17th International Conference on the Applications of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

DISCRETE-CONTINUAL BOUNDARY ELEMENT METHODS OF ANALYSIS FOR TWO-DIMENSIONAL AND THREE-DIMENSIONAL STRUCTURES

A.B. Zolotov*, **P.A. Akimov, and V.N. Sidorov**

* *Moscow State University of Civil Engineering, Moscow*
26, Yaroslavskoe Shosse, 129337, Moscow, RUSSIA
E-mail: pavel.akimov@gmail.com

Keywords: discrete-continual boundary element method, pseudodifferential operators, distributions, discrete-continual boundary element, semianalytical formulations.

Abstract. *The overall objective of this paper is to describe so-called direct and indirect discrete-continual boundary element methods of structural analysis (DDCBEM and IDCBEM). Analytical formulations of the problem in terms of each methods are given. Using fundamental operational relations of direct and indirect approaches after construction of corresponding fundamental matrix-function in a special form convenient for problems of structural mechanics and its application in both cases we obtain resolving set of differential equations with operational coefficients. The discrete-continual design model for structures with constant physical and geometrical parameters in one direction is offered on the basis of so-called discrete-continual boundary elements. Basic pseudodifferential operators are approximated discretely by Fourier series. Fourier transformations and Wavelet analysis can be applied as well. Computational algorithms of DDCBEM, IDCBEM and corresponding software are proposed and described.*

1 INTRODUCTION

The distinctive paper is devoted to basic description of so-called direct discrete-continual boundary element method (DDCBEM) and indirect discrete-continual boundary element method (IDCBEM) of structural analysis. Their field of application comprises structures with invariability of physical and geometrical parameters in some dimensions. We should mention here in particular such objects as beams, thin-walled bars, strip foundations, plates, shells, deep beams, high-rise buildings, extensional buildings, pipelines, rails, dams and others. DDCBEM and IDCBEM come under group of semianalytical methods [3-6,12-13]. Semianalytical formulations are contemporary mathematical models which are becoming realizable at pre-sent due to substantial speed-up of computer productivity. DDCBEM and IDCBEM are based on pseudodifferential boundary equations. Corresponding operators are approximated dis-cretely by Fourier series. Wavelet analysis can be applied as well. Key features of DDCBEM and IDCBEM include double reduction of dimension. Only cross-sectional boundary is under discretization, namely we consider one-dimensional problem. Other advantages of DDCBEM and IDCBEM are allowance of advanced analysis in vital areas, simple data processing, effective computational schemes and computer-oriented algorithms. We consider the second boundary value problem for three-dimensional elastostatics as a specific example of using DDCBEM and IDCBEM (Figure 1).

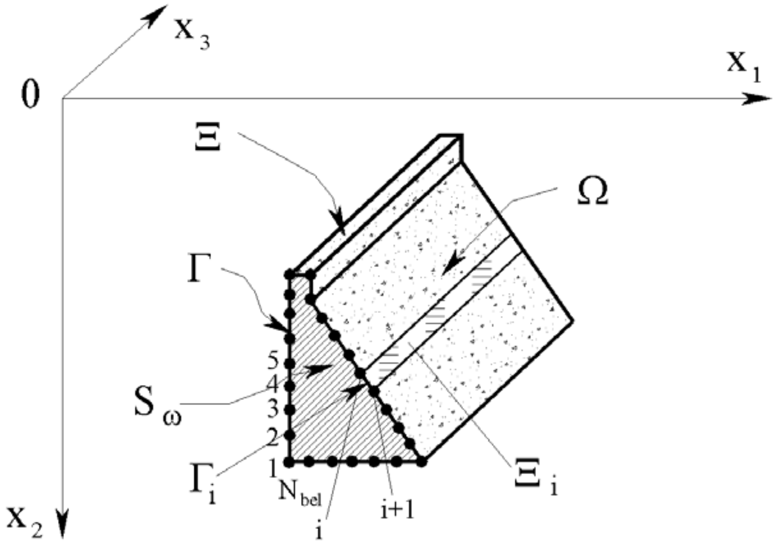

Figure 1. Sample of considering structure

2 ANALYTICAL FORMULATION OF THE PROBLEM IN TERMS OF DDCBEM AND IDCBEM

2.1. Conventional formulation of the problem.

Conventional formulation of the second boundary value problem for elastostatics has the form [14-15]

$$Lu = \sum_{j=1}^{N} \partial_j \sigma_{ij} = -F_i, \ x \in \Omega, \quad lu = \sum_{j=1}^{N} \nu_j \sigma_{ij} = -f_i, \ x \in \partial\Omega, \quad i = 1, \dots N, \qquad (2.1)$$

where Ω is the domain occupied by structure; N is the dimensionality; L is the operator defining conditions in the domain; l is the operator defining conditions at the domain boundary $\partial\Omega$, $\overline{\nu} = [\nu_1 \; ... \; \nu_N]^T$ is its unit normal direction vector with $\nu_N = 0$; σ_{ij} are stress components; $\overline{x}$ is the coordinate vector; $\overline{u}$ is the displacement vector; $\overline{F}$ is the body force vector; $\overline{f}$ is the boundary traction vector.

Hereinafter we will study three-dimensional problems for definiteness.

2.2. Operators defining conditions in the domain and at the domain boundary.

Let x_3 be coordinate axis with invariability of physical and geometrical parameters of structure (basic direction). The reader will have no difficulty in showing for three-dimensional problem that

$$L = -L_2\partial_3^2 + L_1\partial_3 + L_0; \quad l = l_1\partial_3 + l_0, \tag{2.2}$$

$$L_2 = \mu\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma+2 \end{bmatrix}; \quad L_1 = -\mu(\gamma+1)\begin{bmatrix} 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ \partial_1 & \partial_2 & 0 \end{bmatrix}; \tag{2.3}$$

$$L_0 = -\mu\begin{bmatrix} (\gamma+2)\partial_1^2 + \partial_2^2 & (\gamma+1)\partial_1\partial_2 & 0 \\ (\gamma+1)\partial_1\partial_2 & \partial_1^2 + (\gamma+2)\partial_2^2 & 0 \\ 0 & 0 & \partial_1^2 + \partial_2^2 \end{bmatrix}; \tag{2.4}$$

$$l_1 = -\nu_1\mu\begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \nu_2\mu\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 1 & 0 \end{bmatrix}; \tag{2.5}$$

$$l_0 = -\nu_1\mu\begin{bmatrix} (\gamma+2)\partial_1 & \gamma\partial_2 & 0 \\ \partial_2 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \end{bmatrix} - \nu_2\mu\begin{bmatrix} \partial_2 & \partial_1 & 0 \\ \gamma\partial_1 & (\gamma+2)\partial_2 & 0 \\ 0 & 0 & \partial_2 \end{bmatrix}. \tag{2.6}$$

2.3. Operators defining conditions in the domain and at the domain boundary.

2.3.1. Fundamental operational relation of direct approach.

Fundamental operational relation of direct approach has the form [6,11]

$$L\theta\overline{u} = \theta L\overline{u} + \delta_\Xi l\overline{u} - l^*(\delta_\Xi\overline{u}), \tag{2.7}$$

where $\theta(x)$ is the characteristic function of domain Ω; δ_Ξ is the delta function of domain boundary $\partial\Omega$ [6].

2.3.2. Construction of differential equation set of the first order with operational coefficients.

Combining (2.2)-(2.6) and (2.1) we get:

$$-L_2\theta\partial_3^2\overline{u} + L_1\theta\partial_3\overline{u} + L_0\theta\overline{u} = \overline{F} - l_1^*(\delta_\Xi\partial_3\overline{u}) - l_0^*(\delta_\Xi\overline{u}). \tag{2.8}$$

In this case

$$\overline{F} = \theta\overline{F} + \delta_\Xi\overline{f}; \quad \overline{v} = \overline{u}' = \partial_3\overline{u}; \quad \overline{v}' = \partial_3\overline{v}. \tag{2.9}$$

After uniting of (2.8) and (2.9) and corresponding formula translation we obtain the following differential equation set of the first order with respect to x_3

$$\overline{U}' = L_G\overline{U} + \overline{F}_G + l_G^*\delta_\Xi\overline{U}, \tag{2.10}$$

where $\overline{U} = [\theta\overline{u}^T\ \theta\overline{v}^T]^T = [\overline{u}^T\ \overline{v}^T]^T;\quad \overline{U}' = \partial_3\overline{U};\quad \overline{u} = \theta\overline{u};\quad \overline{v} = \theta\overline{v};$ (2.11)

$$L_G = \begin{bmatrix} 0 & E \\ L_2^{-1}L_0 & L_2^{-1}L_1 \end{bmatrix};\quad \overline{F}_G = -\begin{bmatrix} 0 \\ L_2^{-1}\overline{F} \end{bmatrix};\quad l_G^* = \begin{bmatrix} 0 & 0 \\ L_2^{-1}l_0^* & L_2^{-1}l_1^* \end{bmatrix}. \tag{2.12}$$

It is readily seen that all coefficients in (2.10) are pseudodifferential operators and E is identity operator of the corresponding order.

2.3.3. Fundamental matrix-function of differential equations set.

Consider the auxiliary equation set

$$\overline{U}' = L_G\overline{U}. \tag{2.13}$$

Let λ_i be eigenvalue of operator L_G and m_i be multiplicity of λ_i. It can be proved that

$$\lambda_1 = -|\nabla_2|,\ \ \lambda_2 = |\nabla_2|,\ \ m_1 = m_2 = 3;\quad |\nabla_2| = \sqrt{-(\partial_1^2 + \partial_2^2)}. \tag{2.14}$$

Two eigenvectors and root vector corresponding to λ_1 are the following

$$\bar{t}_{11}^e = [-\partial_1, -\partial_2, |\nabla_2|, \partial_1|\nabla_2|, \partial_2|\nabla_2|, -\nabla_2^2]^T;\quad \bar{t}_{12}^e = [\partial_2, -\partial_1, 0, -\partial_2|\nabla_2|, \partial_1|\nabla_2|, 0]^T; \tag{2.15}$$

$$\bar{t}_{11}^r = [0, 0, \frac{\gamma+3}{\gamma+1}, -\partial_1, -\partial_2, -\frac{2}{\gamma+1}|\nabla_2|]^T. \tag{2.16}$$

In exactly the same way for λ_2 we could have written

$$\bar{t}_{21}^e = [\partial_1, \partial_2, |\nabla_2|, \partial_1|\nabla_2|, \partial_2|\nabla_2|, \nabla_2^2]^T;\quad \bar{t}_{22}^e = [\partial_2, -\partial_1, 0, \partial_2|\nabla_2|, -\partial_1|\nabla_2|, 0]^T; \tag{2.17}$$

$$\bar{t}_{21}^r = [0, 0, -\frac{\gamma+3}{\gamma+1}, \partial_1, \partial_2, -\frac{2}{\gamma+1}|\nabla_2|]^T. \tag{2.18}$$

Thus, Jordan decomposition of L_G is defined by formulas:

$$L_G = TJ\tilde{T};\quad T = [\bar{t}_{11}^e, \bar{t}_{11}^r, \bar{t}_{12}^e, \bar{t}_{21}^e, \bar{t}_{21}^r, \bar{t}_{22}^e];\quad \tilde{T} = T^{-1} = [\bar{\tilde{t}}_1, \bar{\tilde{t}}_2, \bar{\tilde{t}}_3, \bar{\tilde{t}}_4, \bar{\tilde{t}}_5, \bar{\tilde{t}}_6]^T; \tag{2.19}$$

