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On Accuracy Order of Fourier Coefficients Computation for Periodic Signal Processing Models

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Abstract. The article is devoted to construction piecewise constant functions for modelling periodic signal. The aim of the paper is to suggest a way to avoid discontinuity at points where waveform values are obtained. One solution is to introduce shifted step function whose middle points within its partial intervals coincide with points of observation. This means that large oscillations of Fourier partial sums move to new jump discontinuities where waveform values are not obtained. Furthermore, any step function chosen to model periodic continuous waveform determines a way to calculate Fourier coefficients. In this case, the technique is certainly a weighted rectangular quadrature rule. Here, the weight is either unit or trigonometric. Another effect of the solution consists in following. The shifted function leads to application midpoint quadrature rules for computing Fourier coefficients. As a result the formula for zero coefficient transforms into trapezoid rule. In the same time, the formulas for other coefficients remain of rectangular type.

1. Introduction

Fourier series is applied to expand function that describes a waveform of periodic signal. As soon as the function is represented with its values obtained at discrete points, we have to substitute it with another function. The process of discretization of continuous wave is known as a time sampling. The new function is usually a step one. Its values in each sampling interval are equal to the values at the left limit of the interval. Trigonometric polynomial as a partial sum of series is a tool for modelling initial waveform [1], [2], [3]. One problem concerning truncated Fourier series consists in increasing amplitude of oscillations at jump discontinuities. The problem is known as the Gibbs phenomenon. There are various points of view to discuss the issue and resolve it [4], [5], [6], [7]. In the paper we consider continuity and discontinuity in points where data are obtained. To avoid the maximum of amplitude at the points of observation we suggest to shift the step function. As a result, the function becomes continuous at these points. In any case, when step function is chosen for sampling, an algorithm of calculation of Fourier coefficients appears as a certain quadrature rule. Suggested shift of step function leads to distinctions between used quadrature rules. As previous works of one of the authors are devoted to numerical integration and its error estimation [8], [9], we emphasize computational aspect of the issue. According to stated, in the article, we consider quadrature rules of order 0 and 1.

This paper is organized as follows.

Firstly, we set initial conditions for the two models. This way, we construct two piecewise constant functions matching the two models. Also we consider formulas for computation Fourier coefficients concerning the two cases.

Secondly, we consider weighted quadrature rules of low algebraic order. At first, we give the rules, that are accurate on all constants, then we perform rules, that are accurate on all linear functions. At the same time, we consider elementary and composite weighted quadrature rules.

Lastly, we show that each formula of Fourier coefficient matches certain quadrature rule. In addition, we discuss the order of approximation.

2. Two models for waveform

In this section, we introduce two step functions. One of them contains points of observed values as jump discontinuities. Another contains points of observed values as points of its continuity.

To begin with, we set the function of signal waveform. Suppose $f^*(t)$ is an unknown function which describes a periodic signal waveform, and $T = 2l$ is its period. Denote by $[a,b]$ the fundamental period of $f^*(t)$, where $b = a + 2l$.

Further, we describe general features of approximating function as a step function. Suppose $f(t)$ is a step function which approximates $f^*(t)$. Let t be an argument that presents a time variable ($t > 0$), and let t_k be the points where observed values of $f^*(t)$ are obtained. Denote by N the number of sampling intervals, therefore the number of points of observation is $N + 1$. By y_k denote the observed values: $y_k = f^*(t_k)$, $k = 0,\ldots,N$. Let t_k be equidistant: $t_{k+1} - t_k = h$, $k = 0,\ldots,N-1$, so $t_k = a + kh$, $k = 0,\ldots,N$. In particular, $t_0 = a$, $t_N = b$.

We give more details of setting step functions in subsections 2.1 and 2.2 in order to specific cases.

2.1. Top-left corner model

In this subsection, we introduce model with piecewise constant function, where the points of observed values are jump discontinuities.

