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Inclusions of characterized subgroups

Giuseppina Barbieri, Anna Giordano Bruno *, Hans Weber

Università degli Studi di Udine – Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Via delle Scienze 206, 33100 Udine, Italy

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A subgroup H of $\mathbb{R}$ is characterized if $H = \tau_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} : u_n x \to 0 \bmod \mathbb{Z}\}$ for some sequence $\mathbf{u}$ in $\mathbb{R}$. Given two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$, we find conditions under which $\tau_{\mathbf{u}}(\mathbb{R})$ is contained or not in $\tau_{\mathbf{v}}(\mathbb{R})$. As a by-product of our main theorems, we find a known result by Eliaš on inclusions of characterized subgroups of $\mathbb{T}$, motivated by problems in harmonic analysis.

1. Introduction

Let G be a topological abelian group and denote by $\widehat{G}$ the group of all continuous characters $\chi : G \to \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is endowed with the compact quotient topology inherited from $\mathbb{R}$. Following [19], for a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\widehat{G}$, let

$$s_{\mathbf{u}}(G) := \{x \in G : u_n(x) \to 0\}.$$

A subgroup H of G is called *characterized* if $H = s_{\mathbf{u}}(G)$ for some sequence $\mathbf{u}$ in $\widehat{G}$.

* Corresponding author.
E-mail addresses: giuseppina.barbieri@uniud.it (G. Barbieri), anna.giordanobruno@uniud.it (A. Giordano Bruno), hans.weber@uniud.it (H. Weber).

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The characterized subgroups were considered almost exclusively for metrizable compact abelian groups (e.g., see [5,18]); only recently, the case of general compact abelian groups was given full attention in [13], and the case of abelian topological groups in [16] (see also [24,25]).

The fundamental and starting case remains when $G = \mathbb{T}$ (e.g., see [6,27,29]); the characterized subgroups of $\mathbb{T}$ were studied also in relation to Diophantine approximation, dynamical systems and ergodic theory (see [6,30,35] and the survey [24]).

Moreover, it is worth pointing out that, since $\widehat{\mathbb{T}}$ can be identified with $\mathbb{Z}$, we may assume that a sequence $\mathbf{u}$ in $\widehat{\mathbb{T}}$ is a sequence in $\mathbb{Z}$. Then $s_{\mathbf{u}}(\mathbb{T})$ coincides with the subgroup

$$t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0\}$$

of all *topologically* $\mathbf{u}$*-torsion elements* of $\mathbb{T}$. The concept of topologically $\mathbf{u}$-torsion element generalizes that of topologically torsion element (for $u_n = n!$) and that of topologically p-torsion element (for $u_n = p^n$), which were introduced to study the structure of topological groups and in particular of locally compact abelian groups (see [2,8,20,33] and the survey [11]). A complete description of the subgroups $t_{\mathbf{u}}(\mathbb{T})$ was found in [17,12] for sequences $\mathbf{u}$ in $\mathbb{N}$ such that u_n divides u_{n+1} for every $n \in \mathbb{N}$.

We consider here the characterized subgroups of $\mathbb{R}$. Since $\widehat{\mathbb{R}}$ is topologically isomorphic to $\mathbb{R}$, we can identify a sequence $\mathbf{u}$ in $\widehat{\mathbb{R}}$ with a sequence in $\mathbb{R}$; then $s_{\mathbf{u}}(\mathbb{R})$ coincides with the subgroup

$$\tau_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} : u_n x \to 0 \mod \mathbb{Z}\}.$$

We would like to underline that, if $\mathbf{u}$ is a sequence in $\mathbb{Z}$, the examination of $\tau_{\mathbf{u}}(\mathbb{R})$ includes that of $t_{\mathbf{u}}(\mathbb{T})$; in fact, in this case $\tau_{\mathbf{u}}(\mathbb{R}) = \pi^{-1}(t_{\mathbf{u}}(\mathbb{T}))$, where $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection.

Characterized subgroups of $\mathbb{R}$ were studied in relation to uniform distribution of sequences modulo $\mathbb{Z}$ by Kuipers and Niederreiter in the book [28], where [28, Theorem 7.8] shows that $\tau_{\mathbf{u}}(\mathbb{R})$ has Lebesgue measure zero if $\mathbf{u}$ is a sequence in $\mathbb{R}$ not converging to 0 in $\mathbb{R}$ (they give credit to Schoenberg for this result, see [32]). Moreover, Borel proved in [7, Proposition 2] that if $H = \tau_{\mathbf{v}}(\mathbb{R})$ is a non-trivial proper characterized subgroup of $\mathbb{R}$, then there exist $\gamma \in \mathbb{R}$ and a strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}$ such that $\gamma H = \tau_{\mathbf{u}}(\mathbb{R})$. This underlines the strict relation between characterized subgroups of $\mathbb{R}$ and characterized subgroups of $\mathbb{T}$. Borel proved also that every countable subgroup of $\mathbb{R}$ is characterized and left open the general question of a complete description of the characterized subgroups of $\mathbb{R}$.

In this paper, under some restrictions on the sequences, we find conditions ensuring the inclusion of one characterized subgroup of $\mathbb{R}$ into another characterized subgroup of $\mathbb{R}$. In particular, we consider characterized subgroups $\tau_{\mathbf{u}}(\mathbb{R})$ of $\mathbb{R}$ always under the assumption that $\mathbf{u}$ is in $\mathbb{R} \setminus \{0\}$ and $|q_n^{\mathbf{u}}| \to +\infty$, where

$$q_n^{\mathbf{u}} := \tfrac{u_n}{u_{n-1}} \ (n \in \mathbb{N}) \text{ and } u_0 = 1.$$

Thus, the cardinality of $\tau_{\mathbf{u}}(\mathbb{R})$ is $\mathfrak{c}$ (see Remark 2.1). Moreover, we can always assume that such sequences are in $\mathbb{R}_+$, since for any sequence $\mathbf{w}$ in $\mathbb{R}$ we have $\tau_{\mathbf{w}}(\mathbb{R}) = \tau_{|\mathbf{w}|}(\mathbb{R})$ where $|\mathbf{w}| := (|w_n|)_{n\in\mathbb{N}}$.

The problem of reciprocal inclusions of characterized subgroups has the following topological motivation, described in more detail in Section 2. Recall that a topological abelian group G is *totally bounded* if for every non-empty open set U in G there exists a finite subset F of G such that $G = U + F$; moreover, G is *precompact* if it is Hausdorff and totally bounded. For two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$ such that $|q_n^{\mathbf{u}}| \to +\infty$, we see in Corollary 2.7 that the condition $\tau_{\mathbf{u}}(\mathbb{R}) \not\subseteq \tau_{\mathbf{v}}(\mathbb{R})$ is equivalent to the existence of a non-metrizable precompact group topology $\mathcal{T}$ on $\mathbb{R}$ such that $u_n \to 0\ (\mathcal{T})$ and $v_n \not\to 0\ (\mathcal{T})$ (i.e., $u_n \to 0$ in $(\mathbb{R}, \mathcal{T})$ and $v_n \not\to 0$ in $(\mathbb{R}, \mathcal{T})$), and also to the fact that $v_n \not\to 0\ (\sigma_{\mathbf{u}})$ where $\sigma_{\mathbf{u}}$ is the finest precompact group topology such that $u_n \to 0\ (\sigma_{\mathbf{u}})$ (see (2.1) for the definition of $\sigma_{\mathbf{u}}$). The topology $\sigma_{\mathbf{u}}$ considered in [4,19] was investigated in [14], where also ss-precompact groups were studied (a precompact group topology $\mathcal{T}$ on an abelian group G is *ss-precompact* if there exists a sequence $\mathbf{u}$ in G such that $\mathcal{T} = \sigma_{\mathbf{u}}$).

Another motivation for considering the reciprocal inclusions of characterized subgroups comes from the following open problem left in [4]. A subgroup H of $\mathbb{T}$ is *factorizable* if $H = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ for proper characterized subgroups $t_{\mathbf{v}}(\mathbb{T})$ and $t_{\mathbf{w}}(\mathbb{T})$ of H. Now, [4, Question 5.1] asks when a given factorizable subgroup is characterized and when a given characterized subgroup is factorizable. A partial answer to this problem is given in [3], where in particular it is proved that $\mathbb{T}$ is factorizable. A first step in approaching this problem appears to be the understanding of when, for two given characterized subgroups, one is contained in the other.

We describe now the main results of the paper, and we start by the following observation. If $\mathbf{u}$ and $\mathbf{v}$ are two strictly increasing sequences in $\mathbb{R}_+$ with $v_1 \geq u_1$, then for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$u_m \leq v_n < u_{m+1},$$

and therefore v_n can be written as

$$v_n = \alpha_m u_m,$$

where α_m is a real number with $1 \leq \alpha_m < q^{\mathbf{u}}_{m+1}$. To simplify the situation we assume that $u_n \leq v_n < u_{n+1}$, i.e., that $v_n = \alpha_n u_n$ with $1 \leq \alpha_n < q^{\mathbf{u}}_{n+1}$ for every $n \in \mathbb{N}$. Under these assumptions, in Section 3 we look for conditions on the sequence $(\alpha_n)_{n\in\mathbb{N}}$ to find out when $\tau_{\mathbf{u}}(\mathbb{R})$ is not contained in $\tau_{\mathbf{v}}(\mathbb{R})$. The first main theorem of the paper is Theorem 3.3, from which we can derive the following result.

Theorem 1.1. *Let $\mathbf{u}$ and $\mathbf{v}$ be sequences in $\mathbb{R}_+$ such that $v_n = \alpha_n u_n$ for every $n \in \mathbb{N}$. If $\alpha_n \to +\infty$ and $\alpha_n \leq \kappa q^{\mathbf{u}}_{n+1}$ eventually for some $\kappa < 1$, then $\tau_{\mathbf{u}}(\mathbb{R}) \not\subseteq \tau_{\mathbf{v}}(\mathbb{R})$.*

The relation $v_n = \alpha_n u_n$ ($n \in \mathbb{N}$) can be written as $\mathbf{v} = A\mathbf{u}$, where A is an infinite diagonal matrix with the values α_n on the diagonal. This suggests to consider the (apparently) more general situation when $\mathbf{u}$ and $\mathbf{v}$ are sequences in $\mathbb{R}$ such that

$$\mathbf{v} = A\mathbf{u},$$

where $A = (a_{i,j})_{i,j\in\mathbb{N}}$ is a row-finite infinite real matrix (here row-finite means that each row of A is eventually null). We do this in Section 4.

Such a situation was considered in different terms by Eliaš in [22,23] (see Theorem 1.2 below) when $\mathbf{u}$ and $\mathbf{v}$ are strictly increasing sequences in $\mathbb{N}$ and A is a particular row-finite infinite integer matrix that always exists by the following argument. First observe that by [23, Theorem 1.1], for a strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}$ and $m \in \mathbb{Z}$ there exists an eventually null sequence $\mathbf{r}$ in $\mathbb{Z}$ (called a *good expansion* of m by $\mathbf{u}$) such that

$$m = \sum_{n\in\mathbb{N}} r_n u_n, \text{ where } \left|\sum_{j\leq n} r_j u_j\right| \leq \frac{u_{n+1}}{2} \text{ for every } n \in \mathbb{N}. \tag{1.1}$$

So, if $\mathbf{v}$ is another strictly increasing sequence in $\mathbb{N}$, for every $i \in \mathbb{N}$ there exists a good expansion $\mathbf{r}_i$ of v_i by $\mathbf{u}$, and $\mathbf{r}_i$ becomes the i-th row of an infinite matrix A such that $\mathbf{v} = A\mathbf{u}$.

