View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Archivio istituzionale della ricerca - Università degli Studi di Udine

arXiv:1311.1788v1 [math.AP] 7 Nov 2013

# Non-critical dimensions for critical problems involving fractional Laplacians

Roberta Musina* and Alexander I. Nazarov†

**Abstract**

We study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m > 0$.

**Keywords:** Fractional Laplace operators, Sobolev inequality, Hardy inequality, critical dimensions.

## 1 Introduction

Let $m, s$ be two given real numbers, with $0 \le s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in $\mathbb{R}^n$ and put

$$2^*_m = \frac{2n}{n-2m}.$$

We study equations

$$(-\Delta)^m u = \lambda(-\Delta)^s u + |u|^{2^*_m-2}u \quad \text{in } \Omega, \tag{1.1}$$

$$(-\Delta)^m u = \lambda|x|^{-2s} u + |u|^{2^*_m-2}u \quad \text{in } \Omega, \tag{1.2}$$

*Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 2009WRJ3W7-001 "Fenomeni di concentrazione e problemi di analisi geometrica".

†St.Petersburg Department of Steklov Institute, Fontanka 27, St.Petersburg, 191023, Russia, and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com. Supported by RFBR grant 11-01-00825 and by St.Petersburg University grant 6.38.670.2013.

under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that $\Omega$ contains the origin. For the definition of fractional Dirichlet–Laplace operators $(-\Delta)^m, (-\Delta)^s$ and for the variational approach to (1.1), (1.2) we refer to the next section.

The celebrated paper [3] by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case $n > 2$, $m = 1$, $s = 0$, that is,

$$-\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter $\lambda$: they proved existence of a nontrivial solution for any small $\lambda > 0$ if $n \geq 4$; in contrast, in the lowest dimension $n = 3$ non-existence phenomena for sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical*[1] for problem (1.3).

Clearly, as larger $s$ is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: *Given $m < \frac{n}{2}$, how large must be $s$ in order to have the existence of a ground state solution, for any arbitrarily small $\lambda > 0$ ?* In case of an affirmative answer, we say that $n$ is <u>not</u> a critical dimension.

We present our main result, that holds for any dimension $n \geq 1$ (see Section 4 for a more precise statement).

**THEOREM.** *If $s \geq 2m - \frac{n}{2}$ then $n$ is not a critical dimension for the Dirichlet boundary value problems associated to equations (1.1) and (1.2).*

We point out some particular cases that are included in this result.

- If $m$ is an integer and $s = m - 1$, then at most the lowest dimension $n = 2m + 1$ is critical.

- For any $n > 2m$ there always exist lower order perturbations of the type $|x|^{-2s}u$ and of the type $(-\Delta)^s u$ such that $n$ is not a critical dimension.

- If $m < 1/4$ then no dimension is critical, for any choice of $s \in [0, m)$.

[1] compare with [13], [8].

After [3], a large number of papers have been focussed on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with $s = 0$, when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case $2 \le m \in \mathbb{N}$, see for instance [13], [6], [2], [10], [7], or the case $m \in (0, 1)$, see [14], [15]. We cite also [4], where equation (1.1) is studied in case $m = 2$, $s = 1$. Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [16]) where equation (1.1) for the so-called Navier-Laplacian is studied in case $m \in (0, 1)$, $s = 0$. For a comparison between the Dirichlet and Navier Laplacians we refer to [12].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2, we provide the main estimates in Section 3. In Section 4 we prove Theorem 1 and point out an existence result for the case $s < 2m - \frac{n}{2}$.

