18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (ed.)
Weimar, Germany, 07–09 July 2009

NON-LINEAR ANALYSIS OF COMPOSITE CROSS-SECTIONS WITH PRE-DEFORMATIONS

E. Raue*

* *Institute of Structural Engineering, Bauhaus-Universität Weimar*
Marienstraße 13, 99423 Weimar, Germany
Fax: +49 3643 584422
E-mail: erich.raue@bauing.uni-weimar.de

Keywords: Non-linear analysis of cross-sections, composite cross-sections, reinforced concrete cross-sections, non-linear optimisation, cross-section capacity

Abstract. *An energy method based on the LAGRANGE Principle of the minimum of total potential energy is presented to calculate the stresses and strains of composite cross-sections. The stress-strain relation of each partition of the cross-section can be an arbitrary piecewise continuous function. The strain energy is transformed into a line integral by GAUSS's integral theorem.*

The total strain of each partition of the cross-section is split into load-dependent strain and pre-strain. Pre-strains have to be taken into account when the cross-section is pre-stressed, retrofitted or influenced by shrinkage, temperature etc.

The unconstrained minimum problem can be solved for each load combination using standard software. The application of the method presented in the paper is demonstrated by means of examples.

1 INTRODUCTION

The analysis of cross-sections is usually based on equations of equilibrium of stresses, equations of compatibility of deformations and on constitutive equations relating the stresses to the strains. Alternatively, the modelling can be based on extremum principles. Fundamental aspects of this methodical conception were explained in [1, 2, 3]. The following paper shows the further development of this alternative approach insofar as it concerns pre-deformations of particular domains of the composite cross-section.

2 BASICS OF MODELLING

The stresses and strains in composite cross-sections loaded by normal force N and bending moments M_y and M_z can be calculated by the energy method based on the LAGRANGE *Principle of minimum of total potential energy*

$$\Pi = \sum_j \Pi_{i,j} - N\varepsilon_0 - M_y\kappa_z - M_z\kappa_y = \sum_j \iint_{B_j} W_j \, dydz - N\varepsilon_0 - M_y\kappa_z - M_z\kappa_y \rightarrow \mathrm{Min} \tag{1}$$

In this equation, ε_0 is the strain in the origin of the co-ordinate system x,y. The curvatures κ_y and κ_z correspond to the slopes of the strain plane with respect to the coordinate axes y and z.

The material law can usually be defined by a non-linear stress-strain relation $\sigma = \sigma(\varepsilon)$. We assume that $\sigma(\varepsilon)$ is a piecewise continuous function. Additionally, we introduce the functions $W(\varepsilon)$, $F(\varepsilon)$ and $\Phi(\varepsilon)$, which all are partial integrals of the stress-strain relation:

$$W(\varepsilon) = \int_0^\varepsilon \sigma(\varepsilon)d\varepsilon, \qquad F(\varepsilon) = \int_0^\varepsilon W(\varepsilon)d\varepsilon, \qquad \Phi(\varepsilon) = \int_0^\varepsilon F(\varepsilon)d\varepsilon. \tag{2}$$

As the stress-strain relation $\sigma = \sigma(\varepsilon)$, the specific strain energy $W(\varepsilon)$ and the functions $F(\varepsilon)$ and $\Phi(\varepsilon)$ completely and equivalently describe the material behaviour.

As shown in [2], the double integral in eq. (1), representing the strain energy Π_{ij} of a partition j of the cross-section can be transformed by GAUSS's theorem into a line integral

$$\Pi_{i,j} = \iint_{B_j} W_j \, dydz = \oint_{L_j} \left(-\frac{\kappa_{zj}}{\kappa_j^2} F_j \, dy + \frac{\kappa_{yj}}{\kappa_j^2} F_j \, dz\right) = -\frac{1}{\kappa_j} \oint_{L_j} F_j \, d\eta \tag{3}$$

where

$$\kappa_j = \sqrt{\kappa_{yj}^2 + \kappa_{zj}^2}. \tag{4}$$

The strain energy of a cross-section with a piecewise linear periphery (polygonal cross-section) explicitly depends on the values σ_{ji}, W_{ji}, F_{ji} und Φ_{ji} at the corner points i of the cross-section and its partitions j.

In the case of cross-sections with *pre-deformations,* the total strains $\varepsilon_j(y,z)$ of the partitions j of the cross-section are divided into two parts: the load-depending strain $\varepsilon(y,z)$ and the pre-strain $\varepsilon_j^{(0)}(y,z)$:

$$\varepsilon_j(y,z) = \varepsilon(y,z) + \varepsilon_j^{(0)}(y,z). \tag{5}$$

According to the BERNOULLI hypothesis, a cross-section perpendicular to the axis of the element with rigid bond between the partitions of the cross-section remains plane during the deformation process. Thus the pre-strain $\varepsilon_j^{(0)}$ of the partition B_j of the cross-section and the load-depending strain $\varepsilon = \varepsilon(y,z)$ of the whole composite cross-section B with rigid bond between all these partitions are linear functions of the coordinates y and z

$$\varepsilon(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z \qquad y,z \in B \tag{6}$$

and

$$\varepsilon_j^{(0)}(y,z) = \varepsilon_{0j}^{(0)} + \kappa_{yj}^{(0)} y + \kappa_{zj}^{(0)} z \qquad y,z \in B_j \tag{7}$$

