18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 07–09 July 2009

THE FOURIER-BESSEL TRANSFORM

F. Brackx, N. De Schepper* and F. Sommen

Clifford Research Group, Department of Mathematical Analysis, Ghent University
Galglaan 2, B-9000 Ghent, Belgium
E-mail: nds@cage.ugent.be

Keywords: multi-dimensional Fourier transform, Clifford analysis.

Abstract. *In this paper we devise a new multi-dimensional integral transform within the Clifford analysis setting, the so-called Fourier-Bessel transform. It appears that in the two-dimensional case, it coincides with the Clifford-Fourier and cylindrical Fourier transforms introduced earlier. We show that this new integral transform satisfies operational formulae which are similar to those of the classical tensorial Fourier transform. Moreover the* L_2*-basis elements consisting of generalized Clifford-Hermite functions appear to be eigenfunctions of the Fourier-Bessel transform.*

1 INTRODUCTION

The *Fourier transform* is by far the most important integral transform. In m-dimensional Euclidean space $\mathbb{R}^m$ it is given by

$$\mathcal{F}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{\xi}, \underline{x} >\right) f(\underline{x})\, dV(\underline{x}) \ ,$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$, $\underline{\xi}$ for $(\xi_1, \ldots, \xi_m)$, $\underline{x}$ for $(x_1, \ldots, x_m)$ and $< \underline{\xi}, \underline{x} >$ for the traditional scalar product in Euclidean space: $< \underline{\xi}, \underline{x} >= \sum_{j=1}^{m} \xi_j x_j$. Since its introduction by Fourier in the early 1800s, it has remained an indispensable and stimulating mathematical concept that is at the core of the highly evolved branch of mathematics called *Fourier analysis*. It has found use in innumerable applications and has become a fundamental tool in engineering sciences, thanks to the generalizations extending the class of Fourier transformable functions and to the development of efficient algorithms for computing the discrete version of it.

The second player in this paper is *Clifford analysis*. It is a function theory for functions defined in Euclidean space $\mathbb{R}^m$ and taking values in the real Clifford algebra $\mathbb{R}_{0,m}$, constructed over $\mathbb{R}^m$. A Clifford algebra is an associative but non-commutative algebra with zero divisors, which combines the algebraic properties of the reals, the complex numbers and the quaternions with the geometric properties of a Grassmann algebra.
During the past 50 years, Clifford analysis has gradually developed into a comprehensive theory offering a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m-dimensional Euclidean space, Clifford analysis focuses on monogenic functions, i.e. null solutions of the Clifford vector-valued Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$, where $(e_1, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^{0,m}$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. The reason is that, as does the Cauchy-Riemann operator in the complex plane, the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator. At the same time, Clifford analysis offers the possibility of generalizing one-dimensional mathematical analysis to higher dimension in a rather natural way by encompassing all dimensions at once, in contrast to the traditional approach, where tensor products of one-dimensional phenomena are taken.

It is precisely this last qualification of Clifford analysis which has been exploited in [2] and [3] to construct a genuine multi-dimensional Fourier transform within the context of Clifford analysis. This so-called *Clifford-Fourier transform* is given in terms of an operator exponential or, alternatively, by a series representation.
Particular attention is directed to the two-dimensional case, since then the Clifford-Fourier kernel can be written in a closed form. Indeed, the two-dimensional Clifford-Fourier transform may be expressed as

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f](\underline{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(\pm(\underline{\xi} \wedge \underline{x})\right) f(\underline{x})\, dV(\underline{x})$$

with $\exp(\underline{\xi} \wedge \underline{x}) = \sum_{r=0}^{\infty} \frac{(\underline{\xi} \wedge \underline{x})^r}{r!}$ and $\underline{\xi} \wedge \underline{x}$ the so-called outer or wedge product (see Section 2) of the Clifford vector variables $\underline{\xi} = \xi_1 e_1 + \xi_2 e_2$ and $\underline{x} = x_1 e_1 + x_2 e_2$. Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension.