## 2 Preliminaries

The fractional Laplacian $(-\Delta)^m u$ of a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined via the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx$$

by the identity

$$\mathcal{F}\left[(-\Delta)^m u\right](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi). \tag{2.1}$$

In particular, Parseval's formula gives

$$\int\limits_{\mathbb{R}^n} (-\Delta)^m u \cdot u \, dx = \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx = \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2 \, d\xi \,.$$

We recall the well known Sobolev inequality

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx \ge \mathcal{S}_m \Big( \int\limits_{\mathbb{R}^n} |u|^{2^*_m} \, dx \Big)^{2/2^*_m}, \tag{2.2}$$

that holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $m < \frac{n}{2}$, see for example [17, 2.8.1/15].

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the Hilbert space obtained by completing $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\|u\|_m^2 = \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx. \tag{2.3}$$

Thanks to (2.2), the space $\mathcal{D}^m(\mathbb{R}^n)$ is continuously embedded into $L^{2^*_m}(\mathbb{R}^n)$. The *best Sobolev constant* $\mathcal{S}_m$ was explicitly computed in [5]. Moreover, it has been proved in [5] that $\mathcal{S}_m$ is attained in $\mathcal{D}^m(\mathbb{R}^n)$ by a unique family of functions, all of them being obtained from

$$\phi(x) = (1 + |x|^2)^{\frac{2m-n}{2}} \tag{2.4}$$

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $R > 0$ it turns out that

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega](\xi)|^2 \, d\xi \quad &= \quad R^{n-2m} \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega(R\cdot)](\xi)|^2 \, d\xi \\ \int\limits_{\mathbb{R}^n} |\omega|^{2^*_m} \, dx \quad &= \quad R^n \int\limits_{\mathbb{R}^n} |\omega(R\cdot)|^{2^*_m} \, dx \, . \end{aligned} \tag{2.5}$$

Finally, we point out that the Hardy inequality

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx \ge \mathcal{H}_m \int\limits_{\mathbb{R}^n} |x|^{-2m} |u|^2 \, dx \tag{2.6}$$

holds for any function $u \in \mathcal{D}^m(\mathbb{R}^n)$. The *best Hardy constant* $\mathcal{H}_m$ was explicitly computed in [11].

The natural ambient space to study the Dirichlet boundary value problems for (1.1), (1.2) is

$$\widetilde{H}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n) \, : \, \operatorname{supp} u \subset \overline{\Omega}\},$$

endowed with the norm $\|u\|_m$. By Theorem 4.3.2/1 [17], for $m + \frac{1}{2} \notin \mathbb{N}$ this space coincides with $H_0^m(\Omega)$ (that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^m(\Omega)$), while for $m + \frac{1}{2} \in \mathbb{N}$ one has $\widetilde{H}^m(\Omega) \subsetneq H_0^m(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\widetilde{H}^m(\Omega)$. Clearly, if $m$ is an integer then $\widetilde{H}^m(\Omega)$ is the standard Sobolev space of functions $u \in H^m(\Omega)$ such that $D^\alpha u = 0$ for every multiindex $\alpha \in \mathbb{N}^n$ with $0 \le |\alpha| < m$.

We agree that $(-\Delta)^0 u = u$, $\widetilde{H}^0(\Omega) = L^2(\Omega)$, since (2.3) reduces to the standard $L^2$ norm in case $m = 0$.

We define (weak) solutions of the Dirichlet problems for (1.1), (1.2) as suitably normalized critical points of the functionals

$$\mathcal{R}^{\Omega}_{\lambda,m,s}[u] = \frac{\int\limits_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx - \lambda \int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx}{\left( \int\limits_{\Omega} |u|^{2^*_m} dx \right)^{2/2^*_m}} \tag{2.7}$$

$$\widetilde{\mathcal{R}}^{\Omega}_{\lambda,m,s}[u] = \frac{\int\limits_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2 \, dx - \lambda \int\limits_{\Omega} |x|^{-2s} |u|^2 \, dx}{\left( \int\limits_{\Omega} |u|^{2^*_m} dx \right)^{2/2^*_m}}, \tag{2.8}$$

respectively. It is easy to see that both functionals (2.7), (2.8) are well defined on $\widetilde{H}^m(\Omega) \setminus \{0\}$.

