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HERMITIAN CLIFFORD–HERMITE WAVELETS

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Abstract. *The one–dimensional continuous wavelet transform is a successful tool for signal and image analysis, with applications in physics and engineering. Standard Clifford analysis offers an appropriate framework for taking wavelets to higher dimension. In the usual orthogonal case Clifford analysis focusses on monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator* $\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$, *where* $(e_1, \ldots, e_m)$ *is an orthogonal basis for the quadratic space* $\mathbb{R}^m$ *underlying the construction of the Clifford algebra* $\mathbb{R}_{0,m}$. *An intrinsic feature of this function theory is that it encompasses all dimensions at once, as opposed to a tensorial approach with products of one–dimensional phenomena. This has allowed for a very specific construction of higher dimensional wavelets and the development of the corresponding theory, based on generalizations of classical orthogonal polynomials on the real line, such as the radial Clifford–Hermite polynomials introduced by Sommen. In this paper, we pass to the Hermitian Clifford setting, i.e. we let the same set of generators produce the complex Clifford algebra* $\mathbb{C}_{2n}$ *with even dimension, and we equip it with a Hermitian conjugation and a Hermitian inner product. Hermitian Clifford analysis then focusses on the null solutions of two mutually conjugate Hermitian Dirac operators which are invariant under the action of the unitary group. In this setting we construct new Clifford–Hermite polynomials, starting in a natural way from a Rodrigues formula which now involves both Dirac operators mentioned. Due to the specific features of the Hermitian setting, four different types of polynomials are obtained, two types of even degree and two types of odd degree. These polynomials are used to introduce a new continuous wavelet transform, after thorough investigation of all necessary properties of the involved polynomials, the mother wavelet and the associated family of wavelet kernels.*

1 INTRODUCTION

The one–dimensional continuous wavelet transform (CWT) is a successful tool for signal analysis and feature detection in signals, with numerous applications in mathematics, physics and engineering (see e.g. [1, 2, 3]). Basically, wavelets constitute a family of functions $\psi_{a,b}$, derived from one single function ψ called the mother wavelet, by a change of scale a (i.e. by dilation) and a change of position b (i.e. by translation):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right), \qquad a > 0, \qquad b \in \mathbb{R}.$$

On the mother wavelet ψ some particular conditions are imposed. First, ψ is requested to be an L_2–function, or in other words, a signal of finite energy, which is well localized both in the time and in the frequency domain. Moreover it has to satisfy the so-called admissibility condition

$$C_\psi \equiv \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(u)|^2}{|u|}\, du < +\infty, \tag{1.1}$$

where $\widehat{\psi} = \mathcal{F}[\psi]$ denotes the Fourier spectrum of ψ. If ψ is also an L_1–function, the latter condition implies that ψ should have ”zero momentum”, i.e.

$$\int_{-\infty}^{+\infty} \psi(x)\, dx = 0$$

which can only be fulfilled if ψ is an oscillating function, explaining the terminology ”wavelet”.

For the applications additional conditions are imposed, in particular a number of vanishing moments, viz

$$\int_{-\infty}^{+\infty} x^n\, \psi(x)\, dx = 0, \qquad n = 0, 1, \ldots, N.$$

This means that the corresponding CWT defined as

$$F(a,b) = \langle \psi_{a,b}, f\rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \overline{\psi}\left(\frac{x-b}{a}\right) f(x)\, dx$$

will filter out polynomial behaviour of the signal f up to degree N, making it adequate at detecting singularities. Furthermore, the CWT is usually requested to be an isometry mapping $L_2(\mathbb{R})$ into an L_2–space on $(a \in \mathbb{R}_+) \times (b \in \mathbb{R})$ with a well-chosen weighted inner product involving the admissibility constant C_ψ, (1.1), viz

$$\langle F, G\rangle = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \overline{F}(a,b)\, G(a,b)\, \frac{da}{a^2}\, db$$

Standard Clifford analysis offers an appropriate framework for developing a higher dimensional CWT–theory. In its most simple yet still useful setting, flat m-dimensional Euclidean space, Clifford analysis focusses on monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

where $(e_1, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^m$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m}$. An intrinsic feature of this function theory is that it encompasses all dimensions at once, as opposed to a tensorial approach with products of one–dimensional phenomena. This has allowed for a very specific construction of higher dimensional wavelets and the development of the corresponding theory, see e.g. [4, 5, 6, 7, 8]. These wavelets are based on Clifford generalizations of classical orthogonal polynomials on the real line. In this context we explicitly mention the radial Clifford–Hermite polynomials, introduced in [9] and applied to wavelet analysis in the orthogonal Clifford setting in [8].