In recent research (see [4], [5] and [6]) we devised a so-called *cylindrical Fourier transform* within the Clifford analysis setting, by taking as a new integral kernel the multi-dimensional generalization of the two-dimensional Clifford-Fourier kernel:

$$\mathcal{F}_{cyl}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(\underline{x} \wedge \underline{\xi})\, f(\underline{x})\, dV(\underline{x}) \ .$$

For a fixed vector $\underline{\xi}$ in the image space, its phase is constant on co-axial cylinders w.r.t. that fixed vector, explaining the name "cylindrical" for this Fourier transform.
Although the cylindrical Fourier transform has a "simple" integral kernel, it satisfies calculation formulae which are substantially more complicated than those of the Clifford-Fourier transform. A similar conclusion holds for the spectrum of an L_2-basis consisting of generalized Clifford-Hermite functions: in case of the Clifford-Fourier transform these basis functions are simply eigenfunctions, while their cylindrical Fourier spectrum is expressed as a sum of generalized hypergeometric series.

In this paper we introduce another promising multi-dimensional integral transform within the language of Clifford analysis, the so-called *Fourier-Bessel transform* (see Section 3). It appears that in the two-dimensional case, it coincides with the above mentioned Clifford-Fourier and cylindrical Fourier transforms. In Section 4 we will also show that the Fourier-Bessel transform satisfies operational formulae which are similar to those of the classical multi-dimensional Fourier transform. Moreover, the L_2-basis elements consisting of generalized Clifford-Hermite functions appear to be eigenfunctions of the Fourier-Bessel transform (see Section 5). To make the paper self-contained a section on definitions and basic properties of Clifford algebra and Clifford analysis is included (Section 2).
In a forthcoming paper [7] we will in the even dimensional case express the Clifford-Fourier transform in terms of the Fourier-Bessel transform, which leads to a closed form of the Clifford-Fourier integral kernel.

2 THE CLIFFORD ANALYSIS TOOLKIT

Clifford analysis (see e.g. [1, 8, 9, 11]) offers a function theory which is a higher dimensional analogue of the theory of the holomorphic functions of one complex variable.

The functions considered are defined in $\mathbb{R}^m$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_{0,m}$ or its complexification $\mathbb{C}_m = \mathbb{R}_{0,m} \otimes \mathbb{C}$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$ is given by all possible products of basis vectors $(e_A : A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra is governed by the rules: $e_j e_k + e_k e_j = -2\delta_{j,k}$ $(j, k = 1, \ldots, m)$.

Conjugation is defined as the anti-involution for which $\overline{e_j} = -e_j$ $(j = 1, \ldots, m)$. In case of $\mathbb{C}_m$, the Hermitean conjugate of an element $\lambda = \sum_A \lambda_A e_A$ $(\lambda_A \in \mathbb{C})$ is defined by $\lambda^\dagger =$

$\sum_A \lambda_A^c \, \overline{e_A}$, where λ_A^c denotes the complex conjugate of λ_A. This Hermitean conjugation leads to a Hermitean inner product and its associated norm on $\mathbb{C}_m$ given respectively by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0 \qquad \text{and} \qquad |\lambda|^2 = [\lambda^\dagger \lambda]_0 = \sum_A |\lambda_A|^2 \ ,$$

where $[\lambda]_0$ denotes the scalar part of the Clifford element λ.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by $\underline{x} = \sum_{j=1}^m e_j x_j$. The product of two vectors splits up into a scalar part (the inner product up to a minus sign) and a so-called bivector part (the wedge product):

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} \ + \underline{x} \wedge \underline{y} \ ,$$

where

$$\underline{x} \cdot \underline{y} = - < \underline{x}, \underline{y} > = - \sum_{j=1}^m x_j y_j \qquad \text{and} \qquad \underline{x} \wedge \underline{y} = \sum_{i=1}^m \sum_{j=i+1}^m e_i e_j (x_i y_j - x_j y_i) \ .$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign: $\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2$.
Moreover, one can verify (see [4]) that for all $\underline{x}, \underline{t} \in \mathbb{R}^m$ the following formula holds:

$$(\underline{x} \wedge \underline{t})^2 = -|\underline{x} \wedge \underline{t}|^2 = (< \underline{x}, \underline{t} >)^2 - |\underline{x}|^2 |\underline{t}|^2 \ . \tag{1}$$

