Polynomial reconstruction of electric charge distribution on the conductive plate caused by external electric field

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Abstract. The paper proposes an original method of calculating the charge distribution on the surface of the conductive plate introduced into the external electrostatic field. The authors managed to obtain the polynomials which allow to solve the integral equation that establishes the relationship between charge distribution of the conductive plate and the potential distribution of the external field and the potential on the surface of the plate. The proposed algorithms solutions are valid in the presence of axial symmetry of the field and the plate. Examples of calculation of conductor charge distribution in the presence of external field by using a polynomial expansion have been presented. The comparisons of results calculated by the polynomial method and by known analytical solutions have been given

1. Introduction

When a conductive plate is placed into the external electrostatic field, the electric charges commence its induction on the plate surface. A correlation, which connects external field potential $U^0(\mathbf{r}_0)$ and potential on the plate surface U_0 with the distribution of the induced charge, can be described as follows [2, 3]

$$-U^0(\mathbf{r}_0)+U_0=\frac{1}{4\pi\varepsilon_0}\int_D \frac{\sigma(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}d\mathbf{r},\ \ \mathbf{r}=\{x,y\},\ \mathbf{r}_0=\{x_0,y_0\},\ \mathbf{r},\mathbf{r}_0\in D\,. \tag{1}$$

Thus, with certain charge distribution $\sigma(\mathbf{r})$ on the conductive plate surface we can determine an algebraic sum of potential distribution of the external field $U^0(\mathbf{r}_0)$ and plate surface potential U_0. In natural experiments we usually know the distribution of the external field potential, the potential of the conductor surface can be measured or kept in direct-current voltage. So the real interest is the inverse problem – definition of the charge distribution on the conductor surface. In accordance with expression (1) we need to solve the Fredholm integral equation of the first kind of required relative value $\sigma(\mathbf{r})$. Value $\sigma(\mathbf{r})$ is total charge distribution in the conductor from different sides of the plate, so the next step is to divide the source-faced surface charge and the opposite one. We need to use the normal voltage component of the resulting field to divide the charges:

$$E_n(\mathbf{r})=-(\nabla U(\mathbf{r}),\mathbf{n})=-\frac{\partial}{\partial z}U(\mathbf{r})\,. \tag{2}$$

Field intensity is related to the correlations charges:

$$2\varepsilon_0 E_n(\mathbf{r})=\sigma_1(\mathbf{r})-\sigma_2(\mathbf{r}),\quad \sigma(\mathbf{r})=\sigma_1(\mathbf{r})+\sigma_2(\mathbf{r})\,, \tag{3}$$

from which we can define:

$$\sigma_1(\mathbf{r}) = \frac{\sigma(\mathbf{r}) + 2\varepsilon_0 E_n(\mathbf{r})}{2}, \qquad \sigma_2(\mathbf{r}) = \frac{\sigma(\mathbf{r}) - 2\varepsilon_0 E_n(\mathbf{r})}{2}. \tag{3}$$

The aim of this work is to replace complicated integral equation (1) with algebraic one with the defined polynomials, in other words, to define analytic correlation, which connects the resulting field potential and the charge distribution in the conductor.

2. Formulation of the problem.

Let us suppose that the external field possesses a symmetry of rotation about the symmetry axis of the disc. Then equation (1) can be described in the following form:

$$U(\rho) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\vartheta' \int_o^R \frac{\sigma(\rho')\rho' d\rho'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\vartheta - \vartheta')}}\ , \tag{4}$$

where $U(\rho) = -U^0(\rho) + U_0$ – resulting potential.

