New effective moduli of isotropic viscoelastic composites. Part I. Theoretical justification

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Abstract. According to the approach based on the commonality of problems of determining effective moduli of composites and viscoelastic solids, which properties are time-inhomogeneous, it is assumed that a viscoelastic solid is a two-component composite. One component displays temporal properties defined by a pair of Castiglianian-type effective moduli, and the other is defined by a pair of Lagrangian-type effective moduli. The Voigt and Reuss averaging is performed for the obtained two-composite solid with the introduction of a time function of volume fraction. In order to determine closer estimates, a method of iterative transformation of time effective moduli is applied to the viscoelastic Voigt–Reuss model. The physical justification of the method is provided. As a result, new time effective moduli of the viscoelastic solid are obtained which give a closer estimate of temporal properties as compared to the known models.

1. Introduction

The problems of calculating the stress-strain state of viscoelastic solids are reduced to the solution of a system of integro-differential equations of equilibrium in the region occupied by the solid, with specified boundary loads and displacements. At the same time, these problems can be considered from the viewpoint of composite mechanics because composite properties are inhomogeneous with respect to coordinates and properties of viscoelastic solids, which are time-inhomogeneous. This makes the application of effective modulus theory to viscoelasticity [1-4] possible. Earlier we derived expressions for time effective moduli of the Lagrangian and Castiglianian type [5] and used them to construct transformations for the Voigt–Reuss model [6]. However, this model gives insufficiently close estimates for approximate solutions, as in the case of elastic composites.

The present paper is aimed at obtainment of iteratively transformed time effective characteristics. Earlier we successfully applied this approach for composites [7]. The requirements to iterative transformations are the following:

(i) classical theorems on minimum of strain energy and additional work functionals must be valid, and

(ii) inequalities for effective moduli (Voigt–Reuss bounds) on each iteration step must be satisfied.

The convergence of iterative transformation of time effective characteristics is verified both numerically and by analytical determination of the limit of the iteration sequence.

2. Derivation of expressions for iterated effective moduli

The constitutive equations of a linearly viscoelastic solid is as follows:

$$s_{ij}(t) = \int_0^t R(t-\tau)\,de_{ij}(\tau) \equiv G^* e_{ij}, \quad \sigma(t) = \int_0^t K(t-\tau)\,d\theta(\tau) \equiv K^*\theta. \tag{2.1}$$

Here, s_{ij}, e_{ij} are the deviatoric stress and strain components, and $\sigma(t) = \sigma_{ii}(t)$, $\theta(t) = \varepsilon_{ii}(t)$, ($i$ = 1, 2, 3), $R(t)$, $K(t)$ are the shear and bulk relaxation functions.

The relations inverse to Equations (2.1) have the following form:

$$e_{ij}(t) = \int_0^t \Pi(t-\tau)\,ds_{ij}(\tau) \equiv G^{*-1} s_{ij}, \quad \theta(t) = \int_0^t \Pi_1(t-\tau)\,d\sigma(\tau) \equiv K^{*-1}\theta, \tag{2.2}$$

$\Pi(t)$, $\Pi_1(t)$ are the shear and bulk creep functions, and G^*, K^*, G^{*-1}, K^{*-1} are the shorthand notations for direct and inverse integral operators.

Approximate constitutive equations can be represented as

$$s_{ij}(t) = g_L(t)e_{ij}(t), \quad e_{ij}(t) = g_C(t)s_{ij}(t), \quad \sigma(t) = k_L(t)\theta(t), \quad \theta(t) = k_C(t)\sigma(t), \tag{2.3}$$

where the Lagrangian and Castiglianian-type time effective moduli are determined by relations [5]:

$$g_L(t) = G^* h, \quad g_C(t) = \left(G^{*-1}h\right)^{-1}, \quad k_L(t) = K^* h, \quad k_C(t) = \left(K^{*-1}h\right)^{-1}, \tag{2.4}$$

h is the Heaviside function.

Using Equations (2.4), we constructed the Voigt–Reuss models of time effective characteristics [6]:

$$G_V(t) = \gamma g_C(t) + (1-\gamma) g_L(t), \quad G_R(t) = \left(\frac{\gamma}{g_C(t)} + \frac{1-\gamma}{g_L(t)}\right)^{-1}, \quad g_C(t) \geq g_L(t), \quad \gamma = 1 - \alpha\frac{g_C(t) - g_L(t)}{g_C(t)}, \tag{2.5}$$

where α is a numerical parameter [6].

Let us introduce expressions for iteration sequences:

$$G_V^{(n)}(t) = \gamma G_V^{(n-1)}(t) + (1-\gamma) G_R^{(n-1)}(t), \quad G_R^{(n)}(t) = \left(\frac{\gamma}{G_V^{(n-1)}(t)} + \frac{1-\gamma}{G_R^{(n-1)}(t)}\right)^{-1}, \quad n = 1, 2, \ldots \tag{2.6}$$

