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Unique determination of a single crack in a uniform simply supported beam in bending vibration

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A B S T R A C T

In this paper we consider one of the basic inverse problems in damage detection based on natural frequency data, namely the identification of a single open crack in a uniform simply supported beam from measurement of the first and the second natural frequency. It is commonly accepted in the literature that the knowledge of this set of spectral data allows for the unique determination of the severity and the position (up to symmetry) of the damage. However, in spite of the fact that many numerical evidences are in support of this property, the result is rigorously proved only when the severity of the crack is small. In this paper we definitely show, by means of an original constructive method, that the above result holds true for any level of crack severity.

1. Introduction

Dynamic methods based on natural frequency measurements are widely used as powerful tool for crack detection in beam structures. Resonant frequencies are often chosen as input data since they are easy to measure in experiments, and are subject to errors less than those affecting other dynamic data, such as, for instance, mode shape components. In addition, in case of ideal undamped systems, it is somewhat more easy to extract information on possible occurrence of damage from natural frequency measurements than from other dynamical parameters, particularly for concentrated damages, such as cracks or notches in beams [1].

After the appearance of the pioneering paper by Adams et al. [2], an extensive literature is nowadays available on the identification of defects in beams structures by frequency measurements; see, among others, the contributions by [3, Chapter 15] and [4] for an introduction to the topic, and [5–7] for recent advances on multiple crack identification in beams and in frames, respectively. In spite of this, several basic, fundamental diagnostic problems are still open. Their study is useful both for the application of dynamic techniques in practice and for the definition of a comprehensive theory of damage detection in structures.

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One of these basic problems is considered here. Specifically, we deal with the inverse problem of determining a single open crack in a uniform simply supported beam from the first two natural bending frequencies. The damage is modelled as a massless linearly elastic rotational spring located at the damaged cross-section [8]. The main contribution of the present research is the rigorous unique determination of the crack position and severity, by means of a constructive algorithm, without any a priori assumption on the smallness of the damage. It should be recalled, in fact, that a well-established theory is available for our problem in case of small crack [9–11]. The smallness of the damage allows us to linearize the inverse problem in a neighborhood of the undamaged configuration, see also [12] for a theory which includes second-order terms in the eigenvalues expansion with respect to the crack severity. Then, taking advantage of the closed-form expression of the eigenpairs of the uniform undamaged beam, it is possible to obtain exact solutions of the linearized inverse problem, with closed form expressions both the position and the severity of the crack in terms of the data. In particular, it turns out that the first two natural frequencies determine uniquely the small crack, up to a symmetrical position with respect to the mid-point of the beam.

When the damage is not small, the linearization is no longer allowed and one has to deal with the full nonlinear crack identification problem. There are strong motivations in support of the extension of the theory to not necessarily small cracks. Firstly, it is not easy to rigorously state when a crack can be considered small. Secondly, the linearized theories by Narkis [9] and Morassi [10] show some limitations when the damage is located near a point of vanishing sensitivity for a vibration mode. Third, it is obviously desirable to have a unifying theory of the diagnostic problem capable to include any severity of the damage.

In this paper we prove, by means of a constructive argument, that the measurement of the first and the second natural frequency of the cracked beam is sufficient for the unique determination of the crack (up to a symmetric position) for any level of severity of the damage. However, unlike the corresponding linearized problem, no closed-form expressions of the damage parameters in terms of the frequency data are available.

Our method differs from that recently presented in [13] for the analysis of the analogous crack identification problem in a uniform beam under longitudinal vibration. The analysis developed in [13] was essentially based on the Frequency Equation Method, that is, a careful study of the solutions of the nonlinear system formed by the frequency equation – which is available in closed form, since the system has constant coefficients – written for the two selected resonant frequencies in terms of the position and severity of the crack. The analysis of the corresponding nonlinear system for the cracked beam in bending turns out to be significantly more difficult, due to the simultaneous presence of harmonic ($\cos$, $\sin$) and hyperbolic ($\cosh, \sinh$) functions. In particular, it has proved difficult to find a complete characterization of the set of admissible natural frequency data for which the existence and uniqueness of the solution to the inverse problem are ensured. In view of these difficulties, we had to follow a different approach that, at the end, resulted in an original constructive algorithm for the identification of the damage parameters.

We now describe the main steps of our approach. The proof of the result is based on three main steps. In a first step, we transform the eigenvalue problem of the cracked beam in an equivalent eigenvalue problem for a simply-supported beam carrying a point mass $m=\frac{1}{K}$ at the cracked cross-section of abscissa s, where K is the stiffness of the linearly elastic rotational spring modelling the crack (see Proposition 3.1 in Section 3). Hence, the crack detection problem is transformed into the equivalent problem of determining the location s and the magnitude m of the point mass from the first two natural frequencies of the beam. In the second step, we study the $\lambda-m$ and $\lambda-s$ curves, that is the functions $\lambda=\lambda(s,\cdot)$ and $\lambda=\lambda(\cdot,m)$, for fixed s and fixed m, respectively, where λ is the first and the second eigenvalue (Section 4). This analysis is based on the determination of the explicit expression of the eigenvalue first partial derivatives with respect to the parameter s and m (Proposition 4.1), and on specific properties of the $\lambda-m$ and $\lambda-s$ curves of the cracked beam (Propositions 4.2 and 4.3). These properties are used in the last step to prove the main result (Section 5). The proof is constructive and leads to an identification method, called λ-Curves Method, which is alternative, although equivalent, to the Frequency Equation Method. It should be noted that the proof relies on a sharp lower bound for the second eigenvalue of the cracked beam (Proposition 3.5). Such a bound plays an important role in our treatment and follows from a careful analysis of the frequency equation of the cracked beam.

Once the existence and uniqueness of the solution to the inverse problem are proved, a specific crack identification problem can be addressed by using either the Frequency Equation Method or the λ-Curves Method. Our experience shows that, although the use of the λ-Curves Method was necessary in the proof of the main theorem, the numerical implementation of the Frequency Equation Method is less onerous. An extensive series of numerical simulations with various positions and severities of the crack support the theory (Section 6).

Finally, for the sake of completeness, we recall that several contributions on the crack identification problem considered in this paper are available. Apart from variational techniques, see, for example, [14–21], the approach generally adopted consists in solving numerically the nonlinear system of the frequency equations written for the first two natural frequencies. We refer, among others, to the studies carried out in [22–26]. All the known results support the conjecture that the inverse problem has positive answer. However, at the best of our knowledge, a rigorous proof of this general property was not available, as the conclusions of the above studies were drawn either on the basis of numerical analysis of specific cases, or on the study of particular experimental situation.

2. Formulation of the inverse problem and main result

Let us consider a uniform simply supported beam under bending vibration with a crack at the cross-section of abscissa z_d, with $0 < z_d < L$, where L is the length of the beam. The crack is assumed to remain open during vibration and is modelled as a rotational linearly elastic spring of stiffness $\widehat{K}$ located at the cracked cross-section. We refer, among other contributions, to [8] for a justification of the localized flexibility model of crack based on Linear Fracture Mechanics arguments, and to [27] for an alternative derivation. The value of $\widehat{K}$ depends on the geometry of the cracked cross-section and on the material properties of the beam. A specific expression for $\widehat{K}$ is considered in Section 6 in the case of rectangular cross-section.

The free undamped bending vibrations of the beam, with radian frequency ω and spatial amplitude $u = u(x)$ $\left(x = \frac{z}{L}\right)$, are governed by the following eigenvalue problem (written in dimensionless form)

$$\begin{cases} u^{IV} - \lambda^4 u = 0, \quad \text{in } (0,s) \cup (s,1), & (1)\\ u(0) = u''(0) = 0, & (2)\\ [\![u(s)]\!] = [\![u''(s)]\!] = [\![u'''(s)]\!] = 0, & (3)\\ K[\![u'(s)]\!] = u''(s), & (4)\\ u(1) = u''(1) = 0, & (5)\end{cases}$$

where $s = \frac{z_d}{L}$, $s \in (0,1)$, $K = \frac{\widehat{K}L}{EI}$, $K \in (0,\infty)$, $\lambda^4 = \omega^2 \frac{\gamma A L^4}{EI}$. Hereinafter, the notation $(\cdot)' = \frac{d(\cdot)}{dx}$ indicates x-differentiation, and the jump of the function $\varphi = \varphi(x)$ at $x = s$ is defined as $[\![\varphi(s)]\!] = (\lim_{x \to s^+} \varphi(x) - \lim_{x \to s^-} \varphi(x))$. In the above equations, E is Young's modulus of the material; γ is the volume mass density; A is the cross-sectional area; I is the second moment of area about the axis through the centroid of the cross-section, at right angles to the plane of vibration. All the above mechanical parameters are assumed to be constant.

The main properties of the eigenpairs of (1)–(5) are recalled in the next proposition.