We conclude this preliminary section with some embedding results.

**Proposition 2.1** *Let $m, s$ be given, with $0 \le s < m < n/2$.*

*i) The space $\widetilde{H}^m(\Omega)$ is compactly embedded into $\widetilde{H}^s(\Omega)$. In particular the infima*

$$\Lambda_1(m,s) := \inf_{\substack{u \in \widetilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\|u\|_s^2} \quad , \qquad \widetilde{\Lambda}_1(m,s) := \inf_{\substack{u \in \widetilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\||x|^{-s} u\|_0^2} \tag{2.9}$$

*are positive and achieved.*

*ii)* $\displaystyle \inf_{\substack{u \in \widetilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\|u\|_{L^{2^*_m}}^2} = \mathcal{S}_m.$

Statement *i)* is well known for $\Lambda_1(m,s)$ and follows from (2.6) for $\widetilde{\Lambda}_1(m,s)$. To check *ii)*, use the inclusion $\widetilde{H}^m(\Omega) \hookrightarrow \mathcal{D}^m(\mathbb{R}^n)$ and a rescaling argument. Clearly, the Sobolev constant $\mathcal{S}_m$ is never achieved on $\widetilde{H}^m(\Omega)$.

## 3 Main estimates

Let $\phi$ be the extremal of the Sobolev inequality (2.2) given by (2.4). In particular, it holds that

$$M := \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}\phi|^2\,dx = \mathcal{S}_m\Big(\int\limits_{\mathbb{R}^n} |\phi|^{2^*_m}\,\,dx\Big)^{2/2^*_m}. \tag{3.1}$$

Fix $\delta > 0$ and a cutoff function $\varphi \in \mathcal{C}^\infty_0(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside $\{|x| < 2\delta\}$. If $\delta$ is sufficiently small, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n}\varphi(x)\phi\left(\frac{x}{\varepsilon}\right) = \varphi(x)\left(\varepsilon^2 + |x|^2\right)^{\frac{2m-n}{2}}$$

has compact support in $\Omega$. Next we define

$$A^\varepsilon_m := \int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}}u_\varepsilon|^2 dx \qquad A^\varepsilon_s := \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 dx$$

$$\widetilde{A}^\varepsilon_s := \int\limits_\Omega |x|^{-2s}|u_\varepsilon|^2 dx \qquad B^\varepsilon := \int\limits_\Omega |u_\varepsilon|^{2^*_m} dx$$

and we denote by $c$ any universal positive constant.

**Lemma 3.1** *It holds that*

$$\begin{cases} A^\varepsilon_m \le \varepsilon^{2m-n}\left(M + c\varepsilon^{n-2m}\right) & (3.2a)\\ A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\varepsilon^{4m-n-2s} & \textit{if } s > 2m - \frac{n}{2} \quad (3.2b)\\ A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\,|\log\varepsilon| & \textit{if } s = 2m - \frac{n}{2} \quad (3.2c)\\ B^\varepsilon \ge \varepsilon^{-n}\left((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\varepsilon^n\right). & (3.2d)\end{cases}$$

**Proof of (3.2a).** First of all, from (2.5) we get

$$A^\varepsilon_m = \varepsilon^{2m-n}\int\limits_{\mathbb{R}^n} |\xi|^{2m}\,|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]|^2\,d\xi. \tag{3.3}$$

Thus

$$\Gamma^\varepsilon_m := \varepsilon^{n-2m}A^\varepsilon_m - M = \int\limits_{\mathbb{R}^n} |\xi|^{2m}\,|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]|^2\,d\xi - \int\limits_{\mathbb{R}^n} |\xi|^{2m}\,|\mathcal{F}[\phi]|^2\,d\xi.$$