2 HERMITIAN CLIFFORD ANALYSIS

When allowing for complex constants, the same set of generators as above $(e_1, \ldots, e_m)$, satisfying the defining relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \qquad j = 1, \ldots, m$$

may in fact also generate the complex Clifford algebra $\mathbb{C}_m$, where moreover the dimension m is taken to be even, say $m = 2n$ (since the odd case is easily seen to reduce to the even one in a direct decomposition). As $\mathbb{C}_{2n}$ is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\,\mathbb{R}_{0,2n}$, any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, leading to the definition of the Hermitian conjugation

$$\lambda^\dagger = (a + ib)^\dagger = \overline{a} - i\overline{b}$$

where $\overline{\cdot}$ denotes the usual conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{C}_{2n}$ given by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0, \qquad |\lambda| = \sqrt{[\lambda^\dagger \lambda]_0}$$

The above framework will be referred to as the Hermitian Clifford setting, as opposed to the traditional orthogonal Clifford setting.

Hermitian Clifford analysis (see e.g. [10],[11]) focusses on the null–solutions of two Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$. These are introduced by means of the so–called Witt basis for the complex Clifford algebra $\mathbb{C}_{2n}$

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(e_j - i\, e_{n+j}), \qquad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(e_j + i\, e_{n+j}), \qquad j = 1, \ldots, n \end{aligned}$$

satisfying the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0, \qquad j, k = 1, \ldots, n$$

and the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j^\dagger = \delta_{jk}, \qquad j, k = 1, \ldots, n$$

The Grassmann algebras generated by $(f_j)_{j=1}^n$ and $(f_j^\dagger)_{j=1}^n$, respectively, are denoted by $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$.

Using this Witt basis, the vector $(X_1, \ldots, X_m) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ is identified with the Clifford vector

$$\underline{X} = \sum_{j=1}^{n}(e_j x_j + e_{n+j} y_j) = \sum_{j=1}^{n} \mathfrak{f}_j z_j - \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger} z_j^c$$

where the complex variables $z_j = x_j + i y_j$ and their complex conjugates $z_j^c = x_j - i y_j$, $j = 1, \ldots, n$ have been introduced. Defining the Hermitian vector variable

$$\underline{z} = \sum_{j=1}^{n} \mathfrak{f}_j z_j$$

and its Hermitian conjugate

$$\underline{z}^{\dagger} = \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger} z_j^c$$

the Clifford vector $\underline{X}$ clearly takes the form

$$\underline{X} = \underline{z} - \underline{z}^{\dagger}$$

This also gives rise to the decomposition of the traditional Dirac operator

$$\partial_{\underline{X}} = \sum_{j=1}^{n} \left(e_j \partial_{x_j} + e_{n+j} \partial_{y_j}\right) = 2 \sum_{j=1}^{n} \left(\mathfrak{f}_j \partial_{z_j^c} - \mathfrak{f}_j^{\dagger} \partial_{z_j}\right) = 2(\partial_{\underline{z}}^{\dagger} - \partial_{\underline{z}})$$

in terms of the Hermitian Dirac operators

$$\partial_{\underline{z}} = \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger} \, \partial_{z_j}$$

and

$$\partial_{\underline{z}}^{\dagger} = (\partial_{\underline{z}})^{\dagger} = \sum_{j=1}^{n} \mathfrak{f}_j \, \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators and their complex conjugates in the complex z_j planes, i.e. $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, $j = 1, \ldots, n$. In fact, also a second Clifford vector must be considered, viz

$$\underline{X}| = \sum_{j=1}^{n}(e_j y_j - e_{n+j} x_j) = \frac{1}{i} \sum_{j=1}^{n}(z_j \mathfrak{f}_j + z_j^c \mathfrak{f}_j^{\dagger})$$

which is orthogonal to $\underline{X}$ and takes the form

$$\underline{X}| = \frac{1}{i}(\underline{z} + \underline{z}^{\dagger})$$

To this Clifford vector variable $\underline{X}|$ a Dirac operator is associated as well, given by

$$\partial_{\underline{X}|} = \sum_{j=1}^{n} \left(e_j \partial_{y_j} - e_{n+j} \partial_{x_j}\right) = \frac{2}{i} \sum_{j=1}^{n} \left(\mathfrak{f}_j \partial_{z_j^c} + \mathfrak{f}_j^{\dagger} \partial_{z_j}\right) = \frac{2}{i} \left(\partial_{\underline{z}}^{\dagger} + \partial_{\underline{z}}\right)$$

We then call a continuously differentiable function g on $\mathbb{R}^m$ with values in $\mathbb{C}_{2n}$ a Hermitian monogenic function if and only if it satisfies the system

$$\partial_{\underline{X}} g = 0 = \partial_{\underline{X}|} g$$

or equivalently

$$\partial_{\underline{z}} g = 0 = \partial_{\underline{z}}^\dagger g$$

The Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ are invariant under the action of a realisation of the unitary group in terms of the Clifford algebra, see [10],[11]. This group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$ is given by

$$\widetilde{\mathrm{U}}(n) = \{ s \in \mathrm{Spin}(2n) \mid \exists \theta \geq 0 : \overline{s} I = e^{(-i\theta)} I \} \tag{2.1}$$

its definition involving the primitive idempotent I, which is introduced as follows. Put

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - i\, e_j e_{n+j}), \qquad j = 1 \ldots, n$$

then I_j are mutually commuting idempotents for which moreover $I_j^\dagger = I_j$. Now, let

$$I = I_1 \ldots I_n = \mathfrak{f}_1 \mathfrak{f}_1^\dagger \mathfrak{f}_2 \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n \mathfrak{f}_n^\dagger$$

then obviously $I^2 = I$ and $I^\dagger = I$.