Proposition 2.1. *Under the above assumptions*:

1. *there exists a numerable sequence of real positive eigenvalues* $\{\lambda_n^4\}_{n=1}^{\infty}$ *of* (1)–(5), *with* $\lim_{n\to\infty} \lambda_n^4 = \infty$. *In particular, denoting by* λ *the fourth root of* λ^4, *the eigenvalues are the* (*positive*) *roots of the frequency equation*

$$\mathcal{P}_K(\lambda, s, K) = 4K \sin\lambda \sinh\lambda + \lambda\big(\sin\lambda(\cosh\lambda - \cosh(\lambda(1-2s)) + \sinh\lambda(\cos\lambda - \cos(\lambda(1-2s))\big) = 0. \qquad (6)$$

2. *The eigenvalues* $\{\lambda_n^4\}_{n=1}^{\infty}$ *of* (1)–(5) *are all simple, e.g.*,

$$0 < \lambda_1^4 < \lambda_2^4 < \cdots < \lambda_n^4 < \cdots, \qquad (7)$$

   and to every eigenvalue λ_n^4 *we can associate a unique eigenfunction* (*e.g., if* u_n *and* $\tilde{u}_n$ *are eigenfunctions associated to the same eigenvalue* λ_n^4, *then* $u_n(x) = C\tilde{u}_n(x)$ *in* $[0,1]$, *where* C *is a no-vanishing constant*). *Every eigenfunction* u_n *is a continuous function in* $[0,1]$ *and belongs to* $C^\infty((0,s) \cup (s,1))$.
3. *The set of eigenfunctions* $\{u_n(x)\}_{n=1}^{\infty}$ *is an orthonormal basis of the space of continuous functions on* $[0,1]$, *vanishing at* $x=0$ *and* $x=1$, *with respect to the usual scalar product* $\langle f, g\rangle = \int_0^1 fg$.
4. *The nth eigenfunction* $u_n(x)$ *has exactly* $(n-1)$ *simple zeros in* $(0,1)$, $n \geq 1$.

Properties 1 and 3 follow from general results for self-adjoint compact operators in Hilbert spaces, see, for example, [28]. Properties 2 and 4 follow from the oscillatory character of the statical Green's function associated to (1)–(5); see, for example, [3].

We are now in position to state the main result of this paper.

Theorem 2.2. *The measurement of the first two natural frequencies of* (1)–(5) *allows for the unique determination of the severity of the damage* K *and the location* s *of the crack, up to the symmetric position* $(1-s)$. *The identification procedure is constructive.*

The rest of the paper is devoted to the proof of Theorem 2.2. As we premised in the Introduction, the proof consists of several steps. In the first step (Section 3), the inverse problem of detecting a single crack is rephrased as the equivalent problem of detecting a point mass m in a simply-supported beam from the first two natural frequencies. It can be shown that this alternative formulation simplifies the study of the dependence of an eigenvalue on the unknown parameters, see Section 4. Finally, in the third and last step (Section 5) the previous results are used to prove the main theorem by means of a constructive procedure.

3. An equivalent eigenvalue problem

The equivalence is stated in the next proposition.

Proposition 3.1.

(i) *Let* (λ^4, u) *be an eigenpair of* (1)–(5). *Then* λ^4 *is an eigenvalue of the problem*

$$\begin{cases} v^{IV} - \lambda^4 v = 0, \quad \text{in } (0,s) \cup (s,1), & (8) \\ v(0) = v''(0) = 0, & (9) \\ [\![v(s)]\!] = [\![v'(s)]\!] = [\![v''(s)]\!] = 0, & (10) \\ [\![v'''(s)]\!] = \lambda^4 m v(s), & (11) \\ v(1) = v''(1) = 0, & (12) \end{cases}$$

where

$$v = -u'' \quad \text{in } (0,s) \cup (s,1), \quad m = K^{-1}. \tag{13}$$

(ii) *Conversely, let* (λ^4, v) *be an eigenpair of* (8)–(12). *Then* λ^4 *is an eigenvalue of the problem* (1)–(5) *with*

$$u = -v'' \quad \text{in } (0,s) \cup (s,1), \quad K = m^{-1}. \tag{14}$$

Proposition 3.1 can be proved by direct calculation.

It should be noted that problem (8)–(12) describes the free bending vibration of a simply supported uniform beam carrying a point mass of intensity $m = \frac{1}{K}$ placed at the cross-section of abscissa $x = s$.

Based on the equivalence between the eigenvalue problems (1)–(5) and (8)–(12), in the following we shall mainly focus on the formulation of the inverse problem in terms of the vibration of the beam with a point mass. Therefore, Theorem 2.2 can be rephrased as follows.

Theorem 3.2. *The measurement of the first two natural frequencies of* (8)–(12) *allows for the unique determination of the intensity m and the location s of the point mass, up to the symmetric position* $(1-s)$. *The identification procedure is constructive.*

The next proposition states some properties of the zeros of the first and the second eigenfunction of (8)–(12), which will be useful in our analysis.

Proposition 3.3. *Let* $v_n(x)$ *be the nth eigenfunction of* (8)–(12). *Then*:

1. $v_1(x)$ *has no zeros in* $(0,1)$ *and* $v_1'(x)$ *has exactly one simple zero in* $(0,1)$.
2. $v_2(x)$ *has exactly one simple zero in* $(0,1)$, *say at* ξ_1, *and* $v_2'(x)$ *has exactly two simple zeros in* $(0,1)$, *say at* η_1, η_2, *with* $\eta_1 < \eta_2$. *Moreover,* $\eta_1 < \xi_1 < \eta_2$.

Proposition 3.3 follows from the property of the zeros of the eigenfunctions of (1)–(5) (Proposition 2.1, point iv) and the definition of v given in (13).

We conclude this section by recalling the weak and variational formulation of the eigenvalue problem (8)–(12). In this respect, it should be noted that the functional space suitable for (8)–(12) is made by functions more regular than those occurring in (1)–(5), e.g., the jump of the first derivative v' at $x = s$ is not allowed, whereas the corresponding u' may be discontinuous at the crack location. It is possible to show that this additional regularity simplifies the study of the $\lambda - s$ and $\lambda - m$ curves, see Section 4. The functional space $\mathcal{H}$ of admissible deformations of the beam for (8)–(12) is

$$\mathcal{H} = \{ f | f \in H^2(0,1), \ f(0) = f(1) = 0 \}, \tag{15}$$

where $H^2(0,1)$ is the Hilbert space of Lebesgue measurable functions $f : (0,1) \to \mathbb{R}$ such that f, and its first and second weak derivatives are square integrable in $(0,1)$, e.g., $\int_0^1 (f^2 + (f')^2 + (f'')^2) < +\infty$.

The weak formulation of (8)–(12) consists in finding $v \in \mathcal{H} \setminus \{0\}$ such that

$$\int_0^1 v'' \varphi'' = \lambda^4 \left(m v(s) \varphi(s) + \int_0^1 v \varphi \right), \quad \text{for every } \varphi \in \mathcal{H}. \tag{16}$$

Rayleigh's Quotient associated to (16) is

$$R[\cdot] : \mathcal{H} \setminus \{0\} \to \mathbb{R}, \quad R[\varphi] = \frac{\int_0^1 (\varphi'')^2}{m \varphi^2(s) + \int_0^1 \varphi^2} \tag{17}$$

and the eigenvalues can be determined by solving the chain of minimum problems

$$R[v_n] = \min_{\varphi \in V_n \setminus \{0\}} R[\varphi] = \lambda_n^4, \tag{18}$$

where

$$V_n = \left\{ \varphi \in \mathcal{H} | m v_i(s)\varphi(s) + \int_0^1 v_i \varphi = 0, \quad i = 1, \ldots, n-1 \right\}. \tag{19}$$

An equivalent formulation follows from the Maximum–Minimum Principle for the eigenvalues of (8)–(12), namely

$$\lambda_n^4 = \max_{l_i \in \mathcal{H}',\ i=1,\ldots,n-1} \left\{ \min_{\varphi \in \mathcal{H}\setminus\{0\},\ l_i(\varphi)=0,\ i=1,\ldots,n-1} R[\varphi] \right\}, \tag{20}$$

where $\mathcal{H}'$ is the dual space of $\mathcal{H}$, that is the space of all the linear and continuous real-valued functionals l_i on $\mathcal{H}$. We refer to [29] for a complete account of the above formulations.

In the next sections we shall compare the eigenpairs of the problem (8)–(12) for finite, no-vanishing m, and $s \in (0,1)$, to those obtained by taking either $m=0$ or $\{s=0, s=1\}$ in (8)–(12). We shall denote by $((\lambda_n^U)^4, v_n^U)$ the nth eigenpair of the *unperturbed* (or undamaged) eigenvalue problem

$$\begin{cases} (v^U)^{IV} - (\lambda^U)^4 v^U = 0, \quad x \in (0,1), & (21) \\ v^U(0) = (v^U)''(0) = 0, & (22) \\ v^U(1) = (v^U)''(1) = 0. & (23) \end{cases}$$

Clearly, Proposition 2.1 continues to hold for $m=0$ (or, equivalently, $K=\infty$), and weak, variational and Maximum–Minimum formulations of (21)–(23) can be deduced by the corresponding formulations (16), (17)–(19) and (20), respectively, by taking formally $m=0$.

Variational and Maximum–Minimum formulations are useful to derive bounds of the perturbed eigenvalues λ_n^4 in terms of unperturbed eigenvalues $(\lambda_n^U)^4$, see [30]. Restricting the attention to the first two eigenvalues, we have the following well-known result.

Proposition 3.4. *For every $s \in (0,1)$ and every $m \in (0,\infty)$, we have*

$$(\lambda_{n-1}^U)^4 \le \lambda_n^4(s,m) \le (\lambda_n^U)^4, \tag{24}$$

where $\lambda_0^U = 0$ and $\lambda_n^U = n\pi$, $n=1,2$.

The right inequality in (24) shows that the addition of a mass m cannot increase, and generally decreases the natural frequencies of the unperturbed beam.

There is another inequality that will play an important role in our analysis: it is the following sharp lower bound of the second eigenvalue λ_2^4.