Suppose $f(t) = y_k$, $t \in [t_k, t_{k+1}] = [a + kh, a + (k+1)h]$, $k = 0,\ldots,N-1$. Then Fourier coefficients for the step function are

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(t)\,\mathrm{d}t = \frac{1}{l}\sum\nolimits_{k=0}^{N-1}\int_{a+kh}^{a+(k+1)h} y_k\,\mathrm{d}t, \tag{2.1}$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{l}\int_a^{a+2l} f(t)\begin{Bmatrix} \cos\dfrac{\pi nt}{l} \\ \sin\dfrac{\pi nt}{l} \end{Bmatrix}\mathrm{d}t = \frac{1}{l}\sum\nolimits_{k=0}^{N-1}\int_{a+kh}^{a+(k+1)h} y_k \begin{Bmatrix} \cos\dfrac{\pi nt}{l} \\ \sin\dfrac{\pi nt}{l} \end{Bmatrix}\mathrm{d}t. \tag{2.2}$$

Integrating right members of equation (2.1) and equation (2.2) by t, we obtain

$$a_0 = \frac{h}{l}\sum\nolimits_{k=0}^{N-1} y_k, \tag{2.3}$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi n}\sum\nolimits_{k=0}^{N-1} y_k \begin{Bmatrix} \sin\dfrac{\pi n(a+(k+1)h)}{l} - \sin\dfrac{\pi n(a+kh)}{l} \\ \cos\dfrac{\pi n(a+kh)}{l} - \cos\dfrac{\pi n(a+(k+1)h)}{l} \end{Bmatrix}. \tag{2.4}$$

In this case, sampling interval coincides with interval of integration. The step function is discontinuous at points t_k and continuous at other points of the interval.

2.2. Midpoint model

In this subsection, we consider a model with a piecewise constant function, where the points of observed values are points of continuity. We shift the function introduced in the subsection 2.1 on half-step leftwards. This yields that length of the very left and the very right intervals decreases on the half. Thus the points of observed values turn out to be the middle points of interval of continuity.

Suppose

$$f(t) = \begin{cases} y_0, & t \in \left[t_0, t_0 + \frac{h}{2}\right], \\ y_k, & t \in \left[t_{k-1} + \frac{h}{2}, t_k + \frac{h}{2}\right], \quad k = 1, \ldots, N-1, \\ y_N, & t \in \left[t_{N-1} + \frac{h}{2}, t_N\right]. \end{cases}$$

According to definition, Fourier coefficients for this function are

$$a_0 = \frac{1}{l}\left(\int_a^{a+\frac{h}{2}} y_0 \, \mathrm{d}t + \sum_{k=1}^{N-1} \int_{a+\frac{h}{2}+(k-1)h}^{a+\frac{h}{2}+kh} y_k \, \mathrm{d}t + \int_{a+\frac{h}{2}+(N-1)h}^{b} y_N \, \mathrm{d}t \right), \tag{2.5}$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{l}\left(\int_a^{a+\frac{h}{2}} y_0 \begin{Bmatrix} \cos\frac{\pi n t}{l} \\ \sin\frac{\pi n t}{l} \end{Bmatrix} \mathrm{d}t + \sum_{k=1}^{N-1} \int_{a+\frac{h}{2}+(k-1)h}^{a+\frac{h}{2}+kh} y_k \begin{Bmatrix} \cos\frac{\pi n t}{l} \\ \sin\frac{\pi n t}{l} \end{Bmatrix} \mathrm{d}t + \int_{a+\frac{h}{2}+(N-1)h}^{b} y_N \begin{Bmatrix} \cos\frac{\pi n t}{l} \\ \sin\frac{\pi n t}{l} \end{Bmatrix} \mathrm{d}t \right). \tag{2.6}$$