Eliaš proved in [22] the following theorem on inclusions between characterized subgroups of $\mathbb{T}$, that we express in our terminology based on our original approach using the row-finite infinite integer matrix A.

Theorem 1.2 *([23, Theorem 1.2]). Let $\mathbf{u}$ and $\mathbf{v}$ be two strictly increasing sequences in $\mathbb{N}$ such that $q^{\mathbf{u}}_n \to +\infty$ and $q^{\mathbf{v}}_n \to +\infty$, and let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be the row-finite integer matrix such that each row $\mathbf{r}_i$ of A is a good expansion of v_i by $\mathbf{u}$ (so $\mathbf{v} = A\mathbf{u}$). Then $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$ if and only if:*

(C) $\mathbf{c}_j$ *is eventually null for each column* $\mathbf{c}_j$ *of* A*;*
(R) $\sup_{i\in\mathbb{N}} \|\mathbf{r}_i\|_1 < +\infty$.

This result was in [23] an important tool to solve a problem on Arbault sets,[1] and more precisely on A-permitted sets (see [31]). In our terminology A is an Arbault set if and only if $A \subseteq \tau_{\mathbf{u}}(\mathbb{R})$ for some strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}$. Arbault introduced this kind of thin sets[2] in studying the sets of absolute convergence of trigonometric series in [1], where he also started the study of permitted sets.

In Section 4, using different techniques from those of Eliaš, we give in Theorem 4.10 a self-contained proof of a more general result; notice that the sequences $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}$ (and not assumed to be integer-valued). We mention here a consequence of Theorem 4.10:

Corollary 1.3. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}\setminus\{0\}$ *such that* $|q_n^{\mathbf{u}}| \to +\infty$*. Let* $A = (a_{i,j})_{i,j\in\mathbb{N}}$ *be a row-finite integer matrix such that there exists* $0 < \kappa < 1$ *with*

$$\left|\sum_{j\leq n} a_{i,j}u_j\right| \leq \kappa\cdot|u_{n+1}| \text{ for every } n, i \in \mathbb{N}. \tag{1.2}$$

If $\mathbf{v} = A\mathbf{u}$*, then* $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ *if and only if (C) and (R) hold.*

The result by Eliaš stated in Theorem 1.2 follows now directly from Corollary 1.3 and (1.1), since under the assumptions of Theorem 1.2 the condition (1.2) is satisfied with $\kappa = \frac{1}{2}$.

To conclude, we leave the following open question suggested by the referee, that deserves to be investigated. For two subgroups $H,\ K$ of an infinite group G say that H is *almost contained* in K if $[H : K \cap H]$ is finite. Similarly, say that H is *weakly contained* in K if $[H : K \cap H]$ is at most countable.

Question 1.4. Do the characterizations for inclusion of characterized subgroups given in this article remain true also for almost inclusion or for weak inclusion in the above sense?

Notations. For a real sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, let $\|\mathbf{a}\|_1 = \sum_{n\in\mathbb{N}} |a_n|$ and $\|\mathbf{a}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$.

Let c_{00} denote the space of all real sequences that are eventually zero and ℓ_∞ the space of all bounded real sequences.

For a matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ we denote by $\mathbf{r}_i$ and $\mathbf{c}_j$ its i-th row and its j-th column, respectively. Moreover, let $\|A\|_\infty = \sup_{i,j\in\mathbb{N}} |a_{i,j}|$.

Let $\|-\| : \mathbb{R} \to \left[0, \frac{1}{2}\right]$ be defined by $\|x\| = \inf\{|x-n| : n \in \mathbb{Z}\}$. For $x \in \mathbb{R}$ we denote by $\{x\} := x - \lfloor x \rfloor$ the fractional part of x and by $\varphi(x)$ the unique number in $\left[-\frac{1}{2}, \frac{1}{2}\right[$ such that $\varphi(x) \equiv_{\mathbb{Z}} x$. In particular, $|\varphi(x)| = \|x\|$.

For $u, \varepsilon \in \mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ consider the following family of intervals of $\mathbb{R}$

$$\mathcal{I}(u,\varepsilon) = \left\{\left[-\frac{\varepsilon}{u}, \frac{\varepsilon}{u}\right] + \frac{n}{u} : n \in \mathbb{Z}\right\}.$$

Thus $\bigcup_{J\in\mathcal{I}(u,\varepsilon)} J = \{x \in \mathbb{R} : \|ux\| \leq \varepsilon\}$.

In the whole paper, let $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}\setminus\{0\}$ and $q_n^{\mathbf{u}} = \frac{u_n}{u_{n-1}}$ $(n \in \mathbb{N})$ with $u_0 = 1$. When we write simply q_n we always mean $q_n^{\mathbf{u}}$.

If $(G, \mathcal{T})$ is a topological group and $\mathbf{u}$ is a sequence in G, by writing shortly $u_n \to 0$ $(\mathcal{T})$ we mean that u_n converges to 0 in G with respect to $\mathcal{T}$.

[1] A set $A \subseteq [0,1]$ is an *Arbault set* if there is a strictly increasing sequence of positive integers $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \sin \pi u_n x = 0$ for all $x \in A$ (see [1]).
[2] See [9] for a survey on thin sets in harmonic analysis.

2. Characterized subgroups and precompact group topologies

We start by the following basic property of the characterized subgroups of $\mathbb{R}$.

Remark 2.1. If $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$, then

$$\tau_{\mathbf{u}}(\mathbb{R}) = \bigcap_{\varepsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \bigcup_{J\in\mathcal{I}(u_k,\varepsilon)} J$$
$$= \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \bigcup_{J\in\mathcal{I}(u_k,\frac{1}{m})} J.$$

Therefore, the characterized subgroup $\tau_{\mathbf{u}}(\mathbb{R})$ is a Borel set, hence either $\tau_{\mathbf{u}}(\mathbb{R})$ is countable or has the cardinality $\mathfrak{c}$ of the continuum (see [26, Theorem 13.6]). The latter occurs whenever $q_n \to +\infty$ (see [4,21] and [34, Theorem 3.4]).

Let G be an abelian group and H a subgroup of $\mathrm{Hom}(G,\mathbb{T})$. It is easy to see that the weakest group topology T_H on G making every character of H continuous is totally bounded. Vice versa, Comfort and Ross proved that any totally bounded group topology is of this type (see [10, Theorem 1.2]).

For all this section, let G be a topological abelian group, denote by $\mathcal{T}$ the group topology of G and consider the group $\widehat{G}$ of all continuous characters of G. Clearly, for $H \leq \mathrm{Hom}(G,\mathbb{T})$,

$$T_H \leq \mathcal{T} \quad \text{if and only if} \quad H \leq \widehat{G}.$$

Therefore, [10, Theorems 1.2, 1.3 and 1.11, Corollary 1.4] (see also [4, Theorem 2.1]) yield the next result.

Theorem 2.2 *(Comfort–Ross). The assignment $H \mapsto T_H$ defines an order preserving isomorphism from the lattice of all subgroups of $\widehat{G}$ onto the lattice of all totally bounded group topologies on G weaker than $\mathcal{T}$. Moreover, for $H \leq \widehat{G}$,*

(a) T_H is Hausdorff if and only if H separates the points of G;
(b) T_H is first countable if and only if H is countable.

Following [19], for a sequence $\mathbf{u}$ in G, let[3]

$$\tau_{\mathbf{u}}(\widehat{G}) := \left\{\chi \in \widehat{G} : \chi(u_n) \to 0\right\} \text{ and } \sigma_{\mathbf{u}} := T_{\tau_{\mathbf{u}}(\widehat{G})}. \tag{2.1}$$

By the definition of the topology T_H on G, we obtain the following equivalence.

Lemma 2.3. [19, Lemma 3.1] *Let $\mathbf{u}$ be a sequence in G and $H \leq \widehat{G}$. Then $u_n \to 0$ (T_H) if and only if $H \leq \tau_{\mathbf{u}}(\widehat{G})$.*

So, in analogy to [4, Corollary 2.3], $\sigma_{\mathbf{u}}$ is the finest totally bounded group topology $\mathcal{T}'$ weaker than $\mathcal{T}$ on G such that $u_n \to 0$ $(\mathcal{T}')$.

[3] If G is a locally compact Hausdorff abelian group, then one can identify G with $\widehat{\widehat{G}}$ in view of Pontryagin–van Kampen duality theorem, so in this case $\tau_{\mathbf{u}}(\widehat{G})$ coincides with $s_{\mathbf{u}}(\widehat{G})$.

Proposition 2.4. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be sequences in* G. *Then the following conditions are equivalent:*

(a) $\tau_{\mathbf{u}}(\widehat{G}) \subseteq \tau_{\mathbf{v}}(\widehat{G})$;
(b) $v_n \to 0\ (\sigma_{\mathbf{u}})$;
(c) for every totally bounded group topology $\mathcal{T}' \leq \mathcal{T}$ *on* G, *if* $u_n \to 0\ (\mathcal{T}')$ *then* $v_n \to 0\ (\mathcal{T}')$.

Proof. (a) ⇔ (b) It follows directly from Lemma 2.3 with $H = \tau_{\mathbf{u}}(\widehat{G})$.

(a) ⇒ (c) Let $\mathcal{T}' \leq \mathcal{T}$ be a totally bounded group topology on G such that $u_n \to 0\ (\mathcal{T}')$. By Theorem 2.2 there exists $H \leq \widehat{G}$ such that $\mathcal{T}' = T_H$. By Lemma 2.3 and the hypothesis, $H \subseteq \tau_{\mathbf{u}}(\widehat{G}) \subseteq \tau_{\mathbf{v}}(\widehat{G})$, hence $v_n \to 0\ (\mathcal{T}')$.

(c) ⇒ (b) Take $\mathcal{T}' = \sigma_{\mathbf{u}}$ and apply again Lemma 2.3. □

We now discuss the case when $G = \mathbb{R}$ and $\mathcal{T} = \mathcal{E}$ is the Euclidean topology. As known, $\widehat{\mathbb{R}}$ is topologically isomorphic to $\mathbb{R}$ itself. For any $\alpha \in \mathbb{R}$ the corresponding continuous character is χ_α, defined by $\chi_\alpha(x) = \pi(\alpha x)$ for every $x \in \mathbb{R}$. So, identifying χ_α with α and $\widehat{\mathbb{R}}$ with $\mathbb{R}$, and taking a sequence $\mathbf{u}$ in $\mathbb{R}$, $\tau_{\mathbf{u}}(\mathbb{R})$ defined in the Introduction coincides with $\tau_{\mathbf{u}}(\widehat{\mathbb{R}})$ defined in this section.

In the proof of the following proposition we use that, for $H \leq \mathbb{R}$, T_H is generated by the family of seminorms $p_\alpha(x) := \|\alpha x\|$ with α belonging to a set of generators of H.

Proposition 2.5. *Let* $H \leq \mathbb{R}$. *Then* T_H *is Hausdorff if and only if* H *is not cyclic.*

Proof. First observe that T_H is Hausdorff if and only if $\overline{\{0\}}^{T_H} = \{0\}$, and that $\overline{\{0\}}^{T_H}$ is a closed subgroup of $(\mathbb{R}, \mathcal{E})$ since $T_H \leq \mathcal{E}$. Therefore, either $\overline{\{0\}}^{T_H} = \mathbb{R}$ or $\overline{\{0\}}^{T_H}$ is cyclic. The first case occurs exactly when $H = 0$ and in this case T_H is not Hausdorff and H is cyclic. So we may assume that $\overline{\{0\}}^{T_H} = \varrho\mathbb{Z}$ for some $\varrho \in \mathbb{R}$.