The invariance of the operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ under the action of $\widetilde{\mathrm{U}}(n)$ is then expressed as

$$[\partial_{\underline{z}}, L(s)] = 0 = [\partial_{\underline{z}}^\dagger, L(s)], \qquad s \in \widetilde{U}(n)$$

where $[\cdot,\cdot]$ denotes the commutator of two operators and $L(s)$ is the so–called l-representation of an arbitrary spin element s, see e.g. [12].

3 THE HERMITIAN CLIFFORD–HERMITE POLYNOMIALS

Our aim is the construction of new Clifford–Hermite polynomials in the Hermitian setting, as a preparatory step for the introduction of a Hermitian CWT; this is established in a rather canonical way, starting from a Rodrigues formula, as has been done for the radial Clifford–Hermite polynomials in the orthogonal case in [9].

However, instead of the single operator $\underline{\partial}$ we now have the Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$, which leads to a natural and suitable adaptation of the original Rodrigues formula, viz

$$D_p(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger) \left[\exp\left(-\frac{|\underline{z}|^2}{2} \right) \right] = H_p(\underline{z}, \underline{z}^\dagger) \exp\left(-\frac{|\underline{z}|^2}{2} \right) \tag{3.1}$$

in order to define the Hermitian Clifford–Hermite polynomials H_p of degree p. Here $D_p(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)$ is a differential operator of order p, being a product of p factors $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$. As, on account of the isotropy of the Witt basis elements, one has that

$$\partial_{\underline{z}}^2 = \partial_{\underline{z}}^{\dagger 2} = 0$$

and moreover, as the Laplace operator Δ_m allows for the decomposition

$$\Delta_m = 4\left(\partial_{\underline{z}}\,\partial_{\underline{z}}^\dagger + \partial_{\underline{z}}^\dagger\partial_{\underline{z}}\right)$$

the proposed form of D_p results into four essentially different types of differential operators, viz two mutually adjoint types of odd order, given by

$$\partial_{\underline{z}}^\dagger\,\Delta_m^p \qquad \text{and} \qquad \partial_{\underline{z}}\,\Delta_m^p$$

as well as two self–adjoint types of even order, given by

$$(\partial_{\underline{z}}\,\partial_{\underline{z}}^\dagger)\,\Delta_m^p \qquad \text{and} \qquad (\partial_{\underline{z}}^\dagger\,\partial_{\underline{z}})\,\Delta_m^p$$

Starting from (3.1) and invoking these explicit forms of D_p, we are lead to four types of Hermitian Clifford–Hermite polynomials, which moreover may be expressed in terms of the Laguerre polynomials on the real line as

$$\begin{aligned} H_{2p+1}^{(1)}(\underline{z},\underline{z}^\dagger) &= (-1)^{p-1}\,2^{p-1}\,p!\,\underline{z}\,L_p^n\left(\frac{1}{2}|\underline{z}|^2\right) \\ H_{2p+1}^{(2)}(\underline{z},\underline{z}^\dagger) &= (-1)^{p-1}\,2^{p-1}\,p!\,\underline{z}^\dagger\,L_p^n\left(\frac{1}{2}|\underline{z}|^2\right) \end{aligned}$$

and

$$\begin{aligned} H_{2p+2}^{(3)}(\underline{z},\underline{z}^\dagger) &= (-1)^{p-1}\,2^{p-1}\,p!\,\left[\beta\,L_p^n(\frac{1}{2}|\underline{z}|^2) - \frac{1}{2}\underline{z}^\dagger\,\underline{z}\,L_p^{n+1}(\frac{1}{2}|\underline{z}|^2)\right] \\ H_{2p+2}^{(4)}(\underline{z},\underline{z}^\dagger) &= (-1)^{p-1}\,2^{p-1}\,p!\,\left[(n-\beta)\,L_p^n(\frac{1}{2}|\underline{z}|^2) - \frac{1}{2}\underline{z}\,\underline{z}^\dagger\,L_p^{n+1}(\frac{1}{2}|\underline{z}|^2)\right] \end{aligned}$$

where moreover β denotes the Clifford number $\sum_{j=1}^n \mathfrak{f}_j\mathfrak{f}_j^\dagger$. Observe that

$$\left(H_{2p+1}^{(1)}(\underline{z},\underline{z}^\dagger)\right)^\dagger = H_{2p+1}^{(2)}(\underline{z},\underline{z}^\dagger)$$

while

$$\left(H_{2p+2}^{(3)}(\underline{z},\underline{z}^\dagger)\right)^\dagger = H_{2p+2}^{(3)}(\underline{z},\underline{z}^\dagger) \qquad \text{and} \qquad \left(H_{2p+2}^{(4)}(\underline{z},\underline{z}^\dagger)\right)^\dagger = H_{2p+2}^{(4)}(\underline{z},\underline{z}^\dagger)$$

which was to be expected, seen the properties of the generating differential operators.