Proposition 3.5. *For every $s \in (0,1)$ and every $m \in (0,\infty)$, we have*

$$(\lambda_2^*)^4 < \lambda_2^4(s,m), \tag{25}$$

where $\lambda_2^* \simeq 3.9266$ is the first positive zero of the equation

$$\tan x = \tanh x. \tag{26}$$

Inequality (26) has been deduced from a careful analysis of the zeros of the frequency equation (6) of (8)–(12). The proof of Proposition 3.5 is rather cumbersome, and it is deferred in the Appendix.

4. Eigenvalue derivatives, $\lambda - m$ and $\lambda - s$ curves

In order to extract quantitative information on the perturbation parameters s and m from the eigenvalues, we found useful to introduce the so-called $\lambda - m$ and $\lambda - s$ curves, that is, the functions $\lambda_n^4 = \lambda_n^4(s,\cdot)$ ($\lambda - m$ curve), $\lambda_n^4 = \lambda_n^4(\cdot, m)$ ($\lambda - s$ curve), $n=1,2$. The constructive proof of the main theorem is based on properties of these λ-curves.

An eigenpair (λ^4, v) of (8)–(12) depends on the perturbation parameters s and m. When necessary, we shall explicitly write this dependence as $\lambda^4 = \lambda^4(s,m)$ and $v = v(x; s, m)$, where $x \in [0,1]$ is the current variable.

Proposition 4.1. *Let (λ^4, v) be an eigenpair of (8)–(12). The eigenvalue is a continuous differentiable function of s and m, and its first-order partial derivatives are*

$$\frac{\partial \lambda^4}{\partial s} = -2\lambda^4 \frac{m v(s) v'(s)}{m v^2(s) + \int_0^1 v^2}, \tag{27}$$

$$\frac{\partial\lambda^4}{\partial m} = -\lambda^4 \frac{v^2(s)}{mv^2(s)+\int_0^1 v^2}, \tag{28}$$

where $v(s) = v(x; s, m)|_{x=s}$, $v'(s) = \left(\frac{dv(x;s,m)}{dx}\right)|_{x=s}$.

A proof of Proposition 4.1 can be obtained along the lines of the proof of the analogous expressions obtained for the case of axially vibrating rods, see [31].

The dependence of the eigenvalue on the parameter m, for a given position s of the point mass, is considered in the next proposition, whose proof can be obtained by adapting a method shown in [31].

Proposition 4.2. *Let* (λ_n^4, v_n), $((\lambda_n^U)^4, v_n^U)$ *be the nth eigenpair of the problem* (8)–(12), (21)–(23), *respectively*, $n=1,2$.

(i) *If* $v_n^U(s_0) = 0$ *for some* $s_0 \in [0,1]$, *then* $\lambda_n^4(s_0, m) = (\lambda_n^U)^4$ *for every finite positive m.*
(ii) *If* $v_n^U(s_0) \neq 0$ *for some* $s_0 \in (0,1)$, *then* $\lambda_n^4 = \lambda_n^4(s_0, m)$ *is a monotonically decreasing function of m in* $[0,\infty)$.
(iii) *If* $\lambda_n^4(s_0, m_0) = (\lambda_n^U)^4$ *for some* $s_0 \in [0,1]$ *and* $m_0 \in (0,\infty)$, *then* $v_n^U(s_0) = 0$.
(iv) *If* $v_n(s_0; s_0, m_0) = 0$ *for some* $s_0 \in [0,1]$ *and* $m_0 \in (0,\infty)$, *then* $v_n^U(s_0) = 0$.

We now pass to the study of the $\lambda - s$ curves. For a simply-supported beam with constant coefficients it is easy to check that the following symmetry property holds true.

Proposition 4.3. *Let* (λ_n^4, v_n) *be the nth eigenpair of* (8)–(12), $n \geq 1$. *Let m be given*, $0 < m < \infty$. *Then*

$$\lambda_n^4(s) = \lambda_n^4(1-s), \quad s \in [0,1]. \tag{29}$$

If $s = \frac{1}{2}$, *then*

$$\text{for } n \text{ odd, we have } v_n(x) = v_n(1-x), \tag{30}$$

and

$$\text{for } n \text{ even, we have } v_n(x) = -v_n(1-x), \quad x \in [0,1]. \tag{31}$$

Corollary 4.4. *Under the assumptions of Proposition 4.3, for every* $n \geq 1$, *we have*

$$\frac{\partial\lambda_n^4}{\partial s}(s) = 0 \quad \text{for } s \in \left\{0, 1, \frac{1}{2}\right\}. \tag{32}$$

The proof of our main theorem and the corresponding constructive diagnostic method are based on the following key result.

Theorem 4.5. *Let* (λ_n^4, v_n), $n = 1,2$, *be the nth eigenpair of* (8)–(12). *Then, for every* $m > 0$ *we have*:

(i) $\lambda_1^4 = \lambda_1^4(s)$ *is a strictly increasing function in* $\left(\frac{1}{2}, 1\right)$;
(ii) *there exists a unique* $\tilde{s} \in \left(\frac{1}{2}, 1\right)$ *such that* $\frac{\partial\lambda_2^4}{\partial s}(\tilde{s}) = 0$, *that is* $\lambda_2^4 = \lambda_2^4(s)$ *is a strictly decreasing function and a strictly increasing function in* $\left(\frac{1}{2}, \tilde{s}\right)$ *and in* $(\tilde{s}, 1)$, *respectively*.

In the proof of Theorem 4.5 we shall use the following *Deformation Lemma*.

Lemma 4.6. *Let* $f_t = f_t(x)$ *be a t-family of real-valued functions of* $x \in [0,1]$ *which are continuous and jointly continuously differentiable in x and in t, where the parameter t belongs to the interval* $[t_1, t_2]$, $-\infty < t_1 < t_2 < +\infty$. *Suppose that for every* $t \in [t_1, t_2]$, *the function* f_t *has a finite number of zeros in* $[0,1]$, *all of which are simple, and has boundary values at* $x=0$ *and* $x=1$ *that are independent of t. Then, the number of zeros of* f_t *is independent of t, for all t satisfying* $t_1 \leq t \leq t_2$.

For a proof of Lemma 4.6 we refer to [32, p. 41]

Proof of Theorem 4.5. A direct calculation shows that, up to a non-vanishing multiplicative factor, the function $v_n'(s) = \left(\frac{dv_n(x;s,m)}{dx}\right)|_{x=s}$, $s \in [0,1]$, takes the following closed form expression:

$$v_n'(s) = \frac{\cos(\lambda_n s)\sin(\lambda_n(1-s))\sinh\lambda_n - \cosh(\lambda_n s)\sin((\lambda_n(1-s))\sin\lambda_n}{\sin((\lambda_n(1-s))\sinh\lambda_n - \sinh(\lambda_n(1-s))\sin\lambda_n}, \tag{33}$$

$n \geq 1$. In order to study the zeros of $v_n' = v_n'(s)$ in $[0,1]$, we found convenient to introduce the change of variable

$$\xi: [0,1] \to \left[-\tfrac{1}{2}, \tfrac{1}{2}\right], \quad \xi(s) = s - \tfrac{1}{2}. \tag{34}$$

Dividing the expression (34) by sinh λ_n and using (34), after a rearrangement of the terms, it can be shown that $v_n'(\xi) = 0$ if and only if the following expression vanishes:

$$f_{\lambda_n}(\xi) = \sin(2\lambda_n\xi) - \frac{\sinh(2\lambda_n\xi)}{\sinh\lambda_n}\sin\lambda_n. \tag{35}$$

Recalling that for $s \in \left(\frac{1}{2}, 1\right)$ we have $\xi \in \left(0, \frac{1}{2}\right)$, and that $\lambda_n \in [\lambda_{n-1}^U = (n-1)\pi, \lambda_n^U = n\pi]$, with $\lambda_0^U = 0$, the domain of definition of the function $f_{\lambda_n}(\xi)$ is the rectangle

$$Q_n = \left[0, \tfrac{1}{2}\right] \times \left[\lambda_{n-1}^U, \lambda_n^U\right], \quad n \geq 1. \tag{36}$$

Proof of Claim (i). By the expression (27) of $\frac{\partial\lambda_1^4}{\partial s}$ and recalling that v_1 does not vanish inside (0, 1) (see Proposition 3.3, point 1), it is enough to prove that the function $f_{\lambda_1}(\xi)$ does not vanish inside Q_1. By the definition of $f_{\lambda_1}(\xi)$ and observing that $0 < 2\lambda_1\xi < \pi$, we have

$$f_{\lambda_1}(0) = 0, \quad f_{\lambda_1}\left(\tfrac{1}{2}\right) = 0, \quad \text{for every } \lambda_1 \in (0, \pi), \tag{37}$$

$$\lim_{\lambda_1 \to 0^+} f_{\lambda_1}(\xi) = 0, \quad \lim_{\lambda_1 \to \pi^-} f_{\lambda_1}(\xi) = \sin(2\pi\xi), \quad \text{for every } \xi \in \left(0, \tfrac{1}{2}\right). \tag{38}$$