Suppose that T_H is not Hausdorff; then $\varrho \neq 0$. If $\alpha \in H \setminus \{0\}$, then from

$$\varrho\mathbb{Z} = \overline{\{0\}}^{T_H} \subseteq \{x \in \mathbb{R} : p_\alpha(x) = 0\} = \frac{1}{\alpha}\mathbb{Z},$$

we derive $\varrho \in \frac{1}{\alpha}\mathbb{Z}$, i.e., $\alpha \in \frac{1}{\varrho}\mathbb{Z}$. Therefore, $H \subseteq \frac{1}{\varrho}\mathbb{Z}$ and so H is cyclic. Suppose now that H is cyclic and $H \neq 0$, and let $\alpha \in \mathbb{R} \setminus \{0\}$ be a generator of H. Since $\overline{\{0\}}^{T_H} = \{x \in \mathbb{R} : p_\alpha(x) = 0\} = \frac{1}{\alpha}\mathbb{Z}$, we get that T_H is not Hausdorff. □

Corollary 2.6. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R} \setminus \{0\}$ *such that* $|q_n^{\mathbf{u}}| \to +\infty$. *Then* $\sigma_{\mathbf{u}} = T_{\tau_{\mathbf{u}}(\mathbb{R})}$ *is Hausdorff (hence, precompact).*

Proof. By Remark 2.1 we get that $|\tau_{\mathbf{u}}(\mathbb{R})| = \mathfrak{c}$. Then apply Proposition 2.5. □

From Theorem 2.2, Proposition 2.4 and Corollary 2.6 we derive a topological interpretation of the non-inclusion of characterized subgroups of $\mathbb{R}$.

Corollary 2.7. *Let* $\mathbf{u}, \mathbf{v}$ *be sequences in* $\mathbb{R} \setminus \{0\}$ *such that* $|q_n^{\mathbf{u}}| \to +\infty$. *Then the following conditions are equivalent:*

(a) $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$;
(b) $v_n \not\to 0\ (\sigma_{\mathbf{u}})$;
(c) there exists a non-metrizable precompact group topology $\mathcal{T}'$ *on* $\mathbb{R}$ *weaker than* $\mathcal{E}$, *such that* $u_n \to 0\ (\mathcal{T}')$ *but* $v_n \not\to 0\ (\mathcal{T}')$.

Recall that a sequence $\mathbf{u}$ in an abelian group G is a *TB-sequence* if there exists a precompact group topology $\mathcal{T}$ on G such that $u_n \to 0$ $(\mathcal{T})$ (see [4]). This notion, studied also in [14], is generalized to topological abelian groups $(G, \mathcal{T})$ in [15], where a sequence $\mathbf{v}$ in G is called a *topological TB-sequence* if there exists a precompact group topology $\mathcal{T}' \leq \mathcal{T}$ such that $u_n \to 0$ $(\mathcal{T}')$ (clearly, every topological TB-sequence of $(G, \mathcal{T})$ is a TB-sequence of G).

With this terminology, a sequence $\mathbf{u}$ in $\mathbb{R}$ is a topological TB-sequence if and only if $\sigma_{\mathbf{u}}$ is Hausdorff, or equivalently, if $\tau_{\mathbf{u}}(\mathbb{R})$ is not cyclic (see Proposition 2.5); in particular, if $\mathbf{u}$ is in $\mathbb{R} \setminus \{0\}$ and $|q_n^{\mathbf{u}}| \to +\infty$, then $\mathbf{u}$ is a topological TB-sequence and $\sigma_{\mathbf{u}}$ is a precompact group topology on $\mathbb{R}$ weaker than $\mathcal{E}$ such that $u_n \to 0$ $(\sigma_{\mathbf{u}})$ (see Corollary 2.6).

3. Comparison of $\tau_{\mathbf{u}}(\mathbb{R})$ and $\tau_{\mathbf{v}}(\mathbb{R})$ whenever $v_n = \alpha_n u_n$

The first main result of this article is Theorem 3.3. In this theorem – and also in Theorems 4.3 and 4.5 – fixed two sequences $\mathbf{v}$ and $\mathbf{u}$ in $\mathbb{R} \setminus \{0\}$ with $|q_n^{\mathbf{u}}| \to +\infty$, we look for sufficient conditions so that $\tau_{\mathbf{u}}(\mathbb{R}) \not\subseteq \tau_{\mathbf{v}}(\mathbb{R})$; in particular, we search an element $x \in \tau_{\mathbf{u}}(\mathbb{R}) \setminus \tau_{\mathbf{v}}(\mathbb{R})$.

The strategy for that consists in constructing:

1. a nested sequence of compact intervals $I_n \in \mathcal{I}(u_n, \varepsilon_n)$, where $0 < \varepsilon_n \downarrow 0$;
2. a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ of $\mathbf{v}$ such that $\|v_{n_k} x\|$ is away from zero for every $x \in I_{n_k}$ and $k \in \mathbb{N}$.

For $x \in \bigcap_{n\in\mathbb{N}} I_n$, the first condition gives that $x \in \tau_{\mathbf{u}}(\mathbb{R})$ and the second that $x \notin \tau_{\mathbf{v}}(\mathbb{R})$.

The following lemma will give the required nested sequence of intervals.

Lemma 3.1. *Let $0 < \varepsilon < \frac{1}{2}$. If $u \in \mathbb{R}_+$ and I is an interval of length $\mu(I) \geq \frac{2}{u}$, then I contains an interval of $\mathcal{I}(u, \varepsilon)$. In particular, if $\mathbf{u}$ is a sequence in $\mathbb{R}_+$ and $n \in \mathbb{N}$, then any $I \in \mathcal{I}(u_n, \varepsilon)$ contains an interval of $\mathcal{I}(u_{n+1}, \varepsilon)$ if $\varepsilon q_{n+1} \geq 1$.*

Proof. To prove the first statement, it is sufficient to observe that the distance between the centers of two successive intervals of $\mathcal{I}(u, \varepsilon)$ is $\frac{1}{u}$, the length of any interval of $\mathcal{I}(u, \varepsilon)$ is $\frac{2\varepsilon}{u}$ and $\frac{1}{u} + \frac{2\varepsilon}{u} \leq \mu(I)$. In particular, if $I \in \mathcal{I}(u_n, \varepsilon)$ and $u = u_{n+1}$, then $\mu(I) = \frac{2\varepsilon}{u_n}$ and thus $\mu(I) \geq \frac{2}{u}$ is equivalent to $\varepsilon q_{n+1} \geq 1$. □

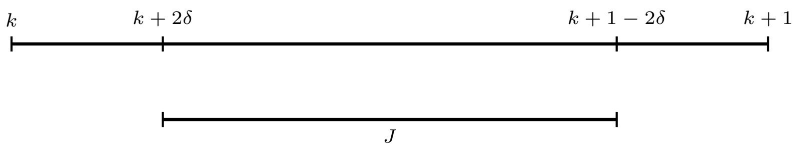
The next lemma is another essential tool in the proof of Theorems 3.3 and 4.3.

Lemma 3.2. *Let $\alpha, u_i, \varepsilon_i \in \mathbb{R}_+$ $(i = 1, 2)$, with*

$$v := \alpha u_1, \quad q_2 := \frac{u_2}{u_1}, \quad 0 < \kappa < 1, \quad \varepsilon_2 \leq \delta := \frac{1-\kappa}{4} \quad \textit{and} \quad \frac{1}{\varepsilon_1} \leq \alpha \leq \kappa q_2.$$

Then, for $I_1 \in \mathcal{I}(u_1, \varepsilon_1)$, there exists $I_2 \in \mathcal{I}(u_2, \varepsilon_2)$ such that $I_2 \subseteq I_1$ and $\|vx\| \geq \delta$ for every $x \in I_2$.

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = vx$. Since $\mu(I_1) = \frac{2\varepsilon_1}{u_1}$, we have $\mu(f(I_1)) = 2\varepsilon_1\alpha \geq 2$. Therefore, there exists $k \in \mathbb{Z}$ such that $[k, k+1] \subseteq f(I_1)$. Let $J := [k + 2\delta, k + 1 - 2\delta]$.



Since for every $i \in \mathbb{N}$ we have

$$\left|f\left(\frac{i}{u_2}\right)-f\left(\frac{i+1}{u_2}\right)\right|=\frac{v}{u_2}=\frac{\alpha}{q_2}\leq\kappa=\mu(J),$$

and the numbers $\frac{i}{u_2}$ are the centers of the intervals of $\mathcal{I}(u_2,\varepsilon_2)$, there exists $I_2\in\mathcal{I}(u_2,\varepsilon_2)$ with center $c=\frac{i}{u_2}$ such that $f(c)\in J$. Therefore, since

$$\frac{\mu(f(I_2))}{2}=v\frac{\mu(I_2)}{2}=v\frac{\varepsilon_2}{u_2}=\frac{\alpha}{q_2}\varepsilon_2\leq\kappa\varepsilon_2\leq\delta,$$

it follows that $f(I_2)\subseteq[k+\delta,k+1-\delta]$.

Consequently, $f(I_2)\subseteq[k+\delta,k+1-\delta]\subseteq f(I_1)$. Hence, $I_2\subseteq I_1$ and $\|vx\|\geq\delta$ for every $x\in I_2$. □

The following first main result of the paper gives sufficient conditions to obtain the required non-inclusion $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}}(\mathbb{R})$. Slightly strengthening the hypotheses (by choosing $\gamma(n)=n$ for every $n\in\mathbb{N}$ and $M=\mathbb{N}$), one can find a simplified reformulation of this result in Theorem 1.1 of the Introduction, and a direct consequence in Corollary 3.4.

Theorem 3.3. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be sequences in* $\mathbb{R}_+$. *Assume that:*

(a) $(q_n)_{n\in\mathbb{N}}$ *has a subsequence* $(q_{\gamma(n)})_{n\in\mathbb{N}}$ *such that* $q_{\gamma(n)}\to+\infty$;
(b) $q_n\in\mathbb{N}$ *for every* $n\in\mathbb{N}\setminus\gamma(\mathbb{N})$.

Let M *be an infinite subset of* $\mathbb{N}$, *set* $\alpha_m:=\frac{v_m}{u_m}$ *for every* $m\in M$ *and let* $0<\kappa<1$. *Assume that:*

(c) $\lim_{M\ni m\to+\infty}\alpha_m=+\infty$;
(d) $\alpha_m\leq\kappa q_{m+1}$ *for every* $m\in M$.

Then $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}}(\mathbb{R})$.

Proof. Let $\delta:=\frac{1-\kappa}{4}$ and $0<\varepsilon_n\downarrow 0$ such that:

(i) $\varepsilon_n q_{n+1}\geq 1$ if $n+1\in\gamma(\mathbb{N})$;
(ii) $\varepsilon_m\alpha_m\geq 1$ if $m\in M$.

Moreover, let $n_0\in\mathbb{N}$ with:

(iii) $\varepsilon_{n_0}\leq\delta$.

We define inductively a decreasing sequence $I_n\in\mathcal{I}(u_n,\varepsilon_n)$ for $n\geq n_0$ such that $\|v_m x\|\geq\delta$ for every $m\in M$ with $m\geq n_0$ and $x\in I_{m+1}$. Then we conclude that $\emptyset\neq\bigcap_{n\geq n_0}I_n\subseteq\tau_{\mathbf{u}}(\mathbb{R})\setminus\tau_{\mathbf{v}}(\mathbb{R})$.

