalized Fourier Family Transform (FGFT)

A Thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Technology

in

Electronic Systems and Communication

by

K Ram Prabhakar

ROLL NO: 212EE1212

Under Guidance of

Dr. (Prof). Supratim Gupta

Department of Electrical Engineering
National Institute of Technology Rourkela
Rourkela, India

May 2014

Declaration of Authorship

I, K Ram PRABHAKAR, declare that this thesis titled, “ Phase-Only Image Synthesis Using Fast Generalized Fourier Family Transform (FGFT)” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

CERTIFICATE

This is to certify that the thesis entitled, **" Phase-only Image Synthesis using Fast Generalized Fourier Family Transform(FGFT)"** submitted by **K Ram Prabhakar** in partial fulfillment of the requirements for the award of Master of Technology Degree in **Electrical Engineering** with specialization in **Electronic Systems and Communication** during 2013-2014 at the National Institute of Technology, Rourkela (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/Institute for the award of any Degree or Diploma.

Date: . . . .

Dr.(Prof). Supratim Gupta
Dept. of Electrical Engineering
National Institute of Technology
Rourkela-769008
Odisha, India

"The three great essentials to achieve anything worthwhile are, first, hard work; second, stick-to-itiveness; third, common sense"

Thomas Edison

Abstract

Phase-Only Image Synthesis Using Fast Generalized Fourier Family Transform (FGFT)

The phase components of complex valued transform coefficients retain the edge information of an image [1]. Fast Generalized Fourier Transform (FGFT) is a complex, non-redundant, progressive resolution, globally referenced phase output, time-frequency representation applicable for non-stationary signals [2]. Though the analysis of forward FGFT framework for 1D signal is present in literature, analysis of synthesis (or reverse) FGFT framework is not available for 1D and 2D signals. An Image (2D signal) synthesized using only phase components of FGFT, does not produce edge information. In the process of FGFT, Fourier Transform (FT) phase coefficients gets mingled together, thus the normalization procedure used to retain phase is ineffective to separate FT phase coefficients from FGFT samples. This thesis proposes an algorithm to effectively separate FT phase coefficients from FGFT samples and thus reconstruct image with edge information from phase-only components of FGFT samples. The amount of information present in phase and magnitude of different transforms is measured. The comparison indicates that FT retains most information than others in phase-only image reconstruction and Curvelet Transform (CT) retains most information in magnitude-only image reconstruction compared to other transforms. In contrast, FGFT retains edge information equally in both magnitude and phase components.

Acknowledgements

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

Foremost, I am greatly indebted to my advisor **Prof. Supratim Gupta** for the continuous support of my M.Tech study and research, for his patience, motivation, enthusiasm and immense knowledge. His guidance helped me in all the time of research. I could not have imagined having a better advisor and mentor for my M.Tech study.

To my mother, **Meenal**, who continues to learn, grow and develop and who has been a source of encouragement and inspiration to me throughout my life, a very special thank you for providing a 'writing space' and for nurturing me through the months of hardships. And also for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realise my potential, and to make this contribution to our world.

To my father, **Kathirvel**, who guided in all phases of my life.

To my brother, **Thivakaran**, for his support.

Last but not least, I thank my fellow labmates in Embedded System & Real Time Lab: Mathew Francis, Mayaluri Zefree Lazarus, Sankarasrinivasan, Sajith Kumar, Sreejith Markkassery and Susant Kumar Panigrahi for the stimulating discussions, for the sleepless nights we were working together before deadlines, and for all the fun we have had at ESRT Lab. I would also like to thank my friends Prathima Addanki, Uma Guguloth and my classmates for happy moments during my project work.

K Ram Prabhakar

Contents

List of Figures

List of Tables

Abbreviations

FT	**F**ourier **T**ransform
TFR	**T**ime **F**requency **R**epresentation
STFT	**S**hort **T**ime **F**ourier **T**ransform
ST	**S**tockwell **T**ransform
FGFT	**F**ast **G**eneralized **F**ourier family **T**ransform
WT	**W**avelet **T**ransform
CT	**C**urvelet **T**ransform
CTW	**C**urvelet **T**ransform with **W**rapping

Dedicated to My Parents

1 Introduction

1.1 Introduction

SIGNAL is a phenomenon that carries information as varying across an independent variable. It is a range of values on an independent variable called domain. The signal measured can be anything, could be voltage measured against time, pressure measured against height or stock market values measured against each day. A signal whose parameters such as mean, standard deviation and frequency are same throughout the duration of signal is called *Stationary signal*. On the other hand, a signal that violates the above property is called *Non-stationary signal*. Most of the real world signals are non-stationary. A signal can store information in different ways in different domains such as time, frequency, space, etc. The two domains that are more interest than others are time and frequency. A same signal presents different information in time and frequency domain. A tool that performs conversion of signal between different domains is called a *transform*.

It is a well proven concept that phase output of several complex valued transform retains edge information of an image (a 2D signal). Among different transforms, absolutely referenced phase output is provided by General Fourier Family Transforms. One among General Fourier Family Transforms is Stockwell Transform (ST), which is a progressive resolution transform. Since the introduction of ST by Stockwell *et al.*[3] immense research is in progress in Windowed Fourier Transforms. However, output of ST for an image is 4-D, thus storing output and processing is impractical for image signals. That was overcome by Fast Generalized Fourier Family Transform (FGFT), which was introduced by R A Brown *et al*[4]. Though, extensive research work has been done on 1-D signal, similar development in 2-D signal is not reported in literature. Although analysis of a signal through FGFT is reported, yet synthesis of a signal through the same is not.

1.2 Motivation

Detecting edges of an image is crucial in many image processing applications. Though many different techniques exist for the purpose, detecting edges using transforms provides more insight into signal properties. Further, image is a 2D non-stationary signal, thus applying TF transform would yield more information than frequency domain transforms. However, many TF transforms are redundant, resulting in 4D output. Though this problem could be eliminated by using FGFT which produces 2D output, synthesizing input image edge information with phase-only FGFT coefficients is erroneous.

1.3 Literature review

TABLE 1.1: Literature review

Authors	Year	Contribution	Limitations
A. V. Oppenheim and J. S. Lim [1]	1981	Proved theoretically and experimentally that FT phase retains more edge information than magnitude for 1D and 2D signals	-
Stockwell *et al.*[3]	1996	Introduced progressive resolution ST for 1D Non-stationary data signals	• Redundant • Explanation about synthesis framework is not attempted
R.A.Brown *et al.*[4]	2010	Introduced Non-redundant ST (i.e) FGFT	Explanation on synthesis framework is not attempted
Naghizadeh *et al.*[5]	2013	Extended 1D FGFT to 2D seismic data signals	Synthesis of signals is not reported

1.4 Objective

- To perform detailed study of FGFT analysis coefficients for 2D input image
- To understand FGFT synthesis framework
- To analyse failure on phase-only image reconstruction using FGFT

- To reconstruct image with edge information from phase-only FGFT coefficients
- To measure amount of edge information in magnitude and phase components of FGFT

1.5 Contributions of the thesis

- Study on FGFT synthesis framework
- Proposed an algorithm to reconstruct edge information from phase-only FGFT coefficients
- Comparison on amount of edge information present in phase and magnitude of different transforms

1.6 Thesis overview

This thesis is divided into six chapters:

- Chapters 1 and 2 discusses the current literature in perspective of this thesis's objective. Chapter 2, covers the broad classfication of frequency transform and Time-Frequency(TF) transform. It also presents the preliminary terms, definitions and concepts necessary to proceed through thesis. Towards end of Chapter 2 comparison of different transforms is presented. It points out the disadvantage in using FT and the reason why TF transforms are preferred.
- Chapter 3, presents the analysis of FGFT, its implementation method and address the reason of non-redundancy in FGFT and further extends the 1D FGFT to 2D FGFT.
- Chapter 4 introduces algorithm to extract edge information from phase-only FGFT coefficients and points out the implementation procedure along with review on algorithm.

- Chapter 5 discusses an application of measuring amount of structural information present in phase and magnitude using proposed algorithm.
- The thesis ends with Chapter 6 concluding the contributions of this thesis and discussing the future scope of this work.
- Appendix A presents short introduction on WT and CT.

2
Background

2.1 Importance of Phase

IN the complex valued transform representation of signals, importance of magnitude and phase differs with respect to application. In many situations, phase part holds more information than magnitude. For example, both phase-only and magnitude-only holograms have been reported in [1]. For phase-only holograms only the phase of the scattered wavefront is recorded and the magnitude is modified to be constant while in the magnitude-only hologram the phase is assumed to be zero and only the magnitude of the scattered wavefront is recorded.

The phase-only hologram reconstruction represents more information about input objects than magnitude-only reconstruction. In one another context where phase information is a necessity is Image processing. The phase component of FT of an image holds the important part of image (i.e) edges. Image reconstructed with magnitude component alone shows only blurred image.

The relative importance of phase over magnitude is demonstrated using two images in Figure 2.1. In Figure 2.1, A and B are input images. Phase information from A is combined with magnitude information from B, and Inverse Fast Fourier Transform (IFFT) of the resulting combination is performed to obtain C. The reverse combination of phase from B combined with magnitude from A, and IFFT of combination results in D. C closely resembles the image from where the original phase is used, i.e., A but not the image from where magnitude is used. Same is true for D also. This verifies the concept that phase retains important information compared to magnitude of FT.

In Phase-only reconstruction, the phase alone is retained and the original signal is reconstructed back. Some applications are: Edge Detection, since all the edge information is stored in phase part, its sufficient that phase alone is processed compared to processing both phase and magnitude.

Another context in which the potential importance of phase-only synthesis has been recognized is in Blind De-Convolution [1], there may be situations that image is blurred by a blurring function, A. In some cases, where the A is due to defocused lens or prolonged exposure to environmental turbulence, A's phase function is zero, then the recovered image has same phase information as that