With respect to the positive Gaussian weight function

$$\exp\left(-\frac{1}{2}|\underline{X}|^2\right) = \exp\left(-\frac{1}{2}|\underline{z}|^2\right) = \exp\left(-\frac{1}{2}|\underline{z}^\dagger|^2\right) = \exp\left(-\frac{1}{2}r^2\right)$$

all Hermitian Clifford–Hermite polynomials are found to be mutually orthogonal in $L_2(\mathbb{R}^{2n})$, i.e. for arbitrary degrees k,l and indices $i,j = 1,2,3,4$, they satisfy the following orthogonality relations:

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\,\left(H_k^{(i)}(\underline{z},\underline{z}^\dagger)\right)^\dagger\,H_l^{(i)}(\underline{z},\underline{z}^\dagger)\,dV &= 0, \qquad k \neq l \\ \int_{\mathbb{R}^{2n}} \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\,\left(H_k^{(i)}(\underline{z},\underline{z}^\dagger)\right)^\dagger\,H_l^{(j)}(\underline{z},\underline{z}^\dagger)\,dV &= 0, \qquad i \neq j \end{aligned}$$

Being apparently not very surprising, the intriguing character of these relations is actually hidden in their proofs. Indeed, all integrals above are eventually found to vanish, however for a number of essentially different reasons: in some cases (part of) the integrand reduces to zero, while in others various recurrence relations have to be applied in order to transform the integral into an equivalent form where either the radial part vanishes, or the angular part turns out to be zero after explicit calculation. For a detailed account, we refer to [13].

4 THE HERMITIAN CLIFFORD–HERMITE WAVELET KERNELS

Following the construction of the four types of Hermitian Clifford–Hermite polynomials in the previous section, also four different families of wavelet kernels with their respective "mother wavelets" may be introduced, the latter being given by

$$\begin{aligned} \psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger) &= \partial^\dagger_{\underline{z}}\, \Delta^p_m \left[\exp\left(-\frac{1}{2}|\underline{z}|^2\right)\right] = \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\, H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger) \qquad (4.1)\\ \psi^{(2)}_{2p+1}(\underline{z}, \underline{z}^\dagger) &= \partial_{\underline{z}}\, \Delta^p_m \left[\exp\left(-\frac{1}{2}|\underline{z}|^2\right)\right] = \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\, H^{(2)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\\ \psi^{(3)}_{2p+2}(\underline{z}, \underline{z}^\dagger) &= (\partial_{\underline{z}}\, \partial^\dagger_{\underline{z}})\, \Delta^p_m \left[\exp\left(-\frac{1}{2}|\underline{z}|^2\right)\right] = \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\, H^{(3)}_{2p+2}(\underline{z}, \underline{z}^\dagger)\\ \psi^{(4)}_{2p+2}(\underline{z}, \underline{z}^\dagger) &= (\partial^\dagger_{\underline{z}}\, \partial_{\underline{z}})\, \Delta^p_m \left[\exp\left(-\frac{1}{2}|\underline{z}|^2\right)\right] = \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\, H^{(4)}_{2p+2}(\underline{z}, \underline{z}^\dagger) \end{aligned}$$

In order not to overload the paper, we will only study the family stemming from $\psi^{(1)}_{2p+1}$ in detail, the necessary properties of the other types being obtained in a similar way. For further use, note that

$$\psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger) = 4^p\, \partial^\dagger_{\underline{z}}\, \left(\partial_{\underline{z}}\, \partial^\dagger_{\underline{z}}\right)^p \left[\exp\left(-\frac{1}{2}|\underline{z}|^2\right)\right] \qquad (4.2)$$

Seeing that $\psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)$ is both integrable and square integrable, the first property to be checked is that $\psi^{(1)}_{2p+1}$ has zero momentum, i.e.

$$\int_{\mathbb{R}^{2n}} \psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\, dV(\underline{X}) \equiv \int_{\mathbb{R}^{2n}} \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\, H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\, dV(\underline{X}) = 0$$

From the form of the generating differential operator $\partial^\dagger_{\underline{z}}\, \Delta^p_m$ one infers that

$$H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger) = (-1)^p \left(-\frac{1}{2}\underline{z}\right)\, \widetilde{H}_{2p}(r)$$

where $\widetilde{H}_{2p}(r)$ is a scalar polynomial of degree p in $r^2 = |\underline{z}|^2$. Passing to spherical co-ordinates, with $\underline{z} = r\Xi$, the desired property then reads

$$\int_0^{+\infty} r^m\, \exp\left(-\frac{1}{2}r^2\right) \widetilde{H}_{2p}(r)\, dr \int_{S^{m-1}} \Xi\, dS(\underline{\Omega}) = 0$$

which indeed is fulfilled, as the integral over the unit sphere S^{m-1} vanishes. In fact, seen this specific reason for the vanishing of the integral, this property also expresses the orthogonality in $L_2(\mathbb{R}^{2n})$ of the polynomials $H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)$ and 1 with respect to any radial weight function ensuring the convergence of the integral's radial part.