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix};\ J_1 = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{12} \end{bmatrix};\ J_2 = \begin{bmatrix} J_{21} & 0 \\ 0 & J_{22} \end{bmatrix}; \tag{2.20}$$

$$J_{11} = \begin{bmatrix} -|\nabla_2| & 1 \\ 0 & -|\nabla_2| \end{bmatrix};\ J_{21} = \begin{bmatrix} |\nabla_2| & 1 \\ 0 & |\nabla_2| \end{bmatrix};\quad J_{12} = -|\nabla_2|;\quad J_{22} = |\nabla_2|; \tag{2.21}$$

$$\bar{\tilde{t}}_1 = [\gamma_1\partial_1|\nabla_2|^{-2}, \gamma_1\partial_2|\nabla_2|^{-2}, \gamma_2|\nabla_2|^{-1}, \gamma_3\partial_1|\nabla_2|^{-3}, \gamma_3\partial_2|\nabla_2|^{-3}, 0]^T; \tag{2.22}$$

$$\bar{\tilde{t}}_2 = [\gamma_4\partial_1|\nabla_2|^{-1}, \gamma_4\partial_2|\nabla_2|^{-1}, \gamma_2, \gamma_2\partial_1|\nabla_2|^{-2}, \gamma_2\partial_2|\nabla_2|^{-2}, \gamma_4|\nabla_2|^{-1}]^T; \tag{2.23}$$

$$\bar{\tilde{t}}_3 = [-\gamma_1\partial_2|\nabla_2|^{-2}, -\gamma_1\partial_1|\nabla_2|^{-2}, 0, \gamma_1\partial_2|\nabla_2|^{-3}, -\gamma_1\partial_1|\nabla_2|^{-3}, 0]^T; \tag{2.24}$$

$$\bar{\tilde{t}}_4 = [-\gamma_1\partial_1|\nabla_2|^{-2}, -\gamma_1\partial_2|\nabla_2|^{-2}, \gamma_2|\nabla_2|^{-1}, \gamma_3\partial_1|\nabla_2|^{-3}, \gamma_3\partial_2|\nabla_2|^{-3}, 0]^T; \tag{2.25}$$

$$\bar{\tilde{t}}_5 = [\gamma_4\partial_1|\nabla_2|^{-1}, \gamma_4\partial_2|\nabla_2|^{-1}, -\gamma_2, -\gamma_2\partial_1|\nabla_2|^{-2}, -\gamma_2\partial_2|\nabla_2|^{-2}, \gamma_4|\nabla_2|^{-1}]^T; \tag{2.26}$$

$$\bar{\tilde{t}}_6 = [-\gamma_1\partial_2|\nabla_2|^{-2}, -\gamma_1\partial_1|\nabla_2|^{-2}, 0, -\gamma_1\partial_2|\nabla_2|^{-3}, \gamma_1\partial_1|\nabla_2|^{-3}, 0]^T. \tag{2.27}$$

We use the following notation:

$$\gamma_1 = \frac{1}{2}; \quad \gamma_2 = \frac{1}{4}\frac{\gamma+1}{\gamma+2}; \quad \gamma_3 = -\frac{1}{4}\frac{\gamma+3}{\gamma+2}; \quad \gamma_4 = \frac{1}{4}(\gamma+1). \tag{2.28}$$

Fundamental matrix-function of (2.16) is the solution of the following set of differential equations:

$$\varepsilon'(x_3) - L_G\varepsilon(x_3) = \delta(x_3)E, \tag{2.29}$$

where $\varepsilon'(x_3) = \partial_3\varepsilon(x_3)$; $\delta(x_3)$ is Dirac delta function. Using [10], we get:

$$\begin{aligned}\varepsilon(x_3) = \exp(-|\nabla_2||x_3|)\widetilde{P}_{1,0} + \operatorname{sign}(x_3)\exp(-|\nabla_2||x_3|)\widetilde{P}_{1,1} + \\ + x_3\exp(-|\nabla_2||x_3|)\widetilde{P}_{2,0} + |x_3|\exp(-|\nabla_2||x_3|)\widetilde{P}_{2,1};\end{aligned} \tag{2.30}$$

$$P_1 = \begin{cases} P_1^+ = T^+\widetilde{T}^+, \ x_3 > 0 \\ P_1^- = T^-\widetilde{T}^-, \ x_3 < 0 \end{cases}; \quad P_2 = \begin{cases} P_2^+ = T^+H\widetilde{T}^+, \ x_3 > 0 \\ P_2^- = T^-H\widetilde{T}^-, \ x_3 < 0 \end{cases}; \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{2.31}$$

$$P_1 = \operatorname{sign}(x_3)\widetilde{P}_{1,0} + \widetilde{P}_{1,1}; \quad P_2 = \operatorname{sign}(x_3)\widetilde{P}_{2,0} + \widetilde{P}_{2,1}; \tag{2.32}$$

$$T^- = [\bar{t}_{21}^e, \bar{t}_{21}^r, \bar{t}_{22}^e]; \quad T^+ = [\bar{t}_{11}^e, \bar{t}_{11}^r, \bar{t}_{12}^e]; \quad \widetilde{T}^- = [\bar{t}_4, \bar{t}_5, \bar{t}_6]^T; \quad \widetilde{T}^+ = [\bar{t}_1, \bar{t}_2, \bar{t}_3]^T. \tag{2.33}$$

Here $\widetilde{P}_{1,0}, \widetilde{P}_{1,1}, \widetilde{P}_{2,0}, \widetilde{P}_{2,1}$ are x_3-independent pseudodifferential operators with respect to x_1, x_2, namely [1]

$$\begin{aligned}\widetilde{P}_{1,0} = |\nabla_2|^{-1} P_{1,0,1} + \partial_1^2|\nabla_2|^{-3} P_{1,0,2} + \partial_1\partial_2|\nabla_2|^{-3} P_{1,0,3} + \partial_2^2|\nabla_2|^{-3} P_{1,0,4} + \partial_1|\nabla_2|^{-1} P_{1,0,5} + \\ + \partial_2|\nabla_2|^{-1} P_{1,0,6} + \partial_1^2|\nabla_2|^{-1} P_{1,0,7} + \partial_1\partial_2|\nabla_2|^{-1} P_{1,0,8} + \partial_2^2|\nabla_2|^{-1} P_{1,0,9} + |\nabla_2| P_{1,0,10};\end{aligned} \tag{2.34}$$

$$\widetilde{P}_{1,1} = 0.5\cdot E; \tag{2.35}$$

$$\begin{aligned}\widetilde{P}_{2,0} = \partial_1|\nabla_2|^{-1} P_{2,0,1} + \partial_2|\nabla_2|^{-1} P_{2,0,2} + |\nabla_2| P_{2,0,3} + \partial_1^2|\nabla_2|^{-1} P_{2,0,4} + \partial_1\partial_2|\nabla_2|^{-1} P_{2,0,5} + \\ + \partial_2^2|\nabla_2|^{-1} P_{2,0,6} + \partial_1|\nabla_2| P_{2,0,7} + \partial_2|\nabla_2| P_{2,0,8};\end{aligned} \tag{2.36}$$

$$\begin{aligned}\widetilde{P}_{2,1} = P_{2,1,1} + \partial_1^2|\nabla_2|^{-2} P_{2,1,2} + \partial_1\partial_2|\nabla_2|^{-2} P_{2,1,3} + \partial_2^2|\nabla_2|^{-2} P_{2,1,4} + \partial_1 P_{2,1,5} + \\ + \partial_2 P_{2,1,6} + \partial_1^2 P_{2,1,7} + \partial_1\partial_2 P_{2,1,8} + \partial_2^2 P_{2,1,9} + |\nabla_2|^2 P_{2,1,10}.\end{aligned} \tag{2.37}$$

Here $P_{i,j,k}$ is numerical matrix coefficient.

2.3.4. Resolving set of operational boundary equations.

After convolution of fundamental matrix-function (2.30) with both sides of (2.10) the result is

$$\begin{aligned}\overline{U} = \exp(-|\nabla_2||x_3|)[\widetilde{P}_{1,0} + \operatorname{sign}(x_3)\widetilde{P}_{1,1} + x_3\widetilde{P}_{2,0} + |x_3|\widetilde{P}_{2,1}] \underset{3}{*} \overline{F}_G + \\ + \exp(-|\nabla_2||x_3|)[\widetilde{Q}_{1,0} + \operatorname{sign}(x_3)\widetilde{Q}_{1,1} + x_3\widetilde{Q}_{2,0} + |x_3|\widetilde{Q}_{2,1}] \underset{3}{*} (\delta_\Xi \overline{U}), \ x \to \Xi + 0;\end{aligned} \tag{2.38}$$

$$\widetilde{Q}_{1,0} = \widetilde{P}_{1,0} l_G^*; \quad \widetilde{Q}_{1,1} = \widetilde{P}_{1,1} l_G^*; \quad \widetilde{Q}_{2,0} = \widetilde{P}_{2,0} l_G^*; \quad \widetilde{Q}_{2,1} = \widetilde{P}_{2,1} l_G^*. \tag{2.39}$$

2.3.5. Reduction of the problem. Reduced resolving set of operational boundary equations.

Major disadvantages of (2.38) are double number of unknowns and high (second) order of pseudodifferential operators with respect to x_1, x_2. However it can be checked that they operate on components u_1, u_2, u_3 only. In this connection it is preferable to exclude u_1, u_2, u_3 as part of reduction procedure. Reduction is based on the following formulas of integration

$$\int \exp(-|\nabla_2||x_3|)dx_3 = |\nabla_2|^{-1}\,\mathrm{sign}(x_3) - |\nabla_2|^{-1}\,\mathrm{sign}(x_3)\exp(-|\nabla_2||x_3|); \tag{2.40}$$

$$\int \mathrm{sign}(x_3)\exp(-|\nabla_2||x_3|)dx_3 = -|\nabla_2|^{-1}\exp(-|\nabla_2||x_3|); \tag{2.41}$$

$$\int x_3 \exp(-|\nabla_2||x_3|)dx_3 = -|\nabla_2|^{-1}\exp(-|\nabla_2||x_3|)[|x_3| + |\nabla_2|^{-1}]; \tag{2.42}$$

$$\int |x_3|\exp(-|\nabla_2||x_3|)dx_3 = -\nabla_2^2\mathrm{sign}(x_3) - |\nabla_2|^{-1}\exp(-|\nabla_2||x_3|)[x_3 + |\nabla_2|^{-1}\,\mathrm{sign}(x_3)] \tag{2.43}$$

and well-known properties of convolution

$$K(x_1,x_2,x_3)\underset{3}{*}u_i(x_1,x_2,x_3) = \left(\int K(x_1,x_2,x_3)dx_3\right)\underset{3}{*}v_i(x_1,x_2,x_3),\ i=1,2,3 \tag{2.44}$$

Here $K(x_1,x_2,x_3)$ is an arbitrary operator from (2.34)-(2.37).

In accordance with such algorithm after numerous transformations we get couple of reduced operational boundary equation sets

$$\begin{aligned}\bar{v} &= \exp(-|\nabla_2||x_3|)[P^r_{1,0} + \mathrm{sign}(x_3)P^r_{1,1} + x_3P^r_{2,0} + |x_3|P^r_{2,1}]\underset{3}{*}\bar{F}^r_G + \\ &+ \exp(-|\nabla_2||x_3|)[Q^r_{1,0} + \mathrm{sign}(x_3)Q^r_{1,1} + x_3Q^r_{2,0} + |x_3|Q^r_{2,1}]\underset{3}{*}(\delta_\Xi\bar{v}),\ x \to \Xi + 0;\end{aligned} \tag{2.45}$$

$$\begin{aligned}\bar{u} &= \exp(-|\nabla_2||x_3|)[\tilde{P}^r_{1,0} + \mathrm{sign}(x_3)\tilde{P}^r_{1,1} + x_3\tilde{P}^r_{2,0} + |x_3|\tilde{P}^r_{2,1}]\underset{3}{*}\bar{F}^r_G + \tilde{Q}^r_s\underset{2}{*}(\delta_\Xi\bar{v}) + \\ &+ \exp(-|\nabla_2||x_3|)[\tilde{Q}^r_{1,0} + \mathrm{sign}(x_3)\tilde{Q}^r_{1,1} + x_3\tilde{Q}^r_{2,0} + |x_3|\tilde{Q}^r_{2,1}]\underset{3}{*}(\delta_\Xi\bar{v}),\ x \to \Xi + 0.\end{aligned} \tag{2.46}$$