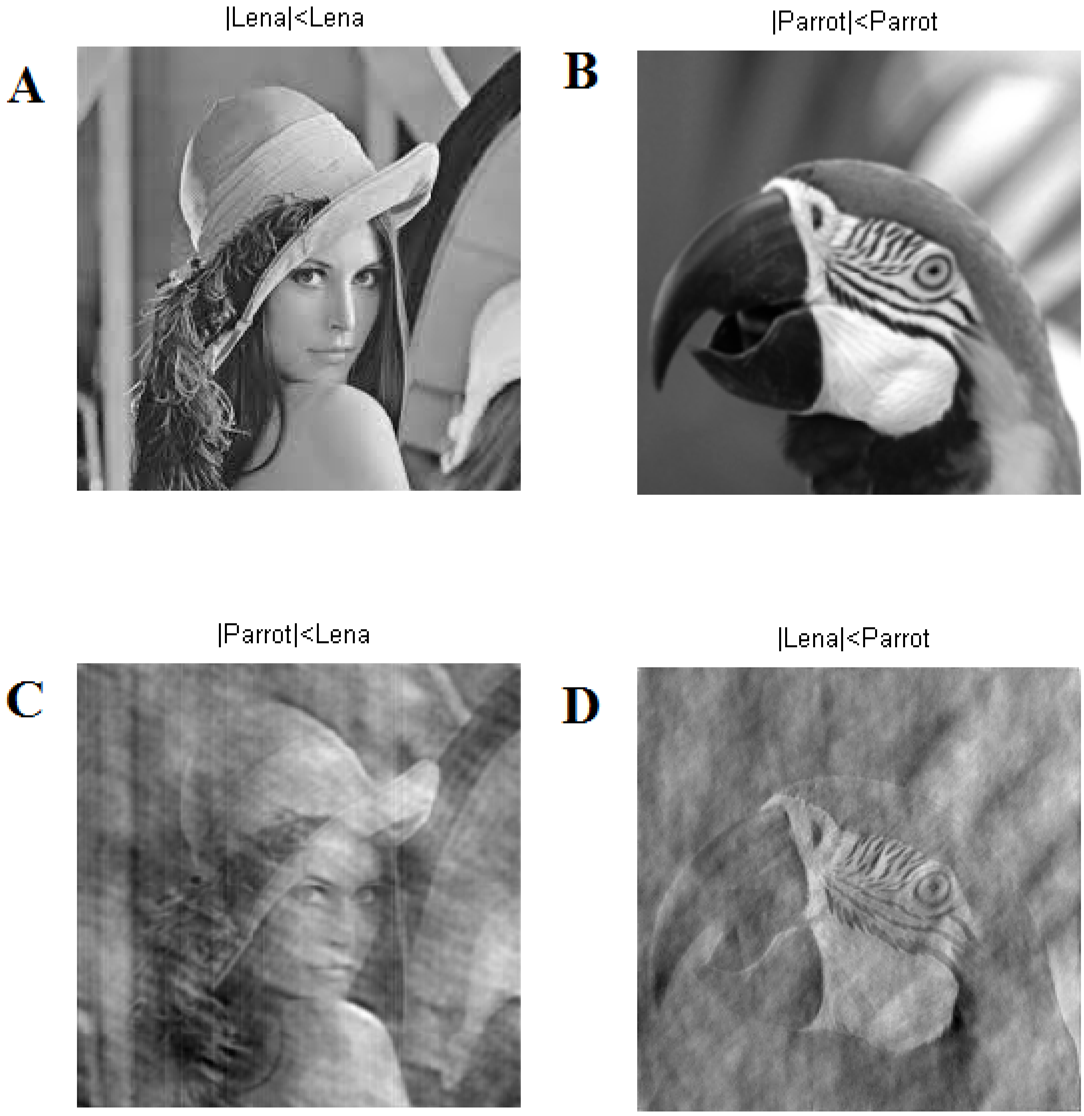


FIGURE 2.1: Phase *vs* Magnitude. A - Lena image. B - Parrot image. C - IFFT of Phase from A and Magnitude from B. D - IFFT of Phase from B and Magnitude from A.

original scene. Here, phase-only reconstruction could be applied to get original scene back.

Next application is, Palm-Print Recognition, it is been reported in literature [6] that many distinct features of palm print could be retrieved from phase information.

In Image coding [1], the magnitude and phase are separately encoded and transferred, in receiver section we will reconstruct image by decoding both magnitude and phase, then apply inverse transform. Phase has to be coded with 2 bits more than magnitude and thus it is sufficient that only phase is transferred and reconstructed at receiver using phase-only reconstruction methodologies. The fore mentioned applications highlight the importance of phase over magnitude of several transforms. There is a loss of information with discarding magnitude part but it is negligible when compared to information stored in phase.

The prime importance of using phase-only reconstruction is to detect edges of input signal. This method of using transform domain coefficients to detect edges is advantageous than time/spatial domain methods, since transforms represent same signal with less coefficients. Thus number of computations required is less while operating with transform domain coefficients.

2.1.1 Extracting edges using phase-normalization technique

This section discusses the procedure to retain phase alone and discard magnitude from the complex output of transforms. The output of complex valued transforms, F has real and imaginary components. F can be expressed as,

$$F = a_k + jb_k$$

where, $j = \sqrt{-1}$, a_k and b_k are real and imaginary parts of the transform output. This method of representation is called *Rectangular form* representation of a complex number. The same can be expressed in terms of *Polar form*, as below,

$$F = Me^{j\phi}$$

where, M is the magnitude of F and ϕ is the angle, which is calculated as follows,

$$M = \sqrt{a_k^2 + b_k^2}$$
$$\phi = \tan^{-1} \frac{b_k}{a_k}$$

The phase information alone is retained by making M=1 or by dividing F with M, which when expressed in polar form,

$$F' = \frac{Me^{j\phi}}{M}$$
$$F' = e^{j\phi}$$

This modified transform coefficients F', when applied with inverse transform will show edge information of input signal.

2.2 Fourier Transform

A signal recorded against time exhibits its characteristics with respect to time. Though this is desirable, signal might have different information in other domains. A technique called transformation is applied on the input signal to produce another signal in different domain. The purpose of using transforms is to provide clear information in one domain which was obscure in another domain. For example, measuring frequencies present in a signal is difficult task in time domain, but becomes very easy once transformed into frequency domain.

It has been well understood that a given signal can be represented in a infinite number of different ways. Different signal representations is suitable for different applications. For example, signals obtained from most engineering applications are usually functions of time. But when studying or designing the system, it is often desired to study signals and systems in the frequency domain. This is because many important features of the signal or system are more easily characterized in the frequency domain than in the time domain.

Although the number of ways of describing a given signal are countless, the most important and fundamental variables in nature are time and frequency. While the time domain function indicates how a signal's amplitude changes over time, the frequency domain function tells how often such changes take place. The bridge between time and frequency domain is the Fourier Transform(FT) developed by J.B.J. Fourier in 18^{th} century. The modified transforms such as Fast Fourier Transform(FFT), Short-time Fourier Transform(STFT) and many other transforms fall under general category called Fourier Family transforms. Transforms can be classified based on their output types and also based on the type of signals that they are applicable for.

In general, the function of any transform is to represent input signal x in terms of set of functions called synthesis functions, e_k.

$$x = \Sigma_k c_k e_k$$

In the geometrical sense, they provide the co-ordinates c_k of the input function with respect to analysis function $\tilde{e}_k$

$$c_k = \langle x, \tilde{e}_k \rangle$$

Analysis functions are related with synthesis functions in following way,

$$\langle e_i, \tilde{e}_j \rangle = c\delta[i - j]$$

where c is an arbitary constant and $\delta[i]$ is an impulse function.

The fundamental idea behind Fourier's original work was to decompose a signal $x(t)$ as the sum of weighted sinusoidal functions. In FT, $e^{-j\omega t}$ is the analysis function, with ω denoting the angular frequency measured in radians per second. The FT coefficients are function of frequencies present in the signal and is calculated by,

$$S_{FT}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.1}$$

The input signal is reconstructed back by inverse FT where $e^{j\omega t}$ is the synthesis function,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{FT}(\omega)e^{j\omega t}d\omega \tag{2.2}$$

FT for 2D signal $x(g, h)$ is as follows,

$$S_{FT}(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(g, h)e^{-j2\pi(ug+vh)}dgdh \tag{2.3}$$

where (u, v) denotes the spatial frequency in case of images. Though FT provides information about frequencies present in a signal, that is inadequate when the signal is real time signals. For example, Electro Encephalogram(EEG) signal recorded for a subject are short duration signals. These type of real world signals are not best represented with infinite duration sinusoids of FT.

Drawback of Fourier transform is that, it produces the time-averaged spectrum [8]. This is adequate for stationary time series in which the characteristics of the time series do not change with time. In most of real world data, however, stationarity is an unrealized abstract entity. As figure 2.2 shows, spectral content of the time series that changes with time, and the time-averaged amplitudes found by Fourier methods are inadequate to describe such phenomena. FT provides information about what frequency is present in signal but not when the frequency is present and how the frequency changes with respect to time.

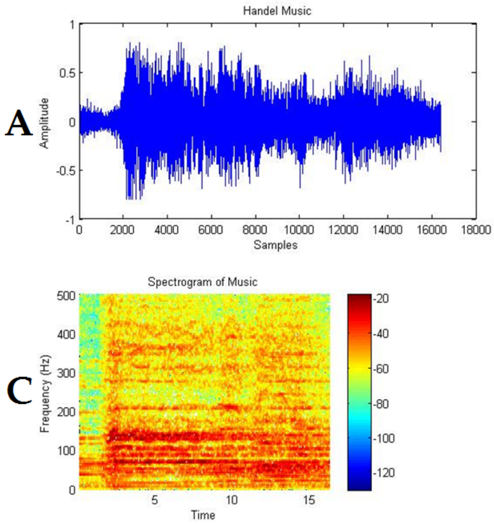
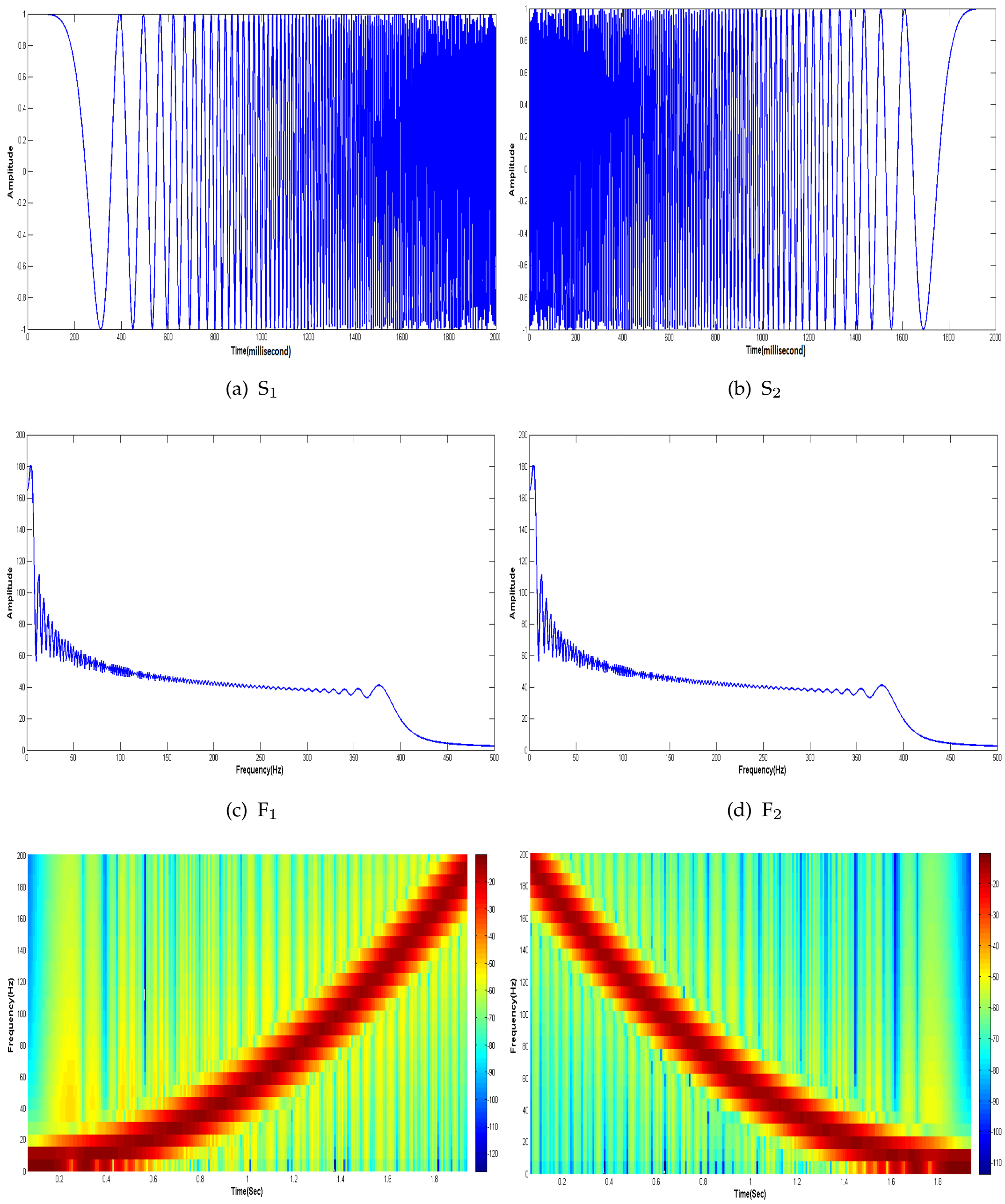


FIGURE 2.2: FT and spectrogram of Non-Stationary signal. A - Non-stationary signal, B - FT of signal and C - Spectrogram of signal

In the figure 2.3, ’a’ is a input chirp signal S_1 whose frequency increase gradually from 0Hz to 200Hz and ’b’ is another chirp signal S_2 whose frequency decrease gradually from 200Hz to 0Hz. Fourier spectrum of both ’a’ and ’b’ are shown in ’c’ and ’d’, though both of the input signals are different their FT is same. In order to overcome the above mentioned limitations, Fourier analysis has given way to more advanced representations known as *Joint Time-Frequency Representations(JTFR)*.

