Intersections of the Hermitian surface with irreducible quadrics in even characteristic

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Abstract

We determine the possible intersection sizes of a Hermitian surface $\mathcal{H}$ with an irreducible quadric of $\mathrm{PG}(3, q^2)$ sharing at least a tangent plane at a common non-singular point when q is even.

Keywords: Hermitian surface; quadrics; functional codes

1 Introduction

The study of intersections of geometric objects is a classical problem in geometry; see e.g. [12, 13]. In the case of combinatorial geometry, it has several possible applications either to characterize configurations or to construct new codes.

Let $\mathcal{C}$ be a projective $[n, k]$-linear code over $\mathrm{GF}(q)$. It is always possible to consider the set of points Ω in $\mathrm{PG}(k-1, q)$ whose coordinates correspond to the columns of any generating matrix for $\mathcal{C}$. Under this setup the problem of determining the minimum weight of $\mathcal{C}$ can be reinterpreted, in a purely geometric setting, as finding the largest hyperplane sections of Ω. More in detail, any codeword $c \in \mathcal{C}$ corresponds to a linear functional evaluated on the points of Ω; see [19, 21]. For examples of applications of these techniques see [4, 5, 6].

Clearly, it is not necessary to restrict the study to hyperplanes. The higher weights of $\mathcal{C}$ correspond to sections of $\mathcal{C}$ with subspaces of codimension larger than 1; see [17] and also [22] for Hermitian varieties.

A different generalization consists in studying codes arising from the evaluation on Ω of functionals of degree $t > 1$; see [19]. These constructions yield, once more, linear codes, whose weight distributions depend on the intersection patterns of Ω with all possible algebraic hypersurfaces of $\mathrm{PG}(k-1, q)$ of degree t.

The case of quadratic functional codes on Hermitian varieties has been extensively investigated in recent years; see [2, 8, 9, 10, 11, 18]. It is however still an open problem to classify all possible intersection numbers and patterns between a quadric surface $\mathcal{Q}$ in $\mathrm{PG}(3, q^2)$ and a Hermitian surface $\mathcal{H} = \mathcal{H}(3, q^2)$.

In [1], we determined the possible intersection numbers between $\mathcal{Q}$ and $\mathcal{H}$ in $\mathrm{PG}(3, q^2)$ under the assumption that q is an odd prime power and $\mathcal{Q}$ and $\mathcal{H}$ share at least one tangent plane. The same problem has been studied independently also in [7] for $\mathcal{Q}$ an elliptic quadric; this latter work contains also some results for q even.

In this paper we fully extend the arguments of [1] to the case of q even. It turns out that the geometric properties being considered, as well as the algebraic conditions to impose, are different and more involved than in the odd q case. Our main result is contained in the following theorem.

Theorem 1.1. *In $\mathrm{PG}(3, q^2)$, with q even, let $\mathcal{H}$ and $\mathcal{Q}$ respectively be a Hermitian surface and an irreducible quadric sharing at least a tangent plane at one common non-singular point P. Then, the possible sizes of the intersection $\mathcal{H} \cap \mathcal{Q}$ are as follows.*

- *For $\mathcal{Q}$ elliptic:*

$$q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1.$$

- *For $\mathcal{Q}$ a quadratic cone:*

$$q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1.$$

- *For $\mathcal{Q}$ hyperbolic:*

$$q^2+1, q^3-q^2+1, q^3-q^2+q+1, q^3-q+1, q^3+1, q^3+q+1, q^3+q^2-q+1, q^3+q^2+1,$$
$$q^3 + 2q^2 - q + 1, q^3 + 2q^2 + 1, q^3 + 3q^2 - q + 1, 2q^3 + q^2 + 1.$$

We remark that, as we are dealing with irreducible quadrics in $PG(3, q^2)$, by quadratic cone (or, in short, cones) we shall always mean in dimension 3 the quadric projecting an irreducible conic contained in a plane π from a point (vertex) $V \notin \pi$.

Our methods are mostly algebraic in nature, based upon the $\mathrm{GF}(q)$–linear representation of vector spaces over $\mathrm{GF}(q^2)$, but in order to rule out some cases some geometric and combinatorial arguments are needed, as well as some considerations on the action of the unitary groups.

For generalities on Hermitian varieties in projective spaces the reader is referred to [3, 16, 15, 20]. Basic notions on quadrics over finite fields are found in [15, 16].

2 Invariants of quadrics

In this section we recall some basic invariants of quadrics in even characteristic; our main reference for these results is [15, §1.1, 1.2], whose notation and approach we closely follow.

Recall that a quadric $\mathcal{Q}$ in $\mathrm{PG}(n,q)$ is just the set of points $(x_0,\ldots,x_n)\in\mathrm{PG}(n,q)$ such that $F(x_0,\ldots,x_n)=0$ for some non-null quadratic form

$$F(x_0,\ldots,x_n)=\sum_{i=0}^{n}a_ix_i^2+\sum_{i<j}a_{ij}x_ix_j.$$

If there is no change of coordinates reducing F to a form in fewer variables, then $\mathcal{Q}$ is called *non-degenerate* or *non-singular*; otherwise $\mathcal{Q}$ is said to be *degenerate* or *singular*. If the polynomial F splits into linear factors in the algebraic closure of $\mathrm{GF}(q)$, then $\mathcal{Q}$ is *reducible*; otherwise $\mathcal{Q}$ is *irreducible.* It is well known that if $\mathcal{Q}$ is reducible, then $F(x_1,\ldots,x_n)=L_1(x_1,\ldots,x_n)L_2(x_1,\ldots,x_n)$ with L_1 and L_2 linear polynomials defined over $\mathrm{GF}(q^2)$.

The minimum number of indeterminates which may appear in an equation for $\mathcal{Q}$ is the *rank* of the quadric, denoted by $\mathrm{rank}(\mathcal{Q})$; see [14, §15.3].

Suppose q even and consider the quadric $\mathcal{Q}$ in $\mathrm{PG}(3,q)$ of equation $\sum_{i=0}^{3}a_ix_i^2+\sum_{i<j}a_{ij}x_ix_j=0$; define

$$A:=\begin{pmatrix}2a_0 & a_{01} & a_{02} & a_{03}\\ a_{01} & 2a_1 & a_{12} & a_{13}\\ a_{02} & a_{12} & 2a_2 & a_{23}\\ a_{03} & a_{13} & a_{23} & 2a_3\end{pmatrix},\quad B:=\begin{pmatrix}0 & a_{01} & a_{02} & a_{03}\\ -a_{01} & 0 & a_{12} & a_{13}\\ -a_{02} & -a_{12} & 0 & a_{23}\\ -a_{03} & -a_{13} & -a_{23} & 0\end{pmatrix}$$

and, for $\det B\neq 0$,

$$\alpha:=\frac{\det A-\det B}{4\det B}. \tag{1}$$

The values $\det A$, $\det B$ and α should be interpreted as follows. In A and B we replace the terms a_i and a_{ij} by indeterminates Z_i and Z_{ij} and we evaluate $\det A$, $\det B$ and α as rational functions over the integer ring $\mathbb{Z}$. Then we specialize Z_i and Z_i to a_i and $a_{i,j}$. Furthermore, as q is even, the quadric $\mathcal{Q}$ induces a symplectic polarity which is non-degenerate if, and only if, $\det B\neq 0$ (this is actually equivalent to $\det A\neq 0$ in odd projective dimension), so the formula (1) giving the invariant α is well defined for non-singular quadrics.