Let us remark that $P^r_{i,j}, \tilde{P}^r_{i,j}$ and $Q^r_{i,j}, \tilde{Q}^r_{i,j}$ are reduced pseudodifferential operators with respect to x_1, x_2 [1]. They can also be visualized as a sum of operators of the form $\partial_1^p\partial_2^q|\nabla_2|^{-s}$ ($p,q,s \in Z$) with numerical matrix coefficients $P^r_{i,j,k}, \tilde{P}^r_{i,j,k}, Q^r_{i,j,k}, \tilde{Q}^r_{i,j,k}$. Moreover operators $Q^r_{i,j,k}$, $\tilde{Q}^r_{i,j,k}$ also delepends on directing vector $\bar{v}$ at the considering point

$$P^r_{1,0} = \partial_1|\nabla_2|^{-1}P^r_{1,0,1} + \partial_2|\nabla_2|^{-1}P^r_{1,0,2};\quad P^r_{1,1} = 0.5E; \tag{2.47}$$

$$P^r_{2,0} = |\nabla_2|P^r_{2,0,1} + \partial_1^2|\nabla_2|^{-1}P^r_{2,0,2} + \partial_1\partial_2|\nabla_2|^{-1}P^r_{2,0,3} + \partial_2^2|\nabla_2|^{-1}P^r_{2,0,4};\quad P^r_{2,0} = \partial_1P^r_{2,1,1} + \partial_2P^r_{2,1,2}; \tag{2.48}$$

$$Q^r_{1,0} = \partial_1|\nabla_2|^{-1}Q^r_{1,0,1} + \partial_2|\nabla_2|^{-1}Q^r_{1,0,2} + \partial_1^2\partial_2|\nabla_2|^{-3}Q^r_{1,0,3} + \partial_1\partial_2^2|\nabla_2|^{-3}Q^r_{1,0,4}; \tag{2.49}$$

$$Q^r_{2,0} = |\nabla_2|Q^r_{2,0,1} + \partial_1^2|\nabla_2|^{-1}Q^r_{2,0,2} + \partial_1\partial_2|\nabla_2|^{-1}Q^r_{2,0,3} + \partial_2^2|\nabla_2|^{-1}Q^r_{2,0,4}; \tag{2.50}$$

$$Q^r_{2,1} = \partial_1Q^r_{2,1,1} + \partial_2Q^r_{2,1,2} + \partial_1^2\partial_2|\nabla_2|^{-2}Q^r_{2,1,3} + \partial_1\partial_2^2|\nabla_2|^{-2}Q^r_{2,1,4}; \tag{2.51}$$

$$\tilde{P}^r_{1,0} = |\nabla_2|^{-1}\tilde{P}^r_{1,0,1} + \partial_1^2|\nabla_2|^{-3}\tilde{P}^r_{1,0,2} + \partial_1\partial_2|\nabla_2|^{-3}\tilde{P}^r_{1,0,3} + \partial_2^2|\nabla_2|^{-3}\tilde{P}^r_{1,0,4}; \tag{2.52}$$

$$\tilde{P}^r_{2,0} = \partial_1|\nabla_2|^{-1}\tilde{P}^r_{2,0,1} + \partial_2|\nabla_2|^{-1}\tilde{P}^r_{2,0,2}; \tag{2.53}$$

$$\tilde{P}^r_{2,1} = \tilde{P}^r_{2,1,1} + \partial_1^2|\nabla_2|^{-2}\tilde{P}^r_{2,1,2} + \partial_1\partial_2|\nabla_2|^{-2}\tilde{P}^r_{2,1,3} + \partial_2^2|\nabla_2|^{-2}\tilde{P}^r_{2,1,4}; \tag{2.54}$$

$$\tilde{Q}^r_{1,0} = |\nabla_2|^{-1}\tilde{Q}^r_{1,0,1} + \partial_1^2|\nabla_2|^{-3}\tilde{Q}^r_{1,0,2} + \partial_1\partial_2|\nabla_2|^{-3}\tilde{Q}^r_{1,0,3} + \partial_2^2|\nabla_2|^{-3}\tilde{Q}^r_{1,0,4}; \tag{2.55}$$

$$\begin{aligned}\tilde{Q}^r_{1,1} &= \partial_1|\nabla_2|^{-2}\tilde{Q}^r_{1,1,1} + \partial_2|\nabla_2|^{-2}\tilde{Q}^r_{1,1,2} + \partial_1^3|\nabla_2|^{-4}\tilde{Q}^r_{1,1,3} + \partial_1^2\partial_2|\nabla_2|^{-4}\tilde{Q}^r_{1,1,4} + \\ &+ \partial_1\partial_2^2|\nabla_2|^{-4}\tilde{Q}^r_{1,1,5} + \partial_2^3|\nabla_2|^{-4}\tilde{Q}^r_{1,1,6};\end{aligned} \tag{2.56}$$

$$\begin{aligned}\tilde{Q}^r_{2,0} &= \partial_1|\nabla_2|^{-1}\tilde{Q}^r_{2,0,1} + \partial_2|\nabla_2|^{-1}\tilde{Q}^r_{2,0,2} + \partial_1^3|\nabla_2|^{-3}\tilde{Q}^r_{2,0,3} + \partial_1^2\partial_2|\nabla_2|^{-3}\tilde{Q}^r_{2,0,4} + \\ &+ \partial_1\partial_2^2|\nabla_2|^{-3}\tilde{Q}^r_{2,0,5} + \partial_2^3|\nabla_2|^{-3}\tilde{Q}^r_{2,0,6};\end{aligned} \tag{2.57}$$

$$\widetilde{Q}^r_{2,1}=\widetilde{Q}^r_{2,1,1}+\partial_1\widetilde{Q}^r_{2,1,2}+\partial_2\widetilde{Q}^r_{2,1,3}+\partial_1^2|\nabla_2|^{-2}\widetilde{Q}^r_{2,1,4}+\partial_1\partial_2|\nabla_2|^{-2}\widetilde{Q}^r_{2,1,5}+\partial_2^2|\nabla_2|^{-2}\widetilde{Q}^r_{2,1,6}; \quad (2.58)$$

$$\widetilde{Q}^r_s=\partial_1|\nabla_2|^{-2}\widetilde{Q}^r_{s,1}+\partial_2|\nabla_2|^{-2}\widetilde{Q}^r_{s,2}+\partial_1^2\partial_2|\nabla_2|^{-4}\widetilde{Q}^r_{s,3}+\partial_1\partial_2^2|\nabla_2|^{-4}\widetilde{Q}^r_{s,4}; \quad (2.59)$$

$$P^r_{1,1}=P^r_{1,1,1};\quad Q^r_{1,1}=Q^r_{1,1,1};\quad \overline{F}^r_G=-L_2^{-1}F. \quad (2.60)$$

2.4. Operational formulation of the problem in terms of IDCBEM.

2.4.1. Fundamental operational relation of indirect approach.

Fundamental operational relation of indirect approach has the form [6,11]

$$L\overline{u}=\overline{F}+\delta_\Xi\overline{q}-l^*(\delta_\Xi\overline{w}), \quad (2.61)$$

where $\overline{q}$ is the abrupt change of boundary conditions in crossing the border Ξ; $\overline{w}$ is the corresponding abrupt change of displacements;

$$\overline{q}=(l\overline{u})_+-(l\overline{u})_-=\Delta l\overline{u};\quad \overline{w}=\overline{u}_+-\overline{u}_-=\Delta\overline{u}. \quad (2.62)$$

2.4.2. Construction of differential equation set of the first order with operational coefficients.

For considering second boundary value problem we define $\overline{w}=0$ and consequently obtain

$$L\overline{u}=\overline{F}+\delta_\Xi\overline{q}. \quad (2.63)$$

If we combine this with (2.2), we get

$$-L_2\partial_3^2\overline{u}+L_1\partial_3\overline{u}+L_0\overline{u}=\overline{F}+\delta_\Xi\overline{q}. \quad (2.64)$$

After uniting of (2.64) and

$$\overline{v}=\overline{u}'=\partial_3\overline{u};\quad \overline{v}'=\partial_3\overline{v} \quad (2.65)$$

with corresponding formula translation we obtain the following differential equation set of the first order with respect to x_3

$$\overline{U}'=L_G\overline{U}+\overline{F}_G+\delta_\Xi\overline{q}_G; \quad (2.66)$$

$$\overline{U}=[\,\overline{u}^T\quad \overline{v}^T\,]^T;\quad \overline{U}'=\partial_3\overline{U}; \quad (2.67)$$

$$L_G=\begin{bmatrix}0 & E\\ L_2^{-1}L_0 & L_2^{-1}L_1\end{bmatrix};\quad \overline{F}_G=-\begin{bmatrix}0\\ L_2^{-1}\overline{F}\end{bmatrix};\quad \overline{q}_G=-\begin{bmatrix}0\\ L_2^{-1}\overline{q}\end{bmatrix};\quad \overline{\overline{q}}=-L_2^{-1}\overline{q}. \quad (2.68)$$

2.4.3. Construction of differential equation set of the first order with operational coefficients.

Fundamental matrix-function of auxiliary differential equation set (2.13) has been already constructed in paragraph 2.3.3 and finally has the form (2.30). After its convolution with both sides of (2.66) and transformations the result is

$$\begin{aligned}\overline{U}=&\exp(-|\nabla_2||x_3|)[P^r_{1,0}+\mathrm{sign}(x_3)P^r_{1,1}+x_3P^r_{2,0}+|x_3|P^r_{2,1}]\underset{3}{*}\overline{F}^r_G+\\&+\exp(-|\nabla_2||x_3|)[P^r_{1,0}+\mathrm{sign}(x_3)P^r_{1,1}+x_3P^r_{2,0}+|x_3|P^r_{2,1}]\underset{3}{*}(\delta_\Xi\overline{\overline{q}}).\end{aligned} \quad (2.69)$$

Note that $P^r_{1,0},P^r_{1,1},P^r_{2,0},P^r_{2,1}$ are reduced pseudodifferential operators corresponding to x_1,x_2, namely [2]

$$P^r_{1,0} = |\nabla_2|^{-1} P^r_{1,0,1} + \partial_1^2 |\nabla_2|^{-3} P^r_{1,0,2} + \partial_1\partial_2 |\nabla_2|^{-3} P^r_{1,0,3} + \\ + \partial_2^2 |\nabla_2|^{-3} P^r_{1,0,4} + \partial_1 |\nabla_2|^{-1} P^r_{1,0,5} + \partial_2 |\nabla_2|^{-1} P^r_{1,0,6}; \quad (2.70)$$

$$P^r_{2,0} = \partial_1 |\nabla_2|^{-1} P^r_{2,0,1} + \partial_2 |\nabla_2|^{-1} P^r_{2,0,2} + |\nabla_2| P^r_{2,0,3} + \partial_1^2 |\nabla_2|^{-1} P^r_{2,0,4} + \\ + \partial_1\partial_2 |\nabla_2|^{-1} P^r_{2,0,5} + \partial_2^2 |\nabla_2|^{-1} P^r_{2,0,6}; \quad (2.71)$$

$$P^r_{2,1} = P^r_{2,1,1} + \partial_1^2 |\nabla_2|^{-2} P^r_{2,1,2} + \partial_1\partial_2 |\nabla_2|^{-2} P^r_{2,1,3} + \partial_2^2 |\nabla_2|^{-2} P^r_{2,1,4} + \partial_1 P^r_{2,1,5} + \partial_2 P^r_{2,1,6}; \quad (2.72)$$

$$P^r_{1,1} = P^r_{1,1,1}; \quad \overline{F}^r_G = -L_2^{-1}\overline{F}. \quad (2.73)$$

It can easily be checked that boundary conditions may be expressed as:

$$l_G \overline{U} = \overline{f}; \quad (2.74)$$

$$l_G = [l_0 \quad l_1]^T; \quad l_G = \nu_1 l_{G,1} + \nu_2 l_{G,2}; \quad (2.75)$$

$$l_{G,1} = -\mu \begin{bmatrix} (\gamma+2)\partial_1 & \gamma\partial_2 & 0 & 0 & 0 & \gamma \\ \partial_2 & \partial_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 1 & 0 & 0 \end{bmatrix}; \quad l_{G,2} = -\mu \begin{bmatrix} \partial_2 & \partial_1 & 0 & 0 & 0 & 0 \\ \gamma\partial_1 & (\gamma+2)\partial_2 & 0 & 0 & 0 & \gamma \\ 0 & 0 & \partial_2 & 0 & 1 & 0 \end{bmatrix}. \quad (2.76)$$