2.3 Joint Time-Frequency Representations(JFTR)

From the discussion on FT in previous section, it is apparent that FT is not suitable for non-stationary signals. The profound reason is that basis function



(a) S_1 (b) S_2

(c) F_1 (d) F_2

(e) $STFT_1$ (f) $STFT_2$

FIGURE 2.3: a - Increasing frequency chirp signal, b - Decreasing frequency chirp signal, c - FT of increasing chirp signal, d - FT of decreasing chirp signal, e - STFT of increasing chirp signal, f - STFT of decreasing chirp signal

used to represent input signal is not localized, meaning that only events or frequencies which are present in entire duration of the signal will be identified. Thus, for non-stationary signals where events occur intermittently and only last for few duration will be overlooked by these infinite duration sinusoidal functions of FT.

In order to circumvent this problem, Time-Frequency(TF) transforms were introduced. The key point is to have a localized basis function in time domain and calculate frequency present only at that duration. Different TF transforms have different method to achieve this objective. This section gives brief overview of basic terminology needed to understand TF transforms.

2.3.1 Resolution and Heisenberg's uncertainty principle

A good time resolution transform is able to identify the time at which change in frequency happens. A good frequency resolution transform is able to identify the finest change in frequency of signal. Time and frequency resolution are related to each other by the following *Heisenberg's uncertainty principle* in equation 2.4. Both good time resolution and good frequency resolution can't be obtained simultaneously; a trade-off has to be made in favour of other. The upper limit of achieving optimum time and frequency resolution is outlined by Heisenberg's uncertainty principle.

$$\Delta t \Delta f \geq \frac{1}{2} \tag{2.4}$$

where, Δt, Δf = time and frequency resolution.
Uncertainty principle could be understood in terms of information theory[9]. The questions that need to answered are,

- How much information is needed to estimate the spectrum of a signal?
- How much time one have to observe the signal to collect such information?

The total amount of information needed to properly estimate power spectrum is equal to the spectral complexity of the signal, C_s. The amount of collected information, $I(t)$ about the signal is the sum of information density of signal, $D(t)$ in that observed duration.

$$I(t) \leq \int_{t_0}^{t_0+\tau_r} D(\epsilon)d\epsilon = C_s \tag{2.5}$$

C_s is the measure of spectral narrowness(amount of peak thickness in TF graph). For example, the spectrum of sinusoid input has a thin ridge, an impulse. Thus its spectral narrowness is so high. The C_s can be calculated as follows,

$$C_s = \sqrt{\max_f(-\frac{\delta^2}{\delta f^2} P_s(f))} \tag{2.6}$$

where, $P_s(f)$ is the power spectral density of the signal. τ_r is the time needed to collect all information to estimate power spectrum.

For the case of stationary signals, the signal measured at time t_1 has high spectral correlation with signal measured at t_2, since stationary signals have same frequency dynamics throughout entire signal duration. Thus longer the signal was observed, the more information could be gathered and thus high spectral narrowness (i.e) high frequency resolution.

For the case of non-stationary signals, spectral correlation of signal at t_1 and at t_2 is less. Since, the frequency dynamics of non-stationary signal keeps on changing, measurement made at t_1 will not help to estimate spectral components at t_2. Any information collected at time other than t_1 must be weighted with non-unitary correlation factor to indicate relativeness of those informations with signal at t_1.

$$I^{t_1}(t) = \int_{t_0}^{t} D(\epsilon)u^{t_1}(\epsilon - t_1)d\epsilon \tag{2.7}$$

where, $u^{t_1}(t_2 - t_1)$ is the non-unitary weighting function. The weighing function included will provide a upper threshold of maximum amount of useful spectral information that can be collected for signal at t_1. This upper limit will be less for non-stationary signals, as signal changes randomly and only few neighbourhood data points contain similar spectral information.

Progressive resolution is a method by which time and frequency resolution changes at any moment according to signal dynamics at that moment.

2.3.2 Short-Time Fourier Transform(STFT)

The problem associated with FT is that, its analysis function uses sinusoids that are not localized in any instant of time. This is easily overcomed by using a analysis function that is defined only for a duration. Time localization is accomplished by introducing a window, $w(t)$ that limits the duration of basis functions. STFT uses same sinusoid basis functions as that of FT. The STFT output for an signal $x(\tau)$ is a 2D function of (τ,ω) and it is given by,

$$S_{STFT}(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(\tau - t)e^{-j\omega t}dt \tag{2.8}$$

The synthesis of $x(\tau)$ from $S_{STFT}(\tau,\omega)$ is given by,

$$x(\tau) = \int_{-\infty}^{\infty} S_{STFT}(\tau,\omega)e^{j\omega\tau}d\omega \tag{2.9}$$

The product of both window and signal, chooses a block of input signal and FT is applied on that block. Thus only the frequency present in that duration is calculated. The window function can be rectangular or gaussian depending on application. Rectangular window provides good time resolution but poor frequency resolution, whereas gaussian window provide optimal time and frequency resolution. The window width can be reduced to get good time resolution but that reduces frequency resolution as per Heisenberg's uncertainty princple. Having fixed resolution(see figure 2.7(b)) is the shortcoming of STFT.

To illustrate the process, in Figure 2.4, a 1-D non-stationary chirp signal is taken, in which frequency changes from 0Hz to 10Hz. The signal is partitioned by a window function $w(t,\sigma)$. Window function can be rectangular function or box car function. A segment of signal is selected by window and FT is applied on that part under the assumption that it is stationary in that segment. Window is translated to other parts of signal and process is repeated. The result which is plotted across frequency Vs time plot, gives the picture of which frequency is present for what duration.

The output of STFT is shown in figure 2.3 'e' and 'f'. The signal in 'e' shows the spectrogram for chirp signal in 'a', it can be seen that spectrogram

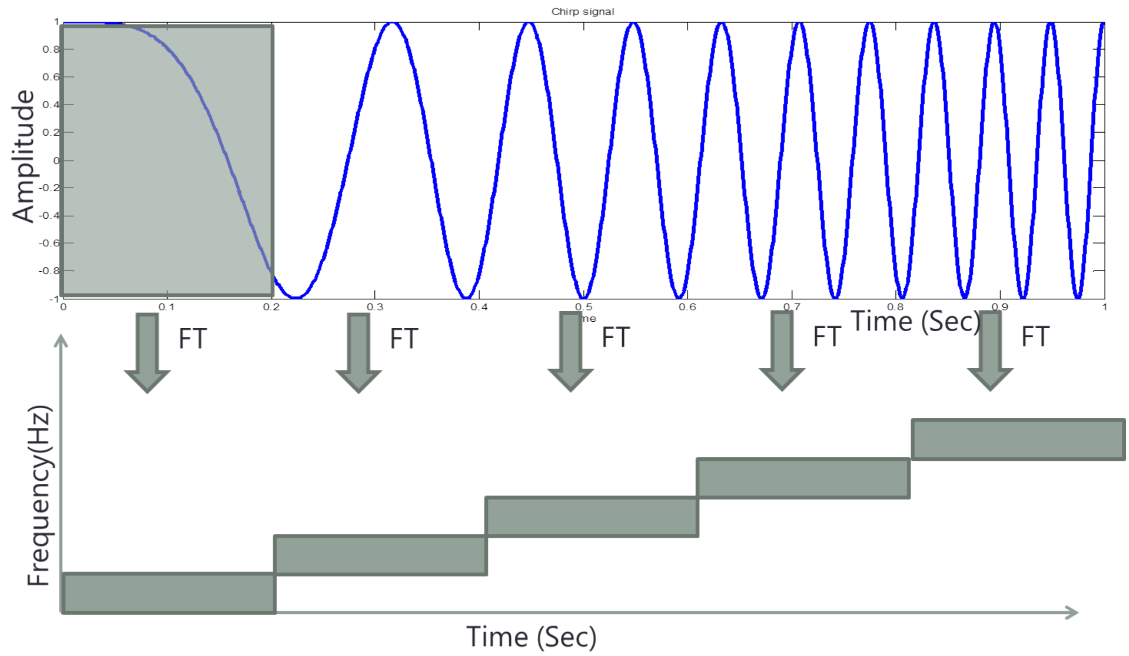


FIGURE 2.4: Demonstration of steps involved in Windowed Fourier Transform.

shows that frequency increases as time progresses and reaches 50Hz at time of 1 second. The same is the case of spectrogram in 'f'.

2.3.3 Stockwell Transform(ST)

The S-transform (ST), introduced by Stockwell *et al.*[3], is similar to STFT except that ST has progressive resolution. ST combines progressive resolution property of WT with globally referenced phase output property of FT. ST also provides frequency invariant amplitude response unlike WT. Absolutely referenced phase information means that the phase information given by the ST refers to the argument of the cosinusoid at zero time (which is the same as phase given by the FT). The progressive resolution is introduced by σ parameter as given in below equation,

$$S(\tau, \upsilon) = \int_{-\infty}^{\infty} g(t) w(t-\tau, \sigma) e^{-i2\pi\upsilon t} dt \tag{2.10}$$

where, τ, υ = shift and frequency coordinates
$g(t)$ = Input signal
$w(t, \sigma)$ = Gaussian window function = $\frac{1}{\sqrt{2\pi}|\sigma|} e^{-\frac{t^2}{2\sigma^2}}$

$\sigma = \frac{1}{v}$, Window scaling parameter, controls the width of window
The window width is adjusted according to the frequency content of the signal, as shown in Figure 2.5, thus high frequency resolution is achieved in low frequency contents and high time resolution is achieved in high frequency contents. The extention of ST for 2D signal $x(g,h)$ yield a 4D output $S_{ST}(\alpha,\beta,u,v)$.