By [15, Theorem 1.2], a non-singular quadric $\mathcal{Q}$ of $PG(3,2^h)$ is hyperbolic or elliptic according as

$$\mathrm{Tr}_q(\alpha)=0 \text{ or } \mathrm{Tr}_q(\alpha)=1,$$

respectively, where Tr_q denotes the absolute trace $\mathrm{GF}(q)\to\mathrm{GF}(2)$ which maps $x\in\mathrm{GF}(q)$ to $x+x^2+x^{2^2}+\ldots+x^{2^{h-1}}$.

3 Some technical tools

In this section we are going to prove a series of lemmas that shall be useful to prove our main result, namely Theorem 1.1.

Henceforth, we shall always assume q to be even; x, y, z will denote affine coordinates in $\mathrm{AG}(3, q^2)$ and the corresponding homogeneous coordinates will be J, X, Y, Z. The hyperplane at infinity of $\mathrm{AG}(3, q^2)$, denoted as Σ_∞, is taken with equation $J = 0$.

Since all non-degenerate Hermitian surfaces of $\mathrm{PG}(3, q^2)$ are projectively equivalent, we can assume, without loss of generality, $\mathcal{H}$ to have affine equation

$$z^q + z = x^{q+1} + y^{q+1}. \tag{2}$$

Since $\mathrm{PGU}(4, q)$ is transitive on $\mathcal{H}$, see [20, §35], we can also suppose that a point where $\mathcal{H}$ and $\mathcal{Q}$ have a common tangent plane is $P = P_\infty(0, 0, 0, 1) \in \mathcal{H}$; so, the tangent plane at P_∞ to $\mathcal{H}$ and $\mathcal{Q}$ is Σ_∞. Under the aforementioned assumptions, $\mathcal{Q}$ has affine equation of the form

$$z = ax^2 + by^2 + cxy + dx + ey + f \tag{3}$$

with $a, b, c, d, e, f \in \mathrm{GF}(q^2)$. A straightforward computation proves that $\mathcal{Q}$ is non-singular if and only if $c \neq 0$; furthermore $\mathcal{Q}$ is hyperbolic or elliptic according as the value of

$$\mathrm{Tr}_{q^2}(ab/c^2)$$

is 0 or 1 respectively. When $c = 0$ and $(a, b) \neq (0, 0)$, the quadric $\mathcal{Q}$ is a cone with vertex a single point V. Write now

$$\mathcal{C}_\infty := \mathcal{Q} \cap \mathcal{H} \cap \Sigma_\infty. \tag{4}$$

If $\mathcal{Q}$ is elliptic, the point P_∞ is, clearly, the only point at infinity of $\mathcal{Q} \cap \mathcal{H}$, that is $\mathcal{C}_\infty = \{P_\infty\}$. The nature of $\mathcal{C}_\infty$ when $\mathcal{Q}$ is either hyperbolic or a cone, is detailed by the following lemma.

Lemma 3.1. *If $\mathcal{Q}$ is a cone, then $\mathcal{C}_\infty$ consists of either 1 point or $q^2 + 1$ points on a line. When $\mathcal{Q}$ is a hyperbolic quadric, then $\mathcal{C}_\infty$ consists of either 1 point, or $q^2 + 1$ points on a line or $2q^2 + 1$ points on two lines. All cases may actually occur.*

Proof. As both $\mathcal{H} \cap \Sigma_\infty$ and $\mathcal{Q} \cap \Sigma_\infty$ split in lines through P_∞, it is straightforward to see that the only possibilities for $\mathcal{C}_\infty$ are those outlined above; in particular, when $\mathcal{Q}$ is hyperbolic, $\mathcal{C}_\infty$ consists of either 1 point or 1 or 2 lines. It is straightforward to see that all cases may actually occur, as given any two lines ℓ, m in $\mathrm{PG}(3, q^2)$ there always exist at least one hyperbolic quadric containing both m and ℓ. Likewise, given a line $\ell \in \Sigma_\infty$ with $P \in \ell$ there always is at least one cone with vertex $V \in \ell$ and $V \neq P$ meeting Σ_∞ just in ℓ. □

Now we are going to use the same group theoretical arguments as in [1, Lemma 2.3] in order to be able to fix the values of some of the parameters in (3) without losing in generality.

Lemma 3.2. *If $\mathcal{Q}$ is a hyperbolic quadric, we can assume without loss of generality:*

1. *$b = 0$, and $a^{q+1} \neq c^{q+1}$ when $\mathcal{C}_\infty$ is just the point P_∞;*
2. *$b = 0$, $a = c$ when $\mathcal{C}_\infty$ is a line;*
3. *$b = \beta a$, $c = (\beta + 1)a$, $a \neq 0$ and $\beta^{q+1} = 1$, with $\beta \neq 1$ when $\mathcal{C}_\infty$ is the union of two lines.*

If $\mathcal{Q}$ is a cone, we can assume without loss of generality:

1. *$b = 0$ when $\mathcal{C}_\infty$ is a point;*
2. *$a = b$ when $\mathcal{C}_\infty$ is a line.*

Proof. Let Λ be the set of all lines of Σ_∞ through P_∞. The action of the stabilizer G of P_∞ in $\mathrm{PGU}(4, q)$ on Λ is the same as the action of $\mathrm{PGU}(2, q)$ on the points of $\mathrm{PG}(1, q^2)$. This can be easily seen by considering the action on $\mathrm{PGU}(2, q)$ on the line ℓ spanned by $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ fixing the equation $X^{q+1} + Y^{q+1} = 0$. Indeed, if M is a 2×2 matrix representing any $\sigma \in \mathrm{PGU}(2, q)$, then $M' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents an element of $\mathrm{PGU}(4, q)$ fixing $P_\infty = (0, 0, 0, 1)$. The action of $\mathrm{PGU}(2, q)$ on ℓ is analyzed in detail in [20, §42]. So, we see that the group G has two orbits on Λ, say Λ_1 and Λ_2 where Λ_1 consists of the totally isotropic lines of $\mathcal{H}$ through P_∞ while Λ_2 contains the remaining $q^2 - q$ lines of Σ_∞ through P_∞. Furthermore, G is doubly transitive on Λ_1 and the stabilizer of any $m \in \Lambda_1$ is transitive on Λ_2.

