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On One Class of Dual Problems of Mechanics of Deformable Solids

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Abstract. Inelastic body's plane deformation is described by two vector fields: vector stress potential (gradient of Airy stress function) and vector displacement field. Conditions for possibility of proceeding to the dual problem, when variables change the roles, are described: stress potential is interpreted as displacement field and vice versa. Both a perfectly plastic body model and its dual model of perfectly solidifying matter are considered.

Introduction

Mathematical models of solids' deformation are based on two general concepts: displacement vector and stress tensor. Vector displacement field yields kinematic characteristics of stress; and stress tensor yields power and dynamic ones. In other words, kinematics and dynamics of the matter are described qualitatively by different mathematical objects. In the first case we have to use a first-rank tensor (vector), in the second case - a second-rank tensor (tensor itself). Accordingly there is an asymmetry in the description. It is possible, however, to find a class of problems, when both power and kinematic descriptions are symmetrical and can be described with the aid of two vector fields. Moreover this class of problems is quite extended. The main idea of this work is that, since we are dealing with objects of identical rank, then after solving the equation, their roles can be interchanged. That is, the stress vector field can be regarded as the kinematic field, and vice versa. Thus we get the solution of a new (dual or conjugate) problem.Mathematical models of solids' deformation are based on two general concepts: displacement vector and stress tensor. Vector displacement field yields kinematic characteristics of stress; and stress tensor yields power and dynamic ones. In other words, kinematics and dynamics of the matter are described qualitatively by different mathematical objects. In the first case we have to use a first-rank tensor (vector), in the second case - a second-rank tensor (tensor itself). Accordingly there is an asymmetry in the description. It is possible, however, to find a class of problems, when both power and kinematic descriptions are symmetrical and can be described with the aid of two vector fields. Moreover this class of problems is quite extended. The main idea of this work is that, since we are dealing with objects of identical rank, then after solving the equation, their roles can be interchanged. That is, the stress vector field can be regarded as the kinematic field, and vice versa. Thus we get the solution of a new (dual or conjugate) problem.

1. Formulation of primal and dual problems.

Let us assume that deformation is plane, and inertial forces and weight can be neglected. Then we have equations for a wide range of deformable continua:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}=0\,,\ \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}=0\,, \tag{1}$$

$$\frac{\partial u_1}{\partial x_1}=a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{13}\sigma_{12}\,,$$
$$\frac{\partial u_2}{\partial x_2}=a_{21}\sigma_{11}+a_{22}\sigma_{22}+a_{23}\sigma_{12}\,, \tag{2}$$
$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=a_{31}\sigma_{11}+a_{32}\sigma_{22}+a_{33}\sigma_{12}\,,$$

where 0x1x2 – Cartesian coordinate system, σ_{ij} and u_i - components of stress, displacement or their velocities, $i,j=1,2,$ coefficients $a_{11},\ldots$ are set. Notations of models (1), (2) are classical. However, in many respects they are "abnormal". [1] For example, five first-order differential equations are reduced to one fourth-order equation only. A natural notation can be obtained by inserting the vector field instead of the stress field $\{p_1,p_2\}$- vector potential of the stress field:

$$\sigma_{11}=\frac{\partial p_2}{\partial x_2},\sigma_{22}=\frac{\partial p_1}{\partial x_1},\sigma_{12}=-\frac{\partial p_2}{\partial x_1}=-\frac{\partial p_1}{\partial x_2}\,, \tag{3}$$

$$\frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1}=0\,,\ \frac{\partial u_1}{\partial x_1}=a_{11}\frac{\partial p_2}{\partial x_2}+a_{12}\frac{\partial p_1}{\partial x_1}-a_{13}\frac{\partial p_1}{\partial x_2}\,,$$
$$\frac{\partial u_2}{\partial x_2}=a_{21}\frac{\partial p_2}{\partial x_2}+a_{22}\frac{\partial p_1}{\partial x_1}-a_{23}\frac{\partial p_1}{\partial x_2}\,, \tag{4}$$
$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=a_{31}\frac{\partial p_2}{\partial x_2}+a_{32}\frac{\partial p_1}{\partial x_1}-a_{33}\frac{\partial p_1}{\partial x_2}\,.$$