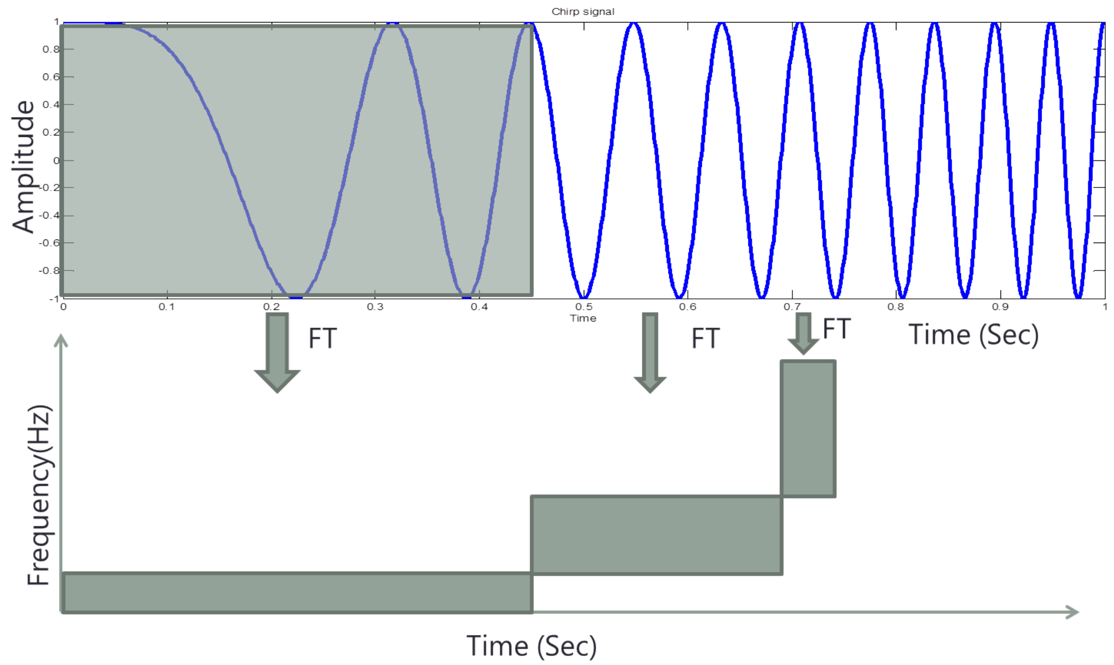


FIGURE 2.5: Progressive Resolution in ST.

For the images, intensity value gets varied across horizontal, g and vertical direction, h. The result is obtained with respect to spatial frequencies (u,v) and window shift parameters (α,β).

$$S_{ST}(\alpha,\beta,u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(g,h)w(\alpha-g,\beta-h,u,v)e^{-j2\pi(ug+vh)}dgdh \tag{2.11}$$

The disadvantage of ST is the redundancy in its output, ST produces 2D output for its 1D input. Redundancy is introduced because ST does not obey uncertainty principle. This point is discussed in detail in next chapter.

2.4 Comparison between different transforms

The Fourier transform (Fourier series counterpart for nonperiodic signals) provides frequency content information which are present for entire duration of signal. Thus the need for localized frequency content information is emphasized. An attempt to create local Fourier bases is the Gabor transform or STFT. A smooth window (mostly Gaussian window) is applied to the signal centered around $t = nT_0$, and a Fourier expansion is applied to windowed signal. The drawback of STFT is due to fixed window size, which restricts the fine tuning of time and frequency resolution. The expansion function of STFT is the combination of window(which can be shifted) and complex exponential function, thus they are function of shift and frequency.

The drawback of STFT was overcome by Stockwell Transform, in which window can be shifted and scaled, thus enabling to have optimal time and frequency resolution. Operations of ST are similar to Wavelet transform (WT), but ST provides global phase reference compared to local phase reference of WT. Mathematically ST is equivalent to WT multiplied by a phase factor[3]. Though ST provides good time and frequency resolution, application of ST in real time scenarios need high computation complexity. Inherently, output of STFT and ST are redundant, since they map a one-variable function (mostly time or space) into a two-variable function (time-frequency or space-spatial frequency). Thus a generalized framework for windowed Fourier transform was introduced as Fast Generalized Fourier transform (FGFT)[4](details of FGFT are given in next chapter). As seen from Figure 2.6 [2–4, 7, 8, 10], any type of windows can be used for FGFT implementation. Time complexity and memory complexity for FGFT is same as that of FFT. For ST and STFT, time and memory complexity are almost squared as that of FGFT. In Figure 2.6, '1+' denotes that, corresponding transform possess that property and '1-' denotes that, corresponding transform does not possess that property. The one-to-one property corresponds to the redundancy of transform.

From figure 2.7, TF resolution for different transforms shows their approach towards representing signals. In figure 2.7(a) represents the TF localization of FT, which infers that FT provides good frequency resolution but loses time localization of signal. In figure 2.7(b) presents TF localization property of STFT and shows that STFT achieves same time and frequency resolution throughout

Fourier Family Transforms				
	General Transform	Time-Frequency Analysis Transforms		
Transform	FFT	STFT	ST	FGFT
Window	1	Rectangular, Gaussian	Gaussian	Any
Time Complexity	O(NLogN)	O(N^2LogN)$	O(N^2LogN)$	O(NLogN)
Output length for input of length N	N	N^2	N^2	N
Output Redundancy	Non-Redundant	Redundant	Redundant	Non-Redundant
Advantages & Disadvantages	1+, 2-, 4+	1+, 2-, 4	1+, 2+, 4-	1+, 2+, 4+
	1) Retains absolute phase reference; 2) Progressive resolution 3) Mother wavelet has zero mean; 4) One-to-One transform; $ - Frequency version implementation			

FIGURE 2.6: Comparison between different transforms.

the duration of signal. For ST and FGFT, TF localization is progressive in nature as shown in figure 2.7(c).

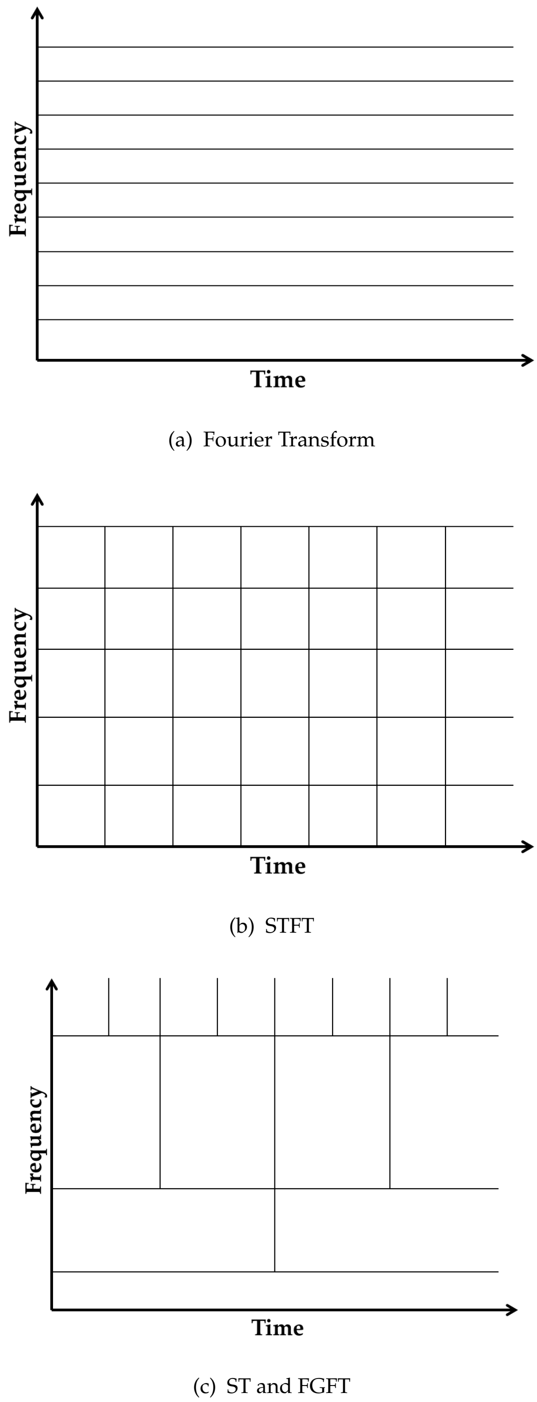


FIGURE 2.7: Time-Frequency resolution of different transforms

3

Analysis of FGFT

3.1 Introduction

FGFT was introduced by R.A Brown *et al*.[4] as an improvement over ST with respect to amount of redundancy and computation time. FGFT compromises in accuracy of resolution over number of computations required. FT, STFT and ST can be realized from FGFT with different window configuration. Initially, FGFT is explained for 1D signal, later all are extended to 2D image input.

3.2 Mathematical formulation

The FGFT coefficients for an input signal $x(t)$ is obtained by localizing the input signal at arbitary duration of time and finding FT on that duration. Later, localized duration is shifted and same process is repeated for whole input sequence. The difference from STFT is that, the localized duration changes in accord with the frequency present in that duration. The FGFT coefficients for input signal $x(t)$ is calculated as,

$$FGFT(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) w(t - \tau, f) e^{-j2\pi f(t-\tau)} df\, dt \tag{3.1}$$

where, $w(t, f)$ is the window function which is translated by τ and scaled by f. Notice the difference between FGFT and ST, ST output is a 2D function of (τ, f) but FGFT output is a 1D function of τ. Different transforms can be obtained with different window functions. The transforms and corresponding window configuration is provided in the table 3.1

TABLE 3.1: Window configurations for different transforms under FGFT

	Window function	**Window length**
FT	1	1
STFT	Boxcar, Sinc	Dyadic
ST	Gaussian	Dyadic

For illustration, Gaussian window with dyadic length is used in this thesis. The term dyadic means incremental size in order of power of two (i.e) 2^N. The Gaussian window is preferred wildly in many applications as they have desirable characteristics, such as

- No side lobes
- Frequency domain counterpart has same shape, thus controlling time and frequency resolution would be easy

The Gaussian window has bell shaped structure, whose width is controlled by the parameter σ,

$$w(t,\sigma) = \frac{1}{\sqrt{2\pi}|\sigma|} e^{-\frac{t^2}{2\sigma^2}} \tag{3.2}$$

Here, σ is made inversely proportional to frequency, thus at high frequency part of input signal, window contracts in time domain to get good time resolution. In other words, at dynamic regions of input, window shrinks to capture those events as shown in figure 3.1. At low frequency part of signal, window width increases in time domain thus capturing many samples to get good frequency resolution. The effect of varying τ and σ can be inferred from figure 3.2. FGFT

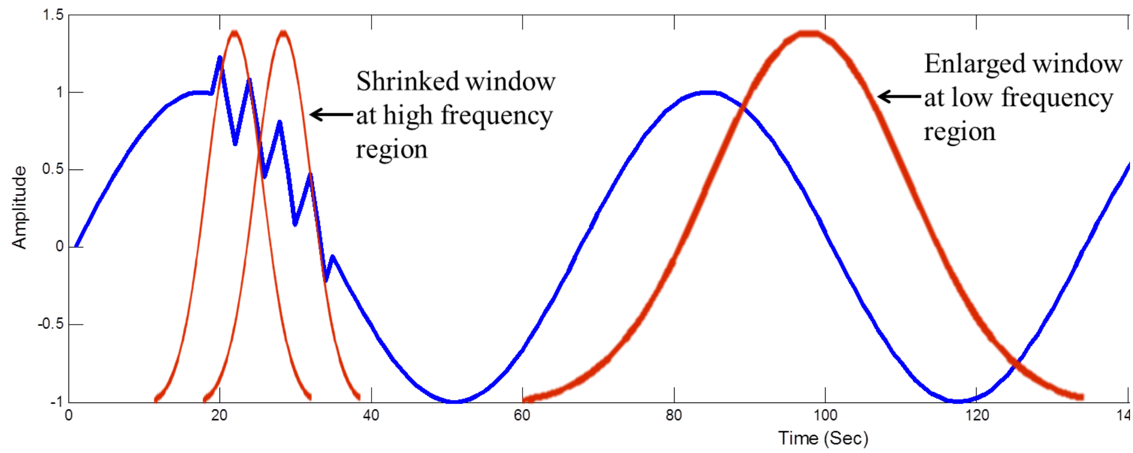


FIGURE 3.1: Variation of scaling parameter according to frequency dynamics of input.

can be derived from ST by taking inverse FT on every frequency component, as follows

$$FGFT(\tau) = \int_{-\infty}^{\infty} ST(\tau, f) e^{j2\pi f\tau} df \tag{3.3}$$