We need to prove that

$$|\Gamma_m^\varepsilon| \le c\varepsilon^{n-2m}. \tag{3.4}$$

If $m \in \mathbb{N}$, the proof of (3.4) has been carried out in [3], [7]. Here we limit ourselves to the more difficult case, namely, when $m$ is not an integer. We denote by $k := \lfloor m \rfloor \ge 0$ the integer part of $m$, so that $m - k > 0$. Then

$$\begin{aligned}\Gamma_m^\varepsilon &= \int\limits_{\mathbb{R}^n} |\xi|^{2k} \mathcal{F}[U_-] \cdot |\xi|^{2(m-k)} \overline{\mathcal{F}[U_+]}\, d\xi \\ &= 2^{2(m-k)+\frac{n}{2}} \frac{\Gamma(m-k+\frac{n}{2})}{\Gamma(-(m-k))} \cdot \int\limits_{\mathbb{R}^n} (-\Delta)^k U_-(x) \cdot V.P. \int\limits_{\mathbb{R}^n} \underbrace{\frac{U_+(x) - U_+(y)}{|x-y|^{n+2(m-k)}}}_{\Psi(x,y)}\, dy\, dx,\end{aligned}$$

where $U_\pm = \varphi(\varepsilon\ \cdot\ )\phi \pm \phi$ (the last equality follows from [9, Ch. 2, Sec. 3]).

We split the interior integral as follows:

$$V.P. \int\limits_{\mathbb{R}^n} \Psi dy = \underbrace{V.P. \int\limits_{|y-x| \le \frac{|x|}{2}} \Psi dy}_{I_1} + \underbrace{\int\limits_{\substack{|y-x| \ge \frac{|x|}{2} \\ |y| \le |x|}} \Psi dy}_{I_2} + \underbrace{\int\limits_{\substack{|y-x| \ge \frac{|x|}{2} \\ |y| \ge |x|}} \Psi dy}_{I_3}.$$

We claim that $|I_j| \le c|x|^{2k-n}$ for $j = 1, 2, 3$. Indeed, the Lagrange formula gives

$$\begin{aligned}|I_1| \le \max_{|y-x| \le \frac{|x|}{2}} |D^2 U_+(y)| \cdot \int\limits_{|z| \le \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)-2}} \\ \le c|x|^{-(n-2m+2)} \cdot |x|^{2-2(m-k)} = c|x|^{2k-n}.\end{aligned}$$

As concerns the last two integrals we estimate

$$|I_2| \le \int\limits_{\substack{|y-x| \ge \frac{|x|}{2} \\ |y| \le |x|}} \frac{c|y|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\, dy \le |x|^{-(n+2(m-k))} \cdot c|x|^{2m} = c|x|^{2k-n}$$

and finally

$$\begin{aligned}|I_3| \le \int\limits_{\substack{|y-x| \ge \frac{|x|}{2} \\ |y| \ge |x|}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\, dy \ &\le\ c|x|^{-(n-2m)} \cdot \int\limits_{|z| \ge \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)}} \\ &\le\ c|x|^{-(n-2m)} \cdot |x|^{-2(m-k)} = c|x|^{2k-n},\end{aligned}$$

and the claim follows. Now, since

$$|(-\Delta)^k U_-(x)| \le \frac{c}{|x|^{n-2(m-k)}}\,\chi_{\{|x|\ge\delta/\varepsilon\}} + \frac{c\varepsilon^{2k}}{|x|^{n-2m}}\,\chi_{\{\delta/\varepsilon\le|x|\le 2\delta/\varepsilon\}},$$

we obtain

$$|\Gamma_m^\varepsilon| \le c \int\limits_{|x|\ge\delta/\varepsilon} \frac{dx}{|x|^{2n-2m}} + c \int\limits_{\delta/\varepsilon\le|x|\le 2\delta/\varepsilon} \frac{\varepsilon^{2k}\,dx}{|x|^{2n-2(m+k)}} \le c\varepsilon^{n-2m},$$

that completes the proof of (3.4) and of (3.2a).