The strains ε_0 and $\varepsilon_{0j}^{(0)}$ at the origin of the coordinate system, as well as the curvatures κ_y, κ_z, $\kappa_{y\,j}^{(0)}$and $\kappa_{z\,j}^{(0)}$, are parameters of the corresponding strain plane.

$$\varepsilon_j(y,z) = \varepsilon(y,z) + \varepsilon_j^{(0)}(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z + \varepsilon_{0j}^{(0)} + \kappa_{yj}^{(0)} y + \kappa_{zj}^{(0)} z = \varepsilon_{0j} + \kappa_{yj} y + \kappa_{zj} z \tag{8}$$

where

$$\varepsilon_{0j} = \varepsilon_0 + \varepsilon_{0j}^{(0)}, \qquad \kappa_{yj} = \kappa_y + \kappa_{yj}^{(0)}, \qquad \kappa_{zj} = \kappa_z + \kappa_{zj}^{(0)} \tag{9}$$

Thus the total strain of the whole cross-section is in general a piecewise linear function.

Pre-deformations have to be taken into account when the cross-section is pre-stressed, a cross-section is retrofitted by additional partitions or when the strains are influenced by temperature, shrinkage etc.

The pre-deformations $\varepsilon_j^{(0)}$ should be related to the state $N = M_y = M_z = 0$. We assume that ILJUSHIN's criterion of simple loading is not violated, i.e. that pre-strains due to different loading can be superposed. All relevant combinations of loading should be investigated separately, taking into account the corresponding pre-deformations.

The strain energy $\Pi_{i,j}$ is determined by the total strains ε_j

$$\Pi_{i,j} = \iint_{B_j} W_j\,[\varepsilon_j(y,z)]\,dydz \tag{10}$$

According to eq.(1), an unrestricted minimum problem can be formulated for each load combination:

$$\Pi = \sum_j \iint_{B_j} W_j\,[\varepsilon_j(y,z)]\,dydz - N\varepsilon_0 - M_y\kappa_z - M_z\kappa_y \rightarrow \text{Min} \tag{11}$$

Variables of the problem are the parameters ε_0, κ_y and κ_z of the corresponding strain plane. If ε_0^*, κ_y^*, κ_z^* is the solution to the problem, the function $\Pi(\varepsilon_0^*,\kappa_y^*,\kappa_z^*)$ has a minimum, and the partial derivatives of Π with respect to ε_0, κ_y und κ_z are zero

$$\frac{\partial \Pi}{\partial \varepsilon_0} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \varepsilon_0} - N = 0, \quad \frac{\partial \Pi}{\partial \kappa_y} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \kappa_y} - M_z = 0, \quad \frac{\partial \Pi}{\partial \kappa_z} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \kappa_z} - M_y = 0\,. \tag{12}$$

The double integral in eq.(12) can be transformed by the LEIBNIZ rule

$$D_t I_x = I_x D_t \tag{13}$$

where D_t is the partial derivative with respect to t and I_x is the integral operator with respect to x over a fixed domain. Thus we obtain

$$\frac{\partial \Pi_i}{\partial \varepsilon_0} = \sum_j \frac{\partial}{\partial \varepsilon_0} \iint_{\mathbf{B}_j} W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]}{\partial \varepsilon_0}\mathrm{d}y\mathrm{d}z =$$
$$= \sum_j \iint_{\mathbf{B}_j} \frac{\mathrm{d}W_j\,(\varepsilon_j)}{\mathrm{d}\varepsilon_j}\frac{\partial \varepsilon_j}{\partial \varepsilon_0}\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y,z)\mathrm{d}y\mathrm{d}z = N \tag{14}$$

$$\frac{\partial \Pi_i}{\partial \kappa_y} = \sum_j \frac{\partial}{\partial \kappa_y} \iint_{\mathbf{B}_j} W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]}{\partial \kappa_y}\mathrm{d}y\mathrm{d}z =$$
$$= \sum_j \iint_{\mathbf{B}_j} \frac{\mathrm{d}W_j\,(\varepsilon_j)}{\mathrm{d}\varepsilon_j}\frac{\partial \varepsilon_j}{\partial \kappa_y}\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y,z)y\mathrm{d}y\mathrm{d}z = M_z \tag{15}$$

$$\frac{\partial \Pi_i}{\partial \kappa_z} = \sum_j \frac{\partial}{\partial \kappa_z} \iint_{\mathbf{B}_j} W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j\,[\varepsilon_j(y,z,\varepsilon_0,\kappa_y,\kappa_z)]}{\partial \kappa_z}\mathrm{d}y\mathrm{d}z =$$
$$= \sum_j \iint_{\mathbf{B}_j} \frac{\mathrm{d}W_j\,(\varepsilon_j)}{\mathrm{d}\varepsilon_j}\frac{\partial \varepsilon_j}{\partial \kappa_z}\mathrm{d}y\mathrm{d}z = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y,z)z\mathrm{d}y\mathrm{d}z = M_y \tag{16}$$

These three equations describe the *conditions of equilibrium*. They are used to check the results and to evaluate the numerical error of the solution.