Next, the mother wavelet should also show a number of vanishing moments, in order to filter out polynomial behaviour. To this end, observe that any polynomial in the variable $\underline{X}$ is a polynomial in $(\underline{z} - \underline{z}^\dagger)$, or still in $(\underline{z}^\dagger \underline{z})^s$, $(\underline{z}\underline{z}^\dagger)^s$, $\underline{z}(\underline{z}^\dagger \underline{z})^s$ and $\underline{z}^\dagger(\underline{z}\underline{z}^\dagger)^s$, $s = 0, 1, 2, \ldots$. Any such polynomial may thus be written as a linear combination of Hermitian Clifford–Hermite polynomials, as one may express

$$\begin{array}{rl} (\underline{z}^\dagger \underline{z})^s & \text{as a combination of} \quad H^{(3)}_{2s}, H^{(3)}_{2s-2}, H^{(3)}_{2s-4}, \ldots \\ (\underline{z}\underline{z}^\dagger)^s & \text{as a combination of} \quad H^{(4)}_{2s}, H^{(4)}_{2s-2}, H^{(4)}_{2s-4}, \ldots \\ \underline{z}\,(\underline{z}^\dagger \underline{z})^s & \text{as a combination of} \quad H^{(1)}_{2s+1}, H^{(1)}_{2s-1}, H^{(1)}_{2s-3}, \ldots \\ \underline{z}^\dagger\,(\underline{z}\underline{z}^\dagger)^s & \text{as a combination of} \quad H^{(2)}_{2s+1}, H^{(2)}_{2s-1}, H^{(2)}_{2s-3}, \ldots \end{array}$$

On account of the orthogonality relations of the Hermitian Clifford–Hermite polynomials obtained in the previous section, it follows that any polynomial P_q of degree q, showing the above form, is orthogonal with respect to the weight function $\exp\left(-\frac{1}{2}|\underline{z}|^2\right)$ to any Hermitian Clifford–Hermite polynomial of degree at least $q+1$. In particular, one has

$$\int_{\mathbb{R}^{2n}} \exp\left(-\frac{1}{2}|\underline{z}|^2\right)\; P_q(\underline{z} - \underline{z}^\dagger)\; H^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X}) = 0 \qquad \text{if } q < 2p+1$$

or

$$\int_{\mathbb{R}^{2n}} P_q(\underline{z} - \underline{z}^\dagger)\; \psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X}) = 0 \qquad \text{if } q < 2p+1$$

This means that

$$\int_{\mathbb{R}^{2n}} \left\{ \begin{array}{c} (\underline{z}^\dagger \underline{z})^t,\ (\underline{z}\underline{z}^\dagger)^t \\ \underline{z}(\underline{z}^\dagger \underline{z})^s,\ \underline{z}^\dagger(\underline{z}\underline{z}^\dagger)^s \end{array} \right\} \psi^{(1)}_{2p+1}(\underline{z}, \underline{z}^\dagger)\; dV(\underline{X}) = 0 \qquad \text{for } s < p,\ t \leq p$$

the latter expressions revealing the exact meaning of the term "vanishing moments" in this Hermtitan context.

For the study of the mother wavelet $\psi^{(1)}_{2p+1}$ in frequency space, we will use the following definition of the standard Fourier transform in $\mathbb{R}^{2n}$:

$$\mathcal{F}[f](\underline{U}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \exp\left(-i\, \langle \underline{U}, \underline{X} \rangle\right)\; f(\underline{X})\; dV(\underline{X})$$

which also takes the form

$$\mathcal{F}[f](\underline{w}, \underline{w}^\dagger) = \frac{i^n}{(4\pi)^n} \int_{\mathbb{C}^n} \exp\left(-2i\, \mathbb{R}e\, (\underline{w}, \underline{z})\right)\; f(\underline{z}, \underline{z}^\dagger)\; d\underline{z} \wedge d\underline{z}^\dagger$$

when passing to Hermitian vector variables, and introducing the notation

$$d\underline{z} \wedge d\underline{z}^\dagger \equiv dz_1 \wedge dz_1^c \wedge \ldots \wedge dz_n \wedge dz_n^c = (-2i)^n\; dV(\underline{X})$$

This Fourier transform shows the basic calculation rules

$$\mathcal{F}\left[\partial_{\underline{z}} f\right] = \frac{i}{2}\, \underline{w}^\dagger\; \mathcal{F}[f]$$

and

$$\mathcal{F}\left[\partial_{\underline{z}}^{\dagger} f\right] = \frac{i}{2}\, \underline{w}\, \mathcal{F}[f]$$

so that, also on account of (4.2),

$$\mathcal{F}\left[\psi_{2p+1}^{(1)}(\underline{z}, \underline{z}^{\dagger})\right](\underline{w}, \underline{w}^{\dagger}) = (-1)^p\, \frac{i}{2}\, \underline{w}\, |\underline{w}|^{2p}\, \exp\left(-\frac{1}{2}|\underline{w}|^2\right) \tag{4.3}$$