Combining (2.74) and (2.69), we obtain

$$\sum_{i=1}^{2} \nu_i \exp(-|\nabla_2||x_3|)[Q^r_{1,0,i} + \mathrm{sign}(x_3)Q^r_{1,1,i} + x_3 Q^r_{2,0,i} + |x_3| Q^r_{2,1,i}] \underset{3}{*} (\delta_\Xi \overline{\overline{q}}) = \\ = \overline{f} - \sum_{i=1}^{2} \nu_i \exp(-|\nabla_2||x_3|)[Q^r_{1,0,i} + \mathrm{sign}(x_3)Q^r_{1,1,i} + x_3 Q^r_{2,0,i} + |x_3| Q^r_{2,1,i}] \underset{3}{*} \overline{F}^r_G, \quad x \to \Xi + 0. \quad (2.77)$$

Each of $Q^r_{i,j,k}$ can be visualized as a sum of operators of the form $\partial_1^p \partial_2^q |\nabla_2|^{-s}$ $(p,q,s \in Z)$ with numerical matrix coefficients.

$$Q^r_{i,j,k} = l_{G,k} P^r_{i,j}, \; i = 1,2; \; j = 0,1; \; k = 1,2; \quad (2.78)$$

$$Q^r_{1,0,1} = \partial_1 |\nabla_2|^{-1} Q^r_{1,0,1,1} + \partial_2 |\nabla_2|^{-1} Q^r_{1,0,1,2} + \partial_1^3 |\nabla_2|^{-3} Q^r_{1,0,1,3} + \partial_1^2\partial_2 |\nabla_2|^{-3} Q^r_{1,0,1,4} + \\ + \partial_1\partial_2^2 |\nabla_2|^{-3} Q^r_{1,0,1,5} + \partial_2^3 |\nabla_2|^{-3} Q^r_{1,0,1,6}; \quad (2.79)$$

$$Q^r_{1,0,2} = \partial_1 |\nabla_2|^{-1} Q^r_{1,0,2,1} + \partial_2 |\nabla_2|^{-1} Q^r_{1,0,2,2} + \partial_1^3 |\nabla_2|^{-3} Q^r_{1,0,2,3} + \partial_1^2\partial_2 |\nabla_2|^{-3} Q^r_{1,0,2,4} + \\ + \partial_1\partial_2^2 |\nabla_2|^{-3} Q^r_{1,0,2,5} + \partial_2^3 |\nabla_2|^{-3} Q^r_{1,0,2,6}; \quad (2.80)$$

$$Q^r_{1,1,1} = Q^r_{1,1,1,1}; \quad Q^r_{1,2,1} = Q^r_{1,2,1,1}; \quad Q^r_{2,0,1} = \partial_1^2 |\nabla_2|^{-1} Q^r_{2,0,1,1} + \partial_1\partial_2 |\nabla_2|^{-1} Q^r_{2,0,1,2}; \quad (2.81)$$

$$Q^r_{2,0,2} = \partial_1\partial_2 |\nabla_2|^{-1} Q^r_{2,0,2,1} + \partial_2^2 |\nabla_2|^{-1} Q^r_{2,0,2,2}; \quad (2.82)$$

$$Q^r_{2,1,1} = \partial_1 Q^r_{2,1,1,1} + \partial_2 Q^r_{2,1,1,2} + \partial_1^3 |\nabla_2|^{-2} Q^r_{2,1,1,3} + \partial_1^2\partial_2 |\nabla_2|^{-2} Q^r_{2,1,1,4} + \partial_1\partial_2^2 |\nabla_2|^{-2} Q^r_{2,1,1,5} + \\ + \partial_2^3 |\nabla_2|^{-2} Q^r_{2,1,1,6}; \quad (2.83)$$

$$Q^r_{2,1,2} = \partial_1 Q^r_{2,1,2,1} + \partial_2 Q^r_{2,1,2,2} + \partial_1^3 |\nabla_2|^{-2} Q^r_{2,1,2,3} + \partial_1^2\partial_2 |\nabla_2|^{-2} Q^r_{2,1,2,4} + \partial_1\partial_2^2 |\nabla_2|^{-2} Q^r_{2,1,2,5} + \\ + \partial_2^3 |\nabla_2|^{-2} Q^r_{2,1,2,6}. \quad (2.84)$$

2.5. Alternative representations of basic pseudodifferential operators of DDCBEM and IDCBEM.

Basic pseudodifferential operators of DDCBEM and IDCBEM presented above can also be formulated with the use convolutions [8-9].

Let $f(x_1, x_2)$ be arbitrary function and

$$r=\sqrt{x_1^2+x_2^2}\,. \quad (2.85)$$

Summary is the following

$$|\nabla_2|f(x_1,x_2)=-\frac{1}{2\pi}\frac{1}{r^3}\underset{x_1,x_2}{*}f(x_1,x_2)\,;\quad |\nabla_2|^{-1}f(x_1,x_2)=\frac{1}{2\pi}\frac{1}{r}\underset{x_1,x_2}{*}f(x_1,x_2)\,; \quad (2.86)$$

$$\nabla_2^{-2}f(x_1,x_2)=-(1/2\pi)\ln r\underset{x_1,x_2}{*}f(x_1,x_2)\,;\quad |\nabla_2|^{-3}f(x_1,x_2)=-(1/2\pi)r\underset{x_1,x_2}{*}f(x_1,x_2)\,; \quad (2.87)$$

$$|\nabla_2|^{-1}\exp(-|\nabla_2||x_3|)f(x_1,x_2)=\frac{1}{2\pi}\frac{1}{\sqrt{r^2+x_3^2}}\underset{x_1,x_2}{*}f(x_1,x_2),\ \ x_3\neq 0; \quad (2.88)$$

$$|\nabla_2|^{-2}\exp(-|\nabla_2||x_3|)f(x_1,x_2)=-\frac{1}{2\pi}\ln(|x_3|+\sqrt{r^2+x_3^2})\underset{x_1,x_2}{*}f(x_1,x_2),\ x_3\neq 0; \quad (2.89)$$

$$|\nabla_2|^{-3}\exp(-|\nabla_2||x_3|)f(x_1,x_2)=-\frac{1}{2\pi}[|x_3|\ln(|x_3|+\sqrt{r^2+x_3^2})-$$
$$-\sqrt{r^2+x_3^2}]\underset{x_1,x_2}{*}f(x_1,x_2),\ \ x_3\neq 0; \quad (2.90)$$

$$\exp(-|\nabla_2||x_3|)f(x_1,x_2)=\frac{1}{2\pi}\frac{|x_3|}{(r^2+x_3^2)^{3/2}}\underset{x_1,x_2}{*}f(x_1,x_2),\ \ x_3\neq 0; \quad (2.91)$$

$$|\nabla_2|\exp(-|\nabla_2||x_3|)f(x_1,x_2)=\frac{1}{2\pi}\left[\frac{3x_3^2}{(r^2+x_3^2)^{5/2}}-\frac{1}{(r^2+x_3^2)^{3/2}}\right]\underset{x_1,x_2}{*}f(x_1,x_2),\ \ x_3\neq 0; \quad (2.92)$$

$$|\nabla_2|^2\exp(-|\nabla_2||x_3|)f(x_1,x_2)=\frac{3}{2\pi}\left[\frac{5|x_3^3|}{(r^2+x_3^2)^{7/2}}-\frac{3x_3}{(r^2+x_3^2)^{5/2}}\right]\underset{x_1,x_2}{*}f(x_1,x_2),\ \ x_3\neq 0. \quad (2.93)$$

2.6. The one approach to regularization of kernels of basic pseudodifferential operators in problems of structural mechanics.

In general, the problem of the solution of the integral and integral-differential equations with kernels of a kind $x^{-k}, |x|^{-k}, k>0$ arises by consideration of various technical problems. The specified kernels not always can be calculated in sense Cauchy. Kernels of type $\ln|x|$ and x^{-1} though are integrated in any sense, in some points they accept infinite values that leads to solution with infinity which make harder enough formulas of numerical integration. The listed functions should be more correctly understand in the generalized sense, i.e. in the form of their regularizations. Many of existing formulas for regularization (canonical, not canonical, etc.) are ambiguous from the point of view of numerical realization, appearing useful, mainly, for theoretical researches. Obviously one can see regularization $Vp(f(x))$ from function f (x) as a derivative of the corresponding order from some continuous function, for example,

$$Vp(1/x^k)=(-1)^{k-1}[x(\ln|x|-1)]^{(k+1)}/(k-1)!\,;\quad Vp(1/|x|)=[|x|(\ln|x|-1)]^{(2)}. \quad (2.94)$$

Thus, after regularization this generalized function can be represented as a finite-difference sequence of the derivatives with the corresponding order with parameter h from some continuous function, i.e.

$$Vp(f(x))=\lim_{h\to 0}D^sF(x)\,, \quad (2.95)$$

where $F(x)=f^{(-s)}(x)$ is the continuously define antiderivative of the function f(x) order "s", D^s is the differential finite-difference operator order "s" with a step equal "h". Values

$Vp(f(x))$ estimated in a point remotely located from the coordinates origins, should almost match with corresponding value of the function $f(x)$, so one can use $f(x)$ itself. Alternative approach to the regularization singular kernels is their approximation by Fourier series.

2.7. Methods of additional regularization of kernels of pseudodifferential operators in terms of DDCBEM and IDCBEM.

Regularization of kernels of pseudodifferential operators implies decrease of their orders. This procedure is especially effective for correctness of computation, better approximation of unknowns and simplification of corresponding discrete model. We can suggest at least two approaches to this problem.

First one is based on well-known properties of convolution (2.59) and intends single or multiple «throwing over» of derivative from the kernel of pseudodifferential operator to unknown vector function. We are of the opinion, for instance, that it is useful to apply such procedure twice in case of IDCBEM.

Alternative approach involves single or multiple integration of boundary operational equations and application of the following property of convolution.

$$\int [K(x_1,x_2,x_3)\underset{3}{*}u_i(x_1,x_2,x_3)]dx_3 = (\int K(x_1,x_2,x_3)dx_3)\underset{3}{*}u_i(x_1,x_2,x_3), i=1,2,3. \quad (2.96)$$

Let us say that double integration in case of IDCBEM is advantageous as well.
In both cases we use formulas of integration:

$$\iint \exp(-|\nabla_2||x_3|)dx_3 = |\nabla_2|^{-2}\exp(-|\nabla_2||x_3|) + |\nabla_2|^{-1}|x_3|; \quad (2.97)$$

$$\iint \mathrm{sign}(x_3)\exp(-|\nabla_2||x_3|)dx = -|\nabla_2|^{-2}\mathrm{sign}(x_3) + |\nabla_2|^{-2}\mathrm{sign}(x_3)\exp(-|\nabla_2||x_3|); \quad (2.98)$$

$$\begin{aligned}\iint x_3\exp(-|\nabla_2||x_3|)dx_3 = -|\nabla_2|^{-3}\mathrm{sign}(x_3) + |\nabla_2|^{-2}x_3\exp(-|\nabla_2||x_3|) + \\ +2|\nabla_2|^{-3}\mathrm{sign}(x)\exp(-|\nabla_2||x_3|);\end{aligned} \quad (2.99)$$

$$\begin{aligned}\iint |x_3|\exp(-|\nabla_2||x_3|)dx_3 = |\nabla_2|^{-2}|x_3|\exp(-|\nabla_2||x_3|) + \\ +2|\nabla_2|^{-3}\exp(-|\nabla_2||x_3|) + |\nabla_2|^{-2}|x_3|.\end{aligned} \quad (2.100)$$

2.8. Allowance for supports restrained by elastic members in terms of DDCBEM and IDCBEM.

If the considering structure has supports restrained by elastic members on the border, they can be taken into account in the stage of initial formulation

$$Lu = \sum_{j=1}^{3}\partial_j\sigma_{ij} = -F_i, \ x\in\Omega; \quad lu = \sum_{j=1}^{3}\nu_j\sigma_{ij} = -f_i + C\overline{u}, \ \ x\in\partial\Omega; \quad i=1,2,3, \quad (2.101)$$

where C is the corresponding matrix function of elastic responses,

$$C = C(x) = \mathrm{diag}\{c_1, c_2, c_3\} \quad (2.102)$$

and c_1, c_2, c_3 are elastic responses in Ox_1, Ox_2, Ox_3 directions.
Subsequent procedure is completely analogues to described above.

