Algorithm of iterative transformation for effective modules of multicomponent isotropic composite

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Abstract. We consider the effective modules of Voigt, Reiss for isotropic elastic composites. We have reformed the method for constructing iterative transformation of the upper and lower estimates of fork (Voigt-Reuss) towards two-component composite in case of an arbitrary number of components. The method is based on the fact that effective modules of Voigt and Reuss can be regarded as elementary symmetric functions introduced by Gauss. The conditions, which the iteratively – transformed efficient modules must fulfill at every iteration, are shown.

1. Introduction

One of the basic problems in the mechanics of composite materials is the determination of effective properties for non-homogeneous elastic bodies. To solve this problem, we apply the theory of the effective module that represents the method of calculation of the elastic characteristics for a homogeneous comparison environment, in which the potential energy is as close as possible to the energy of the elastic composite. At this, the elastic modules of the basic material and the inclusions are taken as inputs to solve the problem.

The solution of this problem allows obtaining estimates of the stress state of structures, without the need for studies related to consideration of the structure of composites. This capability is important for designing the real constructive elements, for example [1, 2].

In addition to the theory of effective module, there is an approach based on the analysis of the geometry for inclusions of elastic composite. This approach is described in numerous publications. Let us name just some of them: [3-6].

As for the theory of effective modules, the models of Voigt and Reuss (VR) [7], Hashin and Shtrikman (HS) [8, 9], as well as model Krishera [10], developed in relation to the problems of thermal conductivity, are the basic mathematical models for definition effective elastic modules of the composite which does not take into account the geometry of the inclusions. The iteratively - converted effective modules for bi-component elastic composites were first introduced in [11, 12]. Despite the fact that the mathematical model has not received enough physical basis, calculations of parameters of the stress-strain state showed greater accuracy of the new model compared to the classical approach.

The main purpose of this paper is to reform the results [11, 12, 14-16] of iterative conversion of the effective modules of Voigt and Reuss for the case of multi-component composites.

2. An algorithm for constructing iterative conversions of the effective modules of isotropic multicomponent composite

An algorithm of constructing the new effective modules for the two-component elastic composite.
Iterative transformations of the upper and lower bounds of Hashin-Shtrikman (H-S) modules were constructed in [11].

We will explain the essence of the method using of the effective modules by Voigt and Reuss. Let us suppose that we have the two-component composite with shear modules $G_1 > G_2$ and volume contents γ_1, γ_2, and $\gamma_1 + \gamma_2 = 1$.

The expressions of Voigt (V) and Reuss (R) modules have the form:

$$G_V = \gamma_1 G_1 + \gamma_2 G_2, \tag{1}$$

$$G_R = \left(\frac{\gamma_1}{G_1} + \frac{\gamma_2}{G_2} \right)^{-1}. \tag{2}$$

The easily verifiable inequalities take place between modules G_1, G_2 and G_v, G_R:

$$G_1 \geq G_V \geq G_R \geq G_2. \tag{3}$$

Thus, model V-R, defined by relations (1), (2), produces modules which are located inside fork $G_1 > G_2$. The question is as follows: can we get a new compression of the fork by applying iterative transformation?
Let us introduce the new modules of such type as Voigt and Reuss:

$$G_V^1 = \gamma_1 G_V + \gamma_2 G_R, \; G_R^1 = \left(\frac{\gamma_1}{G_V} + \frac{\gamma_2}{G_R} \right)^{-1}. \tag{4}$$

We can easily show that the inequalities are performed as follows:

$$G_1 \geq G_V \geq G_V^1 \geq G_R^1 \geq G_R \geq G_2 \tag{5}$$

Thus, expressions (4) produce compression already in the two forks: fork $G_1 > G_2$ and fork $G_V > G_R$. Relationships (4) can be easily generalized to the case of *n* steps of iterative transformations:

$$G_V^n = \gamma_1 G_v^{n-1} + \gamma_2 G_R^{n-1}, \; G_R^n = \left(\frac{\gamma_1}{G_V{}^{n-1}} + \frac{\gamma_2}{G_R{}^{n-1}} \right)^{-1}. \tag{6}$$

We can prove that the chain of inequalities is performed:

$$G_1 \geq G_V \geq G_V^1 \geq \ldots G_V^n \geq G_R^n \geq \ldots \geq G_R^1 \geq G_R \geq G_2. \tag{7}$$

The restrictions which are superimposed on the model of iterative transformation.
When building type sequences (6), we use an approach that is applied in determining the elementary symmetric Gaussian functions [13]. Let us suppose we have $a_0 \geq b_0 \geq 0$. Next, let us find:

$$a_{n+1} = \frac{1}{2}\left(a_n + b_n\right), \; b_{n+1} = \sqrt{a_n b_n}, \; n = 0,1,\ldots.$$

Then, the inequalities of types (5), (7), are valid for a_n, b_n:

$$a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots \geq b_n \geq \ldots \geq b_0$$