The function $f_{\lambda_1}(\xi)$ is continuous and jointly continuous differentiable with respect to ξ and λ_1 in Q_1, and it has vanishing boundary values at $\xi = 0$ and $\xi = \frac{1}{2}$. We prove that for every $\lambda_1 \in (0, \pi]$ all the zeros of $f_{\lambda_1}(\xi)$ in $\left[0, \frac{1}{2}\right]$, if any, are simple zeros. The proof is by contradiction. Suppose that for a given $\lambda_1 = \overline{\lambda} \in (0, \pi]$ the function $f_{\overline{\lambda}}(\xi)$ has a zero at $\overline{\xi}$, $\overline{\xi} \in \left[0, \frac{1}{2}\right]$, with multiplicity greater than one. Then, necessary conditions are

$$\begin{cases} f_{\overline{\lambda}}(\overline{\xi}) = 0, & (39) \\ \dfrac{df_{\overline{\lambda}}}{d\xi}\left(\overline{\xi}\right) = 0, & (40) \end{cases}$$

or equivalently, by (35),

$$\begin{cases} \sin\left(2\overline{\lambda\xi}\right) = \dfrac{\sinh(2\overline{\lambda\xi})}{\sinh\overline{\lambda}}\sin\overline{\lambda} & (41) \\ \cos\left(2\overline{\lambda\xi}\right) = \dfrac{\cosh(2\overline{\lambda\xi})}{\sinh\overline{\lambda}}\sin\overline{\lambda}. & (42) \end{cases}$$

The above two equations imply

$$\tan(2\overline{\lambda\xi}) = \tanh(2\overline{\lambda\xi}), \tag{43}$$

that is

$$2\overline{\lambda\xi} \quad \text{is a positive root of } \tan x = \tanh x. \tag{44}$$

The first positive root of the equation (44) is $\lambda_2^* \simeq 3.9266 \in \left(\pi, \frac{3}{2}\pi\right)$, and then, since $2\overline{\lambda\xi} \in (0, \pi)$ by assumption, we get a contradiction.

We now prove that for every $\lambda_1 \in (0, \pi]$ the function $f_{\lambda_1}(\xi)$ has a finite number of zeros in $\left[0, \frac{1}{2}\right]$. Let us assume that there exists $\tilde{\lambda} \in (0, \pi]$ such that $f_{\tilde{\lambda}}(\xi)$ has an infinite family of zeros $\{z_i\}_{i=1}^{\infty}$ in [0, 1]. Then, by the regularity of $f_{\tilde{\lambda}}(\xi)$, for every $i \geq 2$ there exists ζ_i, $z_{i-1} \leq \zeta_i \leq z_i$, such that $f'_{\tilde{\lambda}}(\zeta_i) = 0$. By Bolzano–Weierstass Theorem, there exists $\overline{z} \in \left[0, \frac{1}{2}\right]$ such that $\lim_{i\to\infty} z_i = \overline{z}$ and $f_{\tilde{\lambda}}(\overline{z}) = 0$. Moreover, we also have $\lim_{i\to\infty}\zeta_i = \overline{z}$ and, by continuity, $f'_{\tilde{\lambda}}(\overline{\zeta}) = 0$. Therefore, $\overline{z}$ is not a simple zero of $f_{\tilde{\lambda}}(\xi)$, a contradiction.

From the above analysis it follows that $f_{\lambda_1}(\xi)$ satisfies all the conditions required by Lemma 4.6 to the function $f_t(x)$, where here the variables t and x are replaced by λ_1 and ξ, respectively. It follows that, for every $\lambda_1 \in (0, \pi]$, the function $f_{\lambda_1}(\xi)$ has the same numbers of zeros of $f_\pi(\xi)$ in $\left[0, \frac{1}{2}\right]$. Since $f_\pi(\xi) = \sin(2\pi\xi) > 0$ for $\xi \in \left[0, \frac{1}{2}\right]$, the function $f_{\lambda_1}(\xi)$ does not vanish in $\left(0, \frac{1}{2}\right) \times (0, \pi)$, and the proof is complete.

Proof of Claim (ii). By Propositions 3.4 and 3.5, we have $\lambda_2^* < \lambda_2(\xi) \leq \lambda_2^U = 2\pi$ for every $\xi \in \left[0, \frac{1}{2}\right]$. Moreover, by Proposition 4.2, we also have $\lambda_2(\xi = 0) = \lambda_2\left(\xi = \frac{1}{2}\right) = \lambda_2^U = 2\pi$. Then, there exists $\tilde{\xi} \in \left(0, \frac{1}{2}\right)$ such that $\frac{\partial\lambda_2^4}{\partial\xi}\left(\tilde{\xi}\right) = 0$. We need to prove that such $\tilde{\xi}$ is unique in the interval $\left(0, \frac{1}{2}\right)$.

By expression (27), the first partial derivative of λ_2^4 vanishes at $\tilde{\xi} \in \left(0, \frac{1}{2}\right)$ if and only if $v'_2(\tilde{\xi}) = 0$ vanishes (here v_2 is expressed in the new variable ξ). In fact, if $v_2(\tilde{\xi}) = 0$ for certain $\tilde{\xi} \in \left(0, \frac{1}{2}\right)$, then, by Proposition 4.2 (point iv), the unperturbed eigenfunction v_2^U must vanish at $\tilde{\xi}$. But v_2^U is proportional to $\sin(2\pi\xi)$ and, therefore, it vanishes only at $\xi = \frac{1}{2}$, which not belongs to the interval $\left(0, \frac{1}{2}\right)$, a contradiction. Therefore, recalling the analysis developed in previous step (i), to prove the statement (ii) it is enough to prove that, for every $\lambda_2 \in (\lambda_2^*, \lambda_2^U)$, the function $f_{\lambda_2}(\xi)$ has only one (simple) zero inside $\left[0, \frac{1}{2}\right]$.

The proof of this property follows the lines of the proof of case (i). Recalling the definition (35) (with $n=2$) of $f_{\lambda_2}(\xi)$ and noticing that $0<2\lambda_2\xi<2\pi$, we have

$$f_{\lambda_2}(0)=0,\quad f_{\lambda_2}\left(\frac{1}{2}\right)=0,\quad \text{for every } \lambda_2\in(\lambda_2^*,2\pi],\tag{45}$$

$$\lim_{\lambda_2\to\lambda_2^{*+}} f_{\lambda_2}(\xi)=f_{\lambda_2^*}(\xi),\qquad \lim_{\lambda_2\to 2\pi^-} f_{\lambda_2}(\xi)=\sin(4\pi\xi).\tag{46}$$

Based on the proof of step (i), to prove (ii) it is enough to show that, for every $\lambda_2\in(\lambda_2^*,2\pi]$, all the zeros of the function $f_{\lambda_2}(\xi)$ in $\left[0,\frac{1}{2}\right]$ are simple.

We proceed by contradiction. Assume that for certain $\lambda_2=\overline{\lambda}\in(\lambda_2^*,2\pi]$ the function $f_{\overline{\lambda}}(\xi)$ has a not simple zero at $\xi=\overline{\xi}\in\left[0,\frac{1}{2}\right]$. Then, we have

$$\begin{cases}\sin\left(2\overline{\lambda\xi}\right)=\dfrac{\sinh(2\overline{\lambda\xi})}{\sinh\overline{\lambda}}\sin\overline{\lambda} & (47)\\[2ex] \cos\left(2\overline{\lambda\xi}\right)=\dfrac{\cosh(2\overline{\lambda\xi})}{\sinh\overline{\lambda}}\sin\overline{\lambda}, & (48)\end{cases}$$

and these conditions imply

$$\tan(2\overline{\lambda\xi})=\tanh(2\overline{\lambda\xi}),\tag{49}$$

that is

$$2\overline{\lambda\xi}=x_1,x_2,x_3,\ldots,\tag{50}$$

where $\{x_i\}$ are the positive roots of $\tan x=\tanh x$. Since $2\overline{\lambda\xi}\in(0,2\pi)$, $x_1=\lambda_2^*\simeq 3.9266$ and $x_2>2\pi$, if $\overline{\xi}\in\left[0,\frac{\lambda_2^*}{4\pi}\simeq 0.31\right]$ then we obtain a contradiction, that is, all the zeros of $f_{\lambda_2}(\xi)$ are simple zeros whenever $\xi\in\left[0,\frac{\lambda_2^*}{4\pi}\right]$ and $\lambda_2\in(\lambda_2^*,2\pi]$.

Let us put $2\overline{\lambda\xi}=\lambda_2^*$ in Eq. (47) (note that (48) coincides with (47) as $2\overline{\lambda\xi}=\lambda_2^*$). We obtain

$$\sin\lambda_2^*=\frac{\sinh\lambda_2^*}{\sinh\overline{\lambda}}\sin\overline{\lambda},\quad \text{for every } \overline{\lambda}\in(\lambda_2^*,2\pi].\tag{51}$$

Let $\tilde{\lambda}_2^*$ be the symmetric point of λ_2^* with respect to $\frac{3}{2}\pi$, e.g., $\tilde{\lambda}_2^*=3\pi-\lambda_2^*$. If $\overline{\lambda}\in(\tilde{\lambda}_2^*,2\pi]$, then $\frac{\sin\lambda_2^*}{\sin\overline{\lambda}}>1$, whereas $\frac{\sinh\lambda_2^*}{\sinh\overline{\lambda}}<1$. By (51), we have the contradiction, and then $f_{\lambda_2}(\xi)$ has only simple zeros in $\left[0,\frac{1}{2}\right]$ for every $\lambda_2\in(\tilde{\lambda}_2^*,2\pi]$.