If the partition j of the cross-section is pre-loaded or pre-stressed, the corresponding parameters of deformation $\varepsilon_{0j}^{(p)}$, $\kappa_{yj}^{(p)}$ and $\kappa_{zj}^{(p)}$ represent the pre-deformations $\varepsilon_{0j}^{(0)}$, $\kappa_{yj}^{(0)}$ and $\kappa_{zj}^{(0)}$ with respect to the following load cases

$$\varepsilon_{0j}^{(0)} = \varepsilon_{0j}^{(p)} \quad \kappa_{yj}^{(0)} = \kappa_{yj}^{(p)} \quad \kappa_{zj}^{(0)} = \kappa_{zj}^{(p)}. \tag{17}$$

If the partition j is subjected to temperature and the deformations $\varepsilon_{0j}^{(T)}$, $\kappa_{yj}^{(T)}$ and $\kappa_{zj}^{(T)}$ are prohibited by the bond between the partitions, the pre-deformations are defined by

$$\varepsilon_{0j}^{(0)} = -\varepsilon_{0j}^{(T)} \quad \kappa_{yj}^{(0)} = -\kappa_{yj}^{(T)} \quad \kappa_{zj}^{(0)} = -\kappa_{zj}^{(T)}. \tag{18}$$

The deformations due to shrinkage $\varepsilon_{0j}^{(sh)}$, $\kappa_{yj}^{(sh)}$ and $\kappa_{zj}^{(sh)}$ are taken into account by the pre-deformations

$$\varepsilon_{0j}^{(0)} = -\varepsilon_{0j}^{(sh)} \quad \kappa_{yj}^{(0)} = -\kappa_{yj}^{(sh)} \quad \kappa_{zj}^{(0)} = -\kappa_{zj}^{(sh)}. \tag{19}$$

Similarly, deformations induced by creep can be incorporated into the model for approximate calculations. For each separate time interval, the deformations due to creep are calculated by iteration.

Table 1: Strain energy and internal forces

Cross-section Type A

$$\Pi^{Q}_{i\,j,i} = \begin{cases} -\dfrac{1}{\kappa_j}\Delta\eta_i \,\dfrac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0,\ \Delta\varepsilon_i \neq 0) \\[2ex] -\dfrac{1}{\kappa_j}\Delta(\eta F)_i & (\kappa_j \neq 0,\ \Delta\varepsilon_i = 0) \\[2ex] \dfrac{W_{im}}{2}(y_i z_{i+1} - y_{i+1} z_i) & (\kappa_j = 0). \end{cases}$$

$$N_{j,i} = \begin{cases} -\dfrac{1}{\kappa_j}\Delta\eta_i \,\dfrac{\Delta F_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0,\ \Delta\varepsilon_i \neq 0) \\[2ex] -\dfrac{1}{\kappa_j}\Delta\eta_i \, W_{im} & (\kappa_j \neq 0,\ \Delta\varepsilon_i = 0) \\[2ex] \dfrac{\sigma_{im}}{2}(y_i z_{i+1} - y_{i+1} z_i) & (\kappa_j = 0) \end{cases}$$

$$M_{yj,i} = \begin{cases} -\dfrac{1}{\kappa_j}\Delta\eta_i \,\dfrac{\Delta(Fz)_i}{\Delta\varepsilon_i} + \dfrac{1}{\kappa_j^2} C_i \,\dfrac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0,\ \Delta\varepsilon_i \neq 0) \\[2ex] -\dfrac{1}{\kappa_j}\Delta\eta_i \,(Wz)_{im} + \dfrac{\kappa_{zj}}{\kappa_j^3}\Delta\eta_i \, F_{im} & (\kappa_j \neq 0,\ \Delta\varepsilon_i = 0) \\[2ex] \dfrac{\sigma_{im}}{6}(y_i z_{i+1} - y_{i+1} z_i)(z_{i+1} + z_i) & (\kappa_j = 0) \end{cases}$$

$$M_{zj,i} = \begin{cases} -\dfrac{1}{\kappa_j}\Delta\eta_i \,\dfrac{\Delta(Fy)_i}{\Delta\varepsilon_i} + \dfrac{1}{\kappa_j^2} D_i \,\dfrac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\[2ex] -\dfrac{1}{\kappa_j}\Delta\eta_i \,(Wy)_{im} + \dfrac{\kappa_{yj}}{\kappa_j^3}\Delta\eta_i \, F_{im} & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\[2ex] \dfrac{\sigma_{im}}{6}(y_i z_{i+1} - y_{i+1} z_i)(y_{i+1} + y_i) & (\kappa_j = 0) \end{cases}$$