The limit of sequences $\{a_n\}$, $\{b_n\}$, [14], is equal to some function $M\left(a_0, b_0\right)$.
For the three positive numbers $a_0 \geq b_0 \geq c_0$, we have:

$$a_{n+1} = \frac{1}{3}\left(a_n + b_n + c_n\right), \; b_{n+1} = \sqrt{\frac{a_n b_n + a_n c_n + b_n c_n}{3}}, \; c_{n+1} = \sqrt[3]{a_n b_n c_n}.$$

The authors of [14] argue that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = M\left(a_0, b_0, c_0\right).$$

Thus, the tasks of transforming sequences of positive numbers and sequences of effective modules in many ways have common patterns. The difference lies in the fact that the effective modules comprise a quantity of positive numbers each of which is multiplied by some weight multiplier of less than 1. Constructions of convergent sequences have the general requirements:

- positivity of each component at each step;
- fulfillment of the inequalities of kinds (5) and (7) at each step of iterative transformations.

In this case, obviously, the convergence of sequences effective modules to the limit, which depends only on the initial value of the sequences (from $\gamma_1 G_1, \gamma_2 G_2$) for the case of a two-component composite, can be proved.

3. An algorithm of constructing the new effective modules for three-, four-, five-component composites

Let us consider a generalization of the mathematical model (6) for the case of multi-component elastic isotropic composites. We will implement all reasoning by the example of the V-R effective modules.

Let us consider the inequality of the form:

$$G_V \geq \sqrt{G_V G_R} \geq G_R \tag{8}$$

For a three-component composite, along with the effective modules of Voigt and Reis:

$$G_V = \gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3,$$

$$G_R = \left(\frac{\gamma_1}{G_1} + \frac{\gamma_2}{G_2} + \frac{\gamma_3}{G_3} \right)^{-1}.$$

We will introduce a module that can be called the geometric mean:

$$G_a = \sqrt{G_V G_R} \tag{9}$$

Next, we will construct a sequence of the iteratively – reformed effective modules for three-component composite. The first iteration will look like:

$$G_V^1 = \gamma_1 G_V + \gamma_2 G_a + \gamma_3 G_R,$$

$$G_R^1 = \left(\frac{\gamma_1}{G_V} + \frac{\gamma_2}{G_a} + \frac{\gamma_3}{G_R} \right)^{-1}, \tag{10}$$

$$G_a^1 = \sqrt{G_V^1 G_R^1}.$$

It is easy to verify the fulfillment of inequalities:

$$G_1 \geq G_V \geq G_V^1 \geq G_a^1 \geq G_a \geq G_R^1 \geq G_R \geq G_3$$

For *n* iterations, we have:

$$G_V^n = \gamma_1 G_v^{n-1} + \gamma_2 G_a^{n-1} + \gamma_3 G_R^{n-1},$$

$$G_R^n = \left(\frac{\gamma_1}{G_V^{n-1}} + \frac{\gamma_2}{G_a^{n-1}} + \frac{\gamma_3}{G_R^{n-1}} \right)^{-1}, \tag{11}$$

$$G_a^n = \sqrt{G_V^n G_R^n}.$$

For the four-component composite, we need the expressions of the two geometric mean modules.
Let us suppose that we have:

$$G_1 \geq G_2 \geq G_3 \geq G_4,$$

$$G_V = \gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3 + \gamma_4 G_4,$$

$$G_R = \left(\frac{\gamma_1}{G_1} + \frac{\gamma_2}{G_2} + \frac{\gamma_3}{G_3} + \frac{\gamma_4}{G_4} \right)^{-1}.$$

Let us introduce:

$$G_a = \sqrt{G_R G_V}\,,\; G_b = \sqrt{G_a G_R}\,. \tag{12}$$

Let us prove the fulfillment of inequalities:

$$G_V \geq G_a \geq G_b \geq G_R\,.$$

We have:

$$\frac{G_V}{G_a} = \frac{G_V}{\sqrt{G_V}\sqrt{G_R}} = \sqrt{\frac{G_V}{G_R}} > 1\,,$$

$$\frac{G_a}{G_b} = \frac{\sqrt{G_V G_R}}{\sqrt{G_R\sqrt{G_R G_V}}} = \frac{\sqrt{G_V}}{\sqrt[4]{G_V G_R}} = \sqrt[4]{\frac{G_V}{G_R}} > 1\,,$$

$$\frac{G_b}{G_R} = \frac{\sqrt{G_R\sqrt{G_R G_V}}}{G_R} = \sqrt[4]{\frac{G_V}{G_R}} > 1\,.$$