It remains to prove that all the zeros of $f_{\lambda_2}(\xi)$ are simple for $\xi\in\left[\frac{\lambda_2^*}{4\pi},\frac{1}{2}\right]$ and $\lambda_2\in(\lambda_2^*,\tilde{\lambda}_2^*]$. We start by showing that

$$\frac{\sin\lambda_2^*}{\sin\overline{\lambda}}>\frac{\sinh\lambda_2^*}{\sinh\overline{\lambda}},\quad \text{for every } \overline{\lambda}\in\left(\lambda_2^*,\tilde{\lambda}_2^*\right].\tag{52}$$

We rewrite (52) as

$$y_1(\overline{\lambda})<y_2(\overline{\lambda}),\quad \text{for every } \overline{\lambda}\in(\lambda_2^*,\tilde{\lambda}_2^*],\tag{53}$$

where

$$y_1\left(\overline{\lambda}\right)=\frac{\sin\overline{\lambda}}{\sin\lambda_2^*},\quad y_2\left(\overline{\lambda}\right)=\frac{\sinh\overline{\lambda}}{\sinh\lambda_2^*}.\tag{54}$$

One can easily check that the two curves $(\overline{\lambda},y_1(\overline{\lambda}))$, $(\overline{\lambda},y_2(\overline{\lambda}))$ are tangent at the point $(\overline{\lambda}=\lambda_2^*,y_1(\lambda_2^*)=y_2(\lambda_2^*)=1)$, with tangent line of equation

$$y_t\left(\overline{\lambda}\right)=1+\frac{1}{\tanh\lambda_2^*}\left(\overline{\lambda}-\lambda_2^*\right)=1+\frac{1}{\tan\lambda_2^*}\left(\overline{\lambda}-\lambda_2^*\right).$$

Moreover, it is easy to see that $y_2(\overline{\lambda})\ge y_t(\overline{\lambda})$ and $y_1(\overline{\lambda})\le y_t(\overline{\lambda})$ for every $\overline{\lambda}\in(\lambda_2^*,\tilde{\lambda}_2^*]$, and that the only point in common between y_1, y_2 and y_t is $(\lambda_2^*,1)$. Then, (53) holds and (52) is satisfied.

Therefore, by (51) and (52) we get a contradiction.

In conclusion, by the above analysis, for every $\lambda_2\in(\lambda_2^*,2\pi]$, the function $f_{\lambda_2}(\xi)$ has a finite number of zeros in $\left[0,\frac{1}{2}\right]$, all of which are simple. From Lemma 4.6 and recalling that $f_{2\pi}(\xi)=\sin(4\pi\xi)$ has only one simple zero at $\xi=1/4$, we deduce that $f_{\lambda_2}(\xi)$ has only one simple zero in $\left(0,\frac{1}{2}\right)$ for every $\lambda_2\in(\lambda_2^*,2\pi]$, and the proof of (ii) is complete.□

5. A constructive algorithm for unique crack identification and proof of the main result

In this section we prove Theorem 2.2. The proof is constructive and leads to an algorithm for the determination of the parameters (s,m) in the problem (8)–(12) (or, equivalently, the position s and the severity K of the crack in (1)–(5)).

Let us denote by $\overline{\lambda}_1^4$, $\overline{\lambda}_2^4$ the measured (e.g., experimental) values of the first two eigenvalues of the cracked beam.

By Propositions 3.4 and 3.5, input data $\overline{\lambda}_1^4$, $\overline{\lambda}_2^4$ are chosen such that

$$0<\overline{\lambda}_1^4<(\lambda_1^U)^4, \quad (\lambda_2^*)^4<\overline{\lambda}_2^4\le(\lambda_2^U)^4. \tag{55}$$

It should be noted that if $\overline{\lambda}_1^4=(\lambda_1^U)^4$, then, by Propositions 3.3 and 4.2, either $s=0$ or $s=1$. Then, avoiding these trivial cases, the strict upper bound on $\overline{\lambda}_1^4$ is assumed in (55).

If $\overline{\lambda}_2^4=(\lambda_2^U)^4$, then, by Propositions 4.2 and 4.3, the point mass is located at $s=\frac{1}{2}$. By Proposition 4.2, the function $\lambda_1^4=\lambda_1^4(\frac{1}{2},m)$ is a monotonically decreasing function of m and, in addition,

$$\lim_{m\to\infty}\lambda_1^4\left(\frac{1}{2},m\right)=0. \tag{56}$$

By (56) and by the monotonicity of the function $\lambda_1^4(\frac{1}{2},m)$ for $m\in(0,\infty)$, we can uniquely determine m by solving the equation $\overline{\lambda}_1^4=\lambda_1^4(\frac{1}{2},m)$.

Taking into account the above considerations, in the remaining of the section we shall consider $\overline{\lambda}_2^4<(\lambda_2^U)^4$ and, by symmetry (see Proposition 4.3), we assume $s\in(0,\frac{1}{2})$.

The main steps of the constructive algorithm are presented in the sequel.

We start by determining the values m_1^-, m_2^-, $0<m_i^-<\infty$, $i=1,2$, of the parameter m such that

$$\overline{\lambda}_1^4=\lambda_1^4(\tfrac{1}{2},m_1^-), \tag{57}$$

$$\overline{\lambda}_2^4=\lambda_2^4(s_{2\min},m_2^-), \tag{58}$$

where (by Theorem 4.5, part ii) $s_{2\min}\in(0,\frac{1}{2})$ is the unique point such that $\frac{\partial\lambda_2^4}{\partial s}(s_{2\min},m_2^-)=0$. Note that $m_1^-\ne m_2^-$ and, clearly, m_1^-, m_2^- are estimates from below of the unknown parameter m:

$$m_2^-\le m, \quad m_1^-<m. \tag{59}$$

We consider preliminarily the special case in which the curve $y=\lambda_1^4(s,m_2^-)$ and the straight line $y=\overline{\lambda}_1^4$ have a (unique) intersection point P with abscissa coinciding with $s_{2\min}$. In this case, obviously, we have $m=m_2^-$ and $s=s_{2\min}$. If the above condition is not satisfied, e.g., if either the intersection point does not exist or its abscissa is different from $s_{2\min}$, then

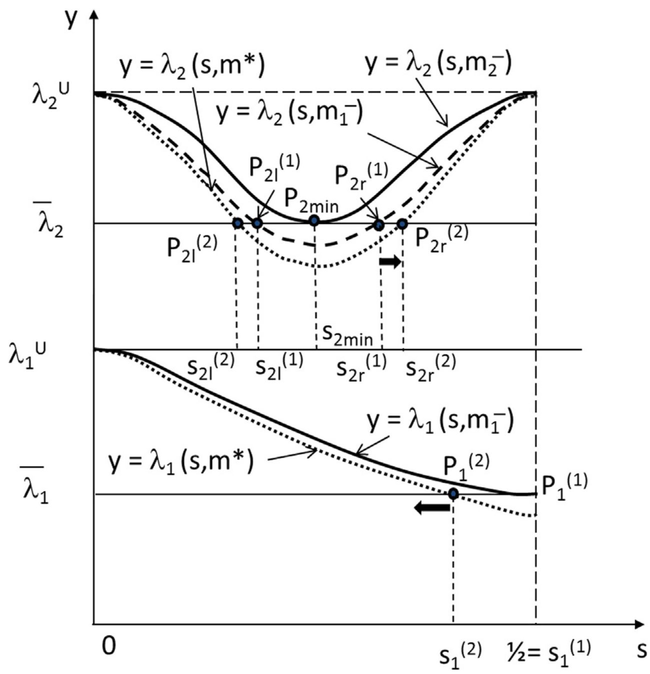

Fig. 1. The λ-curves identification algorithm based on first two resonant frequencies: first case.

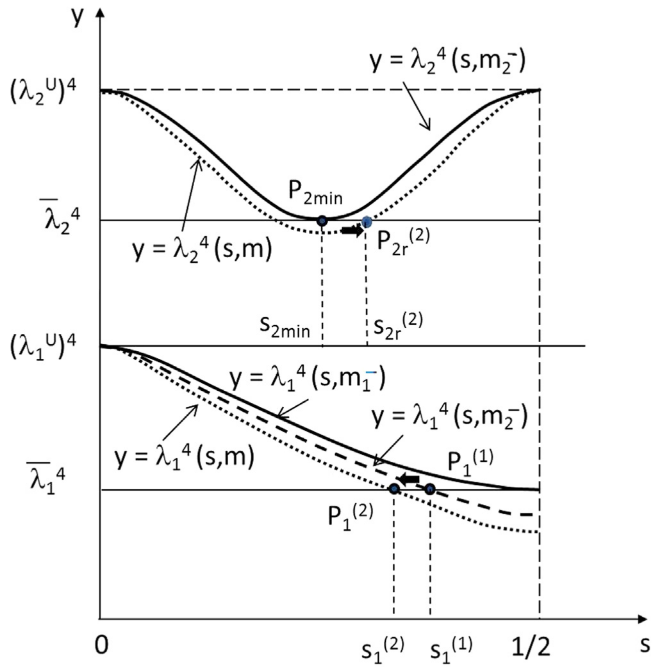

Fig. 2. The λ-curves identification algorithm based on first two resonant frequencies: second case, subcase (a).

$$\max\{m_1^-, m_2^-\} < m \tag{60}$$

and we need to distinguish two main cases.