It is possible to show that sequences $G_V^{(n)}(t)$, $G_R^{(n)}(t)$ converge to the same limit at $n \to \infty$:

$$G_V^{(n)}(t) \to G_R^{(n)}(t) \to \sqrt{G_V(t) G_R(t)}. \tag{2.7}$$

Sequences $K_V^{(n)}(t)$, $K_R^{(n)}(t)$ have the same limit at $n \to \infty$:

$$K_V^{(n)}(t) \to K_R^{(n)}(t) \to \sqrt{K_V(t) K_R(t)}. \tag{2.8}$$

3. Physical justification

Let us consider the strain energy and additional work functionals of a homogeneous isotropic elastic solid with time-dependent elastic moduli:

$$\Pi = \int_V W\left(e_{ij}\right) dv - \int_V \rho f_i u_i dV - \int_{S_t} t_i u_i ds,$$
$$\Psi = \int_V \Lambda\left(\sigma_{ij}\right) dv - \int_{S_u} t_i u_s^i ds,$$
$$W\left(\varepsilon_{ij}\right) = \frac{1}{2}\left[\left(K(t) - \frac{2}{3}G(t)\right)\theta^2 + 2G(t)\varepsilon_{ij}\varepsilon_{ij}\right], \tag{3.1}$$
$$\Lambda\left(\sigma_{ij}\right) = \frac{1}{2}\left[\frac{1}{2G(t)}\sigma_{ij}\sigma_{ij} - \frac{3K(t) - 2G(t)}{6K(t) + 2G(t)}\Sigma^2\right],$$
$$\Sigma = \sigma_{ii,}\ \ \theta = \varepsilon_{ii,}\ \ \left(i, j = 1, 2, 3\right).$$

Here, $K(t)$, $G(t)$ are the elastic bulk compression and shear moduli, t_i, u_s^i are the stresses and displacements specified on portions of boundary $S = S_t U S_u$, σ_{ij}, ε_{ij}, u_i are the stress tensor, strain tensor and displacement vector components, and f_i, t_i are the bulk and surface force vector components. Functionals Π, Ψ assume minimum values on admissible displacement and stress fields that obey equilibrium and compatibility equations.

Now let us consider Voigt–Reuss bounds

$$G_V \geq G_* \geq G_R, \quad K_V \geq K_* \geq K_R\ . \tag{3.2}$$

Here, G_*, K_* are the moduli of the homogeneous elastic medium whose specific potential energy is equal to the corresponding potential of a two-component composite with moduli (2.4). Thus, the Voigt and Reuss inequalities give the upper and lower bounds for moduli G^*, K^*. However, using Equations (3.2) it is impossible to calculate G^* and K^* as well as specific strain energy W_* and additional work Λ_*.

Let us now demonstrate that iteration sequences (2.6) can be used to derive expressions for G_*, K_* and W_*, Λ_*.

Since we consider an elastic medium with constitutive equations (2.4), functionals Π, Ψ and specific potentials W, Λ depend on time t. The same is true of sought moduli G_*, K_*. We assume that minimum values $\Pi_* = \Pi_{\min}$, $\Psi_* = \Psi_{\min}$ correspond to moduli G_*, K_* for each particular value of t. We denote the specific potentials corresponding to $G_V,\ G_V^{(1)}, \ldots, G_V^{(n)}$ and $G_R,\ G_R^{(1)}, \ldots, G_R^{(n)}$ by W, W_1, … , W_n and Λ, Λ_1, … , Λ_n. Then, owing to inequalities

$$G_V \geq G_V^{(1)} \geq \ldots \geq G_V^{(n)}, \quad G_R \leq G_R^{(1)} \leq \ldots \leq G_R^{(n)}, \tag{3.3}$$

the following inequalities are fulfilled:

$$W \geq W_1 \geq \ldots \geq W_n, \quad \Lambda \geq \Lambda_1 \geq \ldots \geq \Lambda_n\ . \tag{3.4}$$

Since moduli $G_V^{(n)}$ and $G_R^{(n)}$ for any n satisfy inequality

$$G_V^{(n)} \geq G_* \geq G_R^{(n)}, \tag{3.5}$$

inequalities

$$W_n \geq W_*, \quad \Lambda_n \geq \Lambda_* \tag{3.6}$$

are also satisfied for the specific potentials.
In the limit at $n \to \infty$ we have:

$$W_n \to W_*, \quad \Lambda_n \to \Lambda_*\ . \tag{3.7}$$

The Voigt–Reuss bounds shrink to a single point in which moduli $G_V^{(n)}$, $G_R^{(n)}$ become equal to the limit (2.7). Thus, the sought limit values of G_*, K_* really correspond to specific potentials $W_*(\varepsilon_{ij})$, $\Lambda_*(\sigma_{ij})$ equal to the specific potential energy and specific additional work of the two-component elastic solid with constitutive equations (2.3).

4. Discussion of results

Expressions for new effective characteristics of viscoelastic solids were derived using the following mathematical procedures:

1. Approximate representation of constitutive equations of viscoelasticity through relations of elastic Hooke's law with time-dependent moduli of Lagrangian and Castiglianian type.
2. Representation of the elastic medium as a two-component one, and formulation of the Voigt–Reuss model for the averaging of properties based on this representation.
3. Construction of iteration sequences for the Voigt–Reuss moduli.
4. Application of minimum variational principles for the functionals of potential strain energy and additional work.

It has been shown that the sequence of iteratively transformed Voigt and Reuss moduli allows the contraction of the Voigt–Reuss bounds. In the limit we obtain effective characteristics of a homogeneous isotropic elastic medium with specific strain and stress potentials equal to the corresponding potentials of a two-component medium. These effective characteristics can be called energetically equivalent.

The reported theoretical findings can be applied to calculate the stress-strain state of structures made of viscoelastic materials, which will be discussed in the second part of this paper [2].

5. Conclusion

Contraction mappings for the Voigt–Reuss bounds were obtained for a two-component homogeneous isotropic elastic medium with time-dependent moduli.

It was shown using theorems on minimum potential strain energy and additional work that the limit of the sequence of iteratively transformed Voigt–Reuss moduli is the effective characteristics of energetically equivalent specific strain and stress potentials of the homogenous medium of comparison. Expressions for these characteristics have been derived.

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