The latter expression can be rewritten with elliptical integral $K(k) = \int_0^{\pi/2} \frac{d\vartheta'}{\sqrt{1 - k\cos(\vartheta')}}$, then correlation (4) transforms into the following expression:

$$U(\rho) = \frac{1}{4\pi\varepsilon_0}\int_0^R \rho' d\rho' \sigma(\rho') \int_0^{2\pi} \frac{d\vartheta'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\vartheta - \vartheta')}} = \frac{1}{\pi\varepsilon_0}\int_0^R \frac{\rho'\sigma(\rho')}{\rho' + \rho} K\left(\frac{2\sqrt{\rho'\rho}}{\rho' + \rho}\right) d\rho'. \tag{5}$$

Taking into account the Landen transformation [2, 3, 4] we can define the following feature of the elliptic integral:

$$\frac{1}{x} K\left(\frac{y}{x}\right) = \int_0^y \frac{ds}{\sqrt{(x^2 - s^2)(y^2 - s^2)}}. \tag{6}$$

This feature allows one to describe an unknown subintegral function with certain potential $U(\rho)$:

$$\sigma(\rho) = -\frac{4\varepsilon_0}{\pi\rho}\frac{d}{d\rho}\int_\rho^R \frac{s\,ds}{\sqrt{s^2 - \rho^2}} \frac{d}{ds}\int_0^s \frac{U(t)t\,dt}{\sqrt{s^2 - t^2}}. \tag{7}$$

Let us find the charge distribution, made by potential

$$U(\rho) = \rho^m \tag{8}$$

After substitution of (8) in (7) we obtain the expressions for charge distribution in the form of polynomials $\psi_m(\rho)$ with even and uneven m accordingly:

$$\sigma(\rho) = \begin{cases} \dfrac{M(1,1)}{2}\dfrac{R}{\sqrt{R^2 - \rho^2}} \qquad \text{when } m = 1, \\[2ex] -N(m,1)\displaystyle\sum_{k=0}^{m/2} c\left(\frac{m}{2}, k\right) \frac{\left(R^2 - \rho^2\right)^{k - \frac{1}{2}}\left(\rho^{-2} m R^2 - m - 2\rho^{-2} k R^2 - 1\right)\rho^{-2k+m}}{2k+1} \\ \qquad \text{when } m - \text{an even} \\[2ex] -\dfrac{M(m,1)}{m+1}\left(\dfrac{-R^m}{\sqrt{R^2 - \rho^2}} + \displaystyle\sum_{k=1}^{m-1}\left(\frac{F2\left(\frac{m+1}{2}, k\right)}{2^k F1\left(\frac{m+1}{2}, k\right)} \cdot \left(2k\rho^{2k-2}\sqrt{R^2 - \rho^2} - \frac{\rho^{2k}}{\sqrt{R^2 - \rho^2}}\right) R^{m-2k}\right)\right) + \\[2ex] -\dfrac{F(m)M(m,1)}{2^{\frac{m+1}{2}}\left(\frac{m+1}{2}\right)!}\left[\ln\left(\dfrac{R + \sqrt{R^2 - \rho^2}}{\rho}\right)\rho^{m-1}(m+1) - \left(\dfrac{1}{\sqrt{R^2 - \rho^2}} \cdot \dfrac{\rho^{(m+1)}}{R + \sqrt{R^2 - \rho^2}} + \rho^{(m+1)}\right)\right] \\ \qquad \text{when } m - \text{an odd.} \end{cases}$$

Here

$$F(m)=\prod_{k=0}^{\frac{m-1}{2}}2k+1,\qquad F1(m,k)=\prod_{s=1}^{k}m-s\,,\qquad F2(m,k)=\prod_{s=1}^{k}2m-2s+1,$$

$$c(m,k)=\frac{m}{(k-m)!k!},$$

$$M(m,s)=\frac{F(m)s^m\pi(m+1)}{2^{\frac{m+3}{2}}\left(\frac{m+1}{2}\right)!},\qquad N(m,s)=(m+1)s^m\sum_{k=0}^{\frac{m}{2}}\frac{(-1)^k}{2k+1}c\left(\frac{m}{2},k\right).$$