First case:

$$\max\{m_1^-, m_2^-\} = m_1^-. \tag{61}$$

In a reference cartesian system (y,s), we determine the curve $y = \lambda_2^4(s, m_1^-)$ (see the dashed curve in Fig. 1). By Proposition 4.2 and Theorem 4.5 (point (ii)), the curve $y = \lambda_2^4(s, m_1^-)$ intersects the straight line $y = \overline{\lambda}_2^4$ exactly at two points, say $P_{2l}^{(1)}$, $P_{2r}^{(1)}$, located to the left ($P_{2l}^{(1)}$) and to the right ($P_{2r}^{(1)}$) of the point $P_{2\min} = (s_{2\min}, \overline{\lambda}_2^4)$. Let us denote by $s_{2l}^{(1)}$, $s_{2r}^{(1)}$ their abscissa, respectively. The curve $y = \lambda_1(s, m_1^-)$ is tangent at $P_1^{(1)} = \left(s_1^{(1)} = \frac{1}{2}, \overline{\lambda}_1^4\right)$ to the straight line $y = \overline{\lambda}_1^4$.

Suppose to increase continuously m from m_1^- to, say, $m^* > m_1^-$, with m^* not too large. We obtain two curves $y = \lambda_2^4(s, m^*)$, $y = \lambda_1^4(s, m^*)$ (see the dotted curves in Fig. 1). The abscissa of the intersection points $P_{2l}^{(2)}$, $P_{2r}^{(2)}$ between $y = \lambda_2^4(s, m^*)$ and $y = \overline{\lambda}_2^4$ is $s_{2l}^{(2)}$, $s_{2r}^{(2)}$, respectively. The point $P_{2r}^{(2)}$ moves to the right of $P_{2r}^{(1)}$, and $P_{2l}^{(2)}$ to the left of $P_{2l}^{(1)}$. The abscissa of the intersection point $P_1^{(2)}$ between $y = \lambda_1^4(s, m^*)$ and $y = \overline{\lambda}_1^4$ is $s_1^{(2)}$, and $P_1^{(2)}$ moves to the left of $P_1^{(1)}$. (Note that, since $s_{2r}^{(1)} < s_1^{(1)} = \frac{1}{2}$, by continuity, such a choice of m^* is always possible.) It follows that, for $m^* > m_1^-$ and m^* not too large, $s_{2r}^{(2)}$ and $s_1^{(2)}$ (with $s_{2r}^{(2)} < s_1^{(2)}$) move one toward each other. By increasing continuously m (from m_1^- to ∞, say), the intersection point P_1 between $y = \lambda_1^4(s, m)$ and the straight line $y = \overline{\lambda}_1^4$ moves from the right to the left, its abscissa $s_1 = s(P_1)$ is a monotonically decreasing function of m and, moreover, $\lim_{m \to \infty} s(P_1) = 0^+$. Simultaneously, the point P_{2r} obtained as the right intersection of $y = \lambda_2^4(s, m)$ and $y = \overline{\lambda}_2^4$ is such that $s_{2r} = s(P_{2r})$ is monotonically increasing as m increases. Then, we can conclude that there exists a unique value $\tilde{m}$ such that $s_{2r} = s_1$. The value $\tilde{m}$ is the intensity m of the point mass, and $s = s_1$ is its position.

Second case:

$$\max\{m_1^-, m_2^-\} = m_2^-. \tag{62}$$

In this case, we determine the curve $y = \lambda_1^4(s, m_2^-)$. This curve has only one point of intersection with the straight line $y = \overline{\lambda}_1^4$, say $P_1^{(1)}$, with abscissa $s_1^{(1)} = s(P_1^{(1)})$ such that $0 < s_1^{(1)} < \frac{1}{2}$. Here, we need to distinguish two subcases, indicated by (a) and (b) in what follows, depending on the relative position of $s_{2\min}$ and $s_1^{(1)}$.

Second case – (a):

$$s_{2\min} \le s_1^{(1)}. \tag{63}$$

The situation is illustrated in Fig. 2. If $s_{2\min} = s_1^{(1)}$, then the inverse problem is solved. If $s_{2\min} < s_1^{(1)}$, then we can repeat the

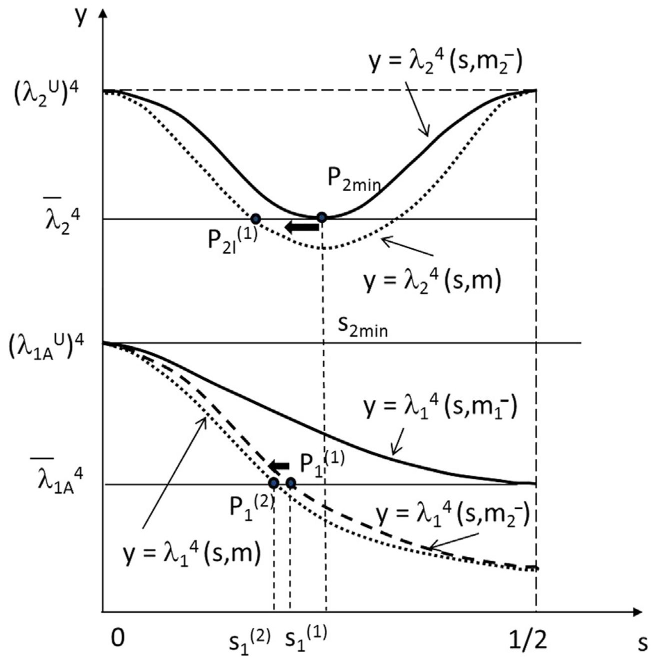

Fig. 3. The λ-curves identification algorithm based on first two resonant frequencies: second case, subcase (b).

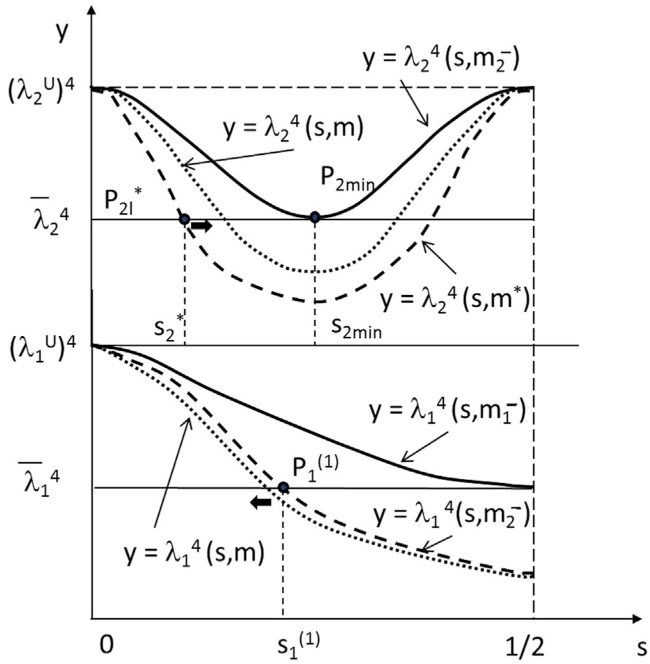

Fig. 4. The λ-curves identification algorithm based on first two resonant frequencies: new argument for second case, subcase (b).

procedure used in the First case. In brief, by increasing continuously m, with $m > m_2^-$, the points P_1 (intersection between $y = \lambda_1^4(s, m)$ and $y = \overline{\lambda}_1^4$) and P_{2r} (right intersection between $y = \lambda_2^4(s, m)$ and $y = \overline{\lambda}_2^4$) shown in Fig. 2 move toward each other. Since $\lim_{m \to \infty} s(P_1) = 0^+$ and $s(P_{2r})$ is increasing with respect to m, there exists exactly one value $\tilde{m}$ such that $s(P_1) = s(P_{2r})$, and the problem is solved.

Second case – (b):

$$s_{2\min} > s_1^{(1)}. \tag{64}$$

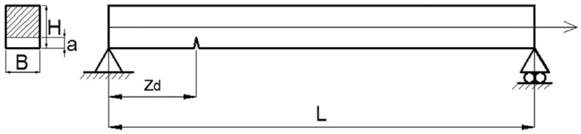

Fig. 5. Cracked simply supported uniform beam in bending.

Table 1
Application of the λ-curves constructive algorithm for the determination of crack severity K ($m=\frac{1}{K}$) and position s in a simply supported uniform beam using exact values of the first two natural frequencies. Crack ratio α=0.1, 0.2, 0.4, 0.6. Time=computing time (in seconds). Percentage errors: $e_s=100\times(s_{est}-s)/s$, $e_K=100\times(K_{est}-K)/K$.