Let now $\mathcal{Q}_\infty = \mathcal{Q} \cap \Sigma_\infty$. If $\mathcal{Q}$ is hyperbolic and $\mathcal{C}_\infty = \{P_\infty\}$ we can assume $\mathcal{Q}_\infty$ to be the union of the line $\ell : J = X = 0$ and another line, say $u : J = aX + cY = 0$ with $a^{q+1} \neq c^{q+1}$. Thus, $b = 0$.

Otherwise, up to the choice of a suitable element $\sigma \in G$, we can always take $\mathcal{Q}_\infty$ as the union of any two lines in $\{\ell, s, t\}$ where

$$\ell\colon\ J = X = 0, \qquad s\colon\ J = X + Y = 0, \qquad t\colon\ J = X + \beta Y = 0$$

with $\beta^{q+1} = 1$ and $\beta \neq 1$.

Actually, when $\mathcal{C}_\infty$ contains just one line we take $\mathcal{Q}_\infty : X(X + Y) = 0$, while if $\mathcal{C}_\infty$ is the union of two lines we have $\mathcal{Q}_\infty : (X + Y)(X + \beta Y) = 0$. When $\mathcal{Q}$ is a cone, we get either $\mathcal{Q}_\infty : X^2 = 0$ or $\mathcal{Q}_\infty : (X + Y)^2 = 0$. The lemma follows. □

4 Proof of Theorem 1.1

We use the same setup as in the previous section. Thus, the Hermitian surface $\mathcal{H}$ has equation (2) whereas the quadric $\mathcal{Q}$ has equation (3). We first determine the number of affine points that $\mathcal{Q}$ and $\mathcal{H}$ have in common, that is the size of $(\mathcal{Q} \cap \mathcal{H}) \setminus \mathcal{C}_\infty$, where $\mathcal{C}_\infty$ is defined in (4). Hence we study the following system of equations

$$\begin{cases} z^q + z = x^{q+1} + y^{q+1} \\ z = ax^2 + by^2 + cxy + dx + ey + f. \end{cases} \tag{5}$$

In order to solve (5), recover the value of z from the second equation and substitute it in the first. This gives

$$a^q x^{2q} + b^q y^{2q} + c^q x^q y^q + d^q x^q + e^q y^q + f^q + ax^2 + by^2 + cxy + dx + ey + f = x^{q+1} + y^{q+1}. \tag{6}$$

Consider $\mathrm{GF}(q^2)$ as a vector space over $\mathrm{GF}(q)$ and fix a basis $\{1, \varepsilon\}$ with $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. Write any element in $\mathrm{GF}(q^2)$ as a linear combination with respect to this basis, that is, for any $x \in \mathrm{GF}(q^2)$ let $x = x_0 + x_1\varepsilon$, where $x_0, x_1 \in \mathrm{GF}(q)$. Analogously write also $a = a_0 + \varepsilon a_1$, $b = b_0 + \varepsilon b_1$ and so on. Thus, (6) can be studied as a quadratic equation over $\mathrm{GF}(q)$ in the indeterminates x_0, x_1, y_0, y_1.

As q is even, it is always possible to choose $\varepsilon \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ such that $\varepsilon^2 + \varepsilon + \nu = 0$, for some $\nu \in \mathrm{GF}(q) \setminus \{1\}$ and $\mathrm{Tr}_q(\nu) = 1$. Then, also, $\varepsilon^{2q} + \varepsilon^q + \nu = 0$. Therefore, $(\varepsilon^q + \varepsilon)^2 + (\varepsilon^q + \varepsilon) = 0$, whence $\varepsilon^q + \varepsilon + 1 = 0$. With this choice of ε, (6) reads as

$$\begin{aligned} &(a_1 + 1)x_0^2 + x_0x_1 + [a_0 + (1+\nu)a_1 + \nu]x_1^2 + (b_1 + 1)y_0^2 + y_0y_1 \\ &\quad + [b_0 + (1+\nu)b_1 + \nu]y_1^2 + c_1x_0y_0 + (c_0 + c_1)x_0y_1 + (c_0 + c_1)x_1y_0 \\ &\quad + [c_0 + (1+\nu)c_1]x_1y_1 + d_1x_0 + (d_0 + d_1)x_1 + e_1y_0 + (e_0 + e_1)y_1 + f_1 = 0. \end{aligned} \tag{7}$$

As (7) is a non-homogeneous quadratic equation in (x_0, x_1, y_0, y_1), its solutions correspond to the affine points of a (possibly degenerate) quadratic hypersurface Ξ of $\mathrm{PG}(4, q)$. Recall that the number N of affine points of Ξ equals the number of points of $\mathcal{H} \cap \mathcal{Q}$ which lie in $\mathrm{AG}(3, q^2)$; we shall use the formulas of [15, §1.5] in order to actually count the number of these points.

To this purpose, we first determine the number of points at infinity of Ξ. These points are those of the quadric Ξ_∞ of $\mathrm{PG}(3, q)$ with equation

$$\begin{aligned} f(x_0, x_1, y_0, y_1) = {} & (a_1 + 1)x_0^2 + x_0x_1 + [a_0 + (1+\nu)a_1 + \nu]x_1^2 \\ & + (b_1 + 1)y_0^2 + y_0y_1 + [b_0 + (1+\nu)b_1 + \nu]y_1^2 + c_1x_0y_0 \\ & + (c_0 + c_1)x_0y_1 + (c_0 + c_1)x_1y_0 + [c_0 + (1+\nu)c_1]x_1y_1 \\ = {} & 0. \end{aligned} \tag{8}$$

Following the approach outlined in Section 2, we write the matrix associated to $\mathcal{Q}_\infty$

$$A_\infty = \begin{pmatrix} 2(a_1+1) & 1 & c_1 & c_0 + c_1 \\ 1 & 2[a_0 + (1+\nu)a_1 + \nu] & c_0 + c_1 & c_0 + (1+\nu)c_1 \\ c_1 & c_0 + c_1 & 2(b_1 + 1) & 1 \\ c_0 + c_1 & c_0 + (1+\nu)c_1 & 1 & 2[b_0 + (1+\nu)b_1 + \nu] \end{pmatrix} \tag{9}$$

As q is even, a direct computation gives

$$\det A_\infty = 1 + c^{2(q+1)};$$

so, the quadric Ξ_∞ is non-singular if and only if $\det A_\infty \neq 0$, that is $c^{q+1} \neq 1$.