**Proof of (3.2b) and (3.2c).** We use the Hardy inequality (2.6) to get

$$\begin{aligned} A_s^\varepsilon &\ge c\widetilde{A}_s^\varepsilon \ge c\varepsilon^{4m-2s-n} \int\limits_{\mathbb{R}^n} |x|^{-2s} |\varphi(\varepsilon\,\cdot)\phi|^2 dx \\ &\ge c\varepsilon^{4m-2s-n} \int\limits_{|x|<\delta/\varepsilon} \frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}\,. \end{aligned}$$

The last integral converges as $\varepsilon \to 0$ if $s > 2m - \frac{n}{2}$, and diverges with speed $|\log\varepsilon|$ if $s = 2m - \frac{n}{2}$.

**Proof of (3.2d).** For $\varepsilon$ small enough we estimate by below

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |u_\varepsilon|^{2_m^*} &= \varepsilon^{-n} \int\limits_{\mathbb{R}^n} |\varphi(\varepsilon\,\cdot)\phi|^{2_m^*}\,dx = \varepsilon^{-n}\Big( \int\limits_{\mathbb{R}^n} |\phi|^{2_m^*}\,dx - \int\limits_{|x|>\delta/\varepsilon} |\varphi(\varepsilon\,\cdot)\phi|^{2_m^*}\,dx \Big) \\ &\ge \varepsilon^{-n}\Big( (M\mathcal{S}_m^{-1})^{2_m^*/2} - c \int\limits_{|x|>\delta/\varepsilon} |x|^{-2n}\,dx \Big) \\ &= \varepsilon^{-n}((M\mathcal{S}_m^{-1})^{2_m^*/2} - c\varepsilon^n) \end{aligned}$$

and the Lemma is completely proved. □

# 4 Two noncompact minimization problems

In this section we deal with the minimization problems

$$\mathcal{S}_\lambda^\Omega(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne 0}} \mathcal{R}_{\lambda,m,s}^\Omega[u]; \qquad \widetilde{\mathcal{S}}_\lambda^\Omega(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne 0}} \widetilde{\mathcal{R}}_{\lambda,m,s}^\Omega[u]\,,$$

where the functionals $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are introduced in (2.7) and (2.8), respectively.

**Lemma 4.1** *The following facts hold for any* $\lambda \in \mathbb{R}$*:*

*i)* $\mathcal{S}_\lambda^\Omega(m,s) \le \mathcal{S}_m$*;*

*ii) If* $\lambda \le 0$ *then* $\mathcal{S}_\lambda^\Omega(m,s) = \mathcal{S}_m$ *and it is not achieved;*

*iii) If* $0 < \mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$*, then* $\mathcal{S}_\lambda^\Omega(m,s)$ *is achieved.*

*The same statements hold for* $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ *instead of* $\mathcal{S}_\lambda^\Omega(m,s)$*.*

**Proof.** The proof is nowdays standard, and is essentially due to Brezis and Nirenberg [3]. We sketch it for the infimum $\mathcal{S}_\lambda^\Omega(m,s)$, for the convenience of the reader.

Fix $\varepsilon > 0$ and take $u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$(\mathcal{S}_m + \varepsilon)\Big( \int\limits_{\mathbb{R}^n} |u|^{2^*_m} dx \Big)^{2/2^*_m} \ge \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \; dx. \tag{4.1}$$

Let $R > 0$ be large enough, so that $u_R(\cdot) := u(R\cdot) \in \mathcal{C}_0^\infty(\Omega)$. Using (2.5) we get

$$\mathcal{S}_\lambda^\Omega(m,s) \le \frac{\|u\|_m^2 - \lambda R^{2(s-m)} \|u\|_s^2}{\|u\|^2_{L^{2^*_m}}} \le (\mathcal{S}_m + \varepsilon)\left(1 + cR^{2(s-m)}\right),$$

where $c$ depends only on $u$ and $\lambda$. Letting $R \to \infty$ we get $\mathcal{S}_\lambda^\Omega(m,s) \le (\mathcal{S}_m + \varepsilon)$ for any $\varepsilon > 0$, and *i*) is proved.