The spin group $\mathrm{Spin}_{\mathbb{R}}(m)$ of the Clifford algebra consists of all products of an even number of unit vectors

$$\mathrm{Spin}_{\mathbb{R}}(m) = \left\{ s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \ ; \ \underline{\omega}_j \in S^{m-1} \ , \ j = 1, \ldots, 2\ell, \ \ell \in \mathbb{N} \right\} \ ,$$

with S^{m-1} the unit sphere in $\mathbb{R}^m$. The spin group doubly covers the rotation group $SO_{\mathbb{R}}(m)$: for $T \in SO_{\mathbb{R}}(m)$, there exists $s \in \mathrm{Spin}_{\mathbb{R}}(m)$ such that $T(\underline{x}) = s\underline{x}\overline{s}$. But then also $T(\underline{x}) = (-s)\underline{x}(-\overline{s})$, explaining the double character of this covering. Note that each spin-element s satisfies: $s\overline{s} = \overline{s}s = 1$.

The central notion in Clifford analysis is the notion of monogenicity, a notion which is the multi-dimensional counterpart to that of holomorphy in the complex plane. A function $F(x_1, \ldots, x_m)$ defined and continuously differentiable in an open region of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$, is called left monogenic in that region if $\partial_{\underline{x}}[F] = 0$. Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$: $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$, an elliptic, rotation-invariant, vector differential operator of the first order, which may be looked upon as the "square root" of the Laplace operator in $\mathbb{R}^m$: $\Delta_{\underline{x}} = -\partial_{\underline{x}}^2$. This factorization of the Laplace operator establishes a special relationship between Clifford analysis and harmonic analysis in that monogenic functions refine the properties of harmonic functions. The notion of right monogenicity is defined in a similar manner by letting act the Dirac operator from the right.

In the sequel the monogenic homogeneous polynomials will play an important role. A left monogenic homogeneous polynomial P_k of degree k ($k \geq 0$) in $\mathbb{R}^m$ is called a left solid inner spherical monogenic of order k. The set of all left solid inner spherical monogenics of order k will be denoted by $M_\ell^+(k)$. The dimension of $M_\ell^+(k)$ is given by

$$\dim\bigl(M_\ell^+(k)\bigr) = \binom{m+k-2}{m-2} = \frac{(m+k-2)!}{(m-2)!\; k!} \ .$$

The set

$$\phi_{s,k,j}(\underline{x}) = \frac{2^{m/4}}{(\gamma_{s,k})^{1/2}}\; H_{s,k}(\sqrt{2}\underline{x})\; P_k^{(j)}(\sqrt{2}\underline{x})\; \exp\left(-\frac{|\underline{x}|^2}{2}\right) \tag{2}$$

$s, k \in \mathbb{N},\ j \leq \dim\bigl(M_\ell^+(k)\bigr)$, constitutes an orthonormal basis for the space $L_2(\mathbb{R}^m)$ of square integrable functions. Here $\bigl\{P_k^{(j)}(\underline{x});\ j \leq \dim\bigl(M_\ell^+(k)\bigr)\bigr\}$ denotes an orthonormal basis of $M_\ell^+(k)$ and $\gamma_{s,k}$ a real constant depending on the parity of s. The polynomials $H_{s,k}(\underline{x})$ are the so-called generalized Clifford-Hermite polynomials introduced by Sommen in [14]; they are a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. Note that $H_{s,k}(\underline{x})$ is a polynomial of degree s in the variable $\underline{x}$ with real coefficients depending on k. More precisely, $H_{2s,k}(\underline{x})$ only contains even powers of $\underline{x}$ and hence is scalar-valued, while $H_{2s+1,k}(\underline{x})$ only contains odd ones and thus is vector-valued. Furthermore, these polynomials can be expressed in terms of the generalized Laguerre polynomials L_ℓ^α on the real line:

$$H_{2p,k}(\underline{x}) = 2^p\, p!\; L_p^{m/2+k-1}\left(\frac{|\underline{x}|^2}{2}\right) \qquad \text{and} \qquad H_{2p+1,k}(\underline{x}) = 2^p\, p!\; L_p^{m/2+k}\left(\frac{|\underline{x}|^2}{2}\right)\, \underline{x} \ ,$$

confirming that $H_{2p,k}$ is scalar-valued, while $H_{2p+1,k}$ is vector-valued.