Lemma 4.1. *If $\mathcal{Q}$ is a cone, then* $\mathrm{rank}(\Xi_\infty) = 4$*. If $\mathcal{Q}$ is non-singular then* $\mathrm{rank}(\Xi_\infty) \geq 2$ *and if* $\mathrm{rank}(\Xi_\infty) = 2$*, then the quadric $\mathcal{Q}$ is hyperbolic.*

Proof. Let $\mathcal{Q}$ be a cone, namely $c = 0$. It turns out that $\det A_\infty \neq 0$ and hence $\mathrm{rank}(\Xi_\infty) = 4$. Now assume that $\mathcal{Q}$ is non-singular. If the equation of Ξ_∞ were to be of the form $f(x_0, x_1, y_0, y_1) = (lx_0 + mx_1 + ny_0 + ry_1)^2$ with l, m, n, r over some extension of $\mathrm{GF}(q)$, then $c = 0$; this is a contradiction. So $\mathrm{rank}(\Xi_\infty) \geq 2$. Finally, suppose $\mathrm{rank}(\Xi_\infty) = 2$, that is Ξ_∞ splits into two planes. We need to prove $\mathrm{Tr}_{q^2}(ab/c^2) = 0$. First observe that $c^{q+1} = 1$ since the quadric Ξ_∞ is degenerate.

Consider now the following 4 intersections $\mathcal{C}_0 : \Xi_\infty \cap [x_0 = 0]$, $\mathcal{C}_1 : \Xi_\infty \cap [x_1 = 0]$, $\mathcal{C}_2 : \Xi_\infty \cap [y_0 = 0]$, $\mathcal{C}_3 : \Xi_\infty \cap [y_1 = 0]$. Clearly, as Ξ_∞ is, by assumption, reducible in the union of two planes, all of these conics are degenerate; thus we get the following four formal equations

$$\frac{1}{2}\det\begin{pmatrix} 2[a_0 + (1+\nu)a_1 + \nu] & c_0 + c_1 & c_0 + (1+\nu)c_1 \\ c_0 + c_1 & 2(b_1 + 1) & 1 \\ c_0 + (1+\nu)c_1 & 1 & 2[b_0 + (1+\nu)b_1 + \nu] \end{pmatrix} = 0,$$

$$\frac{1}{2}\det\begin{pmatrix} 2(a_1 + 1) & c_1 & c_0 + c_1 \\ c_1 & 2(b_1 + 1) & 1 \\ c_0 + c_1 & 1 & 2[b_0 + (1+\nu)b_1 + \nu] \end{pmatrix} = 0,$$

$$\frac{1}{2}\det\begin{pmatrix} 2(a_1 + 1) & 1 & c_0 + c_1 \\ 1 & 2[a_0 + (1+\nu)a_1 + \nu] & c_0 + (1+\nu)c_1 \\ c_0 + c_1 & [c_0 + (1+\nu)c_1] & 2[b_0 + (1+\nu)b_1 + \nu] \end{pmatrix} = 0,$$

$$\frac{1}{2}\det\begin{pmatrix} 2(a_1 + 1) & 1 & c_1 \\ 1 & 2[a_0 + (1+\nu)a_1 + \nu] & c_0 + c_1 \\ c_1 & c_0 + c_1 & 2(b_1 + 1) \end{pmatrix} = 0.$$

Using the condition $c^{q+1} = 1$, these give

$$\begin{cases} a_0 + (1+\nu)a_1 + (c_0^2 + c_1^2)b_0 + \nu(c_0^2 + c_1^2 + c_1^2\nu)b_1 = 0 \\ a_1 + c_1^2 b_0 + (c_0^2 + \nu c_1^2)b_1 = 0 \\ (c_0^2 + c_1^2)a_0 + \nu(c_0^2 + c_1^2 + \nu c_1^2)a_1 + b_0 + (1+\nu)b_1 = 0 \\ c_1^2 a_0 + (c_0^2 + \nu c_1^2)a_1 + b_1 = 0. \end{cases} \tag{10}$$

Since $c^{q+1} = 1$, if $c_1 = 0$, then $c_0 = 1$. Solving (10), we obtain

$$\begin{cases} a_1 = b_1 \\ a_0 + a_1 + b_0 = 0. \end{cases} \tag{11}$$

Therefore $\mathrm{Tr}_{q^2}(ab/c^2) = \mathrm{Tr}_{q^2}((a_0 + \varepsilon a_1)(a_0 + (\varepsilon+1)a_1)) = \mathrm{Tr}_{q^2}(a^{q+1}) = 0$ as $a^{q+1} \in \mathrm{GF}(q)$.

Suppose now $c_1 \neq 0$. Then $c^{q+1} = (c_0^2 + c_0c_1 + \nu c_1^2) = 1$ and, after some elementary algebraic manipulations, System (10) becomes

$$\begin{cases} a_0 = \left(\frac{c_0^2}{c_1^2} + \nu\right) a_1 + \frac{b_1}{c_1^2} \\ b_0 = \frac{a_1}{c_1^2} + \left(\frac{c_0^2}{c_1^2} + \nu\right) b_1; \end{cases} \tag{12}$$

hence,
$$\frac{ab}{c^2} = \frac{(a_1^2+b_1^2)(c_1^2\nu + c_1^2\varepsilon + c_0^2) + a_1b_1(c_1^2\nu + c_1^2\varepsilon + c_0^2+1)^2}{c_1^4(c_0+\varepsilon c_1)^2}.$$
Since $\varepsilon^2 = \varepsilon + \nu$ and $c_1^2\nu + c_0^2 + 1 = c_0c_1$, we get

$$\frac{ab}{c^2} = \frac{a_1^2+b_1^2}{c_1^4} + \frac{a_1b_1}{c_1^2} \in \mathrm{GF}(q),$$

which gives $\mathrm{Tr}_{q^2}(ab/c^2) = 0$ once more. Hence if $\mathrm{rank}(\Xi_\infty) = 2$, then $\mathcal{Q}$ is hyperbolic. $\square$

Lemma 4.2. *Suppose $\mathcal{Q}$ to be a hyperbolic quadric with $\mathcal{C}_\infty$ being the union of two lines. If* $\mathrm{rank}(\Xi_\infty) = 2$*, then $\Xi_\infty = \Pi_1 \cup \Pi_2$ is a plane pair over* $\mathrm{GF}(q)$*.*

Proof. By Lemma 3.2 we can assume that $b = \beta a$, $c = (\beta+1)a$, $a \neq 0$ and $\beta^{q+1} = 1$ with $\beta \neq 1$.