Cross-section Type B

$$\Pi^{Q}_{ii} = \begin{cases} \dfrac{\Delta F_i}{\Delta\varepsilon_i} A_i & (\Delta\varepsilon_i \neq 0) \\[2ex] W_{im} A_i & (\Delta\varepsilon_i = 0). \end{cases}$$

$$N_i = \begin{cases} \dfrac{\Delta W_i}{\Delta\varepsilon_i} A_i & (\Delta\varepsilon_i \neq 0) \\[2ex] \sigma_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$$

$$M_{yi} = \begin{cases} \left(\dfrac{\Delta(Wz)_i}{\Delta\varepsilon_i} - \dfrac{\Delta F_i}{\Delta\varepsilon_i}\cdot\dfrac{\Delta z_i}{\Delta\varepsilon_i}\right) A_i & (\Delta\varepsilon_i \neq 0) \\[2ex] \sigma_{im} z_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$$

$$M_{zi} = \begin{cases} \left(\dfrac{\Delta(Wy)_i}{\Delta\varepsilon_i} - \dfrac{\Delta F_i}{\Delta\varepsilon_i}\cdot\dfrac{\Delta y_i}{\Delta\varepsilon_i}\right) A_i & (\Delta\varepsilon_i \neq 0) \\[2ex] \sigma_{im} y_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$$

Cross-section Type C

$$\Pi^{Q}_{i,i} = \iint_B W(y,z)\,dy\,dz = A_i W_i$$

$$N_i = \sigma_i A_i$$

$$M_{yi} = \sigma_i z_i A_i$$

$$M_{zi} = \sigma_i y_i A_i$$

$$C_i = \frac{\kappa_{zj}}{\kappa_j}\Delta\eta_i - \frac{\kappa_{yj}}{\kappa_j}\Delta\zeta_i + \frac{\kappa_j \Delta\eta_i}{\Delta\varepsilon_i}\Delta z_i$$

$$D_i = \frac{\kappa_{yj}}{\kappa_j}\Delta\eta_i + \frac{\kappa_{zj}}{\kappa_j}\Delta\zeta_i + \frac{\kappa_j \Delta\eta_i}{\Delta\varepsilon_i}\Delta y_i .$$

$$\eta_i = y_i \cos\varphi + z_i \sin\varphi = \frac{\kappa_{zj}}{\kappa_j} y_i - \frac{\kappa_{yj}}{\kappa_j} z_i$$

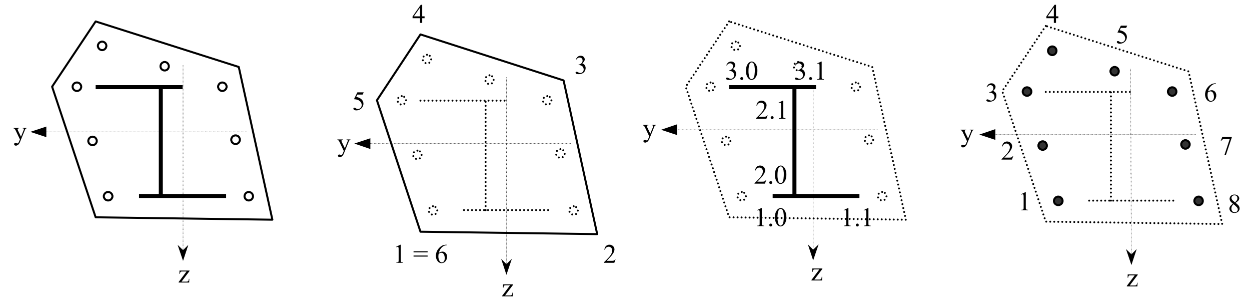
$$\zeta_i = -y_i \sin\varphi + z_i \cos\varphi = \frac{\kappa_{yj}}{\kappa_j} y_i + \frac{\kappa_{zj}}{\kappa_j} z_i$$

$$(\kappa_j \neq 0)$$

3 APPLICATION

The application of the energy method proposed in Chapter 2 will now be demonstrated by means of two examples. We assume that the geometry of the composite cross-section and its partitions correspond to the basic types shown in Fig. 1.

The strain energy Π_i and the internal forces N, M_y and M_z can be calculated by the formulae compiled in Tab. 1.



Composite cross-section   Type A   Type B   Type C

Fig. 1: Composite cross-section and basic types of cross-sections

The numerical solution to the nonlinear optimisation problem can be realised by a programme system for spread-sheet analysis.

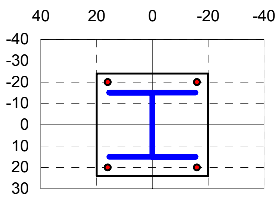
3.1 Example 1

The composite cross-section shown in Tab.2 is produced in two stages. One part of the cross-section consists of concrete and is reinforced by a steel element and four bars in the corners of the cross-section. This part is pre-loaded by the bending moment M_y = 22,000 kNcm.