A result which will be frequently used in Section 5 is the following generalization of the classical Funk-Hecke theorem (see [12]).

Theorem 1 *[Funk-Hecke theorem in space] Let S_k be a spherical harmonic of degree k and $\underline{\eta}$ a fixed point on the unit sphere S^{m-1} in $\mathbb{R}^m$. Denoting $< \underline{\omega}, \underline{\eta} > = \cos(\widehat{\underline{\omega},\underline{\eta}}) = t_{\underline{\eta}}$ for $\underline{\omega} \in S^{m-1}$, $P_{k,m}(t)$ the Legendre polynomial of degree k in m-dimensional Euclidean space and $A_{m-1} = \frac{2\,\pi^{(m-1)/2}}{\Gamma\left(\frac{m-1}{2}\right)}$ the surface area of the unit sphere S^{m-2} in $\mathbb{R}^{m-1}$, one has*

$$\begin{aligned}\int_{\mathbb{R}^m} & g(r)\; f(t_{\underline{\eta}})\; S_k(\underline{\omega})\; dV(\underline{x}) \\ &= A_{m-1}\left(\int_0^{+\infty} g(r)\, r^{m-1}\, dr\right)\left(\int_{-1}^{1} f(t)\,(1-t^2)^{(m-3)/2}\, P_{k,m}(t)\, dt\right) S_k(\underline{\eta}) \ .\end{aligned}$$

As the Legendre polynomials $P_{k,m}(t)$ are even or odd according to the parity of k, we can also state the following corollary.

Corollary 1 *Let S_k be a spherical harmonic of degree k and $\underline{\eta}$ a fixed point on the unit sphere S^{m-1}. Denoting $< \underline{\omega}, \underline{\eta} > = t_{\underline{\eta}}$ for $\underline{\omega} \in S^{m-1}$, the 3D-integral*

$$\int_{\mathbb{R}^m} g(r)\; f(t_{\underline{\eta}})\; S_k(\underline{\omega})\; dV(\underline{x})$$

is zero whenever either f is an odd function and k is even, or f is an even function and k is odd.

3 DEFINITION OF THE FOURIER-BESSEL TRANSFORM

Starting point is the axial exponential function, also called Clifford-Bessel function, given by (see [14]):

$$\mathcal{E}(x_0, \underline{x}) = \exp(x_0)\, \mathcal{E}(\underline{x})$$

with

$$\mathcal{E}(\underline{x}) = 2^{m/2-1}\, \Gamma\left(\frac{m}{2}\right)\, |\underline{x}|^{1-m/2}\left(J_{m/2-1}(|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\, J_{m/2}(|\underline{x}|)\right) ,$$

where J_ν denotes the Bessel function of the first kind:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{\ell=0}^{\infty} \frac{\left(\frac{iz}{2}\right)^{2\ell}}{\ell!\, \Gamma(\nu+\ell+1)} \tag{3}$$

and x_0 stands for an extra real variable.
It is left monogenic in $\mathbb{R}^{m+1}$, i.e. $(\partial_{x_0} + \partial_{\underline{x}})[\mathcal{E}(x_0, \underline{x})] = 0$.
Now replacing formally x_0 by $< \underline{x}, \underline{\xi} >$, $\underline{x}$ by $\underline{x} \wedge \underline{\xi}$ and m by $m-1$ with $\underline{x} = \sum_{j=1}^m x_j e_j$ and $\underline{\xi} = \sum_{j=1}^m \xi_j e_j$, we get

$$\mathcal{E}(< \underline{x}, \underline{\xi} >, \underline{x} \wedge \underline{\xi}) = \exp(< \underline{x}, \underline{\xi} >)\, 2^{(m-3)/2}\, \Gamma\left(\frac{m-1}{2}\right)\, |\underline{x} \wedge \underline{\xi}|^{(3-m)/2}$$
$$\left(J_{(m-3)/2}(|\underline{x} \wedge \underline{\xi}|) + \frac{\underline{x} \wedge \underline{\xi}}{|\underline{x} \wedge \underline{\xi}|}\, J_{(m-1)/2}(|\underline{x} \wedge \underline{\xi}|)\right) .$$