Next, if $\lambda \le 0$ then clearly $\mathcal{S}_\lambda^\Omega(m,s) = \mathcal{S}_m$. If $\lambda = 0$ then $\mathcal{S}_m$ is not achieved. The more it is not achieved for $\lambda < 0$, and *ii*) holds.

Finally, to prove *iii*) take a minimizing sequence $u_h$. It is convenient to normalize $u_h$ with respect to the $L^{2^*_m}$-norm, so that

$$\int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}} u_h|^2 \, dx - \lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx = \mathcal{S}_\lambda^\Omega(m,s) + o(1).$$

We can assume that $u_h \to u$ weakly in $\widetilde{H}^m(\Omega)$ and strongly in $\widetilde{H}^s(\Omega)$ by Proposition 2.1. Since

$$\begin{aligned} \lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u|^2 \, dx &= \lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx + o(1) \\ &= \int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}} u_h|^2 \, dx - \mathcal{S}_\lambda^\Omega(m,s) + o(1) \\ &\ge (\mathcal{S}_m - \mathcal{S}_\lambda^\Omega(m,s)) + o(1), \end{aligned}$$

then $u \neq 0$. By the Brezis–Lieb lemma we get

$$1 = \|u_h\|^{2^*_m}_{L^{2^*_m}} = \|u_h - u\|^{2^*_m}_{L^{2^*_m}} + \|u\|^{2^*_m}_{L^{2^*_m}} + o(1).$$

Thus

$$\begin{aligned}
\mathcal{S}^\Omega_\lambda(m,s) &= \|u_h\|^2_m - \lambda\|u_h\|^2_s + o(1) \\
&= \Big(\|u_h - u\|^2_m + \|u\|^2_m\Big) - \lambda\Big(\|u_h - u\|^2_s + \|u\|^2_s\Big) + o(1) \\
&= \frac{\Big(\|u_h - u\|^2_m - \lambda\|u_h - u\|^2_s\Big) + \Big(\|u\|^2_m - \lambda\|u\|^2_s\Big)}{\Big(\|u_h - u\|^{2^*_m}_{L^{2^*_m}} + \|u\|^{2^*_m}_{L^{2^*_m}}\Big)^{2/2^*_m}} + o(1) \\
&\geq \mathcal{S}^\Omega_\lambda(m,s) \cdot \frac{\xi^2_h + 1}{(\xi^{2^*_m}_h + 1)^{2/2^*_m}} + o(1),
\end{aligned}$$

where we have set

$$\xi_h := \frac{\|u_h - u\|_{L^{2^*_m}}}{\|u\|_{L^{2^*_m}}}.$$

Since $2^*_m > 2$, this implies that $\xi_h \to 0$, that is, $u_h \to u$ in $L^{2^*_m}$ and hence $u$ achieves $\mathcal{S}^\Omega_\lambda(m,s)$. □

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

**Theorem 4.2** *Assume* $s \geq 2m - \frac{n}{2}$.

*i)* *If* $0 < \lambda < \Lambda_1(m,s)$ *then* $\mathcal{S}^\Omega_\lambda(m,s)$ *is achieved and (1.1) has a nontrivial solution in* $\widetilde{H}^m(\Omega)$.

*ii)* *If* $0 < \lambda < \widetilde{\Lambda}_1(m,s)$ *then* $\widetilde{\mathcal{S}}^\Omega_\lambda(m,s)$ *is achieved and (1.2) has a nontrivial solution in* $\widetilde{H}^m(\Omega)$.