Table 2: Parameters of geometry and material (Example 1)

Cross-Section (CS)			Geometry:					Material	Pre-deformation	Effective Cross-section				
j		i	b_{ji}	h_{ji}	y'_{jiS}	z'_{jiS}	A_{ji}	E_j	$\varepsilon_{ji0}^{(0)}$					
			[cm]	[cm]	[cm]	[cm]	[cm²]	[kN/cm²]		LC1	LC2	LC3	LC4	LC5
1	Concrete 1		*40,0*	*48,0*	*0,00*	*0,00*	1.920,0	*2.880*	*0,00*	*1*	*1*	*1*	*1*	*1*
2	Concrete 2		*40,0*	*10,0*	*0,00*	*-29,00*	400,0	*2.880*	*0,00*	*0*	*1*	*1*	*1*	*1*
3	Steel	1	*31,0*	*3,9*			120,9	*20.000*	*0,00*	*1*	*1*	*1*	*1*	*1*
		2	*2,1*	*26,2*	*0,00*	*0,00*	55,0							
		3	*31,0*	*3,9*			120,9							
			n_{ji}	D_{ji}	d_{1y}	d_{1z}	A_{ji}		$\varepsilon_{ji}^{(0)}$					
				[cm]	[cm]	[cm]	[cm²]							
4	Reinforcement 1		*4*	*2,0*	*4,00*	*4,00*	12,6	*20.000*	*0,00*	*1*	*1*	*1*	*1*	*1*
5	Reinforcement 2		*4*	*1,2*	*4,00*	*3,00*	4,5	*20.000*	*0,00*	*0*	*1*	*1*	*1*	*1*

Load Case	N [kN]	M_y [kNcm]	M_z [kNcm]
LC1	*0*	*22.000*	*0*
LC2	*0*	*50.000*	*20.000*
…			



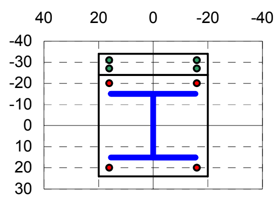


Table 3: Calculation scheme

Load case: **1**

Parameters of deformation						Potentials	
X1: ε_1	-3,92E-04	ε_0	4,82E-05	κ [cm^{-1}]	1,29E-05	Π_a [kN]	-2,85E-01
X2: ε_2	-3,92E-04	κ_y [cm^{-1}]	0,00E+00			Π_i [kN]	1,42E-01
X3: ε_3	3,59E-04	κ_z [cm^{-1}]	1,29E-05			Π [kN]	-1,42E-01

CS j		$\varepsilon_{0j}^{(0)}$ [-]	ε_{0j} [-]	$\kappa_{yj}^{(0)}$ [cm^{-1}]	κ_{yj} [cm^{-1}]	$\kappa_{zj}^{(0)}$ [cm^{-1}]	κ_{zj} [cm^{-1}]	κ_j [cm^{-1}]	Optimisation				N [kN]	M_y [kNcm]	M_z [kNcm]
1	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	0,00E+00	1,29E-05	1,29E-05	Objective function:	$OF = \Pi$:	-1,42E-01	prov	0	22.000	0
2	0	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	Variables:	$X_1 = \varepsilon_1$	-3,92E-04	cal	0	22.000	0
3	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	0,00E+00	1,29E-05	1,29E-05		$X_2 = \varepsilon_2$	-3,92E-04	Difference:	9,04E-04	9,10E-03	0,00E+00
4	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	0,00E+00	1,29E-05	1,29E-05		$X_3 = \varepsilon_3$	3,59E-04				
5	0	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	Restrictions:	no					

j		i	y_i [cm]	z_i [cm]	A_{ji} [cm²]	η_i [cm]	ζ_i [cm]	ε_i [-]	$\varepsilon_{ji}^{(0)}$ [-]	ε_{ji} [-]	σ_{ji} [kN/cm²]	W_{ji} [kN/cm²]	F_{ji} [kN/cm²]	Φ_{ji} [kN/cm²]	$\Pi_{i\,ji}$ [kN/cm²]	N_{ji} [kN]	M_{yji} [kNcm]	M_{zji} [kNcm]
1	Concrete 1	1	20,0	24,0		20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		2	-20,0	24,0		-20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	262
		3	-20,0	-24,0		-20,00	-24,00	-2,62E-04	0,00E+00	-2,62E-04	-7,56E-01	9,92E-05	-8,68E-09	5,69E-13	2,68E-02	-306	5.284	0
		4	20,0	-24,0		20,00	-24,00	-2,62E-04	0,00E+00	-2,62E-04	-7,56E-01	9,92E-05	-8,68E-09	5,69E-13	0,00E+00	0	0	-262
		5=1	20,0	24,0		20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00				
2	Concrete 2	1	20,0	-24,0		20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
	nicht vorh	2	-20,0	-24,0		-20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		3	-20,0	-34,0		-20,00	-34,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		4	20,0	-34,0		20,00	-34,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5=1	20,0	-24,0		20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00				
3	Steel	1.0	15,5	15,1	120,90	15,50	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08		7,14E-02	588	8.845	0
		1.1	-15,5	15,1		-15,50	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08					
		2.0	0,0	15,1	63,21	0,00	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08		9,47E-03	61	1.236	0
		2.1	0,0	-15,1		0,00	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08					
		3.0	15,5	-15,1	120,90	15,50	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08		2,60E-02	-354	5.334	0
		3.1	-15,5	-15,1		-15,50	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08					
4	Reinforcement 1	1	16,0	20,0	3,14	16,00	20,00	3,07E-04	0,00E+00	3,07E-04	6,14E+00	9,43E-04			2,96E-03	19	386	309
		2	-16,0	20,0	3,14	-16,00	20,00	3,07E-04	0,00E+00	3,07E-04	6,14E+00	9,43E-04			2,96E-03	19	386	-309
		3	-16,0	-20,0	3,14	-16,00	-20,00	-2,11E-04	0,00E+00	-2,11E-04	-4,21E+00	4,44E-04			1,39E-03	-13	265	212
		4	16,0	-20,0	3,14	16,00	-20,00	-2,11E-04	0,00E+00	-2,11E-04	-4,21E+00	4,44E-04			1,39E-03	-13	265	-212
5	Reinforcement 2	1	16,0	-27,0	1,13	16,00	-27,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00			0,00E+00	0	0	0
	nicht vorh	2	-16,0	-27,0	1,13	-16,00	-27,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00			0,00E+00	0	0	0
		3	-16,0	-31,0	1,13	-16,00	-31,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00			0,00E+00	0	0	0
		4	16,0	-31,0	1,13	16,00	-31,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00			0,00E+00	0	0	0

