

A COUNTEREXAMPLE TO A THEOREM OF BREMERMANN ON SHILOV BOUNDARIES – REVISITED

BY MAREK JARNICKI AND PETER PFLUG

Abstract. We continue to discuss the example presented in [4]. In particular, we clarify some gaps and complete the description of the Shilov boundary.

For a bounded domain $D \subset \mathbb{C}^n$ let $\mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{D})$) denote the space of all continuous functions $f : \overline{D} \rightarrow \mathbb{C}$ such that $f|_D$ is holomorphic (resp. f extends holomorphically to a neighborhood of \overline{D}). Let $\partial_S D$ (resp. $\partial_B D$) be the *Shilov* (resp. *Bergman*) *boundary* of D , i.e. the minimal compact set $K \subset \overline{D}$ such that $\max_K |f| = \max_{\overline{D}} |f|$ for every $f \in \mathcal{A}(D)$ (resp. $f \in \mathcal{O}(\overline{D})$). Obviously, $\mathcal{O}(\overline{D}) \subset \mathcal{A}(D)$ and hence $\partial_B D \subset \partial_S D \subset \partial D$. Notice that, in general, $\partial_B D \subsetneq \partial_S D$, e.g. for the domain $D := \{(z, w) \in \mathbb{C}^2 : 0 < |z| < 1, |w| < |z|^{-\log |z|}\}$ (cf. [2], § 16).

The algebra $\mathcal{A}(D)$ (resp. $\mathcal{B}(D) :=$ the uniform closure in $\mathcal{A}(D)$ of $\mathcal{O}(\overline{D})$) endowed with the supremum norm is a Banach algebra. Then $\partial_S D$ (resp. $\partial_B D$) coincides with the Shilov boundary of $\mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) in the sense of uniform algebras (cf. [1]). Note that the peak points of $\mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) are dense in $\partial_S D$ (resp. $\partial_B(D)$) (cf. [1]). Recall that a point $a \in \overline{D}$ is called a peak point for $\mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) if there is an $f \in \mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) with $f(a) = 1$ and $|f(z)| < 1$ for all $z \in \overline{D} \setminus \{a\}$; f is called an associated *peak function*.

Assume that the envelope of holomorphy \tilde{D} of D is univalent. In [4] we were interested in answering whether $\partial_S D = \partial_S \tilde{D}$ (resp. $\partial_B D = \partial_B \tilde{D}$).

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REMARK 1. Notice that:

- $\partial_S \tilde{D} \subset \partial_S D$,
- $\partial_B \tilde{D} \subset \partial_B D$,
- if $\mathcal{A}(D) \subset \mathcal{A}(\tilde{D})|_{\overline{D}}$ (resp. $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\tilde{D})|_{\overline{D}}$), then $\partial_S D = \partial_S \tilde{D}$ (resp. $\partial_B D = \partial_B \tilde{D}$).

In [4] we studied the following bounded Hartogs domain $D \subset \mathbb{C}^2$:

$$D := \{(re^{i\varphi}, w) \in \mathbb{C}^2 : \frac{1}{2} < r < 1, (\varphi, |w|) \in Q\},$$

where

$$Q := ((0, \frac{\pi}{2}) \times [0, 1)) \cup ((\frac{\pi}{2}, \frac{3\pi}{2}) \times [0, 3)) \cup ([\frac{3\pi}{2}, 2\pi) \times (2, 3));$$

it is known that D has a univalent envelope of holomorphy \tilde{D} . The main result of [4] is the following theorem.

THEOREM 2. $\partial_S \tilde{D} \subsetneq \partial_S D$, $\partial_B \tilde{D} \subsetneq \partial_B D$, and $\mathcal{O}(\overline{D}) \setminus \mathcal{A}(\tilde{D})|_{\overline{D}} \neq \emptyset$.