Table 1. Description of even and uneven degrees

ρ^{m}	$\psi_m(\rho)$	ρ^m	$\psi_m(\rho)$
1	$\frac{1}{\sqrt{R^2-\rho^2}}$	ρ	$\frac{1}{2}\pi\left(\frac{R}{\sqrt{R^2-\rho^2}}-\ln\left(\frac{R+\sqrt{R^2-\rho^2}}{\rho}\right)\right)$
ρ^2	$-2\frac{R^2-2\rho^2}{\sqrt{R^2-\rho^2}}$	ρ^3	$-\frac{3}{8}\pi\left(\frac{\left(R^2-3\rho^2\right)R}{\sqrt{R^2-\rho^2}}+3\rho^2\ln\left(\frac{R+\sqrt{R^2-\rho^2}}{\rho}\right)\right)$
ρ^4	$-\frac{8}{9}\cdot\frac{4\rho^2R^2-8\rho^4+R^4}{\sqrt{R^2-\rho^2}}$	ρ^5	$-\frac{15}{128}\pi\left(\frac{\left(2R^4+5R^2\rho^2-15\rho^4\right)R}{\sqrt{R^2-\rho^2}}+15\rho^4\ln\left(\frac{R+\sqrt{R^2-\rho^2}}{\rho}\right)\right)$

With these tabular correlations we can define the charge distribution (which is of great interest to us) with the certain potential distribution. We need to describe the distribution of the field potentials in the form of polynomials resolution:

$$U(\rho)=\sum_{m=0}^{N}a_m\rho^m. \tag{9}$$

The coefficients of the expansion of series (9) can be easily defined by the method of the least squares. The charge distribution can be defined with the following correlation:

$$\sigma(\rho)=\frac{4\varepsilon_0}{\pi}\sum_{m=0}^{N}a_m\psi_m(\rho). \tag{10}$$

The kind of polynomial of series (8) is defined with the expression from Table 1.

In case of perforated disc in expression (5) the limits of integration are changed:

$$U(\rho)=\frac{1}{\pi\varepsilon_0}\int_r^{\infty}\frac{\rho'\sigma(\rho')}{\rho'+\rho}K\left(\frac{2\sqrt{\rho'\rho}}{\rho'+\rho}\right)d\rho'. \tag{11}$$

In this case it is unnecessary to calculate new polynomials for expansion of the charge distribution. It is enough to realise a necessary transformation of the coordinate system consisting in projecting of the exterior of the circle on its interior. Replacing of the variables we come from integral equation (11) to integral one (5) with the help of conjugate coordinates:

$$\rho'=\frac{r^2}{t'},\ \rho=\frac{r^2}{t}. \tag{12}$$

The result is the integral equation with new limits of integration

$$U1(t)=\frac{1}{\pi\varepsilon_0}\int_0^{r}\frac{\sigma_1(t')\,t'}{(t'+t)}K\left(\frac{2\sqrt{t\cdot t'}}{t+t'}\right)dt'\ \text{где}\ \sigma_1(t')=\frac{r^2\,\sigma\left(r^2/t'\right)}{t'^3},U_1(t)=U\left(\frac{r^2}{t}\right)\cdot\frac{1}{t} \tag{13}$$

To check the operation of the proposed polynomial algorithm let us consider two examples which have analytical solution [2, 3].

Example 1. Let us define the charge distribution in the grounded disc with radius R = 1m if the disc is in the external field of the point charge, located on the disc axis at a distance of h = 0.3m from the disc surface. Ratio, q/ε_0 had been chosen as 1.

The potential of the point charge in the disc plane and total charge distribution on the disc can be defined with the following expressions accordingly [1]:

$$U(\rho)=\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+h^2}},\quad \sigma(\rho)=-\frac{h}{2\pi^2\left(h^2+\rho^2\right)^{3/2}}\left(arctg\sqrt{\frac{R^2-\rho^2}{h^2+\rho^2}}+\sqrt{\frac{h^2+\rho^2}{R^2-\rho^2}}\right). \qquad (15)$$

Let us describe the potential in the form of the series:

$$U(\rho)=\sum_{m=0}^{N}a_m\rho^m, \qquad (16)$$

Let us define the coefficients of expansion (12), using the method of the least squares. Let us suppose that the number of the expansion coefficients is N = 10.