Load case: **2**

Parameters of deformation						Potentials	
X1: ε_1	-7,44E-04	ε_0	9,91E-05	κ [cm^{-1}]	2,54E-05	Π_a [kN]	########
X2: ε_2	1,64E-04	κ_y [cm^{-1}]	2,27E-05			Π_i [kN]	7,83E-01
X3: ε_3	-8,07E-05	κ_z [cm^{-1}]	1,14E-05			Π [kN]	-2,42E-01

CS j		$\varepsilon_{0j}^{(0)}$ [-]	ε_{0j} [-]	$\kappa_{yj}^{(0)}$ [cm^{-1}]	κ_{yj} [cm^{-1}]	$\kappa_{zj}^{(0)}$ [cm^{-1}]	κ_{zj} [cm^{-1}]	κ_j [cm^{-1}]	Optimisation				N [kN]	M_y [kNcm]	M_z [kNcm]
aus EIN		aus LF1		aus LF1		aus LF1									
1	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05	Objective function:	$OF = \Pi$:	-2,42E-01	prov	0	50.000	20.000
2	1	0,00E+00	9,91E-05	0,00E+00	2,27E-05	0,00E+00	1,14E-05	2,54E-05	Variables:	$X_1 = \varepsilon_1$	-7,44E-04	cal	0	50.000	20.000
3	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05		$X_2 = \varepsilon_2$	1,64E-04	Difference:	8,22E-04	1,03E-02	9,47E-03
4	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05		$X_3 = \varepsilon_3$	-8,07E-05				
5	1	0,00E+00	9,91E-05	0,00E+00	2,27E-05	0,00E+00	1,14E-05	2,54E-05	Restrictions:	no					