The proof consists of the following two parts:

- (1) $\partial_S \tilde{D} \cap (I \times \mathbb{D}(3)) = \emptyset$, where $I := [\frac{1}{2}, 1]$, \mathbb{D} is the unit disc, and $\mathbb{D}(r) := r\mathbb{D}$.
- (2) There exists a function $h \in \mathcal{O}(\overline{D})$ (effectively given) such that

$$h(x, w) = \begin{cases} e^{-2\pi + i \log x}, & \text{if } (x, w) \in I \times \overline{\mathbb{A}}(2, 3) \\ e^{i \log x}, & \text{if } (x, w) \in I \times \overline{\mathbb{D}} \end{cases}$$

and $|h| < 1$ on the remaining part of \overline{D} , where $\mathbb{A}(r_-, r_+) := \mathbb{D}(r_+) \setminus \overline{\mathbb{D}}(r_-)$.

Unfortunately, the proof of (1) contains a gap. The aim of the present note is to close the above gap and to prove some new results related to the Shilov and Bergman boundaries of D and \tilde{D} .

Let

$$A := \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}, \quad I_0 := (\frac{1}{2}, 1), \quad A_0 := A \setminus I_0.$$

By the Cauchy integral formula each function $f \in \mathcal{A}(D)$ extends holomorphically to the domain

$$G = \{(z, w) \in A_0 \times \mathbb{C} : |w|e^{V(z)} < 1\},$$

where

$$V(re^{i\varphi}) := \begin{cases} 0, & \text{if } 0 < \varphi \leq \frac{\pi}{2} \\ -\log 3, & \text{if } \frac{\pi}{2} < \varphi < 2\pi \end{cases}.$$

Hence (by [3], Corollary 3.2.18) the envelope of holomorphy \tilde{D} is univalent and

$$\tilde{D} = \tilde{G} = \{(z, w) \in A_0 \times \mathbb{C} : |w|e^{\tilde{V}(z)} < 1\},$$

where

$$\tilde{V}(z) := \sup\{u \in \mathcal{SH}(A_0) : u \leq V\}.$$

Notice that, by the maximum principle for subharmonic functions, we have $\tilde{V}(z) < 0$, $z \in A_0$. Thus, $\partial D \cap (U \times \mathbb{D}(3)) \subset \tilde{D}$, where $U := \{re^{i\varphi} : r \in I_0, 0 < \varphi < \frac{\pi}{2}\}$. Hence $\partial \tilde{D} \cap (U \times \mathbb{D}(3))$ does not contain points of $\partial_S \tilde{D}$.

We are going to prove the following theorem.

- THEOREM 3.** (a) $\partial_S \tilde{D} \cap (I_0 \times \mathbb{D}(3)) = \emptyset$.
 (b) For any $a \in I$ there exists a $g = g_a \in \mathcal{O}(\overline{D})$ such that $g(a, w) = 1$ for all $w \in \overline{\mathbb{D}}$, and $|g| < 1$ on $\overline{D} \setminus (\{a\} \times \overline{\mathbb{D}})$. In particular, $\partial_B D \cap (\{a\} \times \overline{\mathbb{D}}) \neq \emptyset$.
 (c) $\partial_B D \supset \{a\} \times \mathbb{T}$ for every $a \in I_0$, where $\mathbb{T} := \partial \mathbb{D}$. Therefore, $\partial_B D \supset I \times \mathbb{T}$.
 (d) $\partial_B D \setminus ((iI_0) \times (3\mathbb{T})) = \partial_S D \setminus ((iI_0) \times (3\mathbb{T}))$
 $= \{re^{i\varphi} : r \in \{\frac{1}{2}, 1\}, \frac{\pi}{2} \leq \varphi \leq 2\pi\} \times (3\mathbb{T})$
 $\cup I_0 \times (3\mathbb{T})$
 $\cup I_0 \times \mathbb{T}$
 $\cup \{re^{i\varphi} : r \in \{\frac{1}{2}, 1\}, 0 \leq \varphi \leq \frac{\pi}{2}\} \times \mathbb{T}$
 $=: M_1 \cup M_2 \cup M_3 \cup M_4$.

REMARK 4. (i) Observe that (a) and (b) close gaps in our former proof.

Indeed, we get $\emptyset \neq \partial_B D \setminus \partial_S \tilde{D} \subset (\partial_S D \setminus \partial_S \tilde{D}) \cap (\partial_B D \setminus \partial_B \tilde{D})$. Hence $\partial_S \tilde{D} \subsetneq \partial_S D$ and $\partial_B \tilde{D} \subsetneq \partial_B D$.