Choose arbitrarily $I_{n_0}\in\mathcal{I}(u_{n_0},\varepsilon_{n_0})$. Suppose that $n\geq n_0$ and that $I_n\in\mathcal{I}(u_n,\varepsilon_n)$ is defined. If $n\in M$, Lemma 3.2 yields the existence of $I_{n+1}\in\mathcal{I}(u_{n+1},\varepsilon_{n+1})$ such that $I_{n+1}\subseteq I_n$ and $\|v_n x\|\geq\delta$ for every $x\in I_{n+1}$. Assume that $n\in\mathbb{N}\setminus M$. If $n+1\in\gamma(\mathbb{N})$, there exists $I_{n+1}\in\mathcal{I}(u_{n+1},\varepsilon_{n+1})$ such that $I_{n+1}\subseteq I_n$

by (i) and by Lemma 3.1. If $n+1 \in \mathbb{N}\setminus\gamma(\mathbb{N})$, then $q_{n+1} \in \mathbb{N}$, and so there exists $I_{n+1} \in \mathcal{I}(u_{n+1}, \varepsilon_{n+1})$ such that the center of I_{n+1} coincides with the center of I_n, thus $I_{n+1} \subseteq I_n$. □

Corollary 3.4. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}\setminus\{0\}$ *such that* $|q_n| \to +\infty$. *Consider a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that* $\sup_{n\in\mathbb{N}} |\alpha_n| = +\infty$ *and let* $v_n := \alpha_n u_n$ *for* $n \in \mathbb{N}$. *If there exists* $0 < \kappa < 1$ *such that* $|\alpha_n| \le \kappa|q_{n+1}|$ *eventually, then* $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$.

Proof. It follows from Theorem 3.3 recalling that $\tau_{\mathbf{u}}(\mathbb{R}) = \tau_{|\mathbf{u}|}(\mathbb{R})$ and $\tau_{\mathbf{v}}(\mathbb{R}) = \tau_{|\mathbf{v}|}(\mathbb{R})$. □

An easy modification of the proof of Theorem 3.3 shows that $\tau_{\mathbf{u}}(\mathbb{R})\setminus\tau_{\mathbf{v}}(\mathbb{R})$ contains a Cantor-like set and therefore has size $\mathfrak{c}$. The latter also follows from the next remark under the assumptions of Theorem 3.3 or of Corollary 3.4, since in this case $|\tau_{\mathbf{u}}(\mathbb{R})| = \mathfrak{c}$ by Remark 2.1.

Remark 3.5. Let H, K be infinite subgroups of a group G such that $H \nsubseteq K$. Then $|H \setminus K| = |H|$.

Proof. If $|H \cap K| < |H|$, then the thesis follows from $|H| = |H \setminus K| + |H \cap K|$.

If $|H \cap K| = |H|$ and $x \in H \setminus K$, then $x + (H \cap K) \subseteq H \setminus K$, hence

$$|H| = |H \cap K| = |x + (H \cap K)| \le |H \setminus K|. \qquad \square$$

Now we provide an example which shows that Theorem 3.3 and Corollary 3.4 fail for $\kappa = 1$.

Example 3.6. For $n \in \mathbb{N}$, let $u_n = n!$ and $v_n = (n-1)u_n$. With the notation of Corollary 3.4 we have $q_{n+1} > \alpha_n \to +\infty$, but $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ since $v_n = u_{n+1} - 2u_n$.

As for the assumption $\sup_{n\in\mathbb{N}} \alpha_n = +\infty$ in Corollary 3.4, consider the following observation. If $\mathbf{u}$ and $\mathbf{v}$ are sequences in $\mathbb{R}$ and $v_n = \alpha_n u_n$ where $(\alpha_n)_{n\in\mathbb{N}}$ is a bounded sequence, then obviously $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ if the α_n's are integers. But $\tau_{\mathbf{u}}(\mathbb{R})$ may be not contained in $\tau_{\mathbf{v}}(\mathbb{R})$ if $\alpha_n \in \mathbb{R}\setminus\mathbb{Z}$ as the next example shows.

Example 3.7. For $n \in \mathbb{N}$, let $u_n = n!$ and $v_n = \frac{1}{2}u_n$. Then $e \in \tau_{\mathbf{u}}(\mathbb{R}) \setminus \tau_{\mathbf{v}}(\mathbb{R})$. Indeed, $e = \sum_{n=0}^{+\infty} \frac{1}{n!}$ and for $n \in \mathbb{N}$,

$$u_n e \equiv_{\mathbb{Z}} n! \sum_{i=n+1}^{+\infty} \frac{1}{i!} \le \frac{1}{n+1} \sum_{i=0}^{+\infty} \frac{1}{2^i} = \frac{2}{n+1} \to 0;$$

hence, $\|u_n e\| \to 0$. Analogously, $\|v_{2n} e\| \to \frac{1}{2}$, since

$$v_{2n} e \equiv_{\mathbb{Z}} \frac{1}{2} + (2n)! \sum_{i=2n+1}^{+\infty} \frac{1}{i!} \to \frac{1}{2}.$$

We proceed by proving two technical lemmas, that are used in the rest of the paper.

Lemma 3.8. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}_+$ *with* $q_n \to +\infty$. *Then* $\lim_{n\to+\infty} \sum_{j<n} \frac{u_j}{u_n} = 0$.

Proof. Let $s_n := \sum_{j<n} \frac{u_j}{u_n}$ and $n_0 \in \mathbb{N}$ such that $q_n \ge 2$ for $n \ge n_0$. Then

$$s_{n+1} = \frac{s_n u_n + u_n}{u_{n+1}} = \frac{s_n + 1}{q_{n+1}} \le \frac{s_n + 1}{2}. \tag{3.1}$$

Let $t_n := \max\{1, s_n\}$; then $(t_n)_{n\in\mathbb{N}}$ is a bounded sequence since $t_{n+1} \le t_n$ for every $n \ge n_0$. Hence $(s_n)_{n\in\mathbb{N}}$ is bounded, too, and tends to 0 by (3.1) since $q_n \to +\infty$. □

Lemma 3.9. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}_+$ *with* $q_n \to +\infty$ *and* $A = (a_{i,j})_{i,j\in\mathbb{N}}$ *a row-finite real matrix.*

(a) If $\mathbf{c}_j \in \ell_\infty$ *for every* $j \in \mathbb{N}$*, then*

$$\limsup_{n\to+\infty} \frac{\|\mathbf{c}_n\|_\infty}{q_{n+1}} = \limsup_{n\to+\infty} \sup_{i\in\mathbb{N}} \sum_{j\le n} \frac{|a_{i,j}|u_j}{u_{n+1}} = \limsup_{n\to+\infty} \sup_{i\in\mathbb{N}} \left| \sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}} \right|.$$

(b) If $\sup_{n,i\in\mathbb{N}} \left| \sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}} \right| < +\infty$*, then* $\mathbf{c}_j \in \ell_\infty$ *for every* $j \in \mathbb{N}$*.*

Proof. (i) Let $\limsup_{n\to+\infty} \frac{\|\mathbf{c}_n\|_\infty}{q_{n+1}} < \sigma < \kappa$, for some reals σ and κ, and let $m \in \mathbb{N}$ be such that $\frac{\|\mathbf{c}_j\|_\infty}{q_{j+1}} \le \sigma$ for $j \ge m$. Then, for $i \in \mathbb{N}$ and $n \ge m$,

$$\left| \sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}} \right| \le \sum_{j\le n} \frac{|a_{i,j}|u_j}{u_{n+1}} \le \sum_{j<m} \frac{|a_{i,j}|u_j}{u_{n+1}} + \sigma \sum_{m\le j\le n} \frac{u_{j+1}}{u_{n+1}} =$$

$$= \frac{1}{u_{n+1}} \sum_{j<m} |a_{i,j}|u_j + \sigma \left(1 + \sum_{m<j\le n} \frac{u_j}{u_{n+1}} \right).$$

The first summand tends to 0 for $n \to +\infty$ uniformly on $i \in \mathbb{N}$, and the second one tends to σ by Lemma 3.8. Therefore, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ and $i \in \mathbb{N}$ we have $\sum_{j\le n} \frac{|a_{i,j}|u_j}{u_{n+1}} \le \kappa$. Hence,

$$\limsup_{n\to+\infty} \sup_{i\in\mathbb{N}} \left| \sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}} \right| \le \limsup_{n\to+\infty} \sup_{i\in\mathbb{N}} \sum_{j\le n} \frac{|a_{i,j}|u_j}{u_{n+1}} \le \kappa.$$

(ii) Let now $\limsup_{n\to+\infty} \sup_{i\in\mathbb{N}} |\sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}}| < \sigma < \kappa$ and let $m \in \mathbb{N}$ such that, for $n \ge m$,

$$\sup_{i\in\mathbb{N}} \left| \sum_{j\le n} \frac{a_{i,j}u_j}{u_{n+1}} \right| < \sigma.$$

Then for $n \ge m$ and $i \in \mathbb{N}$ we get

$$|a_{i,n}u_n| \le \left| \sum_{j\le n} a_{i,j}u_j \right| + \left| \sum_{j<n} a_{i,j}u_j \right| \le \sigma u_{n+1} + \sigma u_n.$$

Thus,

$$\sup_{i\in\mathbb{N}} |a_{i,n}| \frac{u_n}{u_{n+1}} \le \sigma + \sigma \frac{u_n}{u_{n+1}} \le \kappa$$

eventually. Hence, $\limsup_{n\to+\infty} \frac{\|\mathbf{c}_n\|_\infty}{q_{n+1}} \le \kappa$.

Then (i) and (ii) give (a). The same reasoning of (ii) applied for $m = 1$ proves (b). □

We give some further consequences of Theorem 3.3.

Theorem 3.10. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}\setminus\{0\}$ *such that* $|q_n|\to+\infty$, *let* $A=(a_{i,j})_{i,j\in\mathbb{N}}$ *be a row-finite real matrix such that* $\mathbf{c}_j\in\ell_\infty$ *for every* $j\in\mathbb{N}$, *and let* $\mathbf{v}=A\mathbf{u}$. *Assume that* $\limsup_{n\to+\infty}\frac{\|\mathbf{c}_n\|_\infty}{|q_{n+1}|}<1$ *and that there exist a sequence* $(r_i)_{i\in\mathbb{N}}$ *in* $\mathbb{N}$ *divergent to* $+\infty$ *and* $l\in\mathbb{N}$ *such that:*

(a) $\sup_{i\in\mathbb{N}}|a_{i,r_i}|=+\infty$;
(b) $a_{i,j}=0$ *if* $j>r_i+l$;
(c) $\{a_{i,j}:i,j\in\mathbb{N},j>r_i\}$ *is a bounded subset of* $\mathbb{Z}$.

Then $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}}(\mathbb{R})$.

Proof. Since $r_i\to+\infty$, $\mathbf{v}$ has a subsequence $\mathbf{v}^*=(v_{i_n})_{n\in\mathbb{N}}$ such that $|a_{i_n,r_{i_n}}|\to+\infty$ and $(r_{i_n})_{n\in\mathbb{N}}$ is strictly increasing. Since $\tau_{\mathbf{v}}(\mathbb{R})\subseteq\tau_{\mathbf{v}^*}(\mathbb{R})$, it suffices to prove that $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}^*}(\mathbb{R})$. To simplify the notation we may assume that $\mathbf{v}^*=\mathbf{v}$, i.e., that $(r_i)_{i\in\mathbb{N}}$ is strictly increasing.