Position s	α=0.1, K=94.2325 (m=0.01061)				α=0.2, K=24.9066 (m=0.04015)				α=0.4, K=5.51319 (m=0.18138)				α=0.6, K=1.52659 (m=0.65506)			
	e_s	e_K	Case	Time	e_s	e_K	Case	Time	e_s	e_K	Case	Time	e_s	e_K	Case	Time
0.03	−0.03	0.06	2b	8.73	−0.01	0.02	2b	9.58	0.00	0.00	2b	9.75	0.00	0.00	2b	11.87
0.08	−0.03	0.05	2b	2.01	0.00	0.01	2b	2.85	0.00	0.00	2b	11.25	0.00	0.00	2b	56.80
0.13	−0.03	0.04	2b	1.99	−0.01	0.01	2b	3.38	0.00	0.00	2b	26.22	0.00	0.00	2b	115.71
0.18	0.00	0.00	2b	1.18	0.00	0.00	2b	10.96	0.00	0.00	2b	11.10	0.00	0.00	2b	33.49
0.23	−0.15	0.06	2b	1.25	−0.06	0.02	2b	1.81	0.00	0.00	2b	0.75	0.00	0.00	2a	133.56
0.28	0.03	0.03	2a	2.08	0.00	0.00	2a	7.13	0.00	0.00	2a	6.14	0.00	0.00	2a	120.23
0.33	0.03	0.08	2a	1.12	0.01	0.02	1	5.44	0.00	0.00	1	14.82	0.00	0.00	1	22.65
0.38	0.01	0.04	1	0.93	0.00	0.02	1	1.73	0.00	0.01	1	24.31	0.00	0.00	1	82.99
0.43	0.01	0.07	1	2.31	0.00	0.00	1	6.62	0.00	0.00	1	25.39	0.00	0.00	1	97.91
0.48	0.00	0.08	1	0.86	0.00	0.01	1	2.89	0.00	0.00	1	4.24	0.00	0.00	1	10.54

In this case, by increasing m (from m_2^- to ∞), both the points P_{2l} (left intersection between $y=\lambda_2^4(s,m)$ and $y=\overline{\lambda}_2^4$) and $P_1^{(2)}$ (intersection between $y=\lambda_1^4(s,m)$ and $y=\overline{\lambda}_1^4$) move to the left, see Fig. 3, and we need to change our arguments. We use the following property: there exist m^*, $m^*>m_2^-$ and m^* large enough, such that the left intersection point P_{2l}^* between $y=\lambda_2(s,m^*)^4$ and $y=\overline{\lambda}_2^4$ is to the left of $P_1^{(1)}$, that is $s_2^*=s(P_{2l}^*)<s(P_1^{(1)})$, see Fig. 4. (In fact, one can prove that $\lim_{m\to\infty}s(P_2^*(m))=0^+$.) Now, by decreasing m (from m^* to m_2^-), the left intersection point $P_{2l}(m)$ between $y=\lambda_2^4(s,m)$ and $y=\overline{\lambda}_2^4$ moves monotonically to the right, and $\lim_{m\to m_2^-}s(P_{2l}(m))=s_{2\min}$, whereas, by increasing m (from m_2^- to ∞), the intersection point $P_1(m)$ between $y=\lambda_1^4(s,m)$ and $y=\overline{\lambda}_1^4$ moves monotonically to the left, and $\lim_{m\to\infty}s(P_1(m))=0^+$. Therefore, there exists a unique value of m, say $\tilde{m}$, with $\tilde{m}\in(m_2^-,m^*)$, such that $s(P_{2l}(\tilde{m}))=s(P_1(\tilde{m}))$, and the problem is solved.

6. Applications

The constructive algorithm described in previous section is tested here to determine position and severity of a single open crack in a uniform simply-supported beam from the first two natural frequencies. Specifically, we shall present a series of numerical simulations for different locations and severities of the crack in a beam of length L with rectangular cross-section $B\times H$, where $H/L=0.1$, see Fig. 5. The open edge crack has front parallel to the side B and depth a, and is located at the cross-section of normalized abscissa $s(=z_d/L)$. Let $\alpha=\frac{a}{H}$ be the crack ratio. The corresponding stiffness of the rotational spring simulating the crack is

$$\widehat{K}=\frac{EI}{L\delta},\tag{65}$$

where the normalized flexibility $\delta=\frac{1}{K}$ can be obtained, for example, according to [33], namely

$$\delta=2\frac{H}{L}\left(\frac{\alpha}{1-\alpha}\right)^2\left(5.93-19.69\alpha+37.14\alpha^2-35.84\alpha^3+13.12\alpha^4\right).\tag{66}$$

The equivalent concentrated mass is given by $m=\frac{1}{K}=\delta$.

The results are collected in Table 1 for crack ratios $\alpha=0.1, 0.2, 0, 4, 0.6$ and for ten damage locations s, from s=0.03 to s=0.48. It can be seen that, when exact data are used as an input, the constructive algorithm leads to virtually exact identification of the crack, both for position and severity. Computing time is minimal and ranges from 1 to 2, 10 to 20, 20 to130 s for α=0.1, 0.4, 0.6, respectively. Large computing-time is typically necessary near the support and in correspondence of the transition between Case 2a and Case 2b.

For the sake of completeness, the identification has been also performed by solving the system formed by the frequency equation (6) evaluated for $\lambda=\overline{\lambda}_1$, $\lambda=\overline{\lambda}_2$, namely

$$\begin{cases}\mathcal{P}_K(\overline{\lambda}_1,s,K)=0, & (67)\\ \mathcal{P}_K(\overline{\lambda}_2,s,K)=0, & (68)\end{cases}$$

where $\overline{\lambda}_1^4$, $\overline{\lambda}_2^4$ are given (e.g., measured) values of the first two eigenvalues of the cracked beam. It is worth noting that Theorem 2.2 ensures that there exists a unique solution (up to symmetry with respect to the mid-point of the beam axis) to (67) and (68) for every pair of admissible frequency data. The resolution of the system (67) and (68) always leads to the exact values of the damage parameters, with computing-time equal to few milliseconds.

The robustness of the identification to possible errors on the data has been tested for the proposed algorithm and for the classical inverse procedure based on the solution of Eqs. (67) and (68). Without going into details, both the approaches turned out to be sensitive to errors on the data and, in particular, estimates were rather inaccurate for crack positions very close to the support (a point of vanishing sensitivity of the resonant frequencies). In general terms, the accuracy improves as the severity of the damage increases, since errors on the frequency data are generally lower than crack-induced shifts in the eigenfrequencies.

7. Conclusions

In this paper we have considered the inverse problem of determining a single open crack of any level of severity in a simply supported uniform beam from the measurement of the first two natural frequencies. We have shown that the crack can be uniquely determined (up to a symmetric position) by the given set of spectral data.

The above result was known to hold in the case of a single small crack, whereas a rigorous proof in the case of not necessarily small crack was lacking.

The approach we used to prove the result is completely different from that adopted in the case of small damage. It also differs from the technique recently used by three of us in [13] in identifying a single open crack in a longitudinally vibrating beam from two resonant frequencies. The present analysis is based on a reduction of the crack identification problem to the equivalent inverse problem of determining a point mass in a simply-supported beam, and on a careful study of the eigenvalues as functions of the mass intensity and position. The proof of the theoretical result leads to a constructive algorithm. The results of an extensive series of numerical simulations are in agreement with the theory.

The method proposed in this paper can be used, in principle, also to deal with beams with a single open crack under different end conditions. The key point in those cases is to deduce qualitative properties of the $\lambda-m$ and $\lambda-s$ curves analogous to those shown in Section 4. This topic is currently under investigation.

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Appendix A

This appendix is devoted to the proof of Proposition 3.5.

We start with some preliminary considerations. By the symmetry of the problem (8)–(12) with respect to $x=1/2$, we have

$$\lambda_2(s,m)=\lambda_2(1-s,m), \quad s\in\left[0,\tfrac{1}{2}\right]. \tag{69}$$

Moreover, we have

$$\lambda_2(0,m)=\lambda_2(1,m)=\lambda_2^U \tag{70}$$

and

$$\lambda_2\left(\tfrac{1}{2},m\right)=\lambda_2^U, \tag{71}$$

for every $m\in(0,\infty)$. Therefore, if $s\in\{0,\frac{1}{2},1\}$, then inequality (25) is always satisfied, and we may restrict the subsequent analysis to the positions $s\in\left(0,\frac{1}{2}\right)$.

We recall that, by Proposition 2.1 (point 1) and Proposition 3.1, the eigenvalues $\lambda_n^4(s,m)$ of (8)–(12) are the (fourth power of) positive roots of the frequency equation

$$\mathcal{P}(\lambda,s,m)=\frac{4}{m}\sin\lambda\sinh\lambda+\lambda(\sin\lambda(\cosh\lambda-\cosh(\lambda(1-2s))+\sinh\lambda(\cos\lambda-\cos(\lambda(1-2s)))=0. \tag{72}$$

By Proposition 3.4 and since the eigenvalues are all simple, there exists exactly one zero of $\mathcal{P}(\lambda,s,m)=0$ in $[\pi,2\pi]$, and

$$\pi\le\lambda_2(s,m)\le 2\pi. \tag{73}$$

By Monotonicity Theorems, if $M > m$, then

$$\lambda_n(s,M) \le \lambda_n(s,m), \quad \text{for every } n \ge 1 \text{ and for every } s \in \left(0,\tfrac{1}{2}\right), \tag{74}$$

where $\{\lambda_n(s,M)\}_{n\ge 1}$ are the positive zeros of the frequency equation (72) with m replaced by M. When $M \to \infty$, by continuity $\lambda_1(s,M) \to 0$ and $\lambda_2(s,M) \to \lambda_2(s,\infty)$, where $\lambda_2(s,\infty)$ is the first positive root of the frequency equation obtained by formally taking $m=\infty$ in (72), namely

$$\mathcal{P}_\infty(\lambda,s) \equiv \mathcal{P}(\lambda,s,\infty) = \lambda\big(\sin\lambda(\cosh\lambda - \cosh(\lambda(1-2s)) + \sinh\lambda(\cos\lambda - \cos(\lambda(1-2s))\big) = 0. \tag{75}$$

Therefore, in order to prove inequality (25), by (74) and by the above considerations, it is enough to show that

$$\lambda_2^* < \lambda_2(s,\infty), \quad \text{for every } s \in \left(0,\tfrac{1}{2}\right). \tag{76}$$