We now introduce a family of wavelet kernels stemming from $\psi_{2p+1}^{(1)}$ by taking into account not only scaling and translation, but also rotation in Euclidean space. Starting from the original Clifford vector $\underline{X}$ and considering a scaling factor $a > 0$ and a translation vector $\underline{B} \in \mathbb{R}^{2n}$, the corresponding operations are easily transferred to the Hermitian setting by

$$\frac{\underline{X} - \underline{B}}{a} = \frac{\underline{z} - \underline{b}}{a} - \frac{\underline{z}^{\dagger} - \underline{b}^{\dagger}}{a}$$

when $\underline{B} = \underline{b} - \underline{b}^{\dagger}$. For the rotation, we will consider spin elements from the "unitary" subgroup $\widetilde{U}(n)$ of Spin$(2n)$, see (2.1), and in particular, the h-transformation associated to those elements, viz

$$h(s):\ a \in \mathbb{C}_{2n} \mapsto h(s)[a] = s\, a\, s^{\dagger} = s\, a\, \overline{s} = s\, a\, s^{-1}$$

This h–representation of $\widetilde{U}(n)$ leaves the k-blades of the Grassmann algebras $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^{\dagger}$ invariant, such that

$$\begin{array}{rcll} h(s)[\underline{z}] & = & s\, \underline{z}\, \overline{s} & \in \mathbb{C}\Lambda_n \cap \mathbb{C}_{2n}^{(1)} \\ h(s)[\underline{z}^{\dagger}] & = & s\, \underline{z}^{\dagger}\, \overline{s} & \in \mathbb{C}\Lambda_n^{\dagger} \cap \mathbb{C}_{2n}^{(1)} \end{array}$$

where $\mathbb{C}_{2n}^{(1)}$ denotes the vectors in $\mathbb{C}^{2n}$. The corresponding operator action on functions, given by

$$H(s)[g(\underline{X})] = s\; g(\overline{s}\, \underline{X}\, s)\; \overline{s}$$

thus takes the form

$$H(s)\left[g(\underline{z}, \underline{z}^{\dagger})\right] = s\; g(\overline{s}\, \underline{z}\, s, \overline{s}\, \underline{z}^{\dagger}\, s)\; \overline{s}$$

We then define the following family of wavelet kernels:

$$\psi_{2p+1}^{(1)}{}^{a,\underline{b},s}(\underline{z}, \underline{z}^{\dagger}) = \frac{1}{a^n}\; s\; \psi_{2p+1}^{(1)}\left(\frac{\overline{s}\,(\underline{z} - \underline{b})\, s}{a}, \frac{\overline{s}\,(\underline{z}^{\dagger} - \underline{b}^{\dagger})\, s}{a}\right)\; \overline{s} \tag{4.4}$$

where a is a positive real number, $\underline{b}$ is a vector from the Grassmann algebra $\mathbb{C}\Lambda_n$ and s is a spin element belonging to the group $\widetilde{U}(n)$. Invoking the basic calculation rules of the Fourier transform for scaling, translation and rotation, applied to the mother wavelet $\psi_{2p+1}^{(1)}$, i.e.

$$\begin{array}{rcl} \mathcal{F}\left[\dfrac{1}{a^n}\, \psi_{2p+1}^{(1)}\left(\dfrac{\underline{z}}{a}, \dfrac{\underline{z}^{\dagger}}{a}\right)\right](\underline{w}, \underline{w}^{\dagger}) & = & a^n\, \mathcal{F}\left[\psi_{2p+1}^{(1)}\right](a\underline{w}, a\underline{w}^{\dagger}) \\ \mathcal{F}\left[\psi_{2p+1}^{(1)}\left(\underline{z} - \underline{b}, \underline{z}^{\dagger} - \underline{b}^{\dagger}\right)\right](\underline{w}, \underline{w}^{\dagger}) & = & \exp\left(-2i\, \mathbb{R}\mathrm{e}\,(\underline{w}, \underline{b})\right)\, \mathcal{F}\left[\psi_{2p+1}^{(1)}\right](\underline{w}, \underline{w}^{\dagger}) \\ \mathcal{F}\left[s\, \psi_{2p+1}^{(1)}\left(\overline{s}\, \underline{z}\, s, \overline{s}\, \underline{z}^{\dagger}\, s\right)\, \overline{s}\right](\underline{w}, \underline{w}^{\dagger}) & = & s\, \mathcal{F}\left[\psi_{2p+1}^{(1)}\right](\overline{s}\, \underline{w}\, s, \overline{s}\, \underline{w}^{\dagger}\, s)\; \overline{s} \end{array}$$

we arrive at

$$\mathcal{F}\left[\psi_{2p+1}^{(1)}{}^{a,\underline{b},s}\right] = a^n\, \exp\left(-2i\, \mathbb{R}\mathrm{e}\,(\underline{w}, \underline{b})\right)\; s\; \mathcal{F}\left[\psi_{2p+1}^{(1)}\right](a\, \overline{s}\, \underline{w}\, s, a\, \overline{s}\, \underline{w}^{\dagger}\, s)\; \overline{s}$$