Furthermore, since $\mathrm{rank}(\Xi_\infty) = 2$, we have $\Xi = \Pi_1 \cup \Pi_2$ where the planes Π_1 and Π_2 have respectively equations $lx_0 + mx_1 + ny_0 + ry_1 = 0$ and $l'x_0 + m'x_1 + n'y_0 + r'y_1 = 0$, for some values of l, m, n, r and l', m', n', r' in $\mathrm{GF}(q^2)$. Clearly, in this case,

$$f(x_0, x_1, y_0, y_1) = (lx_0 + mx_1 + ny_0 + ry_1)(l'x_0 + m'x_1 + n'y_0 + r'y_1). \tag{13}$$

Then, up to a scalar multiple, the following must be satisfied:

$$\begin{cases} ll' = a_1 + 1 \\ lm' + l'm = 1 \\ l'n + ln' = c_1 \\ l'r + lr' = c_0 + c_1 \\ mm' = a_0 + (1+\nu)a_1 + \nu \\ mn' + nm' = c_0 + c_1 \\ mr' + rm' = c_0 + (1+\nu)c_1 \\ nr' + rn' = 1 \\ nn' = b_1 + 1 \\ rr' = b_0 + (1+\nu)b_1 + \nu. \end{cases} \tag{14}$$

If $c_1 = 0$, then $c_0 = 1$ as $c^{q+1} = 1$; in particular, as $c = b + a$, we have $a_0 + b_0 = c_0 = 1$

and, consequently, as (11) holds we get $a_1 = 1 = b_1$. So System (14) becomes

$$\begin{cases} ll' = 0 \\ lm' + l'm = 1 \\ l'n + ln' = 0 \\ l'r + lr' = 1 \\ mm' = a_0 + 1 \\ mn' + nm' = 1 \\ mr' + rm' = 1 \\ nr' + rn' = 1 \\ nn' = 0 \\ rr' = b_0 + 1. \end{cases}$$

We can assume without loss of generality either $l = 0$ or $l' = 0$. Suppose the former; then $l' \neq 0$ and $m = l'^{-1}$. We also have $n = 0$ and $r = l'^{-1}$. It follows that Π_1 is the plane of equation $x_1 + y_1 = 0$. In particular, Π_1 is defined over $\mathrm{GF}(q)$ and, consequently, also Π_2 is. If $l' = 0$, an analogous argument leads to $\Pi_2 : x_1 + y_1 = 0$ and, once more, Ξ_∞ splits into two planes defined over $\mathrm{GF}(q)$.

Now suppose $c_1 \neq 0$. From (14) we get

$$\begin{cases} ll' = a_1 + 1 \\ ln' + l'n = c_1 \\ nn' = b_1 + 1 \\ mm' = a_0 + (1+\nu)a_1 + \nu \\ mr' + rm' = c_0 + (1+\nu)c_1 \\ rr' = b_0 + (1+\nu)b_1 + \nu \\ nr' + rn' = 1 \\ lm' + l'm = 1. \end{cases} \tag{15}$$

We obtain $ll' + nn' = a_1 + b_1 = c_1 = ln' + l'n$ and $mm' + rr' = c_0 + (1+\nu)c_1 = mr' + rm'$. Hence,

$$(l' + n')(l + n) = 0, \qquad (m + r)(m' + r') = 0.$$

There are the following two cases to consider:

1. $l = n$ and $m = r$ or, equivalently, $l' = n'$ and $m' = r'$
2. $l = n$ and $m' = r'$ or, equivalently, $l' = n'$ and $m = r$.

Suppose first $l = n$ and $m = r$; then $n(n' + l') = c_1 \neq 0$; consequently, $n = l \neq 0$ and also $n' \neq l'$. If $m = 0$, then Π_1 has equation $x_0 + x_1 = 0$ and is defined over $\mathrm{GF}(q)$; then also Π_2 is defined over $\mathrm{GF}(q)$ and we are done.

Suppose now $m \neq 0$ (and hence $r \neq 0$). We claim $l/m \in \mathrm{GF}(q)$. This would give that Π_1 is defined over $\mathrm{GF}(q)$, whence the thesis. From (15) we have

$$\begin{cases} n' = \frac{b_1+1}{n} \\ r' = \frac{b_0+(1+\nu)b_1+\nu}{r} \\ nr' + rn' = 1. \end{cases}$$

Replacing the values of n' and r' in the last equation we obtain

$$n^2\left(b_0 + (1+\nu)b_1 + \nu\right) + r^2\left(b_1+1\right) + nr = 0; \tag{16}$$

if we consider

$$\begin{cases} ll' = \frac{a_1+1}{n} \\ lm' + lm' = 1 \\ mm' = a_0 + (1+\nu)a_1 + \nu \end{cases}$$

a similar argument on l, l', m, m' gives

$$l^2\left(a_0 + (1+\nu)a_1 + \nu\right) + m^2(a_1+1) + lm = 0. \tag{17}$$

Since, by assumption, $l = n$ and $m = r$ we get

$$l^2\left(a_0 + b_0 + (1+\nu)(a_1+b_1)\right) + m^2(a_1+b_1) = 0,$$

whence $l^2/m^2 \in \mathrm{GF}(q)$. As q is even, this gives $l/m \in \mathrm{GF}(q)$. The case $l' = n'$ and $m' = r'$ is clearly analogous and can be obtained by switching the roles of Π_1 and Π_2.

Suppose now $l = n$ and $m' = r'$. Since $c_1 \neq 0$, we also have $l \neq 0$; furthermore,

$$n' = \frac{b_1+1}{l}.$$

If $m' = r' = 0$, then Π_2 has equation $(a_1+1)x_0 + (b_1+1)y_0 = 0$ and, consequently, is defined over $\mathrm{GF}(q)$. Suppose then $m' = r' \neq 0$. There are several subcases to consider:

- if $m = 0$, then $m' = r' = l^{-1}$ and Π_2 has equation $(a_1+1)x_0 + x_1 + (b_1+1)y_0 + y_1 = 0$, which is defined over $\mathrm{GF}(q)$;
- if $r = 0$, then $m' = r' = n^{-1} = l^{-1}$ and we deduce, as above, that Π_2 is defined over $\mathrm{GF}(q)$;
- finally, suppose $m \neq 0 \neq r$; then $b_0 + (1+\nu)b_1 + \nu \neq 0$ and from (14) we get

  $$m' = \frac{a_0 + (1+\nu)a_1 + \nu}{m}, \qquad r' = \frac{b_0 + (1+\nu)b_1 + \nu}{r}.$$

  Since $m' = r'$ we deduce

  $$\frac{m}{r} = \frac{a_0 + (1+\nu)a_1 + \nu}{b_0 + (1+\nu)b_1 + \nu} \in \mathrm{GF}(q). \tag{18}$$