3 NUMERICAL IMPLEMENTATION OF DDCBEM AND IDCBEM

3.1. Discrete-continual design model of the border.

Special discrete-continual model [1-2] of the border Ξ is introduced for three-dimensional problems. It presupposes mesh approximation of the cross-section border while in the basic direction (Ox_3) problem remains continual. Thus, border Ξ is divided into so-called discrete-continual boundary elements Ξ_i (Figure 2)

$$\Xi = \bigcup_{i=1}^{N_{bel}} \Xi_i\,; \quad \Xi_i = \{(x_1,x_2,x_3):(x_1,x_2)\in\Gamma_i; x_3\in[-L_3,L_3]\}. \tag{3.1}$$

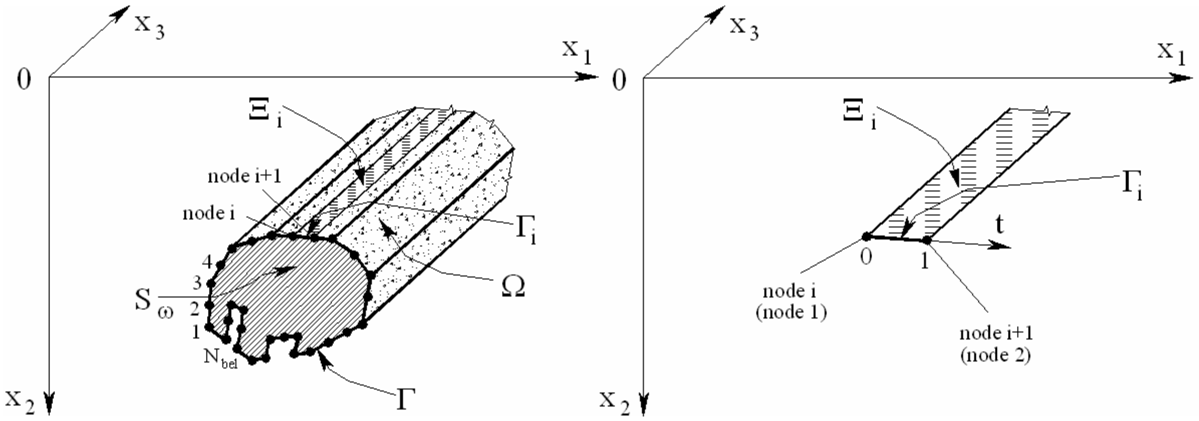

Figure 2. Discrete-continual design model and discrete-continual boundary elements.

3.2. Discrete-continual boundary element (DCBE) and its characteristics.

Consider arbitrary DCBE Ξ_i and its arbitrary cross-section Γ_i. We have (Figure 2)

$$\Gamma_i = \{(x_1,x_2,x_3):\ x_{1,i}\le x_1\le x_{1,i+1};\ x_{2,i}\le x_2\le x_{2,i+1}\}. \tag{3.2}$$

Basic geometrical parameters of DCBE's cross-section Γ_i are defined by formulas

$$h_i=\sqrt{h_{1,i}^2+h_{2,i}^2}\,;\quad h_{1,i}=x_{1,i+1}-x_{1,i}\,;\quad h_{2,i}=x_{2,i+1}-x_{2,i}\,; \tag{3.3}$$

$$\bar{\nu}_i=(\nu_{1,i},\nu_{2,i},\nu_3)\,;\quad \nu_{1,i}=-h_{2,i}/h_i\,;\quad \nu_{2,i}=h_{1,i}/h_i\,;\quad \nu_3=0\,. \tag{3.4}$$

3.3. Element coordinate system.

Local coordinate system is introduced in arbitrary cross-section of DCBE ($t\in[0,1]$, Figure 2). Renumbering of nodes in cross-section of element is performed ($i\Rightarrow 1$; $i+1\Rightarrow 2$).

3.4. Selection of extended domain and orthonormal Fourier basis.

In accordance with distinctive approach the given domain Ω is embordered by extended one ω in the form of a cube,

$$\omega=\{(x_1,x_2,x_3):\ -l_1<x_1<l_1;\ -l_2<x_2<l_2;\ -l_3<x_3<l_3\}. \tag{3.5}$$

We use the following set of functions as the orthonormal Fourier basis in $L^3(\omega)$ ($k_1,k_2,k_3=$ $=0,\pm1,\pm2,...$):

$$\varphi_k(x)=\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\varphi_{k_3}(x_3);\quad \varphi_{k_i}(x_i)=\frac{1}{\sqrt{2l_i}}\exp(\lambda_{k_i}x_i);\quad \lambda_{k_i}=i\frac{k_i\pi}{l_i}. \quad (3.6)$$

In practice we take into account finite quantity of terms of series: $-N_i \le k_i \le N_i, i=1,2,3$.

3.5. Use of Lanczos factors.

Lanczos factors σ_k can be used for convergence acceleration of Fourier series. Let $f(x)$ be arbitrary function. Corresponding formula of approximation has the form

$$f(x)\approx\sum_{k=-N}^{N}f_k\sigma_k\varphi_k(x);\quad \sigma_k=\prod_{i=1}^{3}\sigma_{k_i};\quad \sigma_0=1; \quad (3.7)$$

$$\sigma_{ki}=\frac{iN_i}{2\pi l}\frac{1}{k_i}\left(\exp\left(-i\frac{k_i\pi}{N_i}\right)-\exp\left(i\frac{k_i\pi}{N_i}\right)\right),\ k_i\ne 0; \quad (3.8)$$

3.6. Numerical implementation of IDCBEM.

3.6.1. Approximation of nodal unknown functions.

Basic nodal unknown functions are components of vector function $\overline{q}(x_3)=[q_1,q_2,q_3]^T$ denoted by $\overline{q}_i=[q_{1,i},q_{2,i},q_{3,i}]^T$, $i=1,2,...,\ N_{bel}$. For the sake of being definite, suppose its piecewise constant approximation along Γ and this implies that $\overline{q}_i$ is assumed to be constant within Γ_i.

3.6.2. Approximation of basic pseudodifferential operators.

Let $f=f(x_1,x_2)$ be arbitrary function. Pseudodifferential operators $P^r_{i,j}, i=1,2; j=0,1$ are approximated by Fourier series:

$$P^r_{i,j}f(x_1,x_2)\approx\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}P^r_{i,j,k_1,k_2}f_{k_1k_2}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2); \quad (3.9)$$

$$P^r_{1,0k_1k_2}=T^r_{0,k_1k_2}\widetilde{T}^r_{1,k_1k_2}+T^r_{1,k_1k_2}\widetilde{T}^r_{0,k_1k_2};\quad P^r_{1,1k_1k_2}=T^r_{0,k_1k_2}\widetilde{T}^r_{0,k_1k_2}+T^r_{1,k_1k_2}\widetilde{T}^r_{1,k_1k_2}; \quad (3.10)$$

$$P^r_{2,0,k_1k_2}=T^r_{0,k_1k_2}H\widetilde{T}^r_{1,k_1k_2}+T^r_{1,k_1k_2}H\widetilde{T}^r_{0,k_1k_2};\quad P^r_{2,1,k_1k_2}=T^r_{0,k_1k_2}H\widetilde{T}^r_{0,k_1k_2}+T^r_{1,k_1k_2}H\widetilde{T}^r_{1,k_1k_2}; \quad (3.11)$$

$$T^r_{0,k_1k_2}=\lambda_{k_1k_2}\begin{bmatrix}0&0&\lambda_{k_2}\\0&0&-\lambda_{k_1}\\\lambda_{k_1k_2}&0&0\\\lambda_{k_1}\lambda_{k_1k_2}&0&0\\\lambda_{k_2}\lambda_{k_1k_2}&0&0\\0&\gamma_4\lambda_{k_1k_2}&0\end{bmatrix};\quad \widetilde{T}^r_{0,k_1k_2}=-\frac{1}{\lambda_{k_1k_2}}\begin{bmatrix}\gamma_3\lambda_{k_1}/\lambda^2_{k_1k_2}&\gamma_3\lambda_{k_2}/\lambda^2_{k_1k_2}&0\\0&0&\gamma_4\\0&0&0\end{bmatrix}; \quad (3.12)$$

$$T^r_{1,k_1k_2}=-\begin{bmatrix}\lambda_{k_1}&0&0\\\lambda_{k_2}&0&0\\0&\gamma_5&0\\0&\lambda_{k_1}&-\lambda_{k_2}\lambda_{k_1k_2}\\0&\lambda_{k_2}&\lambda_{k_1}\lambda_{k_1k_2}\\\lambda^2_{k_1k_2}&0&0\end{bmatrix};\quad \widetilde{T}^r_{1,k_1k_2}=\frac{1}{\lambda_{k_1k_2}}\begin{bmatrix}0&0&0\\\gamma_2\lambda_{k_1}/\lambda_{k_1k_2}&\gamma_2\lambda_{k_2}/\lambda_{k_1k_2}&0\\\gamma_1\lambda_{k_1}/\lambda^2_{k_1k_2}&-\gamma_1\lambda_{k_1}/\lambda^2_{k_1k_2}&0\end{bmatrix}; \quad (3.13)$$

$$\lambda_{k_1k_2} = \sqrt{-(\lambda_{k_1}^2 + \lambda_{k_2}^2)} \,. \tag{3.14}$$

We stress that all formulas presented in paragraph 3.6.2 are correct except case $k_1 = k_2 = 0$, which requires exclusive consideration. Corresponding component of solution we are calling "beam" component. Paragraph 3.6.3 is defined to the problem of its definition.

Pseudodifferential operators $Q^r_{i,j,m}, i = 1,2;\; j = 0,1;\; m = 1,2$ in their turn are approximated in accordance with formulas

$$Q^r_{i,j,m} f(x_1,x_2) \approx \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} Q^r_{i,j,m,k_1,k_2} f_{k_1k_2} \varphi_{k_1}(x_1)\varphi_{k_2}(x_2);\quad Q^r_{i,j,m,k_1k_2} = l_{G,m,k_1k_2} P^r_{i,j,k_1k_2}; \tag{3.15}$$

$$l_{G,1,k_1k_2} = \begin{bmatrix} \gamma_5\lambda_{k_1} & \gamma_6\lambda_{k_2} & 0 & 0 & 0 & \gamma_6 \\ \gamma_7\lambda_{k_2} & \gamma_7\lambda_{k_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_7\lambda_{k_1} & \gamma_7 & 0 & 0 \end{bmatrix}; \tag{3.16}$$

$$l_{G,2,k_1k_2} = \begin{bmatrix} \gamma_7\lambda_{k_2} & \gamma_7\lambda_{k_1} & 0 & 0 & 0 & 0 \\ \gamma_6\lambda_{k_1} & \gamma_5\lambda_{k_2} & 0 & 0 & 0 & \gamma_6 \\ 0 & 0 & \gamma_7\lambda_{k_2} & 0 & \gamma_7 & 0 \end{bmatrix}; \tag{3.17}$$

$$\gamma_5 = -\mu(\gamma+2);\quad \gamma_6 = -\mu\gamma;\quad \gamma_7 = -\mu\,. \tag{3.18}$$

3.6.3. Definition of "beam" component.

In the earlier paragraph we let a question concerning so-called "beam" component stand over. Let's get back to this point ($k_1 = k_2 = 0$).

First of all it must be mentioned that for many problems this component of solution is of paramount importance. It characterizes displacements of the whole cross-section.