The system is noted in two vector fields $\overline{p}=\{p_1,p_2\}$ and $\overline{u}=\{u_1,u_2\}$. Suppose that some its solution is built up. As mentioned, the main idea is to interchange roles of vectors $\overline{p}$ and $\overline{u}$: vector $\overline{u}$ correlates with the stress field, and vector $\overline{p}$- with the displacement field of the dual problem. Clearly, this can only be done when $\overline{p}$ and $\overline{u}$ are included in the system in some symmetrical manner. Thus, it is necessary that one of the displacement equations would have the structure of the system's first equation (4). Therefore it is necessary to consider the case of incompressible continuum:

$$\frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1}=0\,,\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0\,,$$
$$\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}=c_1\frac{\partial p_2}{\partial x_2}-c_2\frac{\partial p_1}{\partial x_1}-c_3\frac{\partial p_1}{\partial x_2}\,, \tag{5}$$
$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=d_1\frac{\partial p_2}{\partial x_2}+d_2\frac{\partial p_1}{\partial x_1}-d_3\frac{\partial p_1}{\partial x_2}\,,$$

where coefficients $c_1,\ldots,d_3$ are set.

Let us examine the boundary conditions. Suppose $\bar{n}$ is an external boundary normal, s is its natural parameter, α is an angle between the axis 0x1 and a normal $\bar{n}$. Stress vector components on the area with the normal $\bar{n}$ are equal to:

$$\sigma_{n1} = \sigma_{11}\cos\alpha + \sigma_{12}\sin\alpha, \quad \sigma_{n2} = \sigma_{12}\cos\alpha + \sigma_{22}\cos\alpha \tag{6}$$

Substitution of (3) into (6) gives the following result:

$$\sigma_{n1} = \frac{\partial p_2}{\partial s}, \sigma_{n2} = -\frac{\partial p_1}{\partial s}.$$

Consequently, accurate within a constant, setting of the vector border $\{p_1,p_2\}$ is equivalent to setting of the stress vector $\{\sigma_{n1}, \sigma_{n2}\}$. Here, kinematic boundary conditions remain unchanged:

$u_1 = u_1(s)$, $u_2 = u_2(s)$.

Also more complex conditions, such as dry friction, Winkler elastic foundation and others, are possible. They all are reduced to inter-component relations p_1, p_2, u_1, u_2 , set on the border.

Let us now turn to the dual problem. Relevant variables will be denoted by ˜ "tilde".

$$\tilde{p}_1 = -u_2, \tilde{p}_2 = u_1, \tilde{u}_1 = -p_2, \tilde{u}_2 = p_1,$$
$$\tilde{\sigma}_{11} = \frac{\partial \tilde{p}_2}{\partial x_2}, \tilde{\sigma}_{22} = \frac{\partial \tilde{p}_1}{\partial x_1}, \tilde{\sigma}_{12} = -\frac{\partial \tilde{p}_2}{\partial x_1} = -\frac{\partial \tilde{p}_1}{\partial x_2}. \tag{7}$$

System of equations (5) does not change by such a substitution and takes the following form:

$$\frac{\partial \tilde{u}_2}{\partial x_2} + \frac{\partial \tilde{u}_1}{\partial x_1} = 0, \frac{\partial \tilde{p}_2}{\partial x_1} - \frac{\partial \tilde{p}_1}{\partial x_2} = 0,$$
$$-2\tilde{\sigma}_{12} = -c_1\frac{\partial \tilde{u}_1}{\partial x_2} - c_2\frac{\partial \tilde{u}_2}{\partial x_1} - c_3\frac{\partial \tilde{u}_2}{\partial x_2}, \tag{8}$$
$$\tilde{\sigma}_{11} - \tilde{\sigma}_{22} = -d_1\frac{\partial \tilde{u}_1}{\partial x_2} + d_2\frac{\partial \tilde{u}_2}{\partial x_1} - d_3\frac{\partial \tilde{u}_2}{\partial x_2}.$$

Here one new factor arises. The original system (1), (2) is invariant with reference to rotation of the body as a rigid solid. This is a necessary correctness condition of any problem on solid deformation. Therefore, system (5) will be invariant. However, after the change of roles, invariant variables $\frac{1}{2}\left(\frac{\partial \tilde{u}_2}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_2}\right)$ may not occur with reference to a new turn. The dual problem would be invariant if the system does not depend on $\{\tilde{u}_1, \tilde{u}_2\}$ vector's rotor, that is, on the following combination of derivatives $\frac{\partial \tilde{u}_2}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_2} = \frac{\partial p_1}{\partial x_1} + \frac{\partial p_2}{\partial x_2} = \sigma_{22} + \sigma_{11}$.