Let, for $i\in\mathbb{N}$,

$$v_i'=\sum_{j\le r_i}a_{i,j}u_j\quad\text{and}\quad v_i''=\sum_{j>r_i}a_{i,j}u_j.$$

Then $\mathbf{v}=\mathbf{v}'+\mathbf{v}''$. It easily follows from (b) and (c) that $\tau_{\mathbf{u}}(\mathbb{R})\subseteq\tau_{\mathbf{v}''}(\mathbb{R})$. Therefore, it is enough to show that $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}'}(\mathbb{R})$. Thus, we may assume that $\mathbf{v}'=\mathbf{v}$, i.e., $a_{i,j}=0$ if $i\in\mathbb{N}$ and $j>r_i$.

Set $M:=\{r_i:i\in\mathbb{N}\}$. If $m=r_i$ for some $i\in\mathbb{N}$, choose $\alpha_m\in\mathbb{R}$ such that $v_i=\alpha_m u_m=:w_m$; for $n\in\mathbb{N}\setminus M$, let $w_n=\alpha_n=0$. Obviously, $\tau_{\mathbf{v}}(\mathbb{R})=\tau_{\mathbf{w}}(\mathbb{R})$. With the aid of Corollary 3.4 we show that $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{w}}(\mathbb{R})$. By Lemma 3.9 there exists $0<\kappa<1$ such that, for every $m=r_i\in M$ large enough,

$$||\alpha_m|-|a_{i,m}||\le|\alpha_m-a_{i,m}|=\left|\sum_{j<m}\frac{a_{i,j}u_j}{u_m}\right|\le\kappa.$$

Therefore,

$$\sup_{n\in\mathbb{N}}|\alpha_n|=\lim_{i\to+\infty}|a_{i,r_i}|=+\infty$$

and

$$|\alpha_m|=\left|\frac{v_i}{v_m}\right|=\left|\sum_{j\le m}a_{i,j}\frac{u_j}{u_m}\right|\le|q_{m+1}|\left|\sum_{j\le m}\frac{a_{i,j}u_j}{u_{m+1}}\right|\le|q_{m+1}|\kappa.$$

Now Corollary 3.4 yields $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{w}}(\mathbb{R})$, hence $\tau_{\mathbf{u}}(\mathbb{R})\nsubseteq\tau_{\mathbf{v}}(\mathbb{R})$. □

Remark 3.11. If $A=(a_{i,j})_{i,j\in\mathbb{N}}$ is a row-finite real matrix and $0<\kappa<1$ such that $|a_{i,j}|\le\kappa|q_{j+1}|$ for every $i,j\in\mathbb{N}$ (equivalently, $\sup_{j\in\mathbb{N}}\frac{\|\mathbf{c}_j\|_\infty}{|q_{j+1}|}\le\kappa$), then the assumption $\limsup_{n\to+\infty}\frac{\|\mathbf{c}_n\|_\infty}{|q_{n+1}|}<1$ of the above theorem is satisfied, and moreover $\mathbf{c}_j\in\ell_\infty$ for every $j\in\mathbb{N}$ (see also the assumptions of Theorems 4.3 and 4.10).

In the next consequence of Theorem 3.10 we consider the case when the coefficients of the row-finite infinite matrix A are integer.

Corollary 3.12. *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R}\setminus\{0\}$ *such that* $|q_n|\to+\infty$, *let* $A=(a_{i,j})_{i,j\in\mathbb{N}}$ *be a row-finite integer matrix and* $\mathbf{v}=A\mathbf{u}$. *Assume that there exists* $0<\kappa<1$ *such that* $|a_{i,j}|\le\kappa|q_{j+1}|$ *for every* $i,j\in\mathbb{N}$ *and that:*

(a) $\sup_{i\in\mathbb{N}} |\operatorname{supp} \mathbf{r}_i| < +\infty$;
(b) $\|A\|_\infty = +\infty$.

Then $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$.

Proof. Let $l = \max_{i\in\mathbb{N}} |\operatorname{supp} \mathbf{r}_i|$. Then for every $i \in \mathbb{N}$, there are indices

$$n(i,1) < n(i,2) < \ldots < n(i,l),$$

such that $a_{i,j} = 0$ if $j \notin \{n(i,1), \ldots, n(i,l)\}$.

By (b) there exists $j \in \{1, \ldots, l\}$ such that $\sup_{i\in\mathbb{N}} |a_{i,n(i,j)}| = +\infty$; let j be maximal with this property and put $r_i = n(i,j)$ for $i \in \mathbb{N}$. Then, thanks to Remark 3.11, the hypotheses of Theorem 3.10 are satisfied, thus it yields $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$. □

We see in Theorem 4.3 that the assumption (a) in Corollary 3.12 can be cancelled.

4. Comparison of $\tau_{\mathbf{u}}(\mathbb{R})$ and $\tau_{\mathbf{v}}(\mathbb{R})$ whenever $\mathbf{v} = A\mathbf{u}$

Assumption 4.1. In this section let $\mathbf{u}$ be a sequence in $\mathbb{R} \setminus \{0\}$ such that $|q_n| \to +\infty$, where $q_n := q_n^{\mathbf{u}}$. Moreover, let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be a row-finite integer matrix and let $\mathbf{v} = A\mathbf{u}$. We also recall that we denote by $\mathbf{r}_i$ and $\mathbf{c}_j$ the i-th row and the j-th column of A, respectively.

We preliminarily explain the idea of the proofs of Theorems 4.3 and 4.5. The thesis of these theorems is $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$. As observed before Lemma 3.1, we will construct:

1. a nested sequence of compact intervals $I_n \in \mathcal{I}(u_n, \varepsilon_n)$ where $0 < \varepsilon_n \downarrow 0$;
2. and a subsequence $(v_{m_k})_{k\in\mathbb{N}}$ of $\mathbf{v}$ such that $\inf_{k\in\mathbb{N}} \|v_{m_k} x\| > 0$ for $x \in \bigcap_{n\in\mathbb{N}} I_n$.

The assumption $|q_n| \to +\infty$ will guarantee that there exists a decreasing sequence $I_n \in \mathcal{I}(u_n, \varepsilon_n)$, and therefore $x \in \tau_{\mathbf{u}}(\mathbb{R})$ for $x \in \bigcap_{n\in\mathbb{N}} I_n$.

We briefly describe how to choose the intervals I_n such that in addition $\|v_i x\|$ is large for some i and all $x \in I_{n_1}$ where $v_i = \sum_{j<n_1} a_{i,j} u_j$. Let c_n be the center of the interval I_n, let

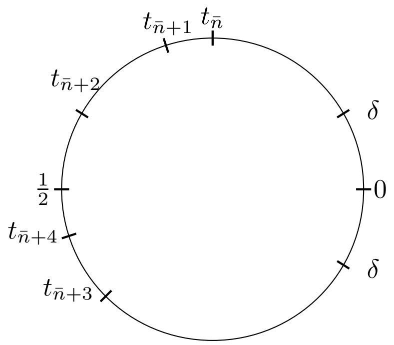
$$C_n = \sum_{j<n} a_{i,j} u_j c_n$$

and let $t_n = \{C_n\} \in [0,1)$; we can identify t_n with a point of $\mathbb{T}$.

In Theorem 4.3 we choose a suitable small $\delta > 0$ and an index i such that one of the coefficients, say $a_{i,\bar{n}}$, of the linear combination $v_i = \sum_{j<n_1} a_{i,j} u_j$ is big enough. As a first step, we show – using an idea of the proof of Theorem 3.10 – that $\|C_{\bar{n}}\| \geq \delta$. Afterwards, we choose I_n ($\bar{n} < n \leq n_1$) in such a way that

$$t_n \geq t_{n-1} \text{ if } t_{n-1} < \tfrac{1}{2} \text{ and } t_n \leq t_{n-1} \text{ otherwise;}$$

in other words, the (finite) sequence t_n first approaches $\frac{1}{2}$ and then remains close to $\frac{1}{2}$. Finally, since any $x \in I_{n_1}$ is close to c_{n_1}, one sees that $\|v_i x\|$ is close to $\|v_i c_{n_1}\| = \|t_{n_1}\|$ which is close to $\frac{1}{2}$ (see Lemma 4.2).



The proof of Theorem 4.5 is based on a similar idea. We choose an index $i \in \mathbb{N}$ such that $\operatorname{supp} \mathbf{r}_i$ is big enough, and I_n in such a way that

$$t_n \geq t_{n-1} \text{ if } t_{n-1} < \tfrac{1}{2} \text{ and } t_n \leq t_{n-1} \text{ otherwise;}$$

finally, a control of the estimation of $t_n - t_{n-1}$ allows us to prove that the (finite) sequence t_n arrives close to $\frac{1}{2}$ and then remains there.

Lemma 4.2. *Assume that* $\mathbf{u}$ *is in* $\mathbb{R}_+$, *that* $\mathbf{c}_j \in \ell_\infty$ *for every* $j \in \mathbb{N}$ *and that* $\limsup_{n\to+\infty} \frac{\|\mathbf{c}_n\|_\infty}{q_{n+1}} < \kappa$ *for some real* κ. *Then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$ *and* $i \in \mathbb{N}$,

$$\sum_{j<n} \frac{|a_{i,j}|u_j}{u_n} \leq \kappa$$

and therefore,

$$\left|\sum_{j<n} a_{i,j} u_j (x-c)\right| \leq \kappa\varepsilon$$

whenever $\varepsilon > 0$, $x \in I \in \mathcal{I}(u_n, \varepsilon)$ *and* c *is the center of* I.

Proof. By Lemma 3.9 there is $n_0 \in \mathbb{N}$ such that $\sum_{j<n} \frac{|a_{i,j}|u_j}{u_n} \leq \kappa$ for every $n \geq n_0$ and $i \in \mathbb{N}$. Hence, for $n \geq n_0$,

$$\left|\sum_{j<n} a_{i,j} u_j (x-c)\right| \leq \sum_{j<n} |a_{i,j}| u_j \cdot \frac{\varepsilon}{u_n} \leq \kappa\varepsilon,$$

where ε, I, c, x are chosen as above. □

As noted in Remark 3.11, the assumptions (b) and (c) of the following theorem are satisfied if there exists $\kappa \in [0,1)$ such that $|a_{i,j}| \leq \kappa |q_{j+1}|$ for every $i, j \in \mathbb{N}$. We consider now the case when the coefficients of the row-finite infinite integer matrix A are unbounded.

Theorem 4.3. *Suppose that:*

(a) $\|A\|_\infty = +\infty$;
(b) $\mathbf{c}_j \in \ell_\infty$ *for every* $j \in \mathbb{N}$;
(c) $\limsup_{j\to+\infty} \frac{\|\mathbf{c}_j\|_\infty}{|q_{j+1}|} < 1$.

Then $\tau_\mathbf{u}(\mathbb{R}) \nsubseteq \tau_\mathbf{v}(\mathbb{R})$.

Proof. Since $\tau_{\mathbf{u}}(\mathbb{R}) = \tau_{|\mathbf{u}|}(\mathbb{R})$ and $\tau_{\mathbf{v}}(\mathbb{R}) = \tau_{|\mathbf{v}|}(\mathbb{R})$ and moreover $|v_i| = \sum_{j\in\mathbb{N}} a_{i,j}\operatorname{sgn}(v_i \cdot u_j)|u_j|$ for all $i, j \in \mathbb{N}$, we may assume that $u_n, v_n \geq 0$ for all $n \in \mathbb{N}$.