Observe that $l' = (a_1+1)l^{-1}$ and also $r' = (b_0 + (1+\nu)b_1 + \nu)r^{-1}$; thus from (14) we obtain

$$l^2(b_0 + (1+\nu)b_1 + \nu) + r^2(a_1+1) + (c_0+c_1)lr = 0. \tag{19}$$

On the other hand, since $lm' + l'm = 1$,

$$\frac{l}{m}(a_0 + (1+\nu)a_1 + \nu) + \frac{m}{l}(a_1+1) = 1;$$

using (18) we obtain

$$\frac{l}{r}(b_0 + (1+\nu)b_1 + \nu) + \frac{r}{l}(a_1+1)\left(\frac{a_0 + (1+\nu)a_1 + 1}{b_0 + (1+\nu)b_1 + 1}\right) = 1,$$

whence

$$l^2(b_0 + (1+\nu)b_1 + \nu) + lr + r^2(a_1+1)\left(\frac{a_0 + (1+\nu)a_1 + 1}{b_0 + (1+\nu)b_1 + 1}\right) = 0; \tag{20}$$

thus, adding (19) to (20), we get

$$\frac{l}{r} = \frac{a_1+1}{c_0+c_1+1}\left(\frac{a_0 + (1+\nu)a_1 + 1}{b_0 + (1+\nu)b_1 + 1}\right) \in \mathrm{GF}(q)$$

and the plane Π_1 is defined over $\mathrm{GF}(q)$. The case $l' = n'$ and $m = r$ is analogous.

□

Lemma 4.3. *Suppose that $\mathcal{Q}$ is a hyperbolic quadric $\mathcal{C}_\infty = \{P_\infty\}$. If $\mathrm{rank}(\Xi_\infty) = 2$, then Ξ_∞ is a line.*

Proof. By Lemma 3.2 we can assume $b = 0$. Since $\mathrm{rank}(\Xi_\infty) = 2$, we have $\det A_\infty = 0$, that is $c^{q+1} = 1$ and (10) holds. If $c_1 = 0$, then solving (10) we obtain (11) and, since $b = 0$, we get $a = 0$. In the case in which $c_1 \neq 0$ from (12) we again obtain $a = 0$. We have now to show that Ξ_∞ is the union of two conjugate planes. In order to obtain this result, it suffices to prove that the coefficients l, m, n, r in (13) belong to some extension of $\mathrm{GF}(q)$ but are not in $\mathrm{GF}(q)$. Since (14) holds we have

$$\begin{cases} ll' = 1 \\ lm' + l'm = 1 \\ mm' = \nu; \end{cases}$$

thus, $\frac{l\nu}{m} + \frac{m}{l} = 1$; hence $\nu l^2 + lm + m^2 = 0$. Since $\mathrm{Tr}_q(\nu) = 1$ this implies that $\frac{l}{m} \notin \mathrm{GF}(q)$. □

Now, set $N = |(\mathcal{H} \cap \mathcal{Q}) \cap \mathrm{AG}(3, q^2)|$. First we observe that $N = |\Xi| - |\Xi_\infty|$. By Lemma 4.1 we see that $\mathrm{rank}(\Xi_\infty) \geq 2$. Thus, the following possibilities for N may occur according as:

(C1) $\mathrm{rank}(\Xi) = 5$ and $\mathrm{rank}(\Xi_\infty) = 4$;

(C1.1) Ξ is a parabolic quadric and Ξ_∞ is a hyperbolic quadric. Then,

$$N = (q+1)(q^2+1) - (q+1)^2 = q^3 - q.$$

(C1.2) Ξ is a parabolic quadric and the quadric Ξ_∞ is elliptic. Then,

$$N = (q+1)(q^2+1) - (q^2+1) = q^3 + q.$$

(C2) $\mathrm{rank}(\Xi) = 5$ and $\mathrm{rank}(\Xi_\infty) = 3$;

Ξ is a parabolic quadric and the hyperplane at infinity is tangent to Ξ whereas Ξ_∞ is a cone comprising the join of a point to a conic. Then,

$$N = (q+1)(q^2+1) - (q^2+q+1) = q^3.$$

(C3) $\mathrm{rank}(\Xi) = 4$ and $\mathrm{rank}(\Xi_\infty) = 4$;

(C3.1) Ξ is a cone projecting a hyperbolic quadric of $\mathrm{PG}(3, q)$ and the quadric Ξ_∞ is hyperbolic. Then,

$$N = q(q+1)^2 + 1 - (q+1)^2 = q^3 + q^2 - q.$$

(C3.2) Ξ is a cone projecting an elliptic quadric of $\mathrm{PG}(3, q)$ and the quadric Ξ_∞ is elliptic. Then,

$$N = q(q^2+1) + 1 - (q^2+1) = q^3 - q^2 + q.$$

(C4) $\mathrm{rank}(\Xi) = 4$, $\mathrm{rank}(\Xi_\infty) = 3$;

(C4.1) Ξ is a cone projecting a hyperbolic quadric and Ξ_∞ is a cone comprising the join of a point to a conic. Then,

$$N = q(q+1)^2 + 1 - [q(q+1)+1] = q^3 + q^2.$$

(C4.2) Ξ is a cone projecting an elliptic quadric and Ξ_∞ is a cone comprising the join of a point to a conic. Then,

$$N = q(q^2+1) + 1 - [q(q+1)+1] = q^3 - q^2.$$

(C5) $\mathrm{rank}(\Xi) = 4$ and $\mathrm{rank}(\Xi_\infty) = 2$;

(C5.1) Ξ is a cone projecting a hyperbolic quadric and Ξ_∞ is the union of two planes defined over $\mathrm{GF}(q)$. Then,

$$N = q(q+1)^2 + 1 - (2q^2+q+1) = q^3.$$

(C5.2) Ξ is a cone projecting an elliptic quadric and and Ξ_∞ is a line (i.e. the union of two planes defined over the extension $\mathrm{GF}(q^2)$ but not over $\mathrm{GF}(q)$). Then,

$$N = q(q^2+1)+1-(q+1) = q^3.$$

(C6) $\mathrm{rank}(\Xi) = \mathrm{rank}(\Xi_\infty) = 3$;

Ξ is the join of a line to a conic and Ξ_∞ is a cone comprising the join of a point to a conic. Then,

$$N = q^3+q^2+q+1-(q^2+q+1) = q^3.$$

(C7) $\mathrm{rank}(\Xi) = 3$, $\mathrm{rank}(\Xi_\infty) = 2$;