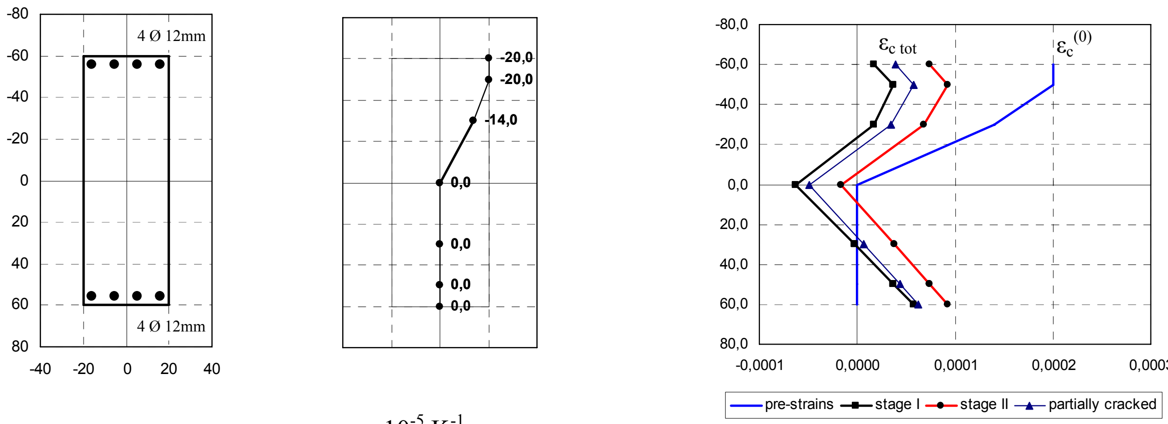
j		i	y_i [cm]	z_i [cm]	A_{ji} [cm²]	η_i [cm]	ζ_i [cm]	ε_i [-]	$\varepsilon_{ji}^{(0)}$ [-]	ε_{ji} [-]	σ_{ji} [kN/cm²]	W_{ji} [kN/cm²]	F_{ji} [kN/cm²]	Φ_{ji} [kN/cm²]	$\Pi_{i\,ji}$ [kN/cm²]	N_{ji} [kN]	M_{yji} [kNcm]	M_{zji} [kNcm]
1	Concrete 1	1	20,0	24,0		-1,72	31,19	8,28E-04	3,59E-04	1,19E-03	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		2	-20,0	24,0		-30,99	3,93	-8,07E-05	3,59E-04	2,78E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	6,37E-02	-286	1.446	5.914
		3	-20,0	-24,0		1,72	-31,19	-6,29E-04	-2,62E-04	-8,92E-04	-2,57E+00	1,15E-03	-3,40E-07	7,59E-11	7,34E-02	-329	7.691	348
		4	20,0	-24,0		30,99	-3,93	2,79E-04	-2,62E-04	1,64E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5	20,0	24,0		-1,72	31,19	8,28E-04	3,59E-04	1,19E-03	0,00E+00	0,00E+00	0,00E+00	0,00E+00				
2	Concrete 2	1	20,0	-24,0		30,43	7,07	2,79E-04	0,00E+00	2,79E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	-1,47E-02	93	-3.002	-188
		2	-20,0	-24,0		12,44	-28,66	-6,29E-04	0,00E+00	-6,29E-04	-1,81E+00	5,70E-04	-1,20E-07	1,88E-11	5,49E-02	-239	5.053	3.339
		3	-20,0	-34,0		21,37	-33,15	-7,44E-04	0,00E+00	-7,44E-04	-2,14E+00	7,96E-04	-1,97E-07	3,67E-11	2,86E-02	-154	6.720	-193
		4	20,0	-34,0		39,36	2,57	1,64E-04	0,00E+00	1,64E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5	20,0	-24,0		30,43	7,07	2,79E-04	0,00E+00	2,79E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00				
3	Steel	1.0	15,5	15,1	120,90	1,08	21,58	6,23E-04	2,43E-04	8,66E-04	1,73E+01	7,50E-03	2,17E-06		3,70E-01	1.243	18.712	4.396
		1.1	-15,5	15,1		-21,60	0,45	-8,08E-05	2,43E-04	1,62E-04	3,25E+00	2,63E-04	1,42E-08					
		2.0	0,0	15,1	63,21	-10,26	11,01	2,71E-04	2,43E-04	5,14E-04	1,03E+01	2,64E-03	4,53E-07		4,21E-02	186	2.327	0
		2.1	0,0	-15,1		10,26	-11,01	-7,30E-05	-1,47E-04	-2,20E-04	-4,39E+00	4,82E-04	-3,53E-08					
		3.0	15,5	-15,1	120,90	21,60	-0,45	2,79E-04	-1,47E-04	1,32E-04	2,65E+00	1,75E-04	7,73E-09		1,08E-01	-531	7.989	4.396
		3.1	-15,5	-15,1		-1,08	-21,58	-4,25E-04	-1,47E-04	-5,71E-04	-1,14E+01	3,27E-03	-6,22E-07					
4	Reinforcement 1	1	16,0	20,0	3,14	-1,92	25,54	6,91E-04	3,07E-04	9,98E-04	2,00E+01	9,96E-03			3,13E-02	63	1.254	1.003
		2	-16,0	20,0	3,14	-25,34	3,73	-3,56E-05	3,07E-04	2,72E-04	5,43E+00	7,38E-04			2,32E-03	17	341	-273
		3	-16,0	-20,0	3,14	1,92	-25,54	-4,93E-04	-2,11E-04	-7,03E-04	-1,41E+01	4,95E-03			1,55E-02	-44	884	707
		4	16,0	-20,0	3,14	25,34	-3,73	2,34E-04	-2,11E-04	2,31E-05	4,61E-01	5,32E-06			1,67E-05	1	-29	23
5	Reinforcement 2	1	16,0	-27,0	1,13	31,31	2,15	1,54E-04	0,00E+00	1,54E-04	3,07E+00	2,36E-04			2,67E-04	3	-94	56
		2	-16,0	-27,0	1,13	16,92	-26,43	-5,73E-04	0,00E+00	-5,73E-04	-1,15E+01	3,28E-03			3,71E-03	-13	350	207
		3	-16,0	-31,0	1,13	20,49	-28,23	-6,19E-04	0,00E+00	-6,19E-04	-1,24E+01	3,83E-03			4,33E-03	-14	434	224
		4	16,0	-31,0	1,13	34,88	0,35	1,08E-04	0,00E+00	1,08E-04	2,16E+00	1,17E-04			1,32E-04	2	-76	39

The second part of the composite cross-section consists of concrete and four reinforcement bars and is added later. The whole cross-section is loaded by biaxial bending (M_y = 50,000kNcm and M_z = 20,000kNcm). The concrete and the reinforcement behave in a linear elastic manner, whereas the concrete of the tension zone is cracked and cannot transmit tensile stresses. Parameters of geometry and material are given in Tab.2

The calculation scheme is shown in Tab.3. The variables of the unconstrained optimisation problem are the strains ε_1, ε_2 and ε_3, which in turn define the parameters ε_0, κ_y und κ_z. The geometry of all partitions j of the cross-section is described by the co-ordinates y_i, z_i and the corresponding areas A_{ji}. The co-ordinates η_i, ζ_i are calculated by the formulae of Tab.1. The strains ε_i and $\varepsilon_{j\,i}^{(0)}$ are obtained from eq.(6) and (7), and the strains ε_{ji} from eq.(5). The stresses σ_{ji}, the values W_{ji}, F_{ji}, Φ_{ji} and the strain energy $\Pi_{i\,ji}$ are determined by Tab.1. By minimising the total potential energy Π, we find the solution to the problem. The differences N_{prov} - N_{calc} etc. indicate the numerical error in satisfying the equations of equilibrium.