Let $\kappa \in \mathbb{R}$ with $\limsup_{j\to+\infty} \frac{\|\mathbf{c}_j\|_\infty}{|q_{j+1}|} < \kappa < 1$ and $\varepsilon_n \downarrow 0$ such that

$$\varepsilon_n q_{n+1} \geq \frac{8}{1-\kappa}, \tag{4.1}$$

for every $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ according to Lemma 4.2; we can assume that moreover $\varepsilon_{n_0} \leq \delta := \frac{1-\kappa}{4}$. We define inductively two strictly increasing sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ of natural numbers, a decreasing sequence of compact intervals $(I_n)_{n\geq n_0}$ and a decreasing sequence $(\bar{\varepsilon}_n)_{n\geq n_0}$ of positive reals such that for every $k \in \mathbb{N} \cup \{0\}$:

(i) $\bar{\varepsilon}_{n_k} = \varepsilon_{n_k}$;
(ii) $I_n \in \mathcal{I}(u_n, \bar{\varepsilon}_n)$ if $n_k \leq n < n_{k+1}$;
(iii) $\|v_{m_k} x\| \geq \frac{\delta}{4}$ if $x \in I_{n_k}$ and $k > 0$.

Then $\varnothing \neq \bigcap_{n\geq n_0} I_n \subseteq \tau_{\mathbf{u}}(\mathbb{R}) \setminus \tau_{\mathbf{v}}(\mathbb{R})$ and this finishes the proof.

For $k = 0$ we choose $\bar{\varepsilon}_{n_0} := \varepsilon_{n_0}$, an arbitrary $I_{n_0} \in \mathcal{I}(u_{n_0}\bar{\varepsilon}_{n_0})$ and $m_0 := 1$. Assume that for some $k \in \mathbb{N} \cup \{0\}$ we have already defined n_k, m_k, and I_n and $\bar{\varepsilon}_n$ for $n_0 \leq n \leq n_k$.

Let $\varepsilon := \bar{\varepsilon}_{n_k}$. Since A is row-finite, by (a) and (b) there exist $m_{k+1} > m_k$ and $j > n_k$ such that $|a_{m_{k+1},j}|\varepsilon \geq 1 + \varepsilon$. Let

$$\bar{n} := \max\{j > n_k : |a_{m_{k+1},j}|\varepsilon \geq 1 + \varepsilon\} \tag{4.2}$$

and $n_{k+1} > \bar{n}$ such that $\varepsilon_{n_{k+1}} < \frac{\varepsilon}{4}$ and $a_{m_{k+1},j} = 0$ for $j \geq n_{k+1}$. Set

$$\bar{\varepsilon}_n := \varepsilon \text{ for } n_k < n \leq \bar{n}, \quad \bar{\varepsilon}_n := \frac{\varepsilon}{4} \text{ for } \bar{n} < n < n_{k+1} \quad \text{and} \quad \bar{\varepsilon}_{n_{k+1}} := \varepsilon_{n_{k+1}}.$$

Thanks to (4.1) and Lemma 3.1 there is, for $n_k < n \leq \bar{n}$, a decreasing sequence of intervals $I_n \in \mathcal{I}(u_n, \bar{\varepsilon}_n)$ contained in I_{n_k}. We show with the aid of Lemma 3.2 that $I_{\bar{n}}$ contains an interval $I_{\bar{n}+1} \in \mathcal{I}(u_{\bar{n}+1}, \bar{\varepsilon}_{\bar{n}+1})$ such that

$$\left\| \sum_{j\leq\bar{n}} a_{m_{k+1},j} u_j x \right\| \geq \delta \text{ for all } x \in I_{\bar{n}+1}. \tag{4.3}$$

For that, let $v := |\sum_{j\leq\bar{n}} a_{m_{k+1},j} u_j|$ and $\alpha \in \mathbb{R}$ with $v = \alpha u_{\bar{n}}$. Then by Lemma 4.2 and (4.2),

$$\kappa q_{\bar{n}+1} \geq \left| \sum_{j\leq\bar{n}} a_{m_{k+1},j} \frac{u_j}{u_{\bar{n}+1}} \right| q_{\bar{n}+1} = \alpha$$

and

$$\alpha = \left| \sum_{j\leq\bar{n}} a_{m_{k+1},j} \frac{u_j}{u_{\bar{n}}} \right| \geq |a_{m_{k+1},\bar{n}}| - \left| \sum_{j<\bar{n}} a_{m_{k+1},j} \frac{u_j}{u_{\bar{n}}} \right| \geq$$

$$\geq |a_{m_{k+1},\bar{n}}| - \kappa \geq \frac{1+\varepsilon}{\varepsilon} - \kappa > \frac{1}{\varepsilon}.$$

Moreover, $\bar{\varepsilon}_{\bar{n}+1} \leq \varepsilon \leq \delta$. It follows from Lemma 3.2 that there exists $I_{\bar{n}+1} \in \mathcal{I}(u_{\bar{n}+1}, \bar{\varepsilon}_{\bar{n}+1})$ such that $I_{\bar{n}+1} \subseteq I_{\bar{n}}$ and $\|vx\| \geq \delta$ for all $x \in I_{\bar{n}+1}$. This proves (4.3).

In what follows we denote by c_n the center of I_n,

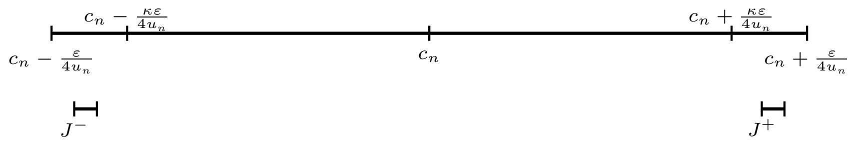
$$C_n := \sum_{j<n} a_{m_{k+1},j} u_j c_n \quad \text{and} \quad t_n := C_n - \lfloor C_n \rfloor = \{C_n\}.$$

Then $\|t_{\bar{n}+1}\| \geq \delta$ by (4.3). If $\bar{n}+1 < n_{k+1}$, we define by induction $I_n \in \mathcal{I}(u_n, \bar{\varepsilon}_n)$ for $\bar{n}+1 < n \leq n_{k+1}$ such that $\|t_n\| \geq \frac{\delta}{2}$. Assume that $\bar{n} < n < n_{k+1}$ and that I_n is defined.

Consider the two subintervals

$$I_n^- := \left[c_n - \frac{\varepsilon}{4u_n}, c_n - \frac{\kappa\varepsilon}{4u_n}\right] \quad \text{and} \quad I_n^+ := \left[c_n + \frac{\kappa\varepsilon}{4u_n}, c_n + \frac{\varepsilon}{4u_n}\right]$$

of the interval $I_n = \left[c_n - \frac{\varepsilon}{4u_n}, c_n + \frac{\varepsilon}{4u_n}\right]$. Both I_n^- and I_n^+ have length $(1-\kappa)\frac{\varepsilon}{4u_n}$, so by Lemma 3.1 and by (4.1) there exist $J^-, J^+ \in \mathcal{I}(u_{n+1}, \bar{\varepsilon}_{n+1})$ such that $J^- \subseteq I_n^-$ and $J^+ \subseteq I_n^+$.



We will choose $I_{n+1} \in \{J^-, J^+\}$ as follows.

Case $a_{m_{k+1},n} \neq 0$. Observe that for any $x \in J^+$, we have $\frac{\kappa\varepsilon}{4} \leq \varphi(u_n x) \leq \frac{\varepsilon}{4}$. Therefore, since $|a_{m_{k+1},n}|\varepsilon \leq 1+\varepsilon$ by (4.2),

$$\left|a_{m_{k+1},n}\right| u_n x \equiv_{\mathbb{Z}} \left|a_{m_{k+1},n}\right| \varphi(u_n x) \in \left[\frac{\kappa\varepsilon}{4}, \frac{1+\varepsilon}{4}\right] \subseteq \left[0, \frac{1}{2}\right[,$$

thus,

$$\operatorname{sgn}(a_{m_{k+1},n})\varphi(a_{m_{k+1},n} u_n x) = \varphi(\left|a_{m_{k+1},n}\right| u_n x) = \left|a_{m_{k+1},n}\right| \varphi(u_n x) \in \left[\frac{\kappa\varepsilon}{4}, \frac{1+\varepsilon}{4}\right].$$

Analogously, $\operatorname{sgn}(a_{m_{k+1},n})\varphi(a_{m_{k+1},n} u_n x) \in \left[-\frac{1+\varepsilon}{4}, -\frac{\kappa\varepsilon}{4}\right]$ for every $x \in J^-$. Therefore, we can choose $I_{n+1} \in \{J^-, J^+\}$ in such a way that

$$\begin{cases} \frac{\kappa\varepsilon}{4} \leq \varphi(a_{m_{k+1},n} u_n x) \leq \frac{1+\varepsilon}{4} & \text{if } 0 \leq t_n < \frac{1}{2}, \\ -\frac{1+\varepsilon}{4} \leq \varphi(a_{m_{k+1},n} u_n x) \leq -\frac{\kappa\varepsilon}{4} & \text{if } \frac{1}{2} \leq t_n < 1. \end{cases}$$

Consider first the case $0 \leq t_n < \frac{1}{2}$. Observe that

$$C_{n+1} = C_n + \left(\sum_{j<n} a_{m_{k+1},j} u_j\right)(c_{n+1} - c_n) + a_{m_{k+1},n} u_n c_{n+1}.$$

Since by Lemma 4.2

$$\left|\sum_{j<n} a_{m_{k+1},j} u_j (c_{n+1} - c_n)\right| \leq \frac{\kappa\varepsilon}{4} \tag{4.4}$$

and $\frac{\kappa\varepsilon}{4} \leq \varphi(a_{m_{k+1},n} u_n c_{n+1}) \leq \frac{1+\varepsilon}{4}$ by the choice of I_{n+1}, we have

$$0 \leq \left(\sum_{j<n} a_{m_{k+1},j} u_j\right)(c_{n+1} - c_n) + \varphi(a_{m_{k+1},n} u_n c_{n+1}) \leq \frac{\kappa\varepsilon}{4} + \frac{1+\varepsilon}{4} \leq \frac{1}{4} + \frac{\delta}{2}.$$

Together with the inductive hypothesis we obtain

$$t_{n+1} = t_n + \left(\sum_{j<n} a_{m_{k+1},j} u_j\right)(c_{n+1} - c_n) + \varphi(a_{m_{k+1},n} u_n c_{n+1}) \in \left[\frac{\delta}{2}, \frac{1}{2} + \frac{1}{4} + \frac{\delta}{2}\right] \subseteq \left[\frac{\delta}{2}, 1 - \frac{\delta}{2}\right],$$

i.e., $\|t_{n+1}\| \geq \frac{\delta}{2}$.

Analogously one shows that $\|t_{n+1}\| \geq \frac{\delta}{2}$ in the case $\frac{1}{2} \leq t_n \leq 1$.

Case $a_{m_{k+1},n} = 0$. Denote by c^- and c^+ the center of J^- and J^+, respectively. Then $c^- < c_n < c^+$ and therefore if one of the sums

$$\sum_{j<n} a_{m_{k+1},j} u_j (c^- - c_n) \quad \text{and} \quad \sum_{j<n} a_{m_{k+1},j} u_j (c^+ - c_n)$$

is positive, the other one is negative. Thus, we can choose $I_{n+1} \in \{J^-, J^+\}$ in such a way that

$$C_{n+1} - C_n = \sum_{j<n} a_{m_{k+1},j} u_j (c_{n+1} - c_n) \begin{cases} \geq 0 & \text{if } 0 \leq t_n < \frac{1}{2}, \\ < 0 & \text{if } \frac{1}{2} \leq t_n < 1. \end{cases}$$

Moreover, $|C_{n+1} - C_n| \leq \frac{\kappa\varepsilon}{4}$ by (4.4). It follows that

$$\begin{cases} \frac{\delta}{2} \leq t_n \leq t_{n+1} \leq t_n + \frac{\kappa\varepsilon}{4} \leq \frac{3}{4} & \text{if } 0 \leq t_n < \frac{1}{2}, \\ 1 - \frac{\delta}{2} \geq t_n \geq t_{n+1} \geq t_n - \frac{\kappa\varepsilon}{4} \geq \frac{1}{4} & \text{if } \frac{1}{2} \leq t_n < 1. \end{cases}$$

Hence, $\|t_{n+1}\| \geq \frac{\delta}{2}$.