(C7.1) Ξ is the join of a line to a conic whereas Ξ_∞ is a pair of planes over $\mathrm{GF}(q)$. Then,

$$N = q^3+q^2+q+1-(2q^2+q+1) = q^3-q^2.$$

(C7.2) Ξ is the join of a line to a conic whereas Ξ_∞ is a line. Then,

$$N = q^3+q^2+q+1-q-1 = q^3+q^2.$$

(C8) $\mathrm{rank}(\Xi) = \mathrm{rank}(\Xi_\infty) = 2$;

(C8.1) Ξ is a pair of solids and Ξ_∞ is a pair of planes over $\mathrm{GF}(q)$. Then,

$$N = 2q^3+q^2+q+1-(2q^2+q+1) = 2q^3-q^2.$$

(C8.2) Ξ is a plane and Ξ_∞ is a line. Then,

$$N = q^2+q+1-(q+1) = q^2.$$

We are going to determine which cases (C1)–(C8) may occur according as $\mathcal{Q}$ is either elliptic or a cone or hyperbolic. In order to do this we need to establish the nature of Ξ_∞, when Ξ_∞ is non-singular. Hence we shall to compute the trace of α as given by (1) where the matrix A is defined as A_∞ in (9) and

$$B = \begin{pmatrix} 0 & 1 & c_1 & c_0+c_1 \\ -1 & 0 & c_0+c_1 & c_0+(1+\nu)c_1 \\ -c_1 & -(c_0+c_1) & 0 & 1 \\ -(c_0+c_1) & -c_0-(1+\nu)c_1 & -1 & 0 \end{pmatrix}.$$

Write $\gamma = (1+c^{q+1}) = (1+c_0^2+\nu c_1^2+c_0c_1)$. A straightforward computation shows

$$\alpha = \frac{a_0+a_1+b_0+b_1}{\gamma} + \frac{1}{\gamma^2}[(1+\nu)(a_1^2+b_1^2)+ \\ (c_0^2+c_1^2\nu)(a_0b_1+a_1b_0)+a_0a_1+b_0b_1+a_0b_0c_1^2+a_1b_1c_0^2]. \quad (21)$$

4.1 The elliptic case

Let $\mathcal{Q}$ be an elliptic quadric. By Lemma 4.1, we have $\mathrm{rank}(\Xi_\infty) \geq 3$ and hence cases (C1), (C2), (C3), (C4) and (C6) may occur. Whence

$$N \in \{q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2\}.$$

In this case $\mathcal{C}_\infty = \{P_\infty\}$, hence

$$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1\}.$$

4.2 The degenerate case

Let $\mathcal{Q}$ be a cone. By Lemma 4.1, N falls in one of cases (C1) or (C3) and hence

$$N \in \{q^3 - q^2 + q, q^3 - q, q^3 + q, q^3 + q^2 - q\}.$$

Here, by Lemma 3.1 $\mathcal{C}_\infty$ is either a point or $q^2 + 1$ points on a line. We distinguish these two cases.

- $\boxed{\mathcal{C}_\infty = \{P_\infty\}.}$ By Lemma 3.2, we can assume $b = 0$ in (3). Thus (21) becomes $\alpha = a_0 + a_1 + (1 + \nu)a_1^2 + a_0 a_1$ and $\mathrm{Tr}_q(\alpha)$ may be either zero or one.

  Hence cases (C1) and (C3) may happen; so

  $$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + q + 1, q^3 + q^2 - q + 1\}.$$

- $\boxed{\mathcal{C}_\infty \text{ is a line.}}$ By Lemma 3.2, we can assume $a = b$ in (3). Furthermore as $\mathcal{Q}$ is a cone $c = 0$ in (3); thus, in this case, (21) gives $\alpha = 0$ that is, $\mathrm{Tr}_q(\alpha) = 0$; this means that only subcases (C1.1) of (C1) and (C3.1) of (C3) may occur. In particular,

  $$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1\}.$$

4.3 The hyperbolic case

Let $\mathcal{Q}$ be a hyperbolic quadric. Then, by Lemma 4.1, $\mathrm{rank}(\Xi_\infty) \geq 2$ and all cases (C1)–(C8) might occur.

$$N \in \{q^2, q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2, 2q^3 - q^2\}.$$

We have three possibilities for $\mathcal{C}_\infty$ from Lemma 3.1, that is $\mathcal{C}_\infty$ is either a point, or $q^2 + 1$ points on a line or $2q^2 + 1$ points in the union of two lines. We now analyze these cases.

- $\boxed{\mathcal{C}_\infty = \{P_\infty\}.}$ We are going to show that some subcases of (C1)–(C8) can be excluded. Indeed, when $\mathrm{rank}(\Xi_\infty) = 2$, from Lemma 4.3 we have that subcases (C7.1) and (C8.1) cannot occur. So,

  $$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{q^2 + 1, q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1\}.$$

- $\boxed{\mathcal{C}_\infty \text{ is one line.}}$ By Lemma 3.2 we can assume $b = 0$ and $a = c$ in (3).

  1. When Ξ_∞ is non-degenerate, only cases (C1) and (C3) may occur. Observe that (21) becomes

  $$\alpha = \frac{c_0 + c_1}{\gamma} + \frac{1}{\gamma^2}[(1+\nu)(c_1^2) + c_0c_1].$$

  Since $\gamma = c_0^2 + \nu c_1^2 + c_0c_1 + 1$ we have $c_1^2 + \nu c_1^2 + c_1c_0 = \gamma + c_0^2 + c_1^2 + 1$. Therefore

  $$\alpha = \frac{(c_0 + c_1)}{\gamma} + \frac{c_0^2 + c_1^2}{\gamma^2} + \frac{1}{\gamma} + \frac{1}{\gamma^2}$$

  and $\mathrm{Tr}_q(\alpha) = 0$. Hence, subcases (C1.2) and (C3.2) cannot happen; so,

  $$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1\}.$$

  2. Assume now that Ξ_∞ is degenerate, that is $2 \leq \mathrm{rank}(\Xi_\infty) \leq 3$. Cases (C2) and (C4)–(C8) occur. We are going to show that $\mathrm{rank}(\Xi_\infty) = 3$. Suppose, on the contrary, $\mathrm{rank}(\Xi_\infty) = 2$; then we end up with considering a system identical to (10), as it appears in the proof of Lemma 4.1, and its consequences (11) and (12); so we shall not repeat explicitly these equations here. First observe that (10) holds. If it were $c_1 = 0$, from (11) we would have $a_1 = a_0 = 0$, that is $a = 0$, which is impossible. So, $c_1 \neq 0$; since we are assuming $b = b_0 + \varepsilon b_1 = 0$, that is, $b_1 = b_0 = 0$, we would now have from (12) $a_1 = a_0 = 0$—again a contradiction.