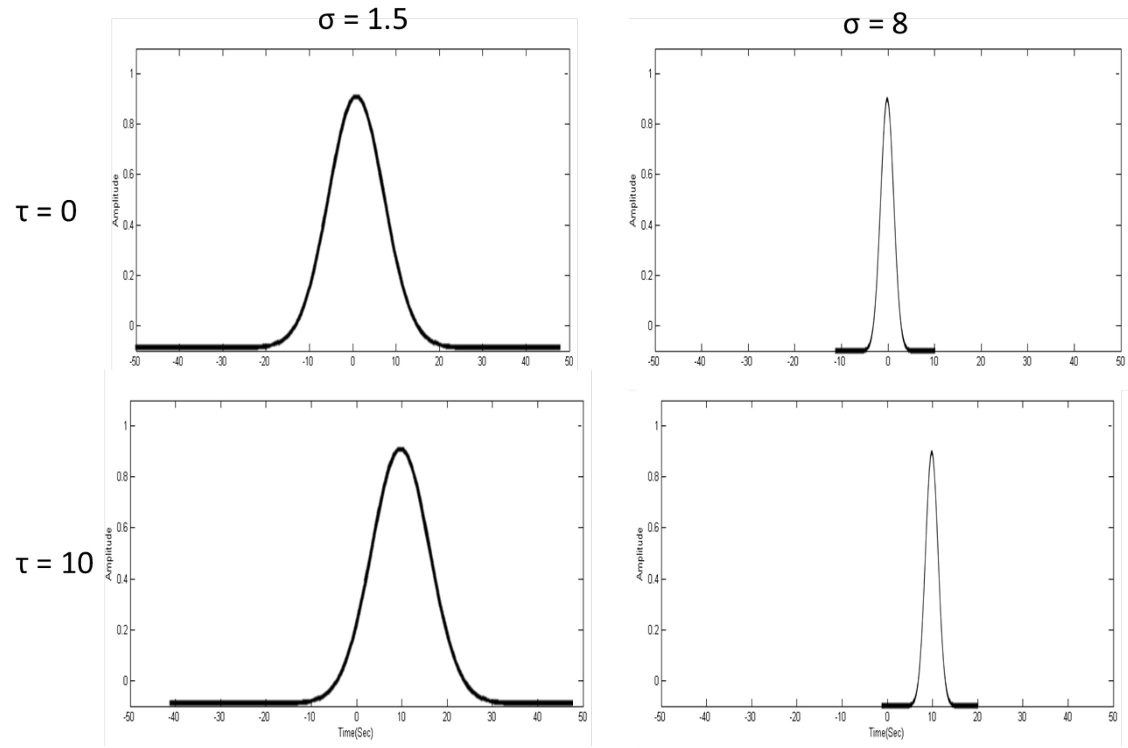


FIGURE 3.2: Effect of varying scale and shift parameters on window shape

Finally, the computation of FGFT becomes easier by making use of Fast FT (FFT). FGFT calculated in frequency domain involves taking FT of input signal, $X(f)$ and multiplying with frequency spectrum of window,

$$FGFT(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\alpha + f)e^{-\frac{2\pi^2\alpha^2}{f^2}} e^{j2\pi(\alpha+f)\tau} d\alpha df \tag{3.4}$$

3.3 Realization of FGFT

The FFT is applied over input image and it is partitioned dyadically in increasing order, 2^N [4]. Each partition is multiplied with FT of window function $w(t, \sigma)$, where σ is inversely proportional to center frequency of that partition. The resulting sequence is applied with IFFT. The inverse FGFT is reverse process of FGFT, except for multiplying with window in forward process, signal is divided by window. The overall process is summarised in Figure 3.3.

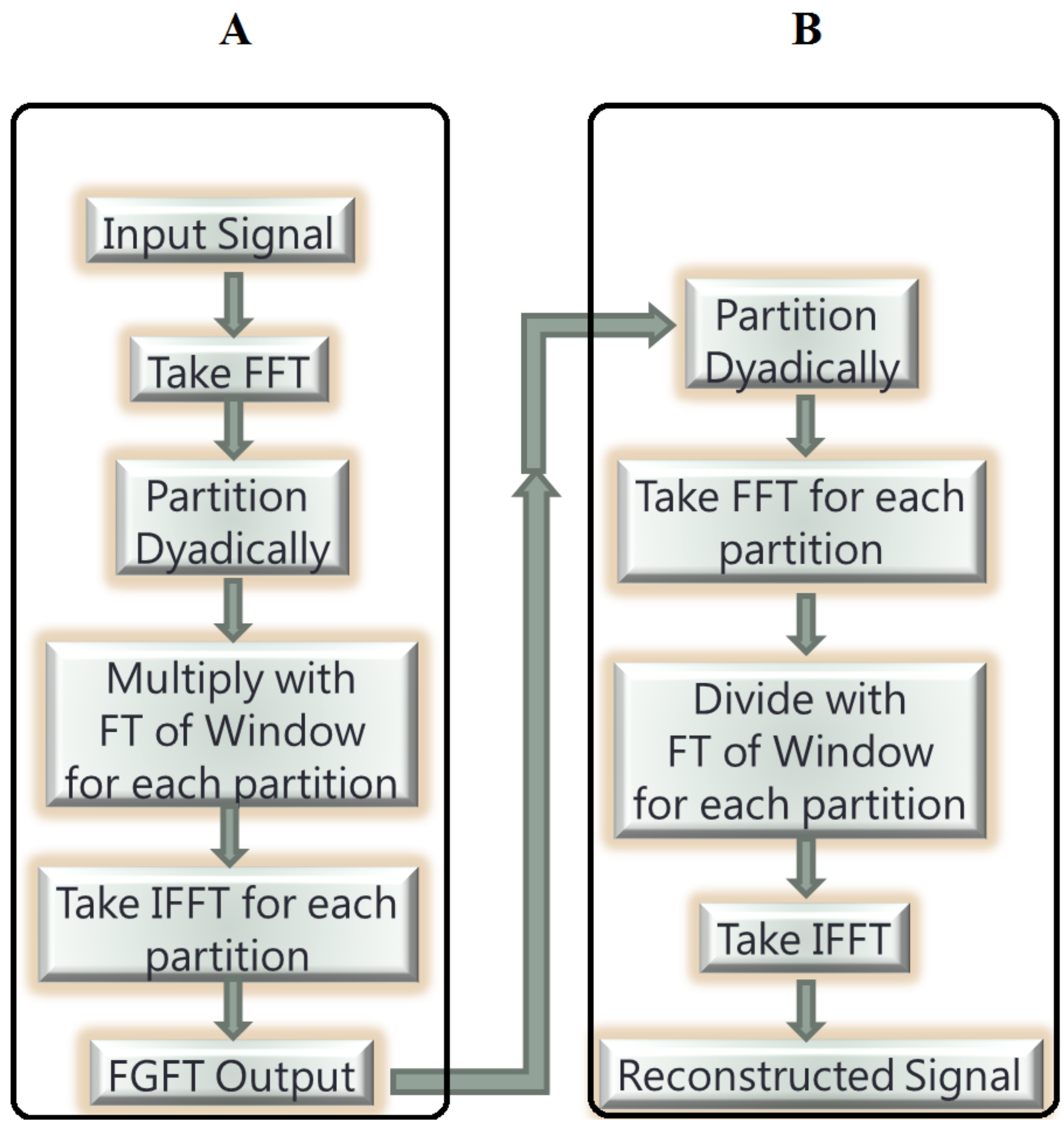


FIGURE 3.3: Steps involved in realization of FGFT. A - Forward FGFT. B - Reverse FGFT [4].

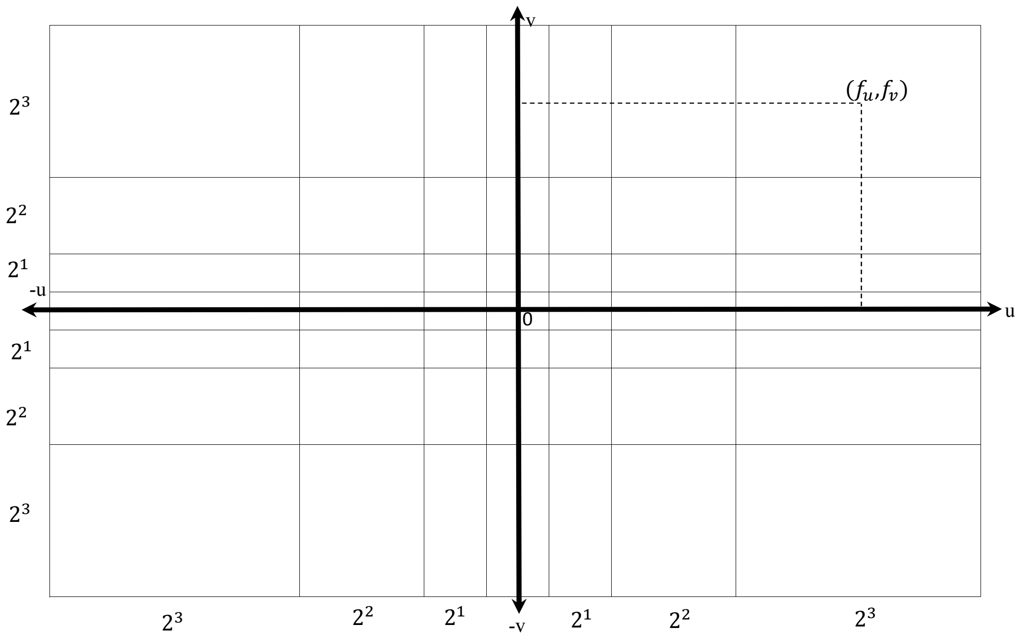
3.4 Non-redundancy in FGFT

The key characterstic of FGFT over ST is reduction in redundancy. Redundancy in ST is due to the fact that, FT of input signal is multiplied with windows calculated for every frequency. Since this method of continuous sampling produces high time and frequency resolution at the same time, ST violates Heisenberg's uncertainty principle. In contrast, in FGFT, FT of input signal is multiplied with windows calculated for only discrete frequencies.

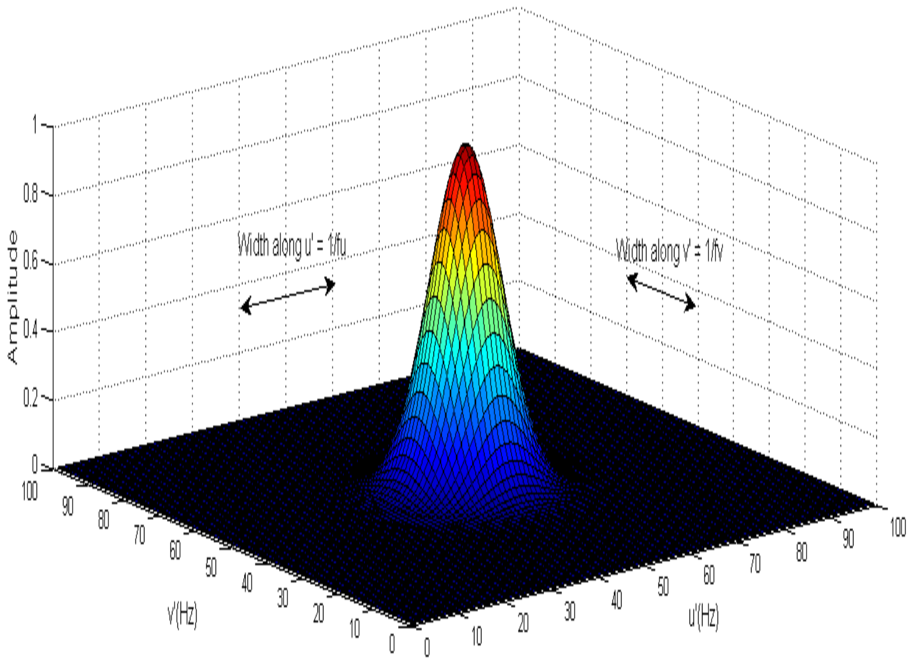
3.5 2D FGFT

The concepts and formulas introduced for 1D signals are extended to 2D signals using separability theorem. As separability theorem depicts, 2D operation can be performed as two step procedure. First, apply FGFT along row wise then secondly, apply in column manner of the result from previous step.