Let's expand unknown vector function $\overline{U}$ into Fourier series with respect to x_1, x_2:

$$\overline{U}(x_1,x_2,x_3) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \overline{U}_{k_1k_2}(x_3)\varphi_{k_1}(x_1)\varphi_{k_2}(x_2). \tag{3.19}$$

If we combine this with (2.66), we get

$$\sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \overline{U}'_{k_1k_2}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \{L_{G,k_1k_2}\overline{U}_{k_1k_2} + \overline{F}_{G,k_1k_2} + \\ + [\delta_{\Xi}\overline{q}_G]_{k_1k_2}\}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2), \tag{3.20}$$

where

$$L_{G,k_1k_2} = \begin{bmatrix} 0 & E \\ L^{-1}_{2,k_1k_2}L_{0,k_1k_2} & L^{-1}_{2,k_1k_2}L_{1,k_1k_2} \end{bmatrix};\quad L_{2,k_1k_2} = \mu\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma+2 \end{bmatrix}; \tag{3.21}$$

$$L_{1,k_1k_2} = -\mu(\gamma+1)\begin{bmatrix} 0 & 0 & \lambda_{k_1} \\ 0 & 0 & \lambda_{k_2} \\ \lambda_{k_1} & \lambda_{k_2} & 0 \end{bmatrix};\quad L_{0,k_1k_2} = -\mu\begin{bmatrix} (\gamma+2)\lambda_{k_1}^2 + \mu\lambda_{k_2}^2 & (\gamma+1)\lambda_{k_1}\lambda_{k_2} & 0 \\ (\gamma+1)\lambda_{k_1}\lambda_{k_2} & \mu\lambda_{k_1}^2 + (\gamma+2)\lambda_{k_2}^2 & 0 \\ 0 & 0 & \mu\lambda_{k_1}^2 + \mu\lambda_{k_2}^2 \end{bmatrix}; \tag{3.22}$$

$$\overline{F}_G(x_1,x_2,x_3)=\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\overline{F}_{G,k_1k_2}(x_3)\varphi_{k_1}(x_1)\varphi_{k_2}(x_2); \quad (3.23)$$

$$[\delta_\Xi\overline{q}_G](x_1,x_2,x_3)=\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}[\delta_\Xi\overline{q}_G]_{k_1k_2}(x_3)\varphi_{k_1}(x_1)\varphi_{k_2}(x_2). \quad (3.24)$$

Multiplying both sides of (3.20) by basis functions $\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)$, $k_1=-N_1,...,N_1$, $k_2=$ $=-N_2,...,N_2$ we obtain differential equation set with respect to x_3

$$\overline{U}'_{k_1k_2}=L_{G,k_1k_2}\overline{U}_{k_1k_2}+\overline{F}_{G,k_1k_2}+[\delta_\Xi\overline{q}_G]_{k_1k_2},\ \ k_1=-N_1,...,N_1;\ \ k_1=-N_2,...,N_2 \quad (3.25)$$

and in particular

$$\overline{U}'_{00}=L_{G,00}\overline{U}_{00}+\overline{F}_{G,00}+[\delta_\Xi\overline{q}_G]_{00}\,. \quad (3.26)$$

$\overline{U}_{00}=\overline{U}_{00}(x_3)$ is basic unknown vector function. It remains to note that matrix $L_{G,00}$ has form

$$L_{G,00}=\begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}. \quad (3.27)$$

Matrix $L_{G,00}$ has single eigenvalue $\lambda=0$ with multiplicity $m=6$. Three eigenvectors and three root vectors corresponding to λ are the following

$$\overline{t}_1^e=[1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0]^T;\ \ \overline{t}_2^e=[0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0]^T; \quad (3.28)$$

$$\overline{t}_3^e=[0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]^T;\ \ \overline{t}_1^r=[0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1]^T; \quad (3.29)$$

$$\overline{t}_2^r=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1]^T;\ \ \overline{t}_3^r=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0]^T \quad (3.30)$$

and $\overline{t}_i^r$ is root vector corresponding to eigenvector $\overline{t}_i^e$.

Thus, Jordan decomposition of $L_{G,00}$ is defined by formulas:

$$L_{G,00}=T_{00}J_{00}\widetilde{T}_{00}; \quad (3.31)$$

$$T_{00}=[\,\overline{t}_1^e\ \ \overline{t}_1^r\ \ \overline{t}_2^e\ \ \overline{t}_2^r\ \ \overline{t}_3^e\ \ \overline{t}_3^r\,]; \quad (3.32)$$

$$\widetilde{T}_{00}=T_{00}^{-1}=\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ -1&0&1&0&0&0\\ 0&0&0&-1&0&1\\ 0&1&-1&0&0&0\\ 0&0&0&0&1&-1 \end{bmatrix};\quad J_{00}=\begin{bmatrix} J_c&0&0\\ 0&J_c&0\\ 0&0&J_c \end{bmatrix};\ \ J_c=\begin{bmatrix} 0&1\\ 0&0 \end{bmatrix}. \quad (3.33)$$

Fundamental matrix function $\varepsilon_{00}(x_3)$ of differential equation set

$$\overline{U}'_{00}=L_{G,00}\overline{U}_{00} \quad (3.34)$$

is defined by formulas

$$\varepsilon_{00}(x_3) = \mathrm{sign}(x_3)P_{1,00} + |x_3|P_{2,00}; \tag{3.35}$$

$$P_{1,00} = 0.5\cdot T\widetilde{T} = 0.5\cdot E; \quad P_{2,00} = 0.5\cdot T\widetilde{H}\widetilde{T} = 0.5\cdot L_{G,00}; \quad \widetilde{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.36}$$

After convolution of (3.35) with both sides of (3.26) and transformations the result is

$$\overline{U}_{00} = [\mathrm{sign}(x_3)P^r_{1,00} + |x_3|P^r_{2,00}] \underset{3}{*} \overline{F}^r_{G,00} + [\mathrm{sign}(x_3)P^r_{1,00} + |x_3|P^r_{2,00}] \underset{3}{*} [\delta_\Xi \overline{\widetilde{q}}]_{00}. \tag{3.37}$$

Here we have

$$[\delta_\Xi \overline{\widetilde{q}}](x_1, x_2, x_3) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} [\delta_\Xi \overline{\widetilde{q}}]_{k_1k_2}(x_3)\varphi_{k_1}(x_1)\varphi_{k_2}(x_2); \tag{3.38}$$

$$\overline{F}^r_G = -L_2^{-1}\overline{F} = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \overline{F}^r_{G,k_1k_2}(x_3)\varphi_{k_1}(x_1)\varphi_{k_2}(x_2); \tag{3.39}$$

$$P_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; P_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.40}$$

If we replace $\overline{U}(x_1, x_2, x_3)$ by (3.19) in boundary conditions (2.74) and expand vector function $\overline{f}(x_1, x_2, x_3)$ into Fourier series with respect to x_1, x_2 we get

$$\sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} l_{G,k_1k_2}\overline{U}_{k_1k_2}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} f_{k_1k_2}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2); \tag{3.41}$$

$$l_{G,k_1k_2} = [\, l_{0,k_1k_2} \quad l_{1,k_1k_2}\,]^T; \quad l_{G,k_1k_2} = \nu_1 l_{G,1,k_1k_2} + \nu_2 l_{G,2,k_1k_2}; \tag{3.42}$$

$$l_{G,1,k_1k_2} = -\mu\begin{bmatrix} (\gamma+2)\lambda_{k_1} & \gamma\lambda_{k_2} & 0 & 0 & 0 & \gamma \\ \lambda_{k_2} & \lambda_{k_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{k_1} & 1 & 0 & 0 \end{bmatrix}; \tag{3.43}$$

$$l_{G,2,k_1k_2} = -\mu\begin{bmatrix} \lambda_{k_2} & \lambda_{k_1} & 0 & 0 & 0 & 0 \\ \gamma\lambda_{k_1} & (\gamma+2)\lambda_{k_2} & 0 & 0 & 0 & \gamma \\ 0 & 0 & \lambda_{k_2} & 0 & 1 & 0 \end{bmatrix} \tag{3.44}$$

and in particular

$$l_{G,1,00} = -\mu\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \quad l_{G,2,00} = -\mu\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.45}$$

Thus, we can define the following matrices

$$Q_1 = l_{G,1,00}P_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & \gamma_6 \\ 0 & 0 & 0 \\ \gamma_7 & 0 & 0 \end{bmatrix}; \quad Q_2 = l_{G,2,00}P_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma_6 \\ 0 & \gamma_7 & 0 \end{bmatrix}. \qquad (3.46)$$

Further note that $l_{G,1,00}P_2 = l_{G,2,00}P_2 = 0$.

3.6.4. Approximation of resolving operational boundary equations set.

We set up systems of boundary equations at points $\tilde{x}_i^+ = (\tilde{x}_{1,i}^+, \tilde{x}_{2,i}^+, x_3) \in \Gamma + 0$, $i = 1,2,...,N_{bel}$,

$$\tilde{x}_{1,i}^+ = x_{1,i} + 0.5 \cdot h_{1,i} + \nu_{1,i} d_i; \quad \tilde{x}_{2,i}^+ = x_{2,i} + 0.5 \cdot h_{2,i} + \nu_{2,i} d_i. \qquad (3.47)$$

Magnitude of d_i is directly related of various factors. On the one hand in accordance with conventional boundary element method we have $d_i = 0.01 h_i$. On the other hand we must avoid Gibbs phenomenon [16-20].

Moreover, due to operation of pseudodifferential operators delta functions and its derivatives may be located on a boundary. This leads to various parasitical effects at approximation stage.

Without loss of generality it can be assumed that the considering structure is subjected to concentrated forces only:

$$\overline{F} = \sum_{p=1}^{N_{vf}} \overline{F}_p \delta(x_1 - x_1^{(p)}) \delta(x_2 - x_2^{(p)}) \delta(x_3 - x_3^{(p)}); \quad \overline{f}_i = \sum_{q=1}^{N_{bf}} \overline{f}_{i,p} \delta(x_3 - x_{3,i}^{(q)}), \ \ i = 1, 2, ..., N_{bel}. \qquad (3.48)$$

Here $\overline{f}_{i,q}$ is the force vector at cross-section of DCBE number *i* with coordinate $x_{3,i}^{(q)}$.

Global vector of unknowns is constructed in the form:

$$\overline{\overline{q}}^G(x_3) = [\ \overline{\overline{q}}_1(x_3) \ \ ... \ \ \overline{\overline{q}}_{N_{bel}}(x_3)\ ]^T; \quad \overline{\overline{q}}^G(x_3) = \sum_{k_3=-N_3}^{N_3} \overline{\overline{q}}_{k_3}^G \varphi_{k_3}(x_3). \qquad (3.49)$$

The result of approximation with the use of Fourier series is the following set of $3N_{bel}$-order systems for Fourier coefficients in (3.49):

$$K_{k_3}^G \overline{\overline{q}}_{k_3}^G = \overline{G}_{k_3}^G, \ k_3 = -N_3,..., N_3; \qquad (3.50)$$

$$K_{k_3}^G = \begin{bmatrix} K_{k_3}^{(1,1)} & \dots & K_{k_3}^{(1,N_{bel})} \\ \dots & \ddots & \dots \\ K_{k_3}^{(N_{bel},1)} & \dots & K_{k_3}^{(N_{bel},N_{bel})} \end{bmatrix}; \quad \overline{G}_{k_3}^G = \begin{bmatrix} \overline{G}_{k_3}^{(1)} \\ \dots \\ \overline{G}_{k_3}^{(N_{bel})} \end{bmatrix}; \qquad (3.51)$$

$$K_{k_3}^{(i,j)} = \frac{\sqrt{2l_3}}{2\sqrt{l_1 l_2}} a_{k_3}^{(5)} \sum_{l=1}^{2} \nu_{l,i} Q_l^r c_{00}^{(j)} + \sqrt{2l_3} \sum_{\substack{k_1=-N_1 \\ k_1 \neq 0 \,\vee\, k_2 \neq 0}}^{N_1} \sum_{k_2=-N_2}^{N_2} A_{k_1 k_2 k_3}^{(i)} \varphi_{k_1}(\tilde{x}_{1,i}^+) \varphi_{k_2}(\tilde{x}_{2,i}^+) c_{k_1 k_2}^{(j)}; \qquad (3.52)$$

$$\begin{aligned} \overline{G}_{k_3}^{(i)} = \sum_{q=1}^{N_{bf}} \overline{f}_{i,q} \varphi_{k_3}(-x_{3,i}^{(q)}) - \frac{\sqrt{2l_3}}{2\sqrt{l_1 l_2}} \sum_{l=1}^{2} \nu_{l,i} [a_{k_3}^{(5)} Q_l^r] \overline{F}_{G,00k_3}^r - \\ - \sqrt{2l_3} \sum_{l=1}^{2} \nu_{i,l} \sum_{\substack{k_1=-N_1 \\ k_1 \neq 0 \,\vee\, k_2 \neq 0}}^{N_1} \sum_{k_2=-N_2}^{N_2} \sum_{p=1}^{N_{vf}} \overline{G}_{p,k_1 k_2 k_3, l} \varphi_{k_1}(\tilde{x}_{1,i}^+) \varphi_{k_2}(\tilde{x}_{2,i}^+); \end{aligned} \qquad (3.53)$$