Let now $x \in I_{n_{k+1}}$. Since $a_{m_{k+1},j} = 0$ for every $j \geq n_{k+1}$, we have $|v_{m_{k+1}} x - C_{n_{k+1}}| \leq \frac{\kappa\varepsilon}{4}$ by Lemma 4.2. Therefore,

$$\|v_{m_{k+1}} x\| \geq \|C_{n_{k+1}}\| - \frac{\kappa\varepsilon}{4} = \|t_{n_{k+1}}\| - \frac{\kappa\varepsilon}{4} \geq \frac{\delta}{2} - \frac{\kappa\varepsilon}{4} \geq \frac{\delta}{4},$$

as required. □

The following example shows that in the above theorem the hypothesis on the matrix A to be integer cannot be dropped.

Example 4.4. Let $u_n = n!$ and $v_n = u_{n+1}$ for $n \in \mathbb{N}$. Since $\mathbf{v}$ is a subsequence of $\mathbf{u}$, we have $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$. We can also write v_n as

$$v_n = \frac{1}{2} n \cdot u_n + \frac{n+2}{2n+2} \cdot u_{n+1},$$

i.e., $\mathbf{v} = A\mathbf{u}$ where A is a matrix satisfying all the hypotheses of Theorem 4.3 except the one to have all entries integer.

We now deal with the case where the coefficients are bounded.

Theorem 4.5. *Assume that:*

(a) $\|A\|_\infty < +\infty$*;*
(b) $\sup_{i\in\mathbb{N}} |\operatorname{supp} \mathbf{r}_i| = +\infty$*.*

Then $\tau_{\mathbf{u}}(\mathbb{R}) \nsubseteq \tau_{\mathbf{v}}(\mathbb{R})$*.*

Proof. As in Theorem 4.3 we may assume that $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}_+^{\mathbb{N}}$. Let $\varepsilon_n \downarrow 0$ be such that

$$\varepsilon_n q_{n+1} \geq 4 \text{ for every } n \in \mathbb{N}. \tag{4.5}$$

Let $s := \|A\|_\infty$. Since $\lim_{n\to+\infty} \sum_{j<n} \frac{u_j}{u_n} = 0$ by Lemma 3.8, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{j<n} \frac{u_j}{u_n} \leq \frac{1}{6s} \text{ for every } n \geq n_0. \tag{4.6}$$

We can also assume that $(s+1)\varepsilon_{n_0} \leq \frac{1}{8}$.

We define inductively two strictly increasing sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ of natural numbers and a decreasing sequence of intervals $(I_n)_{n\geq n_0}$ such that for every $k \in \mathbb{N}\cup\{0\}$:

(i) $I_n \in \mathcal{I}(u_n, \varepsilon_{n_k})$ for every $n_k \leq n < n_{k+1}$;
(ii) $\|v_{m_k} x\| \geq \frac{1}{4}$ if $x \in I_{n_k}$ and $k > 0$.

Then $\varnothing \neq \bigcap_{n\geq n_0} I_n \subseteq \tau_{\mathbf{u}}(\mathbb{R}) \setminus \tau_{\mathbf{v}}(\mathbb{R})$ and this finishes the proof.

For $k = 0$ we choose an arbitrary $I_{n_0} \in \mathcal{I}(u_{n_0}, \varepsilon_{n_0})$ and $m_0 := 1$. Assume that for some $k \in \mathbb{N}\cup\{0\}$ we have already defined n_k, m_k, and I_n for $n_0 \leq n \leq n_k$. Set $\varepsilon := \varepsilon_{n_k}$.

By (b) there exists $m_{k+1} > m_k$ such that

$$|\{j > n_k : a_{m_{k+1},j} \neq 0\}| \cdot \varepsilon \geq 2. \tag{4.7}$$

Let $n_{k+1} > n_k$ such that $a_{m_{k+1},j} = 0$ for every $j \geq n_{k+1}$.

Set $\bar{\varepsilon}_n := \varepsilon$ for $n_k < n < n_{k+1}$ and $\bar{\varepsilon}_{n_{k+1}} := \varepsilon_{n_{k+1}}$. We will define by induction, for $n_k < n \leq n_{k+1}$, a decreasing sequence of intervals $I_n \in \mathcal{I}(u_n, \bar{\varepsilon}_n)$ contained in I_{n_k}. Hereby we denote by c_n the center of I_n, $C_n := \sum_{j<n} a_{m_{k+1},j} u_j c_n$ and $t_n := C_n - \lfloor C_n \rfloor = \{C_n\}$. We will define I_n in such a way that the sequence t_n satisfies

$$\begin{aligned} &t_{n+1} \geq t_n \text{ if } 0 \leq t_n < \frac{1}{2}, \text{ and } t_{n+1} \leq t_n \text{ if } \frac{1}{2} \leq t_n < 1, \\ &\frac{\varepsilon}{3} \leq |t_{n+1} - t_n| \leq (s+1)\varepsilon \text{ if } a_{m_{k+1},n} \neq 0, \text{ and } |t_{n+1} - t_n| \leq \frac{\varepsilon}{6} \text{ if } a_{m_{k+1},n} = 0 \end{aligned} \tag{4.8}$$

for $n_k \leq n < n_{k+1}$.

Now assume to have defined I_n for some $n_k \leq n < n_{k+1}$. We will use several times that, by (4.6), for all $x \in I_n$

$$\left| \sum_{j<n} a_{m_{k+1},j} u_j (x - c_n) \right| \leq s \sum_{j<n} u_j |x - c_n| \leq s \sum_{j<n} u_j \frac{\varepsilon}{u_n} \leq \frac{\varepsilon}{6}. \tag{4.9}$$

Divide I_n in four disjoint subintervals of the same length. Denote by I_n^- and I_n^+, respectively, the first and the fourth of these subintervals. By Lemma 3.1 and (4.5), I_n^- and I_n^+, respectively, contain an interval J^- and J^+ belonging to $\mathcal{I}(u_{n+1}, \bar{\varepsilon}_{n+1})$. We choose $I_{n+1} \in \{J^-, J^+\}$ as follows.

Case $a_{m_{k+1},n} \neq 0$. Observe that for any $x \in I_n^+$, we have $\frac{\varepsilon}{2} \leq \varphi(u_n x) \leq \varepsilon$. Therefore,

$$\left|a_{m_{k+1},n}\right| u_n x \equiv_{\mathbb{Z}} \left|a_{m_{k+1},n}\right| \varphi(u_n x) \in \left[\frac{\varepsilon}{2}, s\varepsilon\right] \subseteq \left[0, \frac{1}{2}\right),$$

thus

$$\operatorname{sgn}(a_{m_{k+1},n})\varphi(a_{m_{k+1},n}u_n x) = \varphi(\left|a_{m_{k+1},n}\right| u_n x) = \left|a_{m_{k+1},n}\right| \varphi(u_n x) \in \left[\frac{\varepsilon}{2}, s\varepsilon\right].$$

Analogously, $\operatorname{sgn}(a_{m_{k+1},n})\varphi(a_{m_{k+1},n}u_n x) \in [-s\varepsilon, -\frac{\varepsilon}{2}]$ for any $x \in I_n^-$. Therefore, we can choose $I_{n+1} \in \{J^-, J^+\}$ in such a way that for any $x \in I_{n+1}$,

$$\begin{cases} \frac{\varepsilon}{2} \leq \varphi(a_{m_{k+1},n}u_n x) \leq s\varepsilon & \text{if } 0 \leq t_n < \frac{1}{2}, \\ -s\varepsilon \leq \varphi(a_{m_{k+1},n}u_n x) \leq -\frac{\varepsilon}{2} & \text{if } \frac{1}{2} \leq t_n < 1. \end{cases} \tag{4.10}$$

Consider first the case $0 \leq t_n < \frac{1}{2}$. Then we have by (4.9) and (4.10)

$$\begin{aligned} C_{n+1} &= C_n + \sum_{j<n} a_{m_{k+1},j}u_j(c_{n+1} - c_n) + a_{m_{k+1},n}u_n c_{n+1} \equiv_{\mathbb{Z}} \\ &\equiv_{\mathbb{Z}} t_n + \sum_{j<n} a_{m_{k+1},j}u_j(c_{n+1} - c_n) + \varphi(a_{m_{k+1},n}u_n c_{n+1}) \\ &\in t_n + \left[-\frac{\varepsilon}{6} + \frac{\varepsilon}{2}, \frac{\varepsilon}{6} + s\varepsilon\right] = t_n + \left[\frac{\varepsilon}{3}, \left(s + \frac{1}{6}\right)\varepsilon\right] \subseteq [0, 1). \end{aligned}$$

Therefore,

$$t_n + \frac{\varepsilon}{3} \leq t_{n+1} \leq t_n + \left(s + \frac{1}{6}\right)\varepsilon.$$

Analogously, one sees that, if $\frac{1}{2} \leq t_n < 1$, then

$$t_n - \left(s + \frac{1}{6}\right)\varepsilon \leq t_{n+1} \leq t_n - \frac{\varepsilon}{3}.$$

Case $a_{m_{k+1},n} = 0$. Denote by c^- and c^+ the center of J^- and J^+, respectively. Then $c^- < c_n < c^+$, and therefore if one of the sums

$$\sum_{j<n} a_{m_{k+1},j}u_j(c^- - c_n) \quad \text{and} \quad \sum_{j<n} a_{m_{k+1},j}u_j(c^+ - c_n)$$

is positive, the other one is negative. Thus, we can choose $I_{n+1} \in \{J^-, J^+\}$ in such a way that, since

$$C_{n+1} - C_n = \sum_{j<n} a_{m_{k+1},j}u_j(c_{n+1} - c_n),$$

$$C_{n+1} - C_n \begin{cases} \geq 0 & \text{if } 0 \leq t_n < \frac{1}{2}, \\ \leq 0 & \text{if } \frac{1}{2} \leq t_n < 1 \end{cases}$$

Moreover, $|C_{n+1} - C_n| \leq \frac{\varepsilon}{6}$ by (4.9). It follows that

$$\begin{cases} t_n \leq t_{n+1} \leq t_n + \frac{\varepsilon}{6} & \text{if } 0 \leq t_n < \frac{1}{2}; \\ t_n \geq t_{n+1} \geq t_n - \frac{\varepsilon}{6} & \text{if } \frac{1}{2} \leq t_n < 1. \end{cases}$$

Let us now prove (ii). It follows, from (4.8) and (4.7), that there exists $\bar{n} \in \mathbb{N}$ with $n_k < \bar{n} < n_{k+1}$ such that $\frac{1}{2}$ lies between $t_{\bar{n}-1}$ and $t_{\bar{n}}$. Then $\left|t_n - \frac{1}{2}\right| \leq (s+1)\varepsilon$ for all $\bar{n} \leq n \leq n_{k+1}$.