Then, integrating right members of equation (2.5) and equation (2.6) by *t*, we have

$$a_0 = \frac{h}{l}\left(\frac{y_0}{2} + \sum_{k=1}^{N-1} y_k + \frac{y_N}{2} \right), \tag{2.7}$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi n}\left(y_0 \begin{Bmatrix} \sin\frac{\pi n\left(a + \frac{h}{2}\right)}{l} - \sin\frac{\pi n a}{l} \\ \cos\frac{\pi n a}{l} - \cos\frac{\pi n\left(a + \frac{h}{2}\right)}{l} \end{Bmatrix} + \sum_{k=1}^{N-1} y_k \begin{Bmatrix} \sin\frac{\pi n\left(a + \frac{h}{2} + kh\right)}{l} - \sin\frac{\pi n\left(a + \frac{h}{2} + (k-1)h\right)}{l} \\ \cos\frac{\pi n\left(a + \frac{h}{2} + (k-1)h\right)}{l} - \cos\frac{\pi n\left(a + \frac{h}{2} + kh\right)}{l} \end{Bmatrix} \right.$$

$$\left. + y_N \begin{Bmatrix} \sin\frac{\pi n b}{l} - \sin\frac{\pi n\left(a + \frac{h}{2} + (N-1)h\right)}{l} \\ \cos\frac{\pi n\left(a + \frac{h}{2} + (N-1)h\right)}{l} - \cos\frac{\pi n b}{l} \end{Bmatrix} \right). \tag{2.8}$$

In this case the interval of integration is shifted left towards the sampling interval. The step function is continuous at points t_k.

3. Quadrature rules of low order

In this section, we consider quadrature rules based on interpolating polynomials of degree 0 and 1. We assume that an integrand is given by tabulating its arguments and values. In this case tabulated arguments match the nodes of interpolation.

Moreover, we consider so called weighted quadrature rules which contain given tabulated function as a factor within integrand. The integrand as a product consists of the given function and some another function called a weight one. In general, the weight function is arbitrary, known, and defined in interval of integration. In certain case, the weight function is given by specific formula.

Each subsection in the section we begin with general form of weighted quadrature rule. Then we give examples with unit and trigonometric weight. This concerns either constant or linear functions.

3.1. Elementary weighted quadrature rules

To begin with, we define the general weighted quadrature rule. As soon as quadrature sum is a linear combination of integrand values, its coefficients are called weights. We should tell the difference between the 'weight function' and the 'weights'. The weight function is factor within integrand, and weights are numerical coefficients in quadrature sum. Denote by M a number of the very right node such that $M + 1$ is a number of nodes, by x_k nodes, by A_k weights, and by $p(x)$ weight function of quadrature rule. Let $f(x)$ be a tabulated function, and $[a,b]$ interval of integration. Then a weighted quadrature formula is defined as following

$$\int_a^b p(x) f(x) \,\mathrm{d}x \approx \sum_{k=0}^{M} A_k f(x_k).$$

Let m be a degree of interpolating polynomial. As known the degree depends on the number of nodes of interpolation as $m = M$. Therefore, quadrature rule accurate on interpolating Lagrange polynomials $L_m(x)$ of degree m is

$$\int_a^b p(x) L_m(x) \,\mathrm{d}x = \sum_{k=0}^{m} A_k L_m(x_k).$$

In following subsubsections we give the quadrature rules with $m = 0.1$.

3.1.1. Accuracy on constant functions. Now suppose that the quadrature rule is accurate on all constant functions $f(x) = L_0(x) = \alpha$, where α is arbitrary real number. So, elementary on $[a,b]$ weighted quadrature rule accurate on polynomials of degree 0 is

$$\int_a^b p(x)\alpha \,\mathrm{d}x = A_0\alpha \quad \Rightarrow \quad \int_a^b p(x) f(x) \,\mathrm{d}x \approx A_0 f(x_0), \quad A_0 = \int_a^b p(x) \,\mathrm{d}x. \tag{3.1}$$

There are three cases for x_0 . The following methods are called as below depending on location of the node within interval of integration: if $x_0 = a$ the rule is called top-left corner method; if $x_0 = \frac{a+b}{2}$ the rule is called midpoint method; if $x_0 = b$ the rule is called top-right corner method. The cases are important for our purpose.