$$m=0\,..\,N,\ n=m,\ c_m=\int_0^R U(\rho)\rho d\rho,\ b_{m,n}=\int_0^R \rho^{m+n+1}dr=\frac{R^{m+n+1}}{m+n+1}.$$

Let us define expansion coefficients a_m by calculating the matrix equation:

$$c_m=b_{m,n}a_n \ \rightarrow\ a_n=\mathrm{Re}\left[\left(b_{m,n}+jI_{m,n}\lambda\right)^{-1}\right]c_m,$$

I – unit matrix, $j=\sqrt{-1}$, $\lambda=10^{-19.25}$– regularisation parameter, decreasing the relative contribution of the noise term in the solution [1, 5, 6].

Table 2. Coefficients of expansion a_m (16) obtained as a result of the solution of the matrix equation

m	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
a_m	**0.265**	**0**	**-1.43**	**-1.438**	**29.7**	**-104.7**	**198.4**	**-229.6**	**162.7**	**-65.06**	**11.273**

A desired charge distribution on surface $\sigma(\rho)$ will be defined by the expression with the calculated coefficients: $\sigma(\rho)=\frac{2\varepsilon_0}{\pi}\sum_{m=0}^{N}a_m\psi_m(\rho)$.

The results are given in the form below:

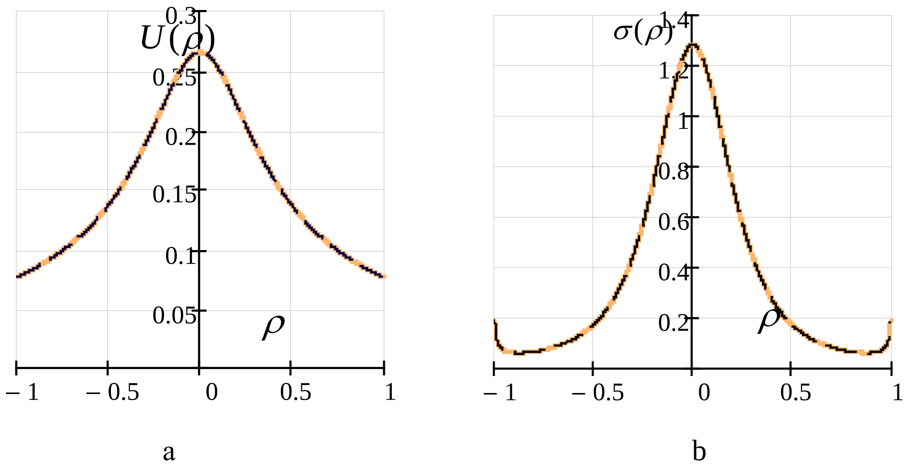

Figure 1. a – potential of the external field as an analytical function and the result of its expansion into the series (12); b – charge distribution as an analytic dependence and the result of the return into the series

There is concurrence of the analytic dependences and polynomial method dependences in the forms.

Example 2. Let us define the charge distribution on the surface of the conductive disc with round perforation r = 0.25m if the disc is located in the external field of the point charge, located on the disc axis at a distance of h = 0.4m from the surface of the disc with radius R = 1m.

The potential of the point charge in the disc plane and total charge distribution on the disc can be defined with the following expressions accordingly [2, 3]:

$$U(\rho)=\begin{cases}\dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{\sqrt{\rho^2+h^2}}, & \rho\ge r\\ 0, & \rho<r\end{cases},\quad \sigma(\rho)=-\frac{h}{\pi^2\left(h^2+\rho^2\right)^{3/2}}\left(arctg\,\frac{h}{r}\sqrt{\frac{\rho^2-r^2}{h^2+\rho^2}}+\frac{r}{h}\sqrt{\frac{h^2+\rho^2}{\rho^2-r^2}}\right). \tag{17}$$