3.2 Example 2

The cross-section of a statically-determined reinforced concrete beam shown in Fig.2 is subjected to temperature. The distribution of the temperature across the height of the cross-section is described by a piecewise linear function. Both concrete and reinforcement behaves in a linear-elastic manner.



$E_c = 28.800 N/mm^2$
$E_s = 200.000 N/mm^2$

$\alpha_{cT} = 10^{-5}\ K^{-1}$
$\alpha_{sT} = 10^{-5}\ K^{-1}$

a. Cross-section b. Temperature [K] c. Strains

Fig. 2: Example 2

Three cases are investigated with respect to the tensile zone:

a. where the concrete is uncracked

b. where the concrete transmits no tensile stresses (see Tab.5)

c. where the stresses in the effective zone around the reinforcement bars are limited to $\beta_{ct} f_{ct}$ ($\beta_{ct} = 0.4$, $f_{ct} = 2.8 N/mm^2$).

Table 5: Calculation schema (Example 2)

Load case:	**T**			**Stage II**						
Parameters of Deformation:						Potentials:		Internal forces		
X1: ε_1	-1,26E-04	ε_0	-1,70E-05			Π_a [kN]	0		N [kN]	M_y [kNcm]
X2: ε_3	9,19E-05	$\kappa=\kappa_z$ [cm^{-1}]	1,82E-06			Π_i [kN]	7,09E-04	prov	0	0
				s_1	1,00E-08	Π [kN]	7,09E-04	cal	0	0
								Difference:	3,08E-05	5,46E-04

j		i	z_i [cm]	A_j [cm²]	ε_i [-]	$\varepsilon_{ji}^{(0)}$ [-]	ε_{ji} [-]	σ_{ji} [kN/cm²]	W_{ji} [kN/cm²]	F_{ji} [kN/cm²]	$\Pi_{i\ ji}$ [kN/cm²]	N_{ji} [kN]	M_{yji} [kNcm]
1	Concrete	1.0 1)	-60,0	400,0	-1,26E-04	2,00E-04	7,41E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		1.11)	-50,0		-1,08E-04	2,00E-04	9,22E-05	0,00E+00	0,00E+00	0,00E+00			
		2.0	-50,0	800,0	-1,08E-04	2,00E-04	9,22E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		2.1	-30,0		-7,15E-05	1,40E-04	6,85E-05	0,00E+00	0,00E+00	0,00E+00			
		3.0	-30,0	1200,0	-7,15E-05	1,40E-04	6,85E-05	0,00E+00	0,00E+00	0,00E+00	3,31E-05	-5,84E+00	1,16E+01
		3.1	0,0		-1,70E-05	0,00E+00	-1,70E-05	-4,90E-02	4,16E-07	-2,36E-12			
		4.0	0,0	1200,0	-1,70E-05	0,00E+00	-1,70E-05	-4,90E-02	4,16E-07	-2,36E-12	5,20E-05	-9,17E+00	-2,86E+01
		4.1	30,0		3,75E-05	0,00E+00	3,75E-05	0,00E+00	0,00E+00	0,00E+00			
		5.0	30,0	800,0	3,75E-05	0,00E+00	3,75E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		5.1	50,0		7,38E-05	0,00E+00	7,38E-05	0,00E+00	0,00E+00	0,00E+00			
		6.0 2)	50,0	400,0	7,38E-05	0,00E+00	7,38E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		6.1 2)	60,0		9,19E-05	0,00E+00	9,19E-05	0,00E+00	0,00E+00	0,00E+00			
2	Reinforcement 1		-56,0	4,5	-1,19E-04	2,00E-04	8,13E-05	1,63E+00	6,61E-05		2,99E-04	7,36E+00	-4,12E+02
3	Reinforcement 2		56,0	4,5	8,47E-05	0,00E+00	8,47E-05	1,69E+00	7,17E-05		3,24E-04	7,66E+00	4,29E+02

The corresponding strains ε_c are shown in Fig. 2c. The behaviour of concrete has a significant influence on the distribution of strains and stresses.

4 CONCLUSIONS

The analysis of strains and stresses can be realised very effectively by the energy method, using an integral formulation of material law. This method is applicable without principal modification for cross-sections with pre-deformations. Strains due to pre-loading, temperature shrinkage etc. are integrated into the model by the decomposition of the total strains of each partition of the composite cross-section.

REFERENCES

[1] Raue, E. Non-linear analysis of cross-sections by mathematical optimisation. (Nichtlineare Querschnittsanalyse als Optimierungsproblem.) Bautechnik 82, Heft 11, 2005. pp. 796-809 (German)

[2] Raue, E. Nichtlineare Analyse von Verbundquerschnitten – ein neuer alternativer Weg. Proceedings 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering. Bauhaus-University Weimar 2006.

[3] Raue, E. Non-linear Analysis of Composite Cross-Sections by non-linear Optimisation. Proceedings 9th International Conference Modern Building Materials, Structures and Techniques. Vilnius 2007.

[4] Raue, E. Timmler, H.-G. Alternative analytical modelling of retrofitted reinforced concrete members. University Cape Town: International Conference on Concrete Repair, Rehabilitation and Retrofitting. 2005.