The partition scheme of 2D FGFT is controlled by two frequency parameters as depicted in figure 3.4(a). The center frequency f_u and f_v decides the width of window along vertical and horizontal directions. The rectangular blocks shown in figure 3.4(a) just shows the boundary of the window in that partition, shape of window inside each of those partition is shown in figure 3.4(b). The 2D FGFT output of an image is shown alongside with frequency spectrum in figure 3.5. From 3.5(c), it is very clear that output of FGFT is in scale space and in same domain(spatial domain) as that of input as compared other transform outputs. The scale that is referred here is the height and width of image.

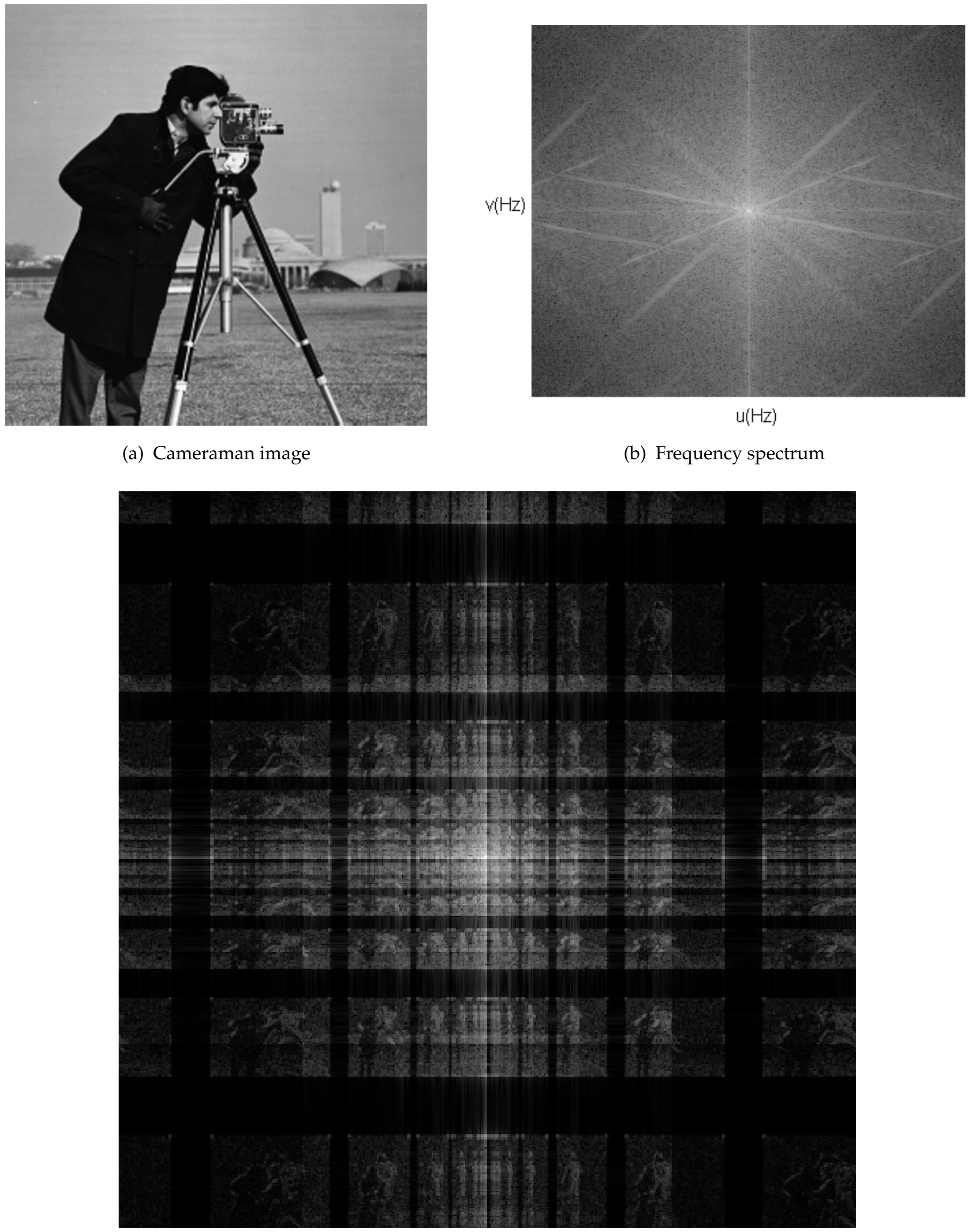


(a) Partition scheme in 2D FGFT



(b) 2D Gaussian window

FIGURE 3.4: 2D partition scheme of FGFT and 2D Gaussian window



(a) Cameraman image

(b) Frequency spectrum

(c) FGFT output of cameraman image in Log scale

FIGURE 3.5: Cameraman image, its frequency spectrum and its FGFT output

4

Proposed Algorithm

4.1 Phase-only reconstruction

IMAGE reconstucted with phase-only FGFT coefficients does not produce edge information as expected. The summary of problem statement is shown in figure 4.1. The prime point for the issue being that output of FGFT is in shift domain, not in frequency domain. This can be seen in frequency domain implementation of FGFT from equation 3.4 [on page 25] and in figure 3.5. If we observe how the FT coefficients are modified during the process of FGFT, then one can conclude that extra phase component is added due to inherent steps involved in FGFT. The addition of extra phase can be seen as phase modulation of FT phase. The objective of this chapter is to identify how the FT phase is getting modulated and to device an algorithm that get back FT phase.

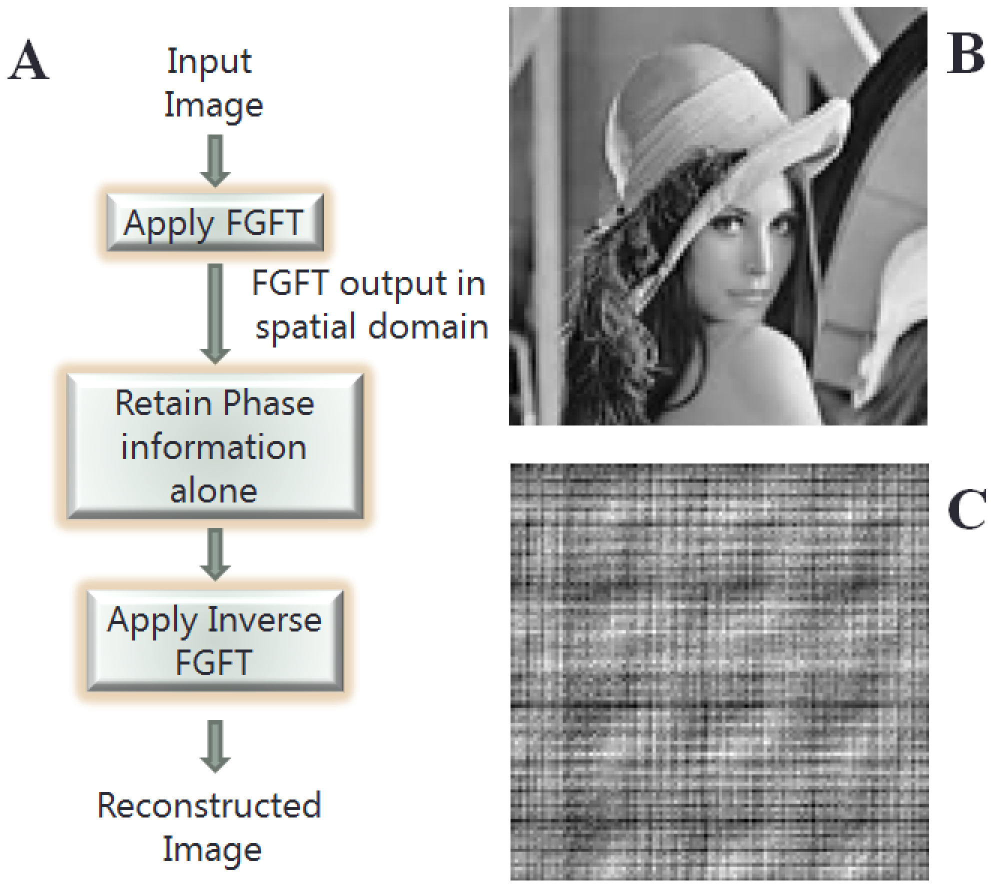


FIGURE 4.1: Problem Statement. A - Phase-only image reconstruction. B - Input image. C - Reconstructed image

In order to understand how FT phase gets modulated, FGFT procedure is applied mathematically on FT coefficients and effect is deduced. The FT coefficients are partitioned dyadically, multiplied with gaussian window and inverse FT is applied on each partition separately. The first FGFT coefficient in each partition is the DC value of Inverse FT of that partition. That coefficient is calculated by sum of product of FT coefficients and window coefficients in that partition. For a partition with P_k FT coefficients, inverse FT of that partition will add new phase components, if hermitian symmetry is not observed in $FT(f)$. Since window has zero phase component, it does not introduce any new phase. Window is assumed to be unity, $W = 1$ and calculations are done for discrete signal case for ease of understanding. The reason of getting extra phase ϕ_m is due to inverse FT operation that is performed on each dyadic partition. Neither partioning nor taking inverse FT can be modified or removed to preserve FT phase. The conclusive solution might be to estimate FT phase from FGFT samples, then retain edge information which is expained in upcoming section.

4.2 Algorithm

The outline of proposed algorithm involves two stages: First, estimating FT coefficients from FGFT coefficients. Second, phase normalizing estimated FT coefficients and generating FGFT coefficients from the result.

The first component of result is FGFT coefficient calculated with zero window shift, which can be obtained by substituting $\tau = 0$,

$$FGFT_k[0] = \sum_{\Omega} W_k X_k$$

which can be expanded as follows,

$$FGFT_k[0] = X_k[0]W_k[0] + X_k[1]W_k[1] + \cdots + X_k[P_k - 1]W_k[P_k - 1]$$

In the process, it is assumed that proposed algorithm does not have access to input sequence x. The objective is to find X_k from above equation, but there are P_k unknowns with only one equation. These type of systems with more unknowns than number of equations are called as under determined systems. They have infinitely many solutions, unless P_k independent equations are available.

The proposed algorithm generates P_k different equations with P_k different windows. Let, W_k^l - Window for partition k with l^{th} frequency element as center frequency, l varies from 1 to P_k.

By solving the P_k independent equations with P_k unknowns, will provide the FT coefficients X_k for each partition. This procedure is repeated for all n_p partitions. The resulting X is in frequency domain, thus phase is retained by normalizing procedure reported in section 2.1.1 [in page 9],

$$\acute{X} = \frac{X}{|X|}$$

$\acute{X}$ is still in frequency domain, which could be converted into FGFT coefficients by dyadically partitioning, multiplying with window and taking inverse FT on each partition.