**Proof.** Since $0 < \lambda < \Lambda_1(m,s)$ then $\mathcal{S}^\Omega_\lambda(m,s)$ is positive, by Proposition 2.1. The main estimates in Lemma 3.1 readily imply $\mathcal{S}^\Omega_\lambda(m,s) < \mathcal{S}_m$. By Lemma 4.1, $\mathcal{S}^\Omega_\lambda(m,s)$ is achieved by a nontrivial $u \in \widetilde{H}^m(\Omega)$, that solves (1.1) after multiplication by a suitable constant. Thus *i*) is proved. For *ii*) argue in the same way. □

In the case $s < 2m - \frac{n}{2}$ the situation is more complicated. We limit ourselves to point out the next simple existence result.

**Theorem 4.3** *Assume* $s < 2m - \frac{n}{2}$.

*i) There exists* $\lambda^* \in [0, \Lambda_1(m,s))$ *such that the infimum* $\mathcal{S}_\lambda^\Omega(m,s)$ *is attained for any* $\lambda \in (\lambda^*, \Lambda_1(m,s))$, *and hence (1.1) has a nontrivial solution.*

*ii) There exists* $\widetilde{\lambda}^* \in [0, \widetilde{\Lambda}_1(m,s))$ *such that the infimum* $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ *is attained for any* $\lambda \in (\widetilde{\lambda}^*, \widetilde{\Lambda}_1(m,s))$, *and hence (1.2) has a nontrivial solution.*

**Proof.** Use Proposition 2.1 to find $\varphi_1 \in \widetilde{H}^m(\Omega)$, $\varphi_1 \neq 0$, such that

$$\int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}}\varphi_1|^2\,dx = \Lambda_1(m,s)\int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}}\varphi_1|^2\,dx\,.$$

Then test $\mathcal{S}_\lambda^\Omega(m,s)$ with $\varphi_1$ to get the strict inequality $\mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$. The first conclusion follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. □

## References


[1] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations **252** (2012), no. 11, 6133–6162.

[2] F. Bernis and H.-C. Grunau, Critical exponents and multiple critical dimensions for polyharmonic operators, J. Differential Equations **117** (1995), no. 2, 469–486.

[3] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[4] P. Caldiroli and R. Musina, On Caffarelli-Kohn-Nirenberg-type inequalities for the weighted biharmonic operator in cones, Milan J. Math. **79** (2011), no. 2, 657–687.

[5] A. Cotsiolis and N. K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. **295** (2004), no. 1, 225–236.

[6] D. E. Edmunds, D. Fortunato and E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator, Arch. Rational Mech. Anal. **112** (1990), no. 3, 269–289.

[7] F. Gazzola, Critical growth problems for polyharmonic operators, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), no. 2, 251–263.

[8] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, 1991, Springer, Berlin, 2010.

[9] I. M. Gelfand and G. E. Shilov, *Generalized functions. V. 1. Properties and operations*, Moscow, FML, 1958 (Russian); English transl.: Boston, MA: Academic press, 1964.

[10] H.-C. Grunau, Critical exponents and multiple critical dimensions for polyharmonic operators. II, Boll. Un. Mat. Ital. B (7) **9** (1995), no. 4, 815–847.

[11] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, Comm. Math. Phys. **53** (1977), no. 3, 285–294.

[12] R. Musina and A. I. Nazarov, On fractional Laplacians, Comm. Partial Differential Equations, to appear. Preprint arXiv:1308.3606 (2013).

[13] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, J. Math. Pures Appl. (9) **69** (1990), no. 1, 55–83.

[14] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, Trans. Amer. Math. Soc., to appear (2012).

[15] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal. **12** (2013), no. 6, 2445–2464.

[16] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, Calc. Var. Partial Differential Equations **42** (2011), no. 1-2, 21–41.

[17] H. Triebel, *Interpolation theory, function spaces, differential operators*, Deutscher Verlag Wissensch., Berlin, 1978.