Moreover, if $a \in I_0$, then $g_a \in \mathcal{O}(\overline{D}) \setminus \mathcal{A}(\tilde{D})|_{\overline{D}}$.

- (ii) It seems to be an *open problem* whether $(iI_0) \times (3\mathbb{T}) \subset \partial_S D$ (resp. $(iI_0) \times (3\mathbb{T}) \subset \partial_B D$).

Proof of Theorem 3. First, let us make the following elementary observation.

(*) Let Σ be an open subset of the boundary $\partial \Omega$ of a bounded domain $\Omega \subset \mathbb{C}^n$. Suppose that $\max_{\partial \Omega} |f| = \max_{\partial \Omega \setminus \Sigma} |f|$ for every $f \in \mathcal{A}(\Omega)$ (resp. $\mathcal{O}(\overline{\Omega})$). Then $\partial_S \Omega \cap \Sigma = \emptyset$ (resp. $\partial_B \Omega \cap \Sigma = \emptyset$).

(a) For all $a \in I_0$ and $f \in \mathcal{A}(\tilde{D})$ the function $f(a, \cdot)$ extends holomorphically to $\mathbb{D}(3)$.

Indeed, we may define $\hat{f}(z, w) := \frac{1}{2\pi i} \int_{|\zeta|=5/2} \frac{f(z, \zeta)}{\zeta - w} d\zeta$, $z \in \mathbb{A}(\frac{1}{2}, 1)$, $\frac{\pi}{2} < \arg z \leq 2\pi$, $|w| < \frac{5}{2}$. Then \hat{f} is holomorphic and coincides with f when $\frac{\pi}{2} < \arg z < \pi$. Hence using identity theorem we see that $f = \hat{f}$ on their common domain of definition. Using continuity of f we get the claimed extension of $f(a, \cdot)$.

In particular, $\max_{\{a\} \times \overline{\mathbb{A}(1,3)}} |f(a, \cdot)| = \max_{\{a\} \times 3\mathbb{T}} |f(a, \cdot)|$. Hence, by (*) with $\Omega := \tilde{D}$ and $\Sigma := I_0 \times \mathbb{A}(1, 3)$, we conclude that $\partial_S \tilde{D} \cap (I_0 \times \mathbb{A}(1, 3)) = \emptyset$.

The same argument shows that $\partial_S \tilde{D} \cap (I_0 \times \mathbb{D}) = \emptyset$. (Note that these first two cases can also be handled using the density of the peak points — see the argument in the next case.)

Suppose that $(z_0, w_0) \in (I_0 \times \mathbb{T}) \cap \partial_S \tilde{D}$. Then there is a peak point (z_1, w_1) nearby. Let $f \in \mathcal{A}(\tilde{D})$ be a function peaking there. The maximum principle excludes the situation where $z_1 \in I_0$. Thus $z_1 \in U$, but we already know that $\partial_S \tilde{D} \cap (U \times \mathbb{D}(3)) = \emptyset$, so it is impossible.

Finally, $\partial_S \tilde{D} \cap (I_0 \times \mathbb{D}(3)) = \emptyset$.

(b) Fix an $a \in I$ and let h be as in (2), $w_0 := e^{i \log a} \in \mathbb{T}$. Define $\varphi(w) := \frac{1}{2w_0}(w + w_0)$, $g := \varphi \circ h$. It is obvious that $g \in \mathcal{O}(\overline{D})$, $g(a, w) = 1$ for all $w \in \overline{\mathbb{D}}$, and $|g| < 1$ on $\overline{D} \setminus (\{a\} \times \overline{\mathbb{D}})$.

(c) Using (a) we have $\partial_B D \cap (I_0 \times \mathbb{D}) = \emptyset$. Hence, by (b), $\partial_B D \cap (\{a\} \times \mathbb{T}) \neq \emptyset$ for every $a \in I_0$. Now using rotational invariance in the second variable of $\partial_B D$ leads to $\{a\} \times \mathbb{T} \subset \partial_B D$ for all $a \in I_0$.