$$A^{(i)}_{k_1k_2k_3} = \nu_{1,i}A_{k_1k_2k_3,1} + \nu_{2,i}A_{k_1k_2k_3,2}\,; \tag{3.54}$$

$$A_{k_1k_2k_3,i} = a^{(1)}_{k_1k_2k_3}Q^{r}_{1,0,i,k_1k_2} + a^{(2)}_{k_1k_2k_3}Q^{r}_{1,1,i,k_1k_2} + a^{(3)}_{k_1k_2k_3}Q^{r}_{2,0,i,k_1k_2} + a^{(4)}_{k_1k_2k_3}Q^{r}_{2,1,i,k_1k_2}, \quad i = 1, 2; \tag{3.55}$$

$$c^{(j)}_{k_1k_2} = 0.5\cdot h_j \exp(\alpha^{(j)}_{k_1k_2})\eta^{(j)}_{k_1k_2} / \sqrt{l_1 l_2}\,; \quad \alpha^{(j)}_{k_1k_2} = -i\pi[k_1 x_{1,j}/l_1 + k_2 x_{2,j}/l_2]; \tag{3.56}$$

$$\eta^{(j)}_{k_1k_2} = \begin{cases} [\exp(\beta^{(j)}_{k_1k_2}) - 1]/\beta^{(j)}_{k_1k_2}, \; \beta^{(j)}_{k_1k_2} \neq 0 \\ 1, \; \beta^{(j)}_{k_1k_2} = 0; \end{cases} \quad \beta^{(j)}_{k_1k_2} = -i\pi\left[k_1\frac{h_{1,j}}{l_1} + k_2\frac{h_{2,j}}{l_2}\right]; \tag{3.57}$$

$$\overline{G}_{p,k_1k_2k_3,l} = -\varphi_{k_3}(-x^{(p)}_3)\phi^{(p)}_{k_1k_2}A_{k_1k_2k_3,l}L_2^{-1}\overline{F}_p\,; \quad \phi^{(p)}_{k_1k_2} = \varphi_{k_1}(-x^{(p)}_1)\varphi_{k_2}(-x^{(p)}_2)\,; \tag{3.58}$$

$$a^{(1)}_{k_1k_2k_3} = \frac{\lambda_{k_1k_2} l_3\sqrt{2l_3}}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}[1 - (-1)^{k_3}\exp(-\lambda_{k_1k_2}l_3)]\,; \tag{3.59}$$

$$a^{(2)}_{k_1k_2k_3} = -\frac{i k_3\pi\sqrt{2l_3}}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}[1 - (-1)^{k_3}\exp(-\lambda_{k_1k_2}l_3)]\,; \tag{3.60}$$

$$a^{(3)}_{k_1k_2k_3} = -\frac{i k_3\pi l_3\sqrt{2l_3}}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}\left[\frac{2\lambda_{k_1k_2}l_3}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2} - (-1)^{k_3}\exp(-\lambda_{k_1k_2}l_3)\left(1 - \frac{2\lambda_{k_1k_2}l_3}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}\right)\right]; \tag{3.61}$$

$$a^{(4)}_{k_1k_2k_3} = \frac{l_3\sqrt{2l_3}}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}\left[\frac{\lambda^2_{k_1k_2}l_3^2 - k_3^2\pi^2}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2} - (-1)^{k_3}\exp(-\lambda_{k_1k_2}l_3)\left(\lambda_{k_1k_2}l_3 + \frac{\lambda^2_{k_1k_2}l_3^2 - k_3^2\pi^2}{\lambda^2_{k_1k_2}l_3^2 + k_3^2\pi^2}\right)\right]; \tag{3.62}$$

$$a^{(5)}_{k_3} = -i(1 - \delta_{i,j})\sqrt{2l_3}[1 - (-1)^{k_3}]/(\pi k_3)\,. \tag{3.63}$$

Now note $\delta_{i,j}$ is Chronicler's symbol.

3.6.4. Calculation of displacements, strains and stresses within domain.

In accordance with foregoing formulas we have

$$\overline{U}(x_1, x_2, x_3) = \sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\sum_{k_3=-N_3}^{N_3}\overline{U}_{k_1k_2k_3}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\varphi_{k_3}(x_3); \tag{3.64}$$

$$\overline{U}_{k_1k_2k_3} = \sqrt{2l_3}\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\left[\sum_{p=1}^{N_{vf}}\overline{S}_{p,k_1k_2k_3} + D_{k_1k_2k_3}\sum_{j=1}^{N_{bel}}c^{(j)}_{k_1k_2}\widetilde{\overline{q}}^{\,\Xi}_{j,k_3}\right]; \tag{3.65}$$

$$\overline{U}_{k_1k_2k_3} = [u_{1,k_1k_2k_3}, u_{2,k_1k_2k_3}, u_{3,k_1k_2k_3}, v_{1,k_1k_2k_3}, v_{2,k_1k_2k_3}, v_{3,k_1k_2k_3}]^T; \tag{3.66}$$

$$D_{k_1k_2k_3} = a^{(1)}_{k_1k_2k_3}\widetilde{P}^{r}_{1,0,k_1k_2} + a^{(2)}_{k_1k_2k_3}\widetilde{P}^{r}_{1,1,k_1k_2} + a^{(3)}_{k_1k_2k_3}\widetilde{P}^{r}_{2,0,k_1k_2} + a^{(4)}_{k_1k_2k_3}\widetilde{P}^{r}_{2,1,k_1k_2}, \quad k_1 \neq 0 \vee k_2 \neq 0; \tag{3.67}$$

$$D_{00k_3} = a^{(5)}_{k_3}\widetilde{P}^{r}_{1,00} + a^{(6)}_{k_3}\widetilde{P}^{r}_{2,00}\,; \quad \overline{S}_{p,k_1k_2k_3} = -\varphi_{k_3}(-x^{(p)}_3)\phi^{(p)}_{k_1k_2}D_{k_1k_2k_3}L_2^{-1}\overline{F}_p\,. \tag{3.68}$$

Let σ_{ij} be stress components and ε_{ij} be displacement components. It is obvious that

$$\sigma_{ij} = \delta_{ij}\lambda\varepsilon + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad i = 1, 2, 3; \quad j = 1, 2, 3\,; \quad \varepsilon = \sum_{i=1}^{3}\varepsilon_{ii}\,. \tag{3.69}$$

This yields that corresponding results can be summarized as follows:

$$\varepsilon_{ij}(x_1, x_2, x_3) = \sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\sum_{k_3=-N_3}^{N_3}\varepsilon_{ij,k_1k_2k_3}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\varphi_{k_3}(x_3), \quad i, j = 1, 2, 3; \tag{3.70}$$

$$\sigma_{ij}(x_1,x_2,x_3)=\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\sum_{k_3=-N_3}^{N_3}\sigma_{ij,k_1k_2k_3}\varphi_{k_1}(x_1)\varphi_{k_2}(x_2)\varphi_{k_3}(x_3),\quad i,j=1,2,3; \qquad (3.71)$$

$$\varepsilon_{11,k_1k_2k_3}=\lambda_{k_1}\sigma_{k_1}u_{11,k_1k_2k_3};\quad \varepsilon_{12,k_1k_2k_3}=\frac{1}{2}(\lambda_{k_1}\sigma_{k_1}u_{2,k_1k_2k_3}+\lambda_{k_2}\sigma_{k_2}u_{1,k_1k_2k_3}); \qquad (3.72)$$

$$\varepsilon_{22,k_1k_2k_3}=\lambda_{k_2}\sigma_{k_2}u_{22,k_1k_2k_3};\quad \varepsilon_{13,k_1k_2k_3}=\frac{1}{2}(\lambda_{k_1}\sigma_{k_1}u_{3,k_1k_2k_3}+v_{1,k_1k_2k_3}); \qquad (3.73)$$

$$\varepsilon_{33,k_1k_2k_3}=v_{3,k_1k_2k_3};\quad \varepsilon_{23,k_1k_2k_3}=\frac{1}{2}(\lambda_{k_2}\sigma_{k_2}u_{3,k_1k_2k_3}+v_{2,k_1k_2k_3}); \qquad (3.74)$$

$$\sigma_{ij,k_1k_2k_3}=\delta_{ij}\lambda\varepsilon_{k_1k_2k_3}+2\mu\varepsilon_{ij,k_1k_2k_3},\ i,j=1,2,3;\quad \varepsilon_{k_1k_2k_3}=\varepsilon_{11,k_1k_2k_3}+\varepsilon_{22,k_1k_2k_3}+\varepsilon_{33,k_1k_2k_3}. \qquad (3.75)$$

3.7. Numerical implementation of DDCBEM.

Numerical implementation of DDCBEM is executed in much the same way as IDCBEM. It is also based on Fourier series approximation. This problem is partially considered in [1] and will not be described in detail here.

3.8. Closing remarks about methods of additional regularizations.

Methods of additional regularization have been already considered above in paragraphs 2.6-2.7. But in view of information from paragraph 3.6.3 it is useful to produce several new integration formulas here:

$$\int \mathrm{sign}(x_3)dx=|x_3|;\quad \iint \mathrm{sign}(x_3)dx=x_3^2\mathrm{sign}(x_3); \qquad (3.76)$$

$$\int |x_3|dx=\frac{1}{2}x_3^2\mathrm{sign}(x_3);\quad \iint |x_3|dx=\frac{1}{6}x_3^3\mathrm{sign}(x_3)=\frac{1}{6}|x_3|^3. \qquad (3.77)$$

However approximation quality of functions $|x_3|^3$ and $x_3^2\mathrm{sign}(x_3)$ by Fourier series will not be passable due to their behavior and behavior of their derivatives nearby points of periodicity $x_{3,j}^{(b)}=jl_3,\quad j=\pm1,\pm3,\pm5,...$

For avoidance of this fact we recommend using functions

$$x_3^3\mathrm{sign}(x_3)-1.5\cdot l_3x_3^2;\quad x_3^2\mathrm{sign}(x_3)-l_3x_3 \qquad (3.78)$$

instead of $|x_3|^3$ and $x_3^2\mathrm{sign}(x_3)$. Functions (3.78) give a better behavior in the specified sense.

3.9. Considerations regarding methods of approximation in terms of DDCBEM and IDCBEM.

We believe that Fourier series approximation with respect to x_1,x_2 is quite standard while its use with respect to x_3 is at least controversial. Fourier transformation [26-28] is the most natural technique in such problems. Furthermore direct Fourier transformation is not complicated operation to a certain extent in this case [17]. However Fourier inversion causes essential difficulties.

Another alternative approach is application of Wavelet analysis [29-41]. Taking into account types of pseudodifferential operators in DDCBEM and IDCBEM in our estimation this method is especially effective.

Prime advantages of Fourier series approximation include relatively simple computational algorithm and demonstrativeness [21-25].

We have also developed version of discrete-continual boundary element method (IDCBEM) based on combined Fourier series and polynomial approximation [7]. Peculiar features of the proposed combined approximation type include algorithmic simplicity and supreme universality. Due to possible presence of finite discontinuities in approximating function exclusive application of Fourier series is apparently undesirable. Finite discontinuities cause so-called Gibbs phenomenon and therefore polynomials are used to avoid this parasitic numerical effect (Figure 3).

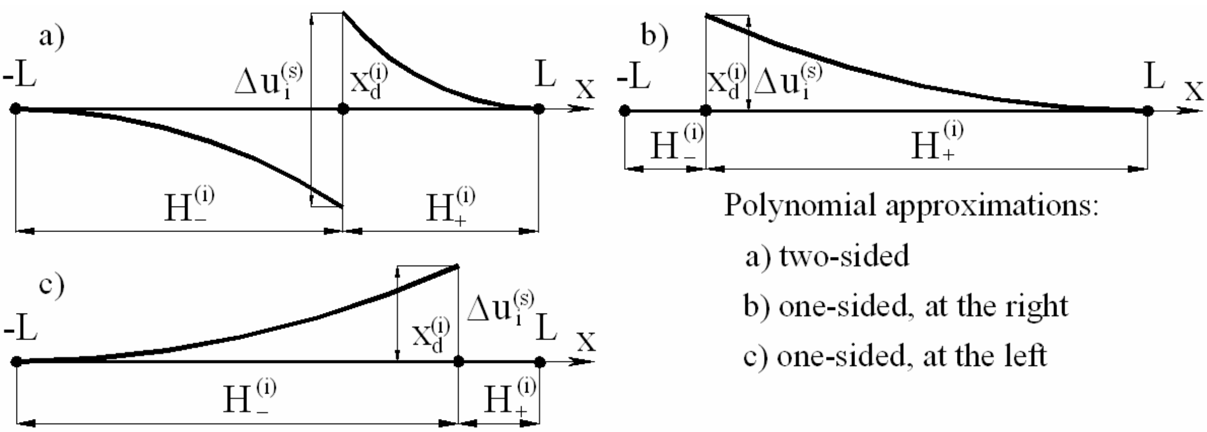

Figure 3. Types of polynomial approximations.