3.1.2. Accuracy on linear functions. Now we turn to the quadrature rule accurate on all linear functions $f(x) = L_1(x) = \beta x + \gamma$, where β and γ are arbitrary real numbers. Then

$$\int_a^b p(x)(\beta x + \gamma) \,\mathrm{d}x = \sum_{k=0}^{1} A_k (\beta x_k + \gamma).$$

Nodes of the formula are set: $x_0 = a$, $x_1 = b$. As the rule is accurate on polynomials of degree 1 and less the following system should be solved to find weights A_k

$$\begin{cases} \int_a^b p(x)\alpha \,\mathrm{d}x = A_0\alpha + A_1\alpha \\ \int_a^b p(x)(\beta x + \gamma) \,\mathrm{d}x = A_0(\beta x_0 + \gamma) + A_1(\beta x_1 + \gamma) \end{cases}$$

Thus, elementary on $[a,b]$ weighted quadrature rule accurate on polynomials of degree 1 with weights A_0 and A_1 is

$$\int_a^b p(x)f(x)\,\mathrm{d}t \approx A_0 f(x_0) + A_1 f(x_1), \quad A_0 = \frac{B - aA}{b - a}, \quad A_1 = \frac{Ab - B}{b - a}, \tag{3.2}$$

$$A = \int_a^b p(x)\,\mathrm{d}x, \quad B = \int_a^b x p(x)\,\mathrm{d}x.$$

To simplify expressions for A_0 and A_1 we assume that $P(x)$ is antiderivative for the weight function $p(x)$. Then integrating A and B we have

$$A_0 = P(b) - \tfrac{1}{b-a}\int_a^b P(x)\,\mathrm{d}x, \quad A_1 = \tfrac{1}{b-a}\int_a^b P(x)\,\mathrm{d}x - P(a). \tag{3.3}$$

This is the quadrature rule with the weight in general form. If the weight is unit, the rule becomes classical trapezoid rule.

3.2. Composite weighted quadrature rules
The interval of integration is sectioned on *N* equal subintervals. An elementary quadrature formula is used on each subinterval. The rule as a result of summing the parts is known as composite quadrature rule. We begin with introducing general forms of lower order rules.

Suppose $h = \frac{b-a}{N}$. Then, applying the rule represented in equation (3.1) to each subinterval, and summing integrals over all subintervals, taking in account we have

$$\int_a^b p(x)f(x)\,\mathrm{d}x = \sum_{k=0}^{N-1}\int_{a+kh}^{a+(k+1)h} p(x)f(x)\,\mathrm{d}x \approx \sum_{k=0}^{N-1} A_{0k}\left\{\begin{matrix} f(a+kh) \\ f\left(a+kh+\frac{h}{2}\right) \\ f(a+(k+1)h) \end{matrix}\right\}. \tag{3.4}$$

Weights in equation (3.4) are $A_{0k} = P(a+(k+1)h) - P(a+kh)$, $k = 0,\dots,N-1$. Double index in A_{0k} means that weight A_0 from equation (3.1) is applied to subinterval whose left limit is numbered by *k*. Thus, equation (3.4) represents quadrature formula accurate on constant functions.

Furthermore, applying equation (3.2) with specifics (3.3) to subinterval and summing, we have

$$\int_a^b p(x)f(x)\,\mathrm{d}x \approx \sum_{k=0}^{N-1}\left(A_{0k} f(a+kh) + A_{1k} f(a+(k+1)h)\right) = B_0 f(a) + \sum_{k=1}^{N-1} B_k f(a+kh) + B_N f(b). \tag{3.5}$$

Weights in equation (3.5) are

$$B_0 = P(a+h) - \tfrac{1}{h}\int_a^{a+h} P(x)\,\mathrm{d}x, \quad B_N = \tfrac{1}{h}\int_{b-h}^b P(x)\,\mathrm{d}x - P(b-h),$$

$$B_k = P(a+(k+1)h) - \tfrac{1}{h}\int_{a+kh}^{a+(k+1)h} P(x)\,\mathrm{d}x + \tfrac{1}{h}\int_{a+(k-1)h}^{a+kh} P(x)\,\mathrm{d}x - P(a+(k-1)h). \tag{3.6}$$

Expression for weights B_k in equation (3.6) is true for all $k = 1,\dots,N-1$. Thus, equation (3.6) represents quadrature formula accurate on linear functions.