(d) Notice that also $\partial_S D$ is invariant under rotations of the second variable.

- Every point from $\partial A \times 3\mathbb{T}$ is a peak point for $\mathcal{O}(\overline{A} \times \mathbb{D}(3))$.

Indeed, fix a point $(a, b) \in \partial A \times 3\mathbb{T}$. Then a is a peak point for $\mathcal{O}(\overline{A})$ and b is a peak point for $\mathcal{O}(\mathbb{D}(3))$. So it suffices to take the product of the corresponding peak functions to see that (a, b) is a peak point for $\mathcal{O}(\overline{A} \times \mathbb{D}(3))$.

Thus $M_1 \subset \partial_B D \subset \partial_S D$.

- Consider the holomorphic function

$$\overline{D} \ni (z, w) \xrightarrow{\Phi} \text{Log}(z) \in R := [-\log 2, 0] \times [0, 2\pi],$$

where Log is a branch of logarithm with $\text{Log}(-1) = \pi$. Note that $\Phi \in \mathcal{O}(\overline{D})$. For every $a \in I_0$ we have $\Phi(a, w) \in (-\log 2, 0) \times \{2\pi\} \in \partial R$ whenever $w \in \overline{\mathbb{A}}(2, 3)$. It is clear that there exists a function $\psi_a \in \mathcal{O}(\overline{R})$ such that $\psi_a(\Phi(a, w)) = 1$, $w \in \overline{\mathbb{A}}(2, 3)$, and $|\psi_a| < 1$ on $\overline{R} \setminus \{\Phi(a, w)\}$. Then the function

$$\overline{D} \ni (z, w) \xrightarrow{f_a} \psi_a(\Phi(z, w))$$

may be considered as a function of class $\mathcal{O}(\overline{D})$. Observe that $f_a(a, w) = 1$ for all $w \in \overline{\mathbb{A}}(2, 3)$, and $|f_a| < 1$ on $\overline{D} \setminus (\{a\} \times \overline{\mathbb{A}}(2, 3))$. Fix a $b \in 3\mathbb{T}$ and define $g(z, w) := f_a(z, w) \frac{1+w/b}{2}$. Then $g \in \mathcal{O}(\overline{D})$ and g peaks at (a, b) . Consequently, $M_2 \subset \partial_B D \subset \partial_S D$.

- By (c), $M_3 \subset \partial_B D \subset \partial_S D$.

• For every $a = re^{i\varphi}$ with $r \in \{\frac{1}{2}, 1\}$, $0 < \varphi < \frac{\pi}{2}$ there exists a function $\psi \in \mathcal{O}(\overline{A})$ such that $\psi(a) = 1$ and $|\psi| < 1$ on $\overline{A} \setminus \{a\}$. Hence $\partial_B D \cap (\{a\} \times \overline{\mathbb{D}}) \neq \emptyset$. Now, by (*) with $\Omega := D$, $\Sigma := \{re^{i\varphi} : r \in \{\frac{1}{2}, 1\}, 0 < \varphi < \frac{\pi}{2}\} \times \mathbb{D}$, we conclude that $M_4 \subset \partial_B D \subset \partial_S D$.

The remaining part of ∂D , i.e. the set $\Sigma := \partial D \setminus (M_1 \cup M_2 \cup M_3 \cup M_4 \cup ((iI_0) \times (3\mathbb{T})))$, is open in ∂D . It remains to use (*). \square

REMARK 5. We will try to complete the description of $\partial_B D$, $\partial_S D$, $\partial_B \tilde{D}$, and $\partial_S \tilde{D}$. Any help by the reader will be appreciated.

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Jagiellonian University
 Faculty of Mathematics and Computer Science
 Institute of Mathematics
 Łojasiewicza 6, 30-348 Kraków
 Poland
e-mail: Marek.Jarnicki@im.uj.edu.pl

Carl von Ossietzky Universität Oldenburg
 Institut für Mathematik
 Postfach 2503
 D-26111 Oldenburg
 Germany
e-mail: Peter.Pflug@uni-oldenburg.de