Hence it follows that $c_1 = c_2 = c$, $d_1 = - d_2 = d$.

Thus, the direct problem is reduced to solving of the equations

$$\frac{\partial p_1}{\partial x_2} - \frac{\partial p_2}{\partial x_1} = 0, \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0,$$
$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = c\cdot\left(\frac{\partial p_2}{\partial x_2} - \frac{\partial p_1}{\partial x_1}\right) + c_3\frac{\partial p_1}{\partial x_2},$$
$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = d\cdot\left(\frac{\partial p_2}{\partial x_2} - \frac{\partial p_1}{\partial x_1}\right) - d_3\frac{\partial p_1}{\partial x_2}.$$

The equations of the dual problem take the form

$$\frac{\partial \tilde{u}_2}{\partial x_2} + \frac{\partial \tilde{u}_1}{\partial x_1} = 0 \ , \frac{\partial \tilde{p}_2}{\partial x_1} - \frac{\partial \tilde{p}_1}{\partial x_2} = 0 \ ,$$

$$\frac{\partial \tilde{p}_2}{\partial x_1} + \frac{\partial \tilde{p}_1}{\partial x_2} = c \cdot \left(-\frac{\partial \tilde{u}_1}{\partial x_2} - \frac{\partial \tilde{u}_2}{\partial x_1} \right) + c_3 \frac{\partial \tilde{u}_2}{\partial x_2},$$

$$\frac{\partial \tilde{p}_2}{\partial x_2} - \frac{\partial \tilde{p}_1}{\partial x_1} = d \cdot \left(-\frac{\partial \tilde{u}_1}{\partial x_2} - \frac{\partial \tilde{u}_2}{\partial x_1} \right) - d_3 \frac{\partial \tilde{u}_2}{\partial x_2} .$$

The limitation, obtained above, is acceptable. It means that in the original problem principal stresses' maximum displacement and orientation should not depend on hydrostatic compression.

2. Model of ideal plasticity.

This model and its generalizations nave been studied completely and are widely used for engineering problems [2,3]. The ideal plasticity's closed equation system can be written down as follows:

$$\frac{\partial p_1}{\partial x_2} - \frac{\partial p_2}{\partial x_1} = 0 \ , \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \ ,$$

$$\left(\frac{\partial p_2}{\partial x_2} - \frac{\partial p_1}{\partial x_1} \right)^2 + \left(\frac{\partial p_1}{\partial x_2} + \frac{\partial p_2}{\partial x_1} \right)^2 = 4\tau_s^2 \ , \qquad (9)$$

$$\frac{\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}}{\dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2}} = -\frac{\dfrac{\partial p_2}{\partial x_1} + \dfrac{\partial p_1}{\partial x_2}}{\dfrac{\partial p_2}{\partial x_2} - \dfrac{\partial p_1}{\partial x_1}}.$$

where u_1 and u_2 are components of velocity vector, $\boldsymbol{\tau}_s$ is yield stress.

Let us consider the dual problem.

$$\frac{\partial \tilde{u}_2}{\partial x_2} + \frac{\partial \tilde{u}_1}{\partial x_1} = 0 \ , \frac{\partial \tilde{p}_2}{\partial x_1} - \frac{\partial \tilde{p}_1}{\partial x_2} = 0 \ ,$$

$$\left(-\frac{\partial \tilde{u}_1}{\partial x_2} - \frac{\partial \tilde{u}_2}{\partial x_1} \right)^2 + \left(\frac{\partial \tilde{u}_2}{\partial x_2} - \frac{\partial \tilde{u}_1}{\partial x_1} \right)^2 = 4\gamma^2 \ , \qquad (10)$$

$$\frac{\tilde{\sigma}_{11} - \tilde{\sigma}_{22}}{2\tilde{\sigma}_{12}} = \frac{\dfrac{\partial \tilde{u}_1}{\partial x_1} - \dfrac{\partial \tilde{u}_2}{\partial x_2}}{\dfrac{\partial \tilde{u}_1}{\partial x_2} + \dfrac{\partial \tilde{u}_2}{\partial x_1}}.$$

Here, the constant τs is denoted by γ. Thus, in the dual problem, conditions of incompressibility, coaxiality and constancy 2γ of deformation field displacement velocity are met.