3.2.1. Unit weight. Now we turn to specific weight functions. We begin with unit weight function. Let $p(x) = 1$, then $P(x) = x$, and $\int_a^b P(x)\mathrm{d}x = \frac{b^2-a^2}{2}$. The expressions are called unit weight parameters. Substituting weights in equation (3.4) by the parameters, we have classical rectangular rule with $A_{0k} = h$, $k = 0,\dots,N-1$

$$\int_a^b f(x)\,\mathrm{d}x \approx h\sum_{k=0}^{N-1}\left\{\begin{matrix} f(a+kh) \\ f\left(a+kh+\frac{h}{2}\right) \\ f(a+(k+1)h) \end{matrix}\right\}. \tag{3.7}$$

In other words, this is a composite unit weighted quadrature formula exact on constant functions. Further, substituting weights in equations (3.5) and (3.6) by unit weight parameters we have classical trapezoid rule with $B_0 = B_N = \frac{h}{2}$, and $B_k = h$, $k = 0,\ldots,N-1$

$$\int_a^b f(x)\,\mathrm{d}\,x \approx h\left[\tfrac{f(a)}{2} + \sum_{k=1}^{N-1} f(a+kh) + \tfrac{f(b)}{2}\right] \tag{3.8}$$

Thus we have two composite quadrature rules with unite weight.

3.2.2. Cosine weigh. We continue with trigonometric weight function. Now we turn to cosine weight. Let $p(x) = \cos\alpha x$, then $P(x) = \frac{1}{\alpha}\sin\alpha x$, and $\int_a^b P(x)\mathrm{d}\,x = \frac{1}{\alpha^2}(\cos\alpha a - \cos\alpha b)$. Here α is an arbitrary real number. Substituting weights in equation (3.4) by this, we have

$$\int_a^b \cos\alpha x f(x)\,\mathrm{d}\,x \approx \sum_{k=0}^{N-1} A_{0k}\left\{\begin{matrix} f(a+kh) \\ f\left(a+kh+\frac{h}{2}\right) \\ f(a+(k+1)h) \end{matrix}\right\}. \tag{3.10}$$

Weights in equation (3.10) are $A_{0k} = \frac{1}{\alpha}[\sin\alpha(a+(k+1)h) - \sin\alpha(a+kh)]$, $k = 0,\ldots,N-1$. Further, substituting weights in equations (3.5) and (3.6) by cosine weight parameters, we have

$$\int_a^b \cos\alpha x f(x)\,\mathrm{d}\,x \approx B_0 f(a) + \sum_{k=1}^{N-1} B_k f(a+kh) + B_N f(b). \tag{3.11}$$

Weights in equation (3.11) are

$$B_0 = \tfrac{1}{\alpha}\sin\alpha(a+h) + \tfrac{1}{h\alpha^2}(\cos\alpha(a+h) - \cos\alpha a),$$

$$B_N = -\tfrac{1}{h\alpha^2}(\cos\alpha b - \cos\alpha(b-h)) - \tfrac{1}{\alpha}\sin\alpha(b-h),$$

$$B_k = \tfrac{1}{\alpha}\sin\alpha(a+(k+1)h) + \tfrac{1}{h\alpha^2}(\cos\alpha(a+(k+1)h) - 2\cos\alpha(a+kh) + \cos\alpha(a+(k-1)h))$$
$$-\tfrac{1}{\alpha}\sin\alpha(a+(k-1)h). \tag{3.12}$$

Expression for weights B_k in equation (3.12) is true for all $k = 1,\ldots,N-1$. Thus we have two composite quadrature rules with cosine weight.