5 THE HERMITIAN CLIFFORD–HERMITE CWT

In this section, we will use the family of functions introduced in (4.4) as kernels for the definition of a new continuous wavelet transform, or CWT. To this end, take $g \in L_2(\mathbb{R}^{2n})$ and define its Hermitian Clifford–Hermite CWT by

$$G(a, \underline{b}, s) = \int_{\mathbb{R}^{2n}} \left({\psi_{2p+1}^{(1)}}^{a,\underline{b},s}(\underline{z}, \underline{z}^\dagger) \right)^\dagger \; g(\underline{z}, \underline{z}^\dagger) \; dV(\underline{X}) \tag{5.1}$$

which in frequency space reads

$$\begin{aligned} G(a, \underline{b}, s) &= \int_{\mathbb{R}^{2n}} \left(\mathcal{F}\left[{\psi_{2p+1}^{(1)}}^{a,\underline{b},s} \right] (\underline{w}, \underline{w}^\dagger) \right)^\dagger \; \mathcal{F}[g](\underline{w}, \underline{w}^\dagger) \; dV(\underline{Y}) \qquad (5.2) \\ &= a^n \; s \; (2\pi)^n \; \mathcal{F}\left[\left(\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] (a\, \overline{s}\, \underline{w}\, s, a\, \overline{s}\, \underline{w}^\dagger\, s) \right)^\dagger \; \overline{s} \; \mathcal{F}[g](\underline{w}, \underline{w}^\dagger) \right] (-\underline{b}, -\underline{b}^\dagger) \end{aligned}$$

As usual, we request the obtained CWT to constitute an isometry from the space of signals g into the space of transforms G, or more precisely from $L_2(\mathbb{R}^{2n})$ into a weighted L_2–space on $(a \in \mathbb{R}_+) \times (\underline{b} \in \mathbb{C}\Lambda^n \cap \mathbb{C}_{2n}^{(1)}) \times (s \in \widetilde{U}(n))$. Thus, equipping both spaces with an appropriate inner product, i.e. putting

$$\langle g_1, g_2 \rangle = \int_{\mathbb{R}^{2n}} (g_1(\underline{z}, \underline{z}^\dagger), g_2(\underline{z}, \underline{z}^\dagger)) \; dV(\underline{X}) = \int_{\mathbb{R}^{2n}} \left[{g_1}^\dagger(\underline{z}, \underline{z}^\dagger) \; g_2(\underline{z}, \underline{z}^\dagger) \right]_0 \; dV(\underline{X})$$

and

$$\begin{aligned} \langle G_1, G_2 \rangle &= \frac{1}{C} \int_{\widetilde{U}(n)} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \int_0^{+\infty} (G_1(a, \underline{b}, s), G_2(a, \underline{b}, s)) \; \frac{da}{a^{2n+1}} \; (d\underline{b} \wedge d\underline{b}^\dagger) \; ds \\ &= \frac{1}{C} \int_{\widetilde{U}(n)} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \int_0^{+\infty} \left[G_1(a, \underline{b}, s)^\dagger \; G_2(a, \underline{b}, s) \right]_0 \; \frac{da}{a^{2n+1}} \; (d\underline{b} \wedge d\underline{b}^\dagger) \; ds \end{aligned}$$

where $(\cdot, \cdot)$ denotes the Hermitian inner product (see §2), we need to determine the constant C such that a Parseval formula holds, viz

$$\langle G_1, G_2 \rangle = \langle g_1, g_2 \rangle$$

Now, as $\overline{s}s = 1$, (5.2) learns that

$$G_1^\dagger \, G_2 = a^{2n} \, (2\pi)^{2n} \; \left(\mathcal{F}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_1] \right] (-\underline{b}, -\underline{b}^\dagger) \right)^\dagger \; \mathcal{F}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_2] \right] (-\underline{b}, -\underline{b}^\dagger)$$

where we have omitted some of the arguments for notational brevity. Singling out the integral over $\mathbb{R}^{2n}$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \left[G_1^\dagger \; G_2 \right]_0 \; d\underline{b} \wedge d\underline{b}^\dagger \\ &\quad = a^{2n}(2\pi)^{2n} \; \langle \mathcal{F}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_1] \right], \mathcal{F}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_2] \right] \rangle \\ &\quad = a^{2n}(2\pi)^{2n} \; \langle \mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_1], \mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \overline{s} \mathcal{F}[g_2] \rangle \\ &\quad = a^{2n}(2\pi)^{2n} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \left[(\mathcal{F}[g_1])^\dagger \; s \mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \left(\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \right)^\dagger \overline{s} \mathcal{F}[g_2] \right]_0 \; d\underline{w} \wedge d\underline{w}^\dagger \\ &\quad = a^{2n}(2\pi)^{2n} \int_{\mathbb{R}^{2n} \cong \mathbb{C}^n} \left[\mathcal{F}\left[\psi_{2p+1}^{(1)} \right] \mathcal{F}\left[\psi_{2p+1}^{(1)} \right]^\dagger \right]_0 \; \left[\mathcal{F}[g_1]^\dagger \mathcal{F}[g_2] \right]_0 \; d\underline{w} \wedge d\underline{w}^\dagger \end{aligned}$$