We will write down the expressions of the iterated effective modules for the *n*-th iteration:

$$G_V^n = \gamma_1 G_V^{n-1} + \gamma_2 G_a^{n-1} + \gamma_3 G_b^{n-1} + \gamma_4 G_R^{n-1}\,,$$

$$G_R^n = \left(\frac{\gamma_1}{G_V^{n-1}} + \frac{\gamma_2}{G_a^{n-1}} + \frac{\gamma_3}{G_b^{n-1}} + \frac{\gamma_4}{G_R^{n-1}}\right)^{-1}\,, \tag{13}$$

$$G_a^n = \sqrt{G_R^n G_V^n}\,,\; G_b^n = \sqrt{G_a^n G_R^n}\,.$$

It is easy to prove the fulfillment of inequalities:

$$G_V^n \geq G_a^n \geq G_b^n \geq G_R^n\,. \tag{14}$$

We will consider a five-component composite. Let us suppose that we have an environment which is characterized by the five modules of shift:

$$G_1 > G_2 > G_3 > G_4 > G_5\,,$$

$$G_V = \gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3 + \gamma_4 G_4 + \gamma_5 G_5\,,$$

$$G_R = \left(\frac{\gamma_1}{G_1} + \frac{\gamma_2}{G_2} + \frac{\gamma_3}{G_3} + \frac{\gamma_4}{G_4} + \frac{\gamma_5}{G_5}\right)^{-1}\,.$$

In addition, we introduce three modules (geometrical mean) of the next kind:

$$G_c = \sqrt{G_V G_R}\,,\; G_a = \sqrt{G_V G_c}\,,\; G_b = \sqrt{G_c G_R}\,.$$

It is easy to prove the inequalities:

$$G_V \geq G_a \geq G_c \geq G_b \geq G_R\,.$$

For *n* iterations, we have:

$$G_V^n = \gamma_1 G_V^{n-1} + \gamma_2 G_a^{n-1} + \gamma_3 G_c^{n-1} + \gamma_4 G_b^{n-1} + \gamma_5 G_R^{n-1}\,,$$

$$G_R^n = \left(\frac{\gamma_1}{G_V^{n-1}} + \frac{\gamma_2}{G_a^{n-1}} + \frac{\gamma_3}{G_c^{n-1}} + \frac{\gamma_4}{G_b^{n-1}} + \frac{\gamma_5}{G_R^{n-1}}\right)^{-1}\,, \tag{15}$$

$$G_c^n = \sqrt{G_V^n G_R^n}\,,\; G_a^n = \sqrt{G_V^n G_c^n}\,,\; G_b^n = \sqrt{G_c^n G_R^n}\,.$$

Similarly, we can construct the sequences of the iterated effective modules for *n* components, where $n > 5$.

4. The discussion of the results

The construction of the sequences of iteratively converted effective modules (EM) of the V-R model, carried out in this paper, includes the following necessary steps.

a. The localization of new effective modules having properties similar to the symmetric Gaussian functions. As shown in item 2, the construction of the inertial sequences of two variables requires two symmetric functions. For this, we can take the expressions of effective modules (V and R). And we must have three symmetric functions in case of three variables, etc. Since the tasks of building of the inertial sequences of some positive rational numbers are the same as the problem of constructing iterative sequences of EM, for the case of three-component composite in addition to the EM of V and R, we need to have one more additional effective module (G_{add}), which should satisfy the inequality:

$$G_V \geq G_{add} \geq G_R$$

For the four- and five-component composites we must have already 2 or 3 additional EM. The modules from the family of the geometric mean act as additional EM. Formulas (10), (11) and (14) give their visual representations.

b. At each stage of the introduction of new additional EM, we have to check the inequalities, which are similar to the inequalities that are performed when constructing of transformations of the symmetric Gaussian functions. These inequalities are shown in the complexes of (11), (13) and (14). The proof of their fulfillment is displayed, too.

The property of convergence of the various sequences to the same limit is one of the most remarkable properties of the iterated effective modules. So, the two-component effective modules of V and R model are reduced to the limit, which can be found by the numerical and analytical ways. The analytical limit of sequences for EM of V- R was found and analyzed by the authors (the mathematical calculations, obtained in deriving the results, are very bulky and this material was not included in this work). It turned out that, first, the limits of analytical and numerical sequences EM V and R coincide completely. Secondly, the calculation of the analytical limit confirmed reasoning of Gauss that the limit of the sequences of symmetric functions and sequences of EM depend only on their initial values (that is, on values $\gamma_1 G_1$, $\gamma_2 G_2$ which characterize the two-component composite). Third, it was found that the analytical limit of two sequences of Voigt and Reuss EM equals to the geometric mean of G_V and G_R which are defined by (4). For the case of three or more components, finding the analytical limit of sequences for iterative-converted EM is a more complicated mathematical problem. However, the latter does not mean that its solution cannot be found.

5. Conclusion

Expressions of the new effective characteristics of multicomponent elastic composites, presented in this paper, are constructed in accordance with the requirements for the iterative transformation of a symmetric mean [14]. They have the following properties:

a. The new efficient characteristics are positive for each step of iterative transformations;

b. The values all of iterated modules lay within the fork, which is formed by the maximum and minimum elastic shear modules of phases.

Recurrence relations, which determine the number of geometric mean modules needed to build iterative transformations, have been obtained. For the three-component composite , we have to have one geometric mean module for the four-component composite – two geometric mean modules, for the five-component composite - three, and so on. Numerical analysis of convergence for sequences of the iterated effective modules and calculations of the stress-strain state of structures with new effective characteristics are given in the second part of this work.

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