3.2.3. Sine weight. Finally we turn to sine weight. Suppose $p(x) = \sin\alpha x$, then $P(x) = -\frac{1}{\alpha}\cos\alpha x$, and $\int_a^b P(x)\mathrm{d}\,x = \frac{1}{\alpha^2}(\sin\alpha a - \sin\alpha b)$. Here α is an arbitrary real number. Substituting weights in equation (3.4) by this, we have

$$\int_a^b \sin\alpha x f(x)\,\mathrm{d}\,x \approx \sum_{k=0}^{N-1} A_{0k}\left\{\begin{matrix} f(a+kh) \\ f\left(a+kh+\frac{h}{2}\right) \\ f(a+(k+1)h) \end{matrix}\right\}. \tag{3.13}$$

Weights in equation (3.13) are $A_{0k} = \frac{1}{\alpha}[\cos\alpha(a+kh) - \cos\alpha(a+(k+1)h)]$, $k = 0,\ldots,N-1$. Further, substituting weights in equations (3.5) and (3.6) by sine weight parameters, we have

$$\int_a^b \sin\alpha x f(x)\,\mathrm{d}\,x \approx B_0 f(a) + \sum_{k=1}^{N-1} B_k f(a+kh) + B_N f(b). \tag{3.14}$$

Weights in equation (3.14) are

$$B_0 = \tfrac{1}{h\alpha^2}\left(\sin\alpha(a+h) - \sin\alpha a\right) - \tfrac{1}{\alpha}\cos\alpha(a+h), \quad B_N = \tfrac{1}{\alpha}\cos\alpha(b-h) - \tfrac{1}{h\alpha^2}\left(\sin\alpha b - \sin\alpha(b-h)\right),$$

$$B_k = \tfrac{1}{\alpha}\cos\alpha(a+(k-1)h) + \tfrac{1}{h\alpha^2}\left(\sin\alpha(a+(k+1)h) - 2\sin\alpha(a+kh) + \sin\alpha(a+(k-1)h)\right) \qquad (3.15)$$
$$- \tfrac{1}{\alpha}\cos\alpha(a+(k+1)h).$$

Expression for weights B_k in equation (3.15) is true for all $k = 1,\ldots,N-1$.Thus we have two composite quadrature rules with sine weight.

4. Fourier coefficients

In this section, we compare formulas for Fourier coefficients of piecewise constant function and quadrature rules of lower orders.

4.1. Case of discontinuity

Let us remember that $y_k = f^*(t_k)$, $k = 0,\ldots,N$. Comparing (2.3) and (3.7), we have

$$a_0 = \frac{1}{l} h \sum_{k=0}^{N-1} f^*(a+kh) \approx \frac{1}{l}\int_a^{a+2l} f^*(t)\,\mathrm{d}t, \qquad (4.1)$$

Here $\alpha = \frac{\pi n}{l}$. Thus, computing formula for zero Fourier coefficient matches composite unit-weighted quadrature rectangular rule for tabulated function $f^*(t)$ in $[a, a+2l]$. Comparing (2.4) and (3.10) (3.13), we have

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{l}\frac{l}{\pi n}\sum_{k=0}^{N-1} f^*(a+kh)\begin{Bmatrix} \sin\frac{\pi n(a+(k+1)h)}{l} - \sin\frac{\pi n(a+kh)}{l} \\ \cos\frac{\pi n(a+kh)}{l} - \cos\frac{\pi n(a+(k+1)h)}{l} \end{Bmatrix} \approx \frac{1}{l}\int_a^{a+2l}\begin{Bmatrix} \cos\frac{\pi n t}{l} \\ \sin\frac{\pi n t}{l} \end{Bmatrix} f^*(t)\,\mathrm{d}t. \qquad (4.2)$$

Thus, computing formulas for *n*th Fourier coefficients match composite cosine- and sine-weighted quadrature rules accurate on constants for tabulated function $f^*(t)$ in $[a, a+2l]$.