  Thus, only cases (C2) and (C4) might happen; in particular,

  $$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + 1, q^3 + q^2 + 1, q^3 + 2q^2 + 1\}.$$

- $\boxed{\mathcal{C}_\infty \text{ consists of two lines.}}$ By Lemma 3.2 we can assume $b = \beta a$, $c = (\beta + 1)a$ where $a \neq 0$ and $\beta^{q+1} = 1$ in (3).

  1. Suppose now Ξ_∞ to be non-degenerate. Cases (C1) and (C3) occur.

  From $c = a + b$ we get $c^{q+1} = a^{q+1} + a^q b + b^q a + b^{q+1}$. Since $b^{q+1} = a^{q+1}$, we have $c^{q+1} = a^q b + b^q a$, that is

  $$a_0b_1 + a_1b_0 = c_0^2 + \nu c_1^2 + c_0c_1,$$

  On the other hand, from $c_0 = a_0 + b_0$ and $c_1 = a_1 + b_1$ we obtain $c_0c_1 = a_0a_1 + a_0b_1 + a_1b_0 + b_0b_1$ that is

  $$a_0a_1 + b_0b_1 = c_0^2 + \nu c_1^2.$$

  Now $(c_0^2 + \nu c_1^2 + c_0c_1)(c_0^2 + \nu c_1^2) = (a_0b_1 + a_1b_0)(a_0a_1 + b_0b_1) = a_0b_0(a_1^2 + b_1^2) + a_1b_1(a_0^2 + b_0^2) = a_0b_0c_1^2 + a_1b_1c_0^2$ and thus (21) becomes

  $$\alpha = \frac{c_0 + c_1}{1 + c_0^2 + \nu c_1^2 + c_0c_1} + \frac{(c_0 + c_1)^2}{(1 + c_0^2 + \nu c_1^2 + c_0c_1)^2};$$

so, α has trace 0.

Hence just subcases (C1.1) and (C3.1) may occur and

$$|\mathcal{Q} \cap \mathcal{H}| = N + 2q^2 + 1 \in \{q^3 + 2q^2 - q + 1, q^3 + 3q^2 - q + 1\}.$$

2. If Ξ_∞ is degenerate, then cases (C2) and (C4)–(C8) may happen.

   When $\mathrm{rank}(\Xi_\infty) = 2$, it follows from Lemma 4.2 that only subcases (C7.1) in (C7) and (C8.1) in (C8) may occur.

   Now, we need a preliminary lemma. Recall that a quadric $\mathcal{Q}$ meeting a Hermitian surface $\mathcal{H}$ in at least 3 lines of a regulus is permutable with $\mathcal{H}$; see [14, §19.3, pag. 124]. We have the following statement.

   Lemma 4.4. *Suppose $\mathcal{Q}$ to be hyperbolic and $\mathcal{C}_\infty$ to be the union of two lines; then $|\mathcal{H} \cap \mathcal{Q}| = q^3 + 3q^2 + 1$ cannot happen.*

   Proof. The case $|\mathcal{H} \cap \mathcal{Q}| = q^3 + 3q^2 + 1$ may happen just for case (C4.1). Let $\mathcal{R}$ be a regulus of $\mathcal{Q}$ and denote by r_1, r_2, r_3 respectively the numbers of 1–tangents, $(q + 1)$–secants and $(q^2 + 1)$-secants to $\mathcal{H}$ in $\mathcal{R}$. A direct counting gives

$$\begin{cases} r_1 + r_2 + r_3 = q^2 + 1 \\ r_1 + (q + 1)r_2 + r_3(q^2 + 1) = q^3 + 3q^2 + 1. \end{cases} \tag{22}$$

   By straightforward algebraic manipulations we obtain

$$qr_1 + (q - 1)r_2 = q(q^2 - q - 1).$$

   In particular, $r_2 = qt$ with $t \leq q$. If it were $t = q$, then $r_1 = -1$ — a contradiction; so $r_2 \leq q(q - 1)$. Solving (22) in r_1 and r_3, we obtain

$$r_3 = \frac{q^2 + 2q - r_2}{q} \geq \frac{q^2 + 2q - q^2 + q}{q} = 3.$$

   In particular, there are at least 3 lines of $\mathcal{R}$ contained in $\mathcal{H}$. This means that $\mathcal{Q}$ is permutable with $\mathcal{H}$, see [14, §19.3, pag. 124] or [20, §86, pag. 154] and $|\mathcal{Q} \cap \mathcal{H}| = 2q^3 + q^2 + 1$, a contradiction. □

   So, by Lemma 4.4 only subcase (C4.2) in (C4) is possible.

   Thus we get

$$|\mathcal{Q} \cap \mathcal{H}| = N + 2q^2 + 1 \in \{q^3 + q^2 + 1, q^3 + 2q^2 + 1, 2q^3 + q^2 + 1\}$$

   and the proof is completed.

It is straightforward to see, by means of a computer aided computation for small values of q, that all the cardinalities enumerated above may occur.

5 Extremal configurations

As in the case of odd characteristic, it is possible to provide a geometric description of the intersection configuration when the size is either q^2+1 or $2q^3+q^2+1$. These values are respectively the minimum and the maximum yielded by Theorem 1.1. and they can happen only when $\mathcal{Q}$ is an hyperbolic quadric. Throughout this section we assume that the hypotheses of Theorem 1.1 hold, namely that $\mathcal{H}$ and $\mathcal{Q}$ share a tangent plane at some point P.

Theorem 5.1. *Suppose $|\mathcal{H}\cap\mathcal{Q}|=q^2+1$. Then, $\mathcal{Q}$ is a hyperbolic quadric and $\Omega=\mathcal{H}\cap\mathcal{Q}$ is an ovoid of $\mathcal{Q}$.*

Proof. By Theorem 1.1, $\mathcal{Q}$ is hyperbolic. Fix a regulus $\mathcal{R}$ on $\mathcal{Q}$. The q^2+1 generators of $\mathcal{Q}$ in $\mathcal{R}$ are pairwise disjoint and each has non-empty intersection with $\mathcal{H}$; so there can be at most one point of $\mathcal{H}$ on each of them. It follows that $\mathcal{H}\cap\mathcal{Q}$ is an ovoid. In particular, by the above argument, any generator of $\mathcal{Q}$ through a point of Ω must be tangent to $\mathcal{H}$. Thus, at all points of Ω the tangent planes to $\mathcal{H}$ and to $\mathcal{Q}$ are the same. □

Theorem 5.2. *Suppose $|\mathcal{H}\cap\mathcal{Q}|=2q^3+q^2+1$. Then, $\mathcal{Q}$ is a hyperbolic quadric permutable with $\mathcal{H}$.*

Theorem 5.2 can be obtained as a consequence of the analysis contained in [8, §5.2.1], in light of [14, Lemma 19.3.1].

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