4.3 Implementation

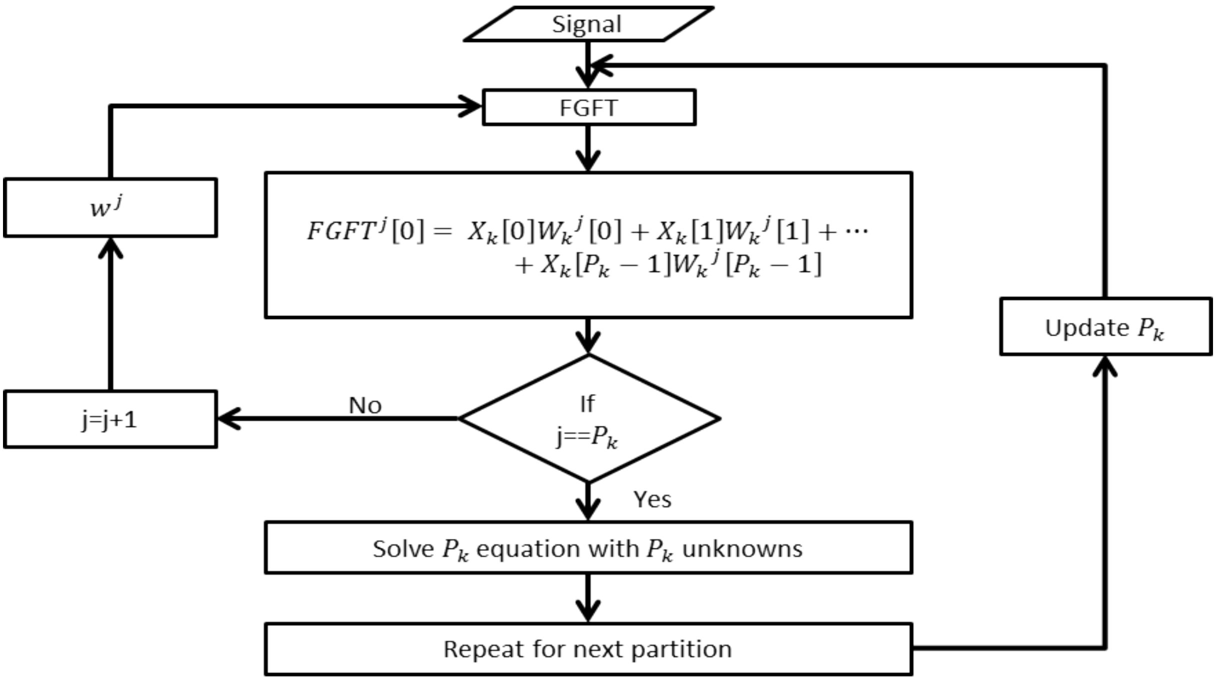


FIGURE 4.2: Overview of Proposed Algorithm

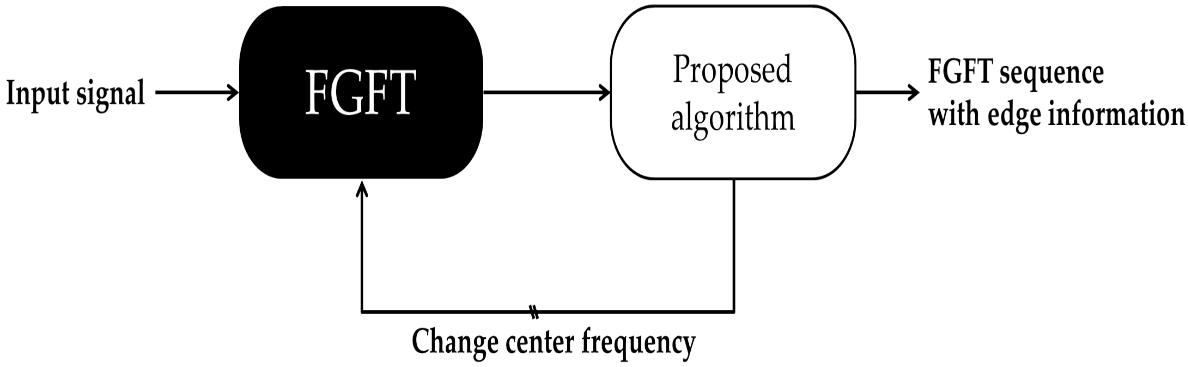


FIGURE 4.3: Structure of proposed algorithm

As shown in figure 4.3, input to proposed algorithm is length of signal and FGFT coefficients calculated at different window configuration for each partition. The working of proposed algorithm assumes black box idea about FGFT.

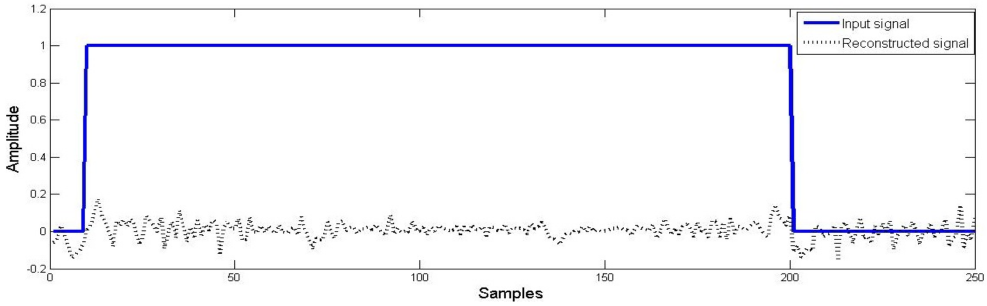
4.4 Result & Discussion

The proposed algorithm in previous section is applied on 1D and 2D signals and results are presented in this section. For illustration, results for two 1D signals and two images are presented here.

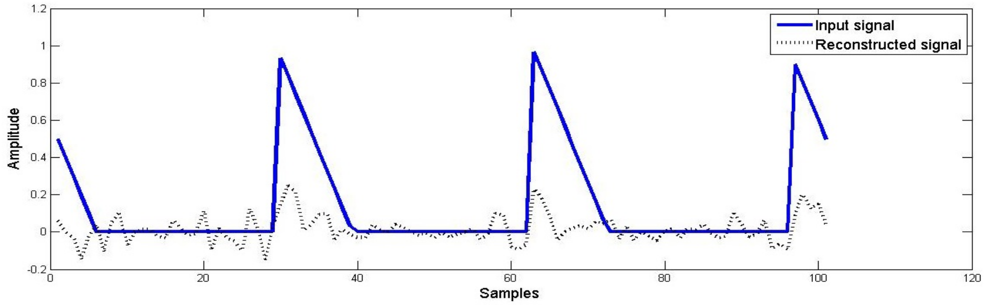
In Figure 4.4(a), input signal (signal in blue legend) is an 1D rectangular pulse of length 250 samples. The dotted signal is the edges detected using proposed algorithm. The amplitude of detected edges are very small as the phase retains only the change in amplitude of signal not the amplitude itself. In Figure 4.4(b), a triangular pulse of length 100 samples is used. Figure 4.4(c) shows performance of proposed algorithm on a 2D image of size 225 $\times$ 225. Figure 4.4(d) shows the extracted edge information from a image of size 398 $\times$ 341.

4.5 Remarks on algorithm

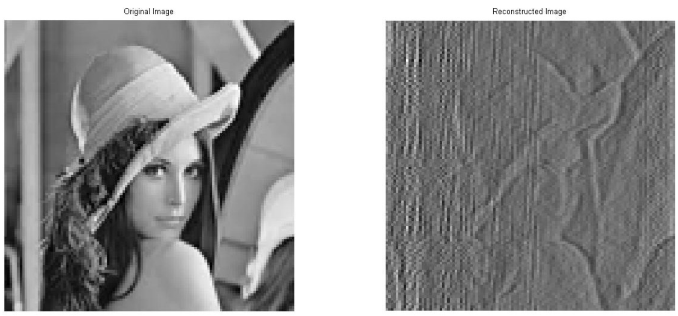
- The accuracy of estimated FT coefficients becomes less when there are huge number of equations to solve (occurs when the input size is large) and absolute value of the coefficients are very less. Though this does not



(a) Rectangular pulse



(b) Triangular pulse



(c) Lena image

(d) Cameraman image

FIGURE 4.4: Reconstructed edge information from different inputs

have huge impact in detecting edges, it produces a background noise that affects the smooth region.

- The execution time for algorithm is approximately around 25 seconds (For a image of size 512×512 on a computer with Intel core i5$^{\mathrm{TM}}$processor and 4GB RAM) as compared to usual normalization procedure which executes within 8 milliseconds. The execution time increases exponentially when size of input is doubled. This is due to huge number of equations to be solved for estimating FT coefficients.

5

Amount of information in Phase and Magnitude

ONE possible application using proposed algorithm would be to measure and compare amount of information in magnitude and phase of different transforms. The idea of phase retaining more edge information is verified for frequency domain transforms, but not for TF transforms. Transforms compared are broadly categorized as: Frequency domain (FT) and TF domain(FGFT, Wavelet, Curvelet and Curvelet with Wrapping [1]) transforms. In this experiment, following questions are answered.

1. Which transform retains more edge information in magnitude and phase components?
2. Whether each transform stores more information in magnitude or phase component?

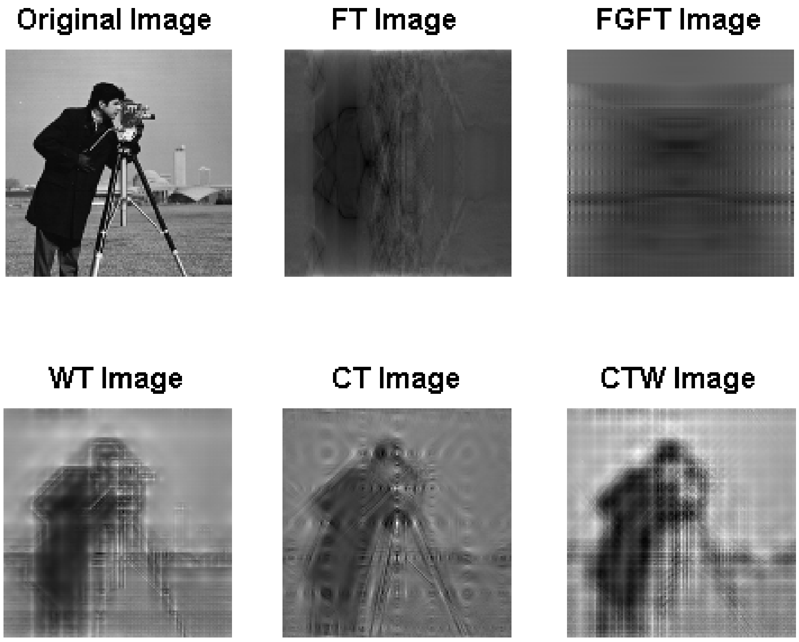
Information measured is mainly the structural correlation between input image and reconstructed image with phase-only and magnitude-only components of different transforms. Figure 5.1(a) shows the output images reconstructed with retaining only magnitude components of all transforms(by making phase zero). From the figure, it is clear that among all, CT retains more structural information. The output of CT shows only smoothness in image and sharp edge transistions are stored in its phase components. Figure 5.1(b) shows the images reconstructed with phase-only components of all transforms. Among all, FT shows clear edge information.