4.1.1. Case of continuity. Comparing (2.7) and (3.8), we have

$$a_0 = \frac{1}{l} h\left[\frac{f^*(a)}{2} + \sum_{k=1}^{N-1} f^*(a+kh) + \frac{f^*(b)}{2}\right] \approx \frac{1}{l}\int_a^{a+2l} f^*(t)\,\mathrm{d}t. \qquad (4.3)$$

Thus, computing formula for zero Fourier coefficient matches composite unit-weighted quadrature trapezoid rule for tabulated function $f^*(t)$ in $[a, a+2l]$. In other words, in comparison with the top-left corner model the midpoint model provides computation with quadrature rule of higher order.

Looking at (2.8) and (3.8), we see that

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{l}\frac{l}{\pi n}\left(f^*(a)\begin{Bmatrix} \sin\frac{\pi n\left(a+\frac{h}{2}\right)}{l} - \sin\frac{\pi n a}{l} \\ \cos\frac{\pi n a}{l} - \cos\frac{\pi n\left(a+\frac{h}{2}\right)}{l} \end{Bmatrix} + f^*(b)\begin{Bmatrix} \sin\frac{\pi n b}{l} - \sin\frac{\pi n\left(a+\frac{h}{2}+(N-1)h\right)}{l} \\ \cos\frac{\pi n\left(a+\frac{h}{2}+(N-1)h\right)}{l} - \cos\frac{\pi n b}{l} \end{Bmatrix}\right.$$
$$\left. + \sum_{k=1}^{N-1} f^*(a+kh)\begin{Bmatrix} \sin\frac{\pi n\left(a+\frac{h}{2}+kh\right)}{l} - \sin\frac{\pi n\left(a+\frac{h}{2}+(k-1)h\right)}{l} \\ \cos\frac{\pi n\left(a+\frac{h}{2}+(k-1)h\right)}{l} - \cos\frac{\pi n\left(a+\frac{h}{2}+kh\right)}{l} \end{Bmatrix}\right). \qquad (4.4)$$

Comparing, we see that, although expressions have similar structure, weight coefficients do not provide higher order of quadrature rule. So, we have to write

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{l}\left[B_0 f(a) + \sum_{k=1}^{N-1} B_k f(a+kh) + B_N f(b)\right]$$

$$\approx \frac{1}{l}\left[\int_a^{a+\frac{h}{2}} \begin{Bmatrix} \cos\dfrac{\pi n t}{l} \\ \sin\dfrac{\pi n t}{l} \end{Bmatrix} f^*(t)\,\mathrm{d}t + \int_{a+\frac{h}{2}}^{b-\frac{h}{2}} \begin{Bmatrix} \cos\dfrac{\pi n t}{l} \\ \sin\dfrac{\pi n t}{l} \end{Bmatrix} f^*(t)\,\mathrm{d}t + \int_{b-\frac{h}{2}}^{b} \begin{Bmatrix} \cos\dfrac{\pi n t}{l} \\ \sin\dfrac{\pi n t}{l} \end{Bmatrix} f^*(t)\,\mathrm{d}t\right]. \quad (4.5)$$

Expression in square brackets before approximate equality sign means following. The first part is an elementary top-left corner quadrature rectangular term, the middle part is a composite midpoint quadrature rectangular sum, and the last part is an elementary top-right corner term. Expression in square brackets after approximate equality sign matches trigonometric-weighted definite integrals of tabulated waveform function. We see that order of quadrature rule remains the same in this case.

5. Conclusion

Taking the set of tabulated function values as an initial data for calculating Fourier coefficients, we have a task to compute approximately some definite integrals. The way to complete the task is use of appropriate quadrature rules. If the modelling function is a piecewise constant, the rules appear to be composite of rectangular or trapezoid weighted type.

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