In this case we use equation (5) with the replacement of the variables:

$$\rho'=\frac{r^2}{t'},\ \ \rho=\frac{r^2}{t}. \tag{18}$$

The result is the integral equation with new limits of integration

$$U\left(\frac{r^2}{t}\right)=\frac{1}{\pi\varepsilon_0}\int_0^r\frac{\frac{r^2}{t'}\sigma\left(\frac{r^2}{t'}\right)}{\frac{r^2}{t'}+\frac{r^2}{t}}K\left(\frac{2\sqrt{\frac{r^2}{t'}\cdot\frac{r^2}{t}}}{\frac{r^2}{t'}+\frac{r^2}{t}}\right)d\frac{r^2}{t'}. \tag{19}$$

Let us simplify equation (19) to obtain the following result:

$$U1(t)=\frac{1}{\pi\varepsilon_0}\int_0^r\frac{\sigma1(t')\,t'}{(t'+t)}K\left(\frac{2\sqrt{t\cdot t'}}{t+t'}\right)dt',\ \text{where}\ \ \sigma1(t')=\frac{r^2\,\sigma\left(\frac{r^2}{t'}\right)}{t'^3},\ \ U1(t)=U\left(\frac{r^2}{t}\right)\cdot\frac{1}{t}. \tag{20}$$

Let us describe field potential $U1(t)$ in the form of series

$$U1(t)=\frac{q}{4\pi\varepsilon_0}\sum_{m=0}^{N}a_m t^m \tag{21}$$

Using the same method we will define expansion coefficients a_m, and then using the polynomial table and replacement of the variable in form of $t=r^2/\rho$, let us define

$$\sigma1(\rho)=\frac{2\varepsilon_0}{\pi}\sum_{m=0}^{N}a_m\psi_m(\rho),\quad \sigma\left(\frac{r^2}{t}\right)=\sigma1(t)\cdot\frac{t^3}{r^2},\quad \sigma(\rho)=\sigma1\left(\frac{r^2}{\rho}\right)\cdot\frac{r^4}{\rho^3}. \tag{22}$$

Defined charge distribution $\sigma(\rho)$ on the conductor surface is described in form 2, there is also a result of the analytical formula [2, 3].

$$\sigma(\rho)=-\frac{h}{\pi^2\left(h^2+\rho^2\right)^{3/2}}\left(arctg\,\frac{h}{r}\sqrt{\frac{\rho^2-r^2}{h^2+\rho^2}}+\frac{r}{h}\sqrt{\frac{h^2+\rho^2}{\rho^2-r^2}}\right).$$

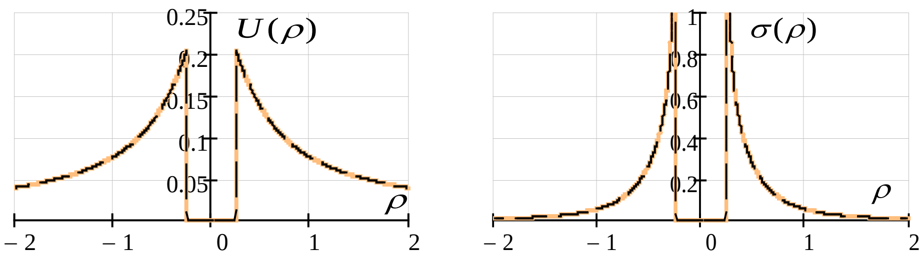

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Figure 2. a – Potential of the external field as an analytical function and the result of its expansion into the series (21), b – Charge distribution as an analytic dependence and the result of the return into the series (22), where $\lambda=10^{-12}$ – a regularization parameter

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Conclusion

The authors managed to define the polynomials, leading the integral equation, which connects the charge distribution on the plate with the external static field to a more simplistic algebraic equation. A polynomial algorithm holds with an axial-symmetric external field and an axial-symmetric shape of the plate. In case of using the polynomial method it necessary to do regularization of the expansion coefficients for decreasing the algorithm noises influence.

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