TABLE 5.1: List of Image databases used

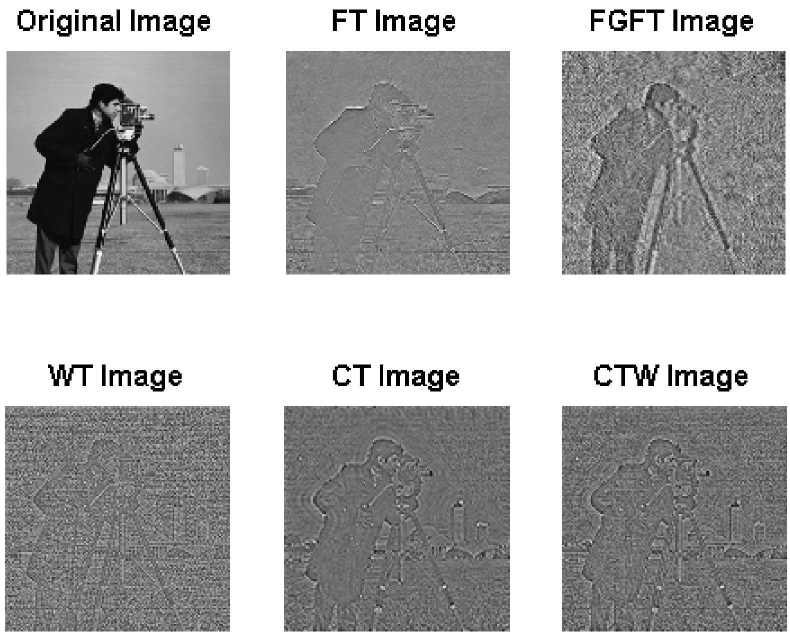
Databases	**Number of Images**	**Image type**
Berkeley Segmentation Dataset(BSDS) [12]	300	JPEG
CalTech Background Image Dataset[13]	451	JPEG
CMU Pose, Illumination, and Expression(PIE) database[14]	208	BMP
Laboratory for Image and Video Engineering(LIVE) image database[15]	262	JPEG2000

It is observed that except for FGFT all other transforms retain more structural information in phase than magnitude. For FGFT, more information is stored in phase for CMU PIE and CalTech databases and more information

[1] Refer Appendix A for short review on WT and CT.



(a) Magnitude-only reconstruction



(b) Phase-only reconstruction

FIGURE 5.1: Reconstructed images with magnitude-only and phase-only components for different transforms

stored in magnitude for BSDS300 and LIVE databases. Thus, FGFT almost have same amount of information in magnitude and phase. This is due to fact that structural information is mingled in both magnitude and phase of FGFT, and indicates that magnitude also posses equal importance as that of phase. This is different from the conventional idea that phase alone retains more structural information. From the above experimental results, it can be inferred that any application that require edge information from FGFT have to operate on both magnitude and phase.

6

Concluding remarks and future scope

Non-stationarity property of a signal limits the use of FT, which paved way for JTFR that was developed exclusively for these type of signals. The STFT and WT have few disadvantages. Though those disadvantages are overcomed by ST, it is redundant. This led to the development of FGFT, which is a general framework to realize FT, STFT and ST with quicker computations. The output of FGFT is complex with magnitude and phase components. Phase component of FGFT do not retain edge information of input image. This thesis proposed a method (in Chapter 4) to extract edge information from phase components of FGFT. The analysis of results from experiments outlined in this thesis, suggests the following conclusions

- The edge information of input image is spread in both magnitude and phase of FGFT. Thus, usual phase normalization technique is ineffective to segregate edge information from FGFT samples
- The edge information of image is extracted from phase-only FGFT coefficients using proposed algorithm
- Among FT, FGFT, WT and CT, FT retains more edge information in phase part and CT retains more edge information in magnitude part
- FGFT stores almost equal structural information in both magnitude and phase components

The future scope of this work would be to,

- Reduce the computation time and complexity of proposed algorithm as to make it suitable for real time applications
- Improve the detected edges by increasing accuracy of estimated FT coefficients and thus suppress jitter in smooth regions of reconstructed image
- Investigate on Space - Spatial Frequency characteristics of FGFT output for an image
- Analysis and application of FGFT scale-space representation of an image

A

Wavelet and Curvelet Transforms

In this appendix a short review on both Wavelet Transform(WT) and Curvelet Transform(CT) is presented.

A.1 Wavelet Transform

WAVELETS are short duration pulses with high frequency contents to provide high time resolution at high frequencies and high frequency resolution at low frequencies (also called as Progressive resolution). Continuous WT(CWT) was developed by Mallat in 1984 as an alternative to fixed resolution STFT. The input signal $x(t)$ is expanded as functions of shifted and streched real valued bandpass wavelets $\psi(t)$ as given by,

$$W(u,s) = \int_{-\infty}^{\infty} x(t)\frac{1}{\sqrt{s}}\psi^*(\frac{t-u}{s})dt \tag{A.1}$$

The output of WT is a 2D function of shift and scale (inverse of frequency). The functions $\psi(t)$ are also called as *Mother Wavelets*. A function to be used as mother wavelet(zero average pulses) should satisfy below admissibility condition,

$$\int_{0}^{\infty} \frac{|\Psi(\omega)|^2}{\omega} d\omega < \infty$$

The WT is very similar to ST, except for a phase correcting term. WT provides locally referenced phase as compared to globally refrenced phase by ST. WT basis function(mother wavelet) shifts along the signal and thus the reference point to calculate phase also shifts. In contrast, for ST, window shifts while basis function(sinusoids) is not moving, thus the reference point is t=0. The primary application on WT include signal compression and image denoising.

The CWT is redundant as it is calculated for huge scale and shift values. Redundancy is reduced by discretizing s and u, which results in Discrete WT(DWT). DWT suffers from following disadvantages,

- Lack of shift invariance - translation of input causes sporadic changes in output.
- Lack of symmetry of mother wavelet
- Poor directional selectivity

Implementation of DWT was introduced by Mallat, as several stages of filter banks. Every stage produces approximation and detail coefficients. Approximation coefficients are calculated by filtering the original signal. Detail coefficients are calculated by subsampling the original signal [16].

One of the particularity of DWT is that, amplitude response is deteriorated for high frequencies. This is not observed in FT, because the basis function is a complex exponential extending from negative infinity to positive infinity. Thus to overcome the disadvantage of DWT, a transform with complex mother wavelet was implemented by Kingsbury *et al.* namely Dual Tree based Complex WT(DT CWT). In Kingsbury et. al.[17] paper , it is outlined that shift invariance in DWT can be eliminated by increasing sampling rate by 2. This idea is implemented by having two trees with separate mother wavelet(one is hilbert transform of other). Dual Tree based CWT(DT CWT) is a complex-valued transform, meaning that coefficients from 1^{st} and 2^{nd} tree is the real and imaginary part of a single DWT. The implementations of several WT can be found at [18] [19] [20].

A.2 Curvelet Transform

FT expansion of a signal cannot accurately model the point discontunities, which was possible in WT. WT suffers from line discontinuties like curves. Thus Curvelet Transform(CT) was introduced by Emmanuel Candès et. al.[21] to effectively represent curves in images(or any 2D and higher dimensional signals). CT follows anisotropic scaling which is not possible in WT. Curvelet coefficients are calculated as [21],

$$C^D(j,l,k) = \sum_{0 \le t_1, t_2 < n} f[t_1, t_2]\overline{\varphi^D_{j,l,k}[t_1, t_2]} \tag{A.2}$$

where, $\varphi^D_{j,l,k}[t_1, t_2]$ is digital curvelet waveform. The output of CT is a function of scale(j), orientation(l) and space(k). The CT implementations are performed in frequency domain in two variants as given below:

1. Digital CT based on Unequally-Spaced fast FT(USFFT)
2. Digital CT with wrapping

Both of the above mentioned methods differ by spatial window in frequency domain on which curvelet coefficients are calculated. Wrapping method is computed faster than USFFT method.

Implementation of curvelet based on USFFT is as follows [21],

- Find frequency spectrum of image using 2D FFT
- Interpolate for each angle/scale pair
- Multiply interpolated samples with parabolic window
- Apply inverse 2D FFT on the result from previous step

The steps involved in calculating CT with wrapping are [21]:

- Find frequency spectrum of image using 2D FFT
- Partition frequency spectrum into different wedge shaped regions

- Wrap each of the wedges into rectangular region around the origin
- Apply inverse 2D FFT on the wrapped window

The MATLAB implementations of different version of CT is available at [22].

Bibliography

[1] A. V. Oppenheim and J. S. Lim, "The importance of phase in signals," *Proceedings of the IEEE*, vol. 69, no. 5, pp. 529–541, 1981.

[2] R. A. Brown and R. Frayne, "A fast discrete s-transform for biomedical signal processing," in *Engineering in Medicine and Biology Society, 2008. EMBS 2008. 30th Annual International Conference of the IEEE*. IEEE, 2008, pp. 2586–2589.

[3] R. G. Stockwell, L. Mansinha, and R. Lowe, "Localization of the complex spectrum: The s transform," *IEEE Transactions on Signal Processing*, vol. 44, no. 4, pp. 998–1001, 1996.

[4] R. A. Brown, M. L. Lauzon, and R. Frayne, "A general description of linear time-frequency transforms and formulation of a fast, invertible transform that samples the continuous s-transform spectrum nonredundantly," *IEEE Tran., Signal Processing*, vol. 58, no. 1, pp. 281–290, 2010.

[5] M. Naghizadeh and K. A. Innanen, "Two-dimensional fast generalized fourier interpolation of seismic records," *Geophysical Prospecting*, vol. 61, no. s1, pp. 62–76, 2013.

[6] G. Badrinath and P. Gupta, "Stockwell transform based palm-print recognition," *Applied Soft Computing*, vol. 11, no. 7, pp. 4267–4281, 2011.

[7] S. Qian and D. Chen, *Joint Time-Frequency Analysis, Methods and Applications*. PTR Prentice Hall, 1996.

[8] R. G. Stockwell, "Why use the s-transform?" *Pseudo-Differential Operators: Partial Differential Equations and Time Frequency Analysis by Fields Institute and American Mathematical Society*, vol. 52.

[9] B. Boashash, *Time-Frequency Analysis and Processing, A Comprehensive Reference*. Elsevier, 2003.

[10] M. Naghizadeh and K. A. Innanen, "Seismic data interpolation using a fast generalized fourier transform," *Geophysics*, vol. 76, no. 1, pp. V1–V10, 2011.

[11] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, "Image quality assessment: from error visibility to structural similarity," *IEEE Transactions on Image Processing*, vol. 13, no. 4, pp. 600–612, 2004.

[12] D. Martin, C. Fowlkes, D. Tal, and J. Malik, "A database of human segmented natural images and its application to evaluating segmentation algorithms and measuring ecological statistics," in *ICCV 2001. Proceedings. Eighth IEEE International Conference on Computer Vision, 2001*, vol. 2. IEEE, 2001, pp. 416–423.

[13] M. Weber. Database of background images. [Online]. Available: http://www.vision.caltech.edu/html-files/archive.html

[14] S. Baker. The cmu pose, illumination, and expression (pie) database. [Online]. Available: http://vasc.ri.cmu.edu/idb/html/face/

[15] L. C. H.R. Sheikh, Z. Wang and A. Bovik. Live image quality assessment database release 2. [Online]. Available: http://live.ece.utexas.edu/research/quality

[16] I. Adam, "Complex wavelet transform: application to denoising," Ph.D. dissertation, University of Rennes, 2006.

[17] N. G. Kingsbury, "The dual-tree complex wavelet transform: a new technique for shift invariance and directional filters," in *Proc. 8th IEEE DSP workshop*, vol. 8. Citeseer, 1998, p. 86.

[18] D. Donoho and et. al. Wavelab 850. [Online]. Available: http://www-stat.stanford.edu/~wavelab

[19] P. G. Francois Auger, Olivier Lemoine and P. Flandrin. Time frequency toolbox. [Online]. Available: http://tftb.nongnu.org/

[20] J. C. O'Neill. Discrete time frequency distributions toolbox. [Online]. Available: http://tfd.sourceforge.net/

[21] E. Candes, L. Demanet, D. Donoho, and L. Ying, "Fast discrete curvelet transforms," *Multiscale Modeling & Simulation*, vol. 5, no. 3, pp. 861–899, 2006.

[22] E. Candes and et. al. Curvelab 2.1.2. [Online]. Available: http://www.curvelet.org/software.html