so that the inner product in the space of transforms becomes

$$\langle G_1, G_2\rangle$$
$$= \frac{1}{C}\int_{\widetilde{U}(n)}\int_0^{+\infty}(2\pi)^{2n}\int_{\mathbb{R}^{2n}\cong\mathbb{C}^n}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)}\right]\mathcal{F}\left[\psi_{2p+1}^{(1)}\right]^\dagger\right]_0 \left[\mathcal{F}[g_1]^\dagger\mathcal{F}[g_2]\right]_0 (d\underline{w}\wedge d\underline{w}^\dagger)\frac{da}{a}ds$$

which will lead to the desired Parseval formula, if we can define the "admissibility" constant C to be

$$C = (2\pi)^{2n}\int_{\widetilde{U}(n)}\int_0^{+\infty}\left[\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](a\overline{s}\underline{w}s, a\overline{s}\underline{w}^\dagger s)\left(\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](a\overline{s}\underline{w}s, a\overline{s}\underline{w}^\dagger s)\right)^\dagger\right]_0\frac{da}{a}\,ds$$

assuming for a moment that the involved integral is convergent. Indeed, in that case

$$\langle G_1, G_2\rangle = \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n}\left[\mathcal{F}[g_1]^\dagger\mathcal{F}[g_2]\right]_0\, d\underline{w}\wedge d\underline{w}^\dagger = \int_{\mathbb{R}^{2n}\cong\mathbb{C}^n}\left[g_1^\dagger g_2\right]_0\, d\underline{z}\wedge d\underline{z}^\dagger = \langle g_1, g_2\rangle$$

For further computation of the admissibility constant C, we first execute the substitution

$$\underline{w} = \frac{r}{a}\,\underline{\xi} \qquad \text{with } |\underline{\xi}| = 1$$

leading to

$$C = (2\pi)^{2n}\int_{\widetilde{U}(n)}\int_0^{+\infty}\left|\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](r\overline{s}\underline{\xi}s, r\overline{s}\underline{\xi}^\dagger s)\right|^2\,\frac{dr}{r}\,ds$$

followed by the subsitution

$$\overline{s}\underline{\xi}s = \underline{\eta}, \quad \overline{s}\underline{\xi}^\dagger s = \underline{\eta}^\dagger$$

which turns the integration over $\widetilde{U}(n)$ into an integration over the unit sphere S^{2n-1} of $\mathbb{R}^{2n}$, since $|\underline{\eta}| = |\underline{\eta}^\dagger| = 1$. We consecutively obtain

$$\begin{aligned} C &= (2\pi)^{2n}\int_{S^{2n-1}}\int_0^{+\infty}\left|\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](r\underline{\eta}, r\underline{\eta}^\dagger)\right|^2\,\frac{dr}{r}\,dS(\underline{\eta}) \\ &= (2\pi)^{2n}\int_{R^{2n}}\frac{\left|\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](r\underline{\eta}, r\underline{\eta}^\dagger)\right|^2}{r^{2n}}\,dV(\underline{X}) \\ &= (2\pi)^{2n}\int_{R^{2n}}\frac{\left|\mathcal{F}\left[\psi_{2p+1}^{(1)}\right](\underline{w}, \underline{w}^\dagger)\right|^2}{|\underline{w}|^{2n}}\,dV(\underline{X}) \end{aligned}$$

Invoking (4.3) we finally arrive at

$$C = (2\pi)^{2n}\int_{\mathbb{R}^{2n}}\frac{1}{4}|\underline{w}|^{4p-2n+2}\exp\left(-|\underline{w}|^2\right)\,dV = (2\pi)^{2n}\frac{(2p)!}{8}\,a_{2n}$$

which not only shows that the considered integral converges and hence that the mother wavelet $\psi_{2p+1}^{(1)}$ satisfies a suitably adapted admissibility condition, but also explicitly determines the admissibility constant C. Here a_{2n} denotes the area of the unit sphere S^{2n-1}.

6 CONCLUDING REMARKS

We have shown that the Hermitian Clifford–Hermite wavelet functions do constitute "good" kernel functions for a continuous wavelet transform. However, the Hermitian setting of the Clifford analysis framework necessitated two major adaptations as compared to the Clifford–Hermite wavelet kernels established in the orthogonal setting of Clifford analysis. First, the notion of "vanishing moments" had to be redefined in terms of powers of the Hermitian vector variables $\underline{z}$ and $\underline{z}^\dagger$. Secondly, the Parseval formula, which expresses the norm–preserving character of the continuous wavelet transform, one of the fundamental properties in wavelet theory, had to be reformulated in terms of suitably adapted scalar inner products on the respective L_2–spaces involved. Nevertheless, these first examples of Hermitian wavelet kernels presented in this paper, pave the way for the study of the continuous wavelet transform in the new framework of Hermitian Clifford analysis.

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