4 COMPUTER REALIZATION OF DDCBEM, IDCBEM AND SOFTWARE

4.1. Computer realization of DDCBEM. Program system DDCBEM3D.

All methods and algorithms of DDCBEM considered in the distinctive paper have been realized in program system DDCBEM3D. Its main purpose is analysis of three-dimensional problems with the use of DDCBEM. We use Microsoft Fortran PowerStation 4.0 Professional, Compaq Visual Fortran 6.6B Professional and Intel Fortran Compiler 8.0 as programming environments. Program is designed for Microsoft Windows 95/98/NT/2000/ME/XP/2003.

4.2. Computer realization of IDCBEM. Program system DDCBEM3D.

All methods and algorithms of IDCBEM considered in the distinctive paper have been realized in program system IDCBEM3D. Its main purpose is analysis of three-dimensional problems with the use of IDCBEM. We use Microsoft Fortran PowerStation 4.0 Professional, Compaq Visual Fortran 6.6B Professional and Intel Fortran Compiler 8.0 as programming environments. Program is designed for Microsoft Windows 95/98/NT/2000/ME/XP/2003.

ACKNOWLEDGMENTS

The material presented here is based on the work supported by the Grant of the President of Russian Federation for Young Doctors of Sciences, Grant Number MD-1785.2006.8.

REFERENCES

[1] P.A. Akimov, A.B. Zolotov. Direct Discrete-continual Boundary Element Method of Structural Analysis for Three-dimensional Problem of Elastic Theory. *Proceedings of 20-th International Conference "Mathematical Modeling in Solid Mechanics. Boundary & Finite Elements Methods".* Saint Petersburg, Russia, 2003, pp. 6-12.

[2] A.B. Zolotov, P.A. Akimov. Indirect Discrete-continual Boundary Element Method of Structural Analysis for Three-dimensional Problem of Elastic Theory. *Proceedings of 20-th International Conference "Mathematical Modeling in Solid Mechanics. Boundary & Finite Elements Methods".* Saint Petersburg, Russia, 2003, pp. 213-220.

[3] V.N. Sidorov, A.B. Zolotov, P.A. Akimov. Discrete-continual Boundary Element Methods of Structural Analysis. *International Journal for Computational Civil and Structural Engineering*. Volume 1, Number 5, Begell House Inc. Publishers & ASV, 2003, pp. 84-99.

[4] A.B. Zolotov, P.A. Akimov. Discrete-continual Methods of Structural Analysis with the use of exact analytical solutions. *Proceedings of International Sci-tech Conference "Computational Mechanics of Rigid Bodies".* Moscow, MIIT, Volume 1, 2006, pp. 168-175.

[5] P.A. Akimov, A.B. Zolotov. Semianalytical Methods of Structural Analysis: Outlook and Comparison. *Computer-aided Design and Graphics*, Volume 1, 2005, pp. 78-82.

[6] A.B. Zolotov, P.A. Akimov. *Some Semianalytical Methods of Analysis of Boundary problems of Structural Mechanics*. Moscow, ASV, 2004, 200 pages.

[7] P.A. Akimov, A.B. Zolotov. Discrete-continual Boundary Element Method Based on Combined Fourier Series and Polynomial Approximation for Structural Analysis of Infinite Strip on Elastic Foundation. *Proceedings of 13-th of Slovak-Polish-Russian Seminar "The theory of construction"*, Bratislava, Slovakia, 2004, pp. 21-26.

[8] P.A. Akimov, A.B. Zolotov. Formulation of Resolving Set of Integro-differential Boundary Equations with Operational Coefficients for Three-dimensional Theory of Elasticity. *Proceedings of 12-th of Polish-Russian Seminar "The theory of construction"*, Nizhniy Novgorod, Russia, 2003, pp. 65-74.

[9] P.A. Akimov, A.B. Zolotov, Shirinsky V.I., Lopatinskaya E.L. Basic Integro-differential Operators of Discrete-continual Boundary Element Method for Second Boundary Value Problem of Elastostatics. *Problems of Applied Mathematics and Computational Mechanics*, Moscow, 2003, pp. 74-84.

[10] A.B. Zolotov, P.A. Akimov, M.L. Mozgaleva, V.I. Shirinsky. Analytical Solution of Multipoint Boundary Problem for System of Ordinary Differential Equations with Jordan Cells in Matrix of Coefficients. *Problems of Applied Mathematics and Computational Mechanics*, Moscow, 2000, pp. 61-73.

[11] A.B. Zolotov, A.V. Larionov, M.L. Mozgaleva, J.I. Mskhalaya. *Formulation and Approximation of Boundary Problems with the Use of Method of Extended Domain*. MSCEU, Moscow, 1992, 88 pages.

[12] A.B. Zolotov, P.A. Akimov. Semianalytical Finite Element Method for Two-dimensional and Three-dimensional Problems of Structural Analysis. *Proceedings of the International Symposium LSCE 2002 organized by Polish Chapter of IASS*, Warsaw, Poland, 2002, pp. 431-440.

[13] A.B. Zolotov, A.Yu. Abdurashitov, P.A. Akimov. Use of Semianalytical Method of Finite Elements for Evaluation of Stress-strained State of Rail. *VESTNIK of the All-Russia Railway Research Institute. Scientific & Technical Journal*. 4/2001., pp. 26-32.

[14] P.K. Banerjee, R. Butterfield. *Boundary Element Methods in Engineering Science*. Mc Graw-Hill Book Company, Inc. New York, 1981.

[15] A.I. Tseytlin, Petrosyan L.G. *Boundary Element Methods in Structural Mechanics*. Erevan, Luys, 1987.

[16] Gottlieb D., Shu C. On the Gibbs phenomena IV: Recovering Exponential Accuracy in a Subinterval from a Gegenbauer Partial Sum of a Piecewise Analytic Function. ICASE Rep. No. 94-33 and *Math. of Comp*. Vol.64, #211, pp.1081-1096, July 1995.

[17] Eckhoff K.S. On a High Order Numerical Method for Functions with Singularities, *Math. Comp. 67*, 1998, pp. 1063–1087.

[18] Geer J., Banerjee N.S. Exponentially Accurate Approximations to Piecewise Smooth Periodic Functions, *J. Sci. Comput*. 12, 1997, pp. 253–287.

[19] Kvernadze G., Hagstrom T., Shapiro H. Locating Discontinuities of a Bounded Function by the Partial Sums of its Fourier Series, *J. Sci. Comput*. 14, 1999, pp. 301-327.

[20] Driscoll T.A., Fornberg B. A Padé-based Algorithm for Overcoming the Gibbs Phenomenon. *Numerical Algorithms* 26, 2001, pp. 77-92.

[21] Deng J.-G., Cheng F.-P. Fourier Series Method for Plane Elastic Problems of Polygonal Domain. // *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 35, pp. 4569-4585(17), 2001.

[22] Beekman G. *Applied Partial Differential Equations: With Fourier Series and Boundary Value Problems*, Prentice Hall, 2003.

[23] Maksimovich V.N., Tsybul'skii O.A. Application of Fourier Series Methods and Integral Equations for Solving Nonstationary Nonaxisymmetric Heat Conduction Problems for Bodies of Revolution. *Journal of Engineering Physics and Thermophysics*, Volume 68, Number 6, 2002.

[24] Li W.L., Daniels M. A Fourier Series Method For The Vibrations Of Elastically Restrained Plates Arbitrarily Loaded With Springs And Masses. // *Journal of Sound and Vibration*, May 2002, vol. 252, no. 4, pp. 768-781(14).

[25] Fan S., Zheng D.Y., Au F.T.K. Gibbs-Phenomenon-Free Fourier Series for Vibration and Stability of Complex Beams. // *AIAA Journal*, 2001, vol. 39, no. 10, pp. 1977-1984.

[26] Rahman M. Fourier-transform Solution to the Problem of Steady-state Transonic Motion of a Line Load Across the Surface of an Elastic Half-space. *Appl Math Lett*, 14, 2001, pp. 437-441.

[27] Linkov A.M., Zoubkov V.V., Sylla M., al Heib M. Spectral BEM for Multi-layered Media with Cracks or/and Openings. // *Proc. Symposium IABEM-2000. Brescia: Int. Association for Boundary Element Methods*, 2000, pp. 141-142.

[28] Penmetsa R.C., Grandhi R.V. Structural Failure Probability Prediction Using Fast Fourier Transformations for High Accuracy. *Journal of Finite Elements in Analysis and Design*, Vol.39, No. 5-6, March 2003, pp. 473-486.

[29] Salajegheh E. Dynamic Analysis of Structures Against Earthquake by Combined Wavelet Transform and Fast Fourier Transform. *Asian Journal of Civil Engineering*, Volume 3, Number 3, September 2002.

[30] Wong L.A., Chen J.C. Nonlinear and Chaotic Behavior of Structural System Investigated by Wavelet Transform Techniques. *Int. J. Non-Linear Mech.*, 36, 2001, pp. 221–235.

[31] Hou Z., Noori M., St. Amand R. Wavelet-Based Approach for Structural Damage Detection. *ASCE Journal of Engineering Mechanics*, Volume 126, Number 7, July 2000, pp. 677-683.

[32] Spanos P.D., Rao V.R.S. Random Field Representation in a Biorthogonal Wavelets Basis. *Journal of Engineering Mechanics*, ASCE, 127, 2001, pp. 194-205.

[33] Gurley K., Kijewski T., Kareem A. First- and Higher-Order Correlation Detection Using Wavelet Transforms. *Journal of Engineering Mechanics*, ASCE, 129(2), 2003, pp. 188-201.

[34] Cohen A. Wavelet Methods in Numerical Analysis. *in P. G. Ciarlet, J. L. Lions eds. Handbook of Numerical Analysis* Vol. 7, North-Holland, 2000, pp. 417-711.

[35] Dahmen W., Kurdila A., Oswald P. *Multiresolution Analysis and Wavelets for the Numerical Solution of Partial Differential Equations*. Academic Press, 1997.

[36] Monasse P., Perrier V. Orthonormal Wavelet Bases Adapted for Partial Differential Equations with Boundary Conditions. // *SIAM J. Math. Anal.*, 29(4), 1998, pp. 1040-1065.

[37] Dahmen W., Kunoth A., Urban K. Biorthogonal Spline–wavelets on the Interval – Stability and Moment Conditions, *Appl. Comp. Harm. Anal.*, 6, 1999, pp. 132–196.

[38] Castro L.M.S., Freitas J.A.T. Wavelets in Hybrid-mixed Stress Elements. *Computer Methods in Applied Mechanics and Engineering*, vol. 190/31, 2001, pp. 3977-3998.

[39] Zhou Y., Wang J., Zheng X. Application of Wavelets Galerkin FEM to Bending of Beam and Plate Structures. *Applied Mathematics and Mechanics*, 19 (8), 1998, pp. 697-706.

[40] Naldi G., Venini P., Urban K. Wavelet Based Methods in Elastoplasticity and Damage Analysis. *WCCM V Fifth World Congress on Computational Mechanics*. July 7–12, 2002, Vienna, Austria. Eds.: H.A. Mang, F.G. Rammerstorfer, J. Eberhardsteiner.

[41] Fasano D., Naldi G., Venini P. Computational Softening Plasticity by Wavelet Bases. *in B. H. V. Topping ed. Computational techniques for materials, composites anc composite structures*, CIVILCOMP Ltd., Edinburgh, Scotland, 2000, pp. 39-45.