Some similarities (9), (10) can be seen in a simple example. Let us take (9) a uniform stress distribution and a linear velocity distribution, i.e. consider a case of perfectly plastic body affine deformation:

$$\sigma_{11} = \frac{\partial p_2}{\partial x_2} = \tau_s, \sigma_{22} = \frac{\partial p_1}{\partial x_1} = -\tau_s, \sigma_{12} = -\frac{\partial p_2}{\partial x_1} = -\frac{\partial p_1}{\partial x_2} = 0.$$

From (9) it follows that

$$p_2 = \tau_s x_2,\ p_1 = -\tau_s x_1,\ u_1 = \gamma \cdot x_1 - \Omega \cdot x_2,\ u_2 = \Omega \cdot x_1 - \gamma \cdot x_2. \quad (11)$$

Here 2γ is deformation velocity, which (for a perfectly plastic body) is set with border conditions, Ω is a rate of rotation. In [4] it is shown that kinematics (11) is realized under complex loading, when the body is continuously rotated at a rate of Ω in the direction of expansion and contraction.

Such deformations occur in cases of tidal deformation when the body rotates continuously towards the disturbing body.

In addition, study of such strains is of interest for pressure processing of inelastic materials [5,6]. In [4] it is shown that if Ω >γ, elliptic area self-converting corresponds to kinematics (11).

Wherein, on the ellipse border, velocity is directed tangentially to the border; and its value is subjected to Kepler's Law: the radius vector, connecting the ellipse center and the boundary point, encloses equal areas for equal time. Let us proceed to the dual problem. Substituting (7) into (11) gives the following result: $\tilde{u}_1 = -\tau_s x_2, \tilde{u}_2 = \tau_s x_1, \tilde{p}_1 = -\Omega \cdot x_1 + \gamma \cdot x_2,\ \tilde{p}_2 = \gamma \cdot x_1 - \Omega \cdot x_2.$

$$\tilde{\sigma}_{11} = \frac{\partial \tilde{p}_2}{\partial x_2} = -\Omega, \tilde{\sigma}_{22} = \frac{\partial \tilde{p}_1}{\partial x_1} = -\Omega, \tilde{\sigma}_{12} = -\frac{\partial \tilde{p}_2}{\partial x_1} = -\frac{\partial \tilde{p}_1}{\partial x_2} = -\gamma.$$

Thus, rotation uncertainty Ω in the original problem corresponds to additive hydrostatic compression uncertainty in the dual task; shear uncertainty 2γ in a perfectly plastic body corresponds to shear stress uncertainty in a perfectly solidifying body. Conditions of incompressibility and coaxiality are retained in both problems. More trivial examples can be constructed, using exact solutions, outlined in [7] and [8].

3. Summary

The concept of the vector potential was introduced in a number of works under different names. First it appears in [9], after independently (and for other reasons) in [1,10]. A more general case is examined in [11]. Issues of duality are considered in [9,11,12,13]. Despite the fundamental character of these works, potential possibilities of this field of research are far from being exhausted. Thus, condition of shear velocity constancy (the second equation in (10)) is the key in problems of optimal blasting fragmentation of rocks [14,15]. In [16,12] condition of maximum shear velocity constancy has been studied in the framework of the theory of ideal solidified substance with use of duality with reference to equations of ideal plasticity. In [17,18], this model has been studied in connection with problems of brittle fracture. These articles also contain additional lists of scientific publications.

4. Conclusions

1. Description of inelastic body plane deformation can be reduced to a system of four differential equations coupling components of vector potential stress and displacement vector field.

2. Under certain conditions it is possible to proceed to the dual problem when roles of the noted vector fields are mutually changed.

3. The model of perfectly solidifying body is dual in relation to the model of perfect plasticity.

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