Since $v_{m_{k+1}} c_{n_{k+1}} = C_{n_{k+1}}$, it follows that

$$\left\| v_{m_{k+1}} c_{n_{k+1}} - \frac{1}{2} \right\| = \left\| t_{n_{k+1}} - \frac{1}{2} \right\| \leq (s+1)\varepsilon.$$

Moreover, for all $x \in I_{n_{k+1}}$ (compare (4.9))

$$|v_{m_{k+1}} x - v_{m_{k+1}} c_{n_{k+1}}| = \left| \sum_{j < n_{k+1}} a_{m_{k+1},j} u_j (x - c_{n_{k+1}}) \right| \leq \frac{\varepsilon_{n_{k+1}}}{6}.$$

Hence,

$$\left\| v_{m_{k+1}} x - \frac{1}{2} \right\| \leq \left\| v_{m_{k+1}} x - v_{m_{k+1}} c_{n_{k+1}} \right\| + \left\| v_{m_{k+1}} c_{n_{k+1}} - \frac{1}{2} \right\| \leq \frac{\varepsilon_{n_{k+1}}}{6} + (s+1)\varepsilon \leq \frac{1}{4},$$

and thus $\left\| v_{m_{k+1}} x \right\| \geq \frac{1}{4}$ for every $x \in I_{n_{k+1}}$. □

The following example shows that in the above theorem the hypothesis for the matrix A to be integer cannot be dropped.

Example 4.6. For $n \in \mathbb{N}$, let $u_n := n!$ and $v_n := 2u_{2n}$. Then $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$. Moreover,

$$v_n = \sum_{j=n}^{2n} \alpha_n u_j, \quad \text{where} \quad \alpha_n = 2\frac{(2n)!}{n! + \cdots + (2n)!} \in [1, 2].$$

Then $\mathbf{v} = A\mathbf{u}$, where the coefficients of the matrix $A = (a_{i,j})_{i,j}$ are $a_{i,j} = \alpha_i$ if $i \leq j \leq 2i$ and $a_{i,j} = 0$ otherwise.

It is worth stressing that in Theorems 4.3 and 4.5 the condition $\lim_{n\to+\infty} |q_n| = +\infty$ cannot be replaced by the milder one $\sup_{n\in\mathbb{N}} |q_n| = +\infty$, as items (a) and (b) in the next example witness, respectively.

Example 4.7.

(a) Choose a sequence $\mathbf{u}$ in $\mathbb{R}_+$ such that $\sup_{n\in\mathbb{N}} q_n = +\infty$ and infinitely many times $u_{n+1} = u_n + 1$. Let A be a diagonal matrix with $a_{i,i} = \frac{1}{2} q_{i+1}$ for every $i \in \mathbb{N}$ and $\mathbf{v} = A\mathbf{u}$. Then $\tau_{\mathbf{u}}(\mathbb{R}) = \{0\} \subseteq \tau_{\mathbf{v}}(\mathbb{R})$.

(b) Put

$$A := \begin{pmatrix} 210000000000000\ldots \\ 000211000000000\ldots \\ 000000021110000\ldots \\ \ldots \end{pmatrix}.$$

Denote by k_n the indices of the columns of A which are identically zero, that is, $k_n = k_{n-1} + n + 2$ for $n \geq 1$, where $k_0 := 0$. Let $\mathbb{N} \ni p_n \to +\infty$. Moreover, let $q_m = p_n$ if $m = k_n + 1$ for some $n \in \mathbb{N}$ and $q_m = 2$ otherwise. Finally, $u_n := \prod_{i=1}^{n} q_i$ for $n \in \mathbb{N}$. Then for $\mathbf{v} := A\mathbf{u}$ we have $v_n = u_{k_n}$, since $v_n = \sum_{j=k_{n-1}+1}^{k_n - 1} a_{n,j} u_j$ and $u_{j+1} = 2u_j$ for every $j = k_{n-1}+1, \dots, k_n - 1$. Therefore, $\mathbf{v}$ is a subsequence of $\mathbf{u}$ and thus $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$.

Our next aim is Theorem 4.10. One implication of the equivalence given there is obvious:

Lemma 4.8 *(see [23, Lemma 2.1]). If the matrix A satisfies the conditions*

(C) $\mathbf{c}_j \in c_{00}$ *for every* $j \in \mathbb{N}$,
(R) $\sup_{i\in\mathbb{N}} \|\mathbf{r}_i\|_1 < +\infty$,

then $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$.

Proof. Let $x \in \tau_{\mathbf{u}}(\mathbb{R})$ and $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\|u_j x\| \leq \varepsilon$ for all $j \geq n_0$. By (C) there exists $k_0 \in \mathbb{N}$ such that $a_{i,j} = 0$ for $i \geq k_0$ and $j < n_0$. Therefore, for all $i \geq k_0$ we have $\|v_i x\| = \|\sum_{j\in\mathbb{N}} a_{i,j} u_j x\| = \|\sum_{j\geq n_0} a_{i,j} u_j x\| \leq \|\mathbf{r}_i\|_1 \cdot \varepsilon \leq s\varepsilon$ where $s := \sup_{i\in\mathbb{N}} \|\mathbf{r}_i\|_1$. This proves that $x \in \tau_{\mathbf{v}}(\mathbb{R})$. □

The next proposition gives assumptions under which condition (R) in Lemma 4.8 implies condition (C).

Proposition 4.9. *Assume that the matrix A satisfies the condition*

$$\textit{for all } i, n \in \mathbb{N},\ \sum_{j\leq n} a_{i,j} u_j = 0 \textit{ implies } a_{i,j} = 0 \textit{ for every } j \leq n,$$

If $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$, *then (R) implies (C).*

Proof. Since $\tau_{\mathbf{u}}(\mathbb{R})$ has size $\mathfrak{c}$ by Remark 2.1, it contains a number $x \neq 0$ such that $\sum_{j\leq n} a_{i,j} u_j x \neq b$ for all $n, i \in \mathbb{N}$ and $b \in \mathbb{Z} \setminus \{0\}$. By (R) there exists $m \in \mathbb{N}$ with $m \geq \sup_{i\in\mathbb{N}} \|\mathbf{r}_i\|_1$. Therefore,

$$a_{i,j} \in \{-m, -m+1, \dots, 0, 1, \dots, m-1, m\} \text{ for every } i, j \in \mathbb{N}. \tag{4.11}$$

By way of contradiction suppose that there exists $n \in \mathbb{N}$ such that $\mathbf{c}_n \notin c_{00}$. Starting with $A^{(0)} := A$ we define matrices $A^{(i)}$ $(i \leq l)$ such that $A^{(i)}$ is obtained by erasing rows from $A^{(i-1)}$ and $A^{(i)}$ has still infinitely many rows. Let n_1 be the smallest index with $\mathbf{c}_{n_1} \notin c_{00}$. It follows from (4.11) that $\mathbf{c}_{n_1}$ has infinitely many entries equal to the same integer $a_{n_1} \neq 0$. We can erase some rows of A in such a way that the entries of the column with index n_1 of the new matrix $A^{(1)}$ are all equal to a_{n_1} and $A^{(1)}$ has still infinitely many rows. Let n_2 be the smallest index $> n_1$ such that the column of $A^{(1)}$ with index n_2 does not belong to c_{00} if such a column exists. As before define $A^{(2)}$ by erasing rows from $A^{(1)}$ in such a way that the entries of the column with index n_2 of the new matrix $A^{(2)}$ are all equal to an integer $a_{n_2} \neq 0$ and $A^{(2)}$ has still infinitely many rows. It follows from (R) that this process stops after l steps for some $l \leq m$, i.e., the columns of $A^{(l)}$ with index $n \neq n_k$ $(k = 1, \dots, l)$ belong to c_{00}. Observe that $\mathbf{w} := A^{(l)}\mathbf{u}$ is a subsequence of $\mathbf{v}$, thus $x \in \tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R}) \subseteq \tau_{\mathbf{w}}(\mathbb{R})$. Therefore, we may assume, to simplify the notation, that $A = A^{(l)}$, i.e., that for every $k = 1, \dots, l$ the entries of $\mathbf{c}_{n_k}$ are all equal to a_{n_k} and $\mathbf{c}_j \in c_{00}$ for $j \neq n_k$.

Let $\varepsilon > 0$ and $n_0 > n_l$ such that $\|u_j x\| \leq \frac{\varepsilon}{m}$ for $j \geq n_0$. Choose $i \in \mathbb{N}$ such that $\|v_i x\| \leq \varepsilon$ and $a_{i,j} = 0$ for $j \leq n_0$ different from $n_1, \dots, n_l$. Then

$$\left\| \left(\sum_{k=1}^{l} a_{n_k} u_{n_k} \right) x \right\| = \left\| \sum_{j<n_0} a_{i,j} u_j x \right\| \leq \|v_i x\| + \left\| \sum_{j\geq n_0} a_{i,j} u_j x \right\| \leq \|v_i x\| + \|\mathbf{r}_i\|_1 \frac{\varepsilon}{m} \leq 2\varepsilon.$$

This shows that $\|(\sum_{k=1}^{l} a_{n_k} u_{n_k})x\| \leq \varepsilon$ for every $\varepsilon > 0$.

Therefore, $\|(\sum_{k=1}^{l} a_{n_k} u_{n_k})x\| = 0$, consequently we have $(\sum_{k=1}^{l} a_{n_k} u_{n_k})x \in \mathbb{Z}$. Thus, $\sum_{k=1}^{l} a_{n_k} u_{n_k} = 0$ by the choice of x. By hypothesis we obtain $a_{n_k} = 0$ for every $k = 1, \ldots, l$, a contradiction. □

We are now in position to prove the second main result of the paper.

Theorem 4.10. *Assume that:*

(a) for all $i, n \in \mathbb{N}$, $\sum_{j \leq n} a_{i,j} u_j = 0$ implies $a_{i,j} = 0$ for every $j \leq n$;
(b) $\mathbf{c}_j \in \ell_\infty$ for every $j \in \mathbb{N}$;
(c) $\limsup_{j \to +\infty} \frac{\|\mathbf{c}_j\|_\infty}{|q_{j+1}|} < 1$.

Then $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ if and only if (C) and (R) hold.

Proof. If (C) and (R) hold, then $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ by Lemma 4.8.

Now assume that $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$. By Proposition 4.9 it is enough to show that condition (R) holds true. It follows from Theorem 4.3 that $\|A\|_\infty = \sup_{i,j \in \mathbb{N}} |a_{i,j}| < +\infty$. This together with Theorem 4.5 yields $\sup_{i \in \mathbb{N}} |\operatorname{supp} \mathbf{r}_i| < +\infty$. These two conditions imply (R). □

Corollary 4.11. *Assume that there exists $0 < \kappa < 1$ such that*

$$\left| \sum_{j \leq n} a_{i,j} u_j \right| \leq \kappa \cdot |u_{n+1}|$$

for every n, $i \in \mathbb{N}$. Then $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ if and only if (C) and (R) hold.

Proof. We have to check that the hypotheses (a), (b) and (c) of Theorem 4.10 are satisfied. By Lemma 3.9, items (b) and (c) hold; it remains to verify (a). To this end, let $i, n \in \mathbb{N}$ and assume that $\sum_{j \leq n} a_{i,j} u_j = 0$ and $a_{i,j} \neq 0$ for some $j \leq n$. Let $r = \max\{j \leq n : a_{i,j} \neq 0\}$. Then

$$0 = \sum_{j \leq r} a_{i,j} \frac{u_j}{u_r} = \sum_{j < r} a_{i,j} \frac{u_j}{u_r} + a_{i,r},$$

hence $a_{i,r} = -\sum_{j<r} a_{i,j} \frac{u_j}{u_r}$. Since $\left| \sum_{j<r} a_{i,j} \frac{u_j}{u_r} \right| \leq \kappa < 1$ and $a_{i,r} \in \mathbb{Z}$, it follows that $a_{i,r} = 0$, a contradiction. □

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