We rewrite $\mathcal{P}_\infty(\lambda,s)$ dividing by $\lambda\cos\lambda\cosh\lambda$, and adding and subtracting $\tanh\lambda$. After simple algebra, the zeros of $\mathcal{P}_\infty(\lambda,s)$ are the roots of the following equation:

$$\tan\lambda = G(\lambda,s), \tag{77}$$

with

$$G(\lambda,s) = \tanh\lambda + \tanh\lambda \cdot \left(\frac{\dfrac{\cos((\lambda(1-2s))}{\cos\lambda} - 1}{1 - \dfrac{\cosh((\lambda(1-2s))}{\cosh\lambda}} - 1 \right), \tag{78}$$

for $s \in \left(0,\tfrac{1}{2}\right)$. It is easy to see that

$$0 < 1 - \frac{\cosh((\lambda(1-2s))}{\cosh\lambda} < 1 - \frac{1}{\cosh\lambda}, \quad s \in \left(0,\frac{1}{2}\right), \quad \lambda \in [\pi, 2\pi]. \tag{79}$$

Therefore, the function $G(\lambda,s)$ may be singular inside $(\pi,2\pi)$ only at the point $\lambda = \frac{3}{2}\pi$. A direct inspection of the term $\cos(\lambda(1-2s))$ near $\lambda = \frac{3}{2}\pi$ shows that we need to distinguish different behaviors of $G(\lambda,s)$ corresponding to the following three cases:

$$\text{Case (a):} \tfrac{1}{3} < s < \tfrac{1}{2}; \tag{80}$$

$$\text{Case (b):} s < \tfrac{1}{3}; \tag{81}$$

$$\text{Case (c):} s = \tfrac{1}{3}. \tag{82}$$

Case (a): The function $G(\lambda,s)$ has vertical asymptotes at $\lambda = \frac{\pi}{2}$ and $\lambda = \frac{3\pi}{2}$, with

$$\lim_{\lambda \to \frac{\pi}{2}^-} G(\lambda,s) = +\infty, \quad \lim_{\lambda \to \frac{\pi}{2}^+} G(\lambda,s) = -\infty, \tag{83}$$

$$\lim_{\lambda \to \frac{3\pi}{2}^-} G(\lambda,s) = -\infty, \quad \lim_{\lambda \to \frac{3\pi}{2}^+} G(\lambda,s) = +\infty. \tag{84}$$

Moreover, a tedious but straightforward calculation shows that, for every $s \in \left(\tfrac{1}{3},\tfrac{1}{2}\right)$, we have

$$G(\pi,s) < 0, \quad G(2\pi,s) < 0. \tag{85}$$

By (83)–(85) and taking into account that, by (73) and (74), there exists exactly one zero of (77) in $(\pi,2\pi)$, the qualitative

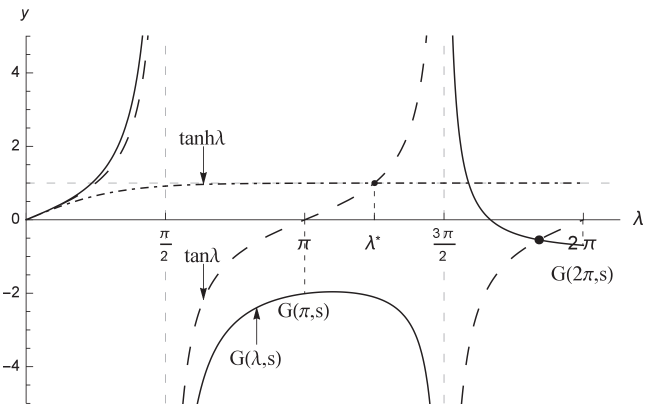

Fig. 6. Proof of Proposition 3.5: case (a).

behavior of $G(\lambda, s)$ compared to $\tan \lambda$ (e.g., the left hand side of (77)), is sketched in Fig. 6. We can deduce that the single root of (77) in $[\pi, 2\pi]$ must belong to the interval $(\frac{3\pi}{2}, 2\pi)$, that is $\lambda_2(s, \infty) \in (\frac{3\pi}{2}, 2\pi)$, and inequality (25) is satisfied.

Case (*b*): Calculations similar to those performed in Case (a) show that the function $G(\lambda, s)$ has vertical asymptotes at $\lambda = \frac{\pi}{2}$ and $\lambda = \frac{3\pi}{2}$, with

$$\lim_{\lambda \to \frac{\pi}{2}^-} G(\lambda, s) = +\infty, \quad \lim_{\lambda \to \frac{\pi}{2}^+} G(\lambda, s) = -\infty, \tag{86}$$

$$\lim_{\lambda \to \frac{3\pi}{2}^-} G(\lambda, s) = +\infty, \quad \lim_{\lambda \to \frac{3\pi}{2}^+} G(\lambda, s) = -\infty \tag{87}$$

and

$$G(\pi, s) < 0, \quad G(2\pi, s) < 0, \tag{88}$$

see Fig. 7. If the curves $\tan \lambda$ and $G(\lambda, s)$ intersect at $\lambda_2 \in (\frac{3\pi}{2}, 2\pi)$, then inequality (76) is satisfied, and there is nothing to prove. If the two curves do not intersect at a point belonging to the interval $(\frac{3\pi}{2}, 2\pi)$, to prove (76), by (88) (left) it is enough to prove that

$$G(\lambda_2^*, s) < \tan(\lambda_2^* s), \quad \text{for every } s \in \left(0, \frac{1}{3}\right). \tag{89}$$

Since $\tan \lambda_2^* = \tanh \lambda_2^*$ by definition, (89) is satisfied if and only if

$$\frac{\cos((\lambda_2^*(1-2s))}{\cos \lambda_2^*} + \frac{\cosh((\lambda_2^*(1-2s))}{\cosh \lambda_2^*} < 2, \quad \text{for every } s \in \left(0, \frac{1}{3}\right). \tag{90}$$

Inequality (90) is equivalent to

$$w_1(t) < w_2(t), \quad \text{for every } t \in \left(\frac{\lambda_2^*}{3}, \lambda_2^*\right), \tag{91}$$

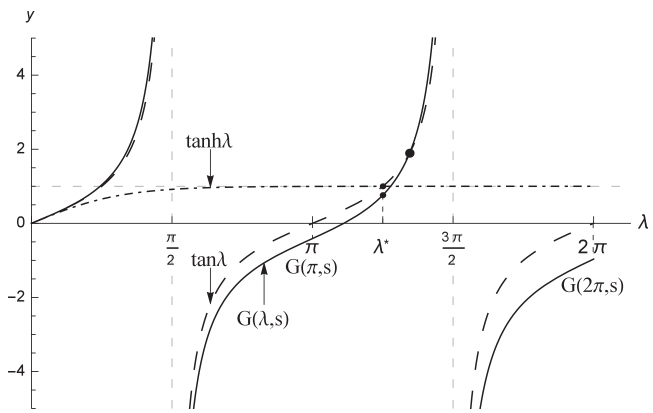

Fig. 7. Proof of Proposition 3.5: case (b).

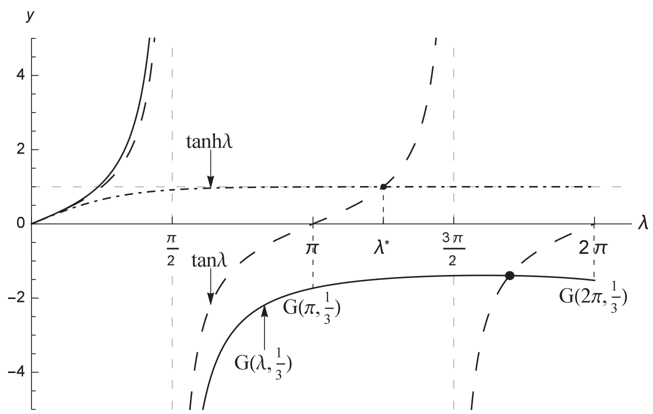

Fig. 8. Proof of Proposition 3.5: case (c).

where

$$t=\lambda_2^*(1-2s), \quad w_1(t)=\frac{\cos\, t}{\cos\, \lambda_2^*}-1, \quad w_2(t)=1-\frac{\cosh\, t}{\cosh\, \lambda_2^*}. \tag{92}$$

By writing $w_1(t)$ and $w_2(t)$ as power series around $t=\lambda_2^*$ and taking the difference, we easily get

$$w_2(t)-w_1(t)=-2\sum_{k=0}^{\infty}\frac{(t-\lambda_2^*)^{3+4k}}{(3+4k)!}\left(\tan\, \lambda_2^*+\frac{t-\lambda_2^*}{4(1+k)}\right). \tag{93}$$

All the terms of the series in (93) are strictly negative in $\left(\frac{\lambda_2^*}{3},\lambda_2^*\right)$. Therefore, inequality (91) is satisfied, and the proof of (76) is complete.

Case (c): In this case, the singularity of $G\left(\lambda,\frac{1}{3}\right)$ at $\frac{3\pi}{2}$ is removable and we have

$$\lim_{\lambda\to\frac{\pi}{2}^-} G\left(\lambda,\frac{1}{3}\right)=+\infty, \quad \lim_{\lambda\to\frac{\pi}{2}^+} G\left(\lambda,\frac{1}{3}\right)=-\infty, \tag{94}$$

$$G\left(\pi,\frac{1}{3}\right)<-\frac{3}{2}\tanh\pi, \quad G\left(\frac{3\pi}{2},\frac{1}{3}\right)<-\frac{1}{3}\tanh\left(\frac{3\pi}{2}\right), \quad G\left(2\pi,\frac{1}{3}\right)<-\frac{1}{2}\tanh(2\pi), \tag{95}$$

see Fig. 8. It is easy to prove that the unique root of (77) in $(\pi,2\pi)$ must belong to the interval $\left(\frac{3\pi}{2},2\pi\right)$, and the proof of Proposition 3.5 is complete.

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