This so-called "Bessel-exponential" function is left monogenic in $\underline{x}$ and right monogenic in $\underline{\xi}$:

$$\partial_{\underline{x}}[\mathcal{E}(< \underline{x}, \underline{\xi} >, \underline{x} \wedge \underline{\xi})] = [\mathcal{E}(< \underline{x}, \underline{\xi} >, \underline{x} \wedge \underline{\xi})]\partial_{\underline{\xi}} = 0 \ . \tag{4}$$

The left monogenicity in $\underline{x}$ for example, is proved as follows. First, we have that

$$\partial_{\underline{x}}[\underline{x} \wedge \underline{\xi}] = \partial_{\underline{x}}[\underline{x}\underline{\xi} + < \underline{x}, \underline{\xi} >] = -m\underline{\xi} + \underline{\xi} = (1-m)\underline{\xi} \ . \tag{5}$$

Furthermore, in view of (1) we also find that

$$\begin{aligned} \partial_{\underline{x}}[|\underline{x} \wedge \underline{\xi}|^2] &= \partial_{\underline{x}}[|\underline{x}|^2|\underline{\xi}|^2 - (< \underline{x}, \underline{\xi} >)^2] \ = \ 2\underline{x}\, |\underline{\xi}|^2 - 2 < \underline{x}, \underline{\xi} > \underline{\xi} \\ &= -2\underline{\xi}\, (\underline{\xi}\underline{x} + < \underline{x}, \underline{\xi} >) \ = \ -2\underline{\xi}\, (\underline{\xi} \wedge \underline{x}) \ = \ 2\underline{\xi}\, (\underline{x} \wedge \underline{\xi}) \ . \end{aligned}$$

Combining the above result with

$$\partial_{\underline{x}}[|\underline{x} \wedge \underline{\xi}|^2] = 2|\underline{x} \wedge \underline{\xi}|\, \partial_{\underline{x}}[|\underline{x} \wedge \underline{\xi}|] \ ,$$

we obtain

$$\partial_{\underline{x}}[|\underline{x} \wedge \underline{\xi}|] = \frac{\underline{\xi}(\underline{x} \wedge \underline{\xi})}{|\underline{x} \wedge \underline{\xi}|} \ . \tag{6}$$

Taking into account (5) and (6), we arrive at

$$\partial_{\underline{x}}\Bigg[\exp\left(<\underline{x},\underline{\xi}>\right)\Bigg(|\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\,J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)+(\underline{x}\wedge\underline{\xi})\;|\underline{x}\wedge\underline{\xi}|^{(1-m)/2}$$
$$J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\Bigg)\Bigg]=\exp\left(<\underline{x},\underline{\xi}>\right)\underline{\xi}\,\Bigg(|\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\,J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)+(\underline{x}\wedge\underline{\xi})$$
$$|\underline{x}\wedge\underline{\xi}|^{(1-m)/2}\,J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)+\left(\frac{3-m}{2}\right)\,|\underline{x}\wedge\underline{\xi}|^{(-1-m)/2}\,(\underline{x}\wedge\underline{\xi})\,J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)$$
$$+\,|\underline{x}\wedge\underline{\xi}|^{(1-m)/2}\,J'_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)\,(\underline{x}\wedge\underline{\xi})-|\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\,J'_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)$$
$$+\,\frac{(1-m)}{2}\,|\underline{x}\wedge\underline{\xi}|^{(1-m)/2}\,J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\Bigg)\;.$$

Using the recurrence relations (see for e.g. [13])

$$J'_\nu(z)=-J_{\nu+1}(z)+\frac{\nu}{z}J_\nu(z)\qquad\text{and}\qquad zJ'_\nu(z)=zJ_{\nu-1}(z)-\nu J_\nu(z)\;,$$

indeed yields:

$$\partial_{\underline{x}}\Bigg[\exp\left(<\underline{x},\underline{\xi}>\right)\Bigg(|\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\,J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)+(\underline{x}\wedge\underline{\xi})$$
$$|\underline{x}\wedge\underline{\xi}|^{(1-m)/2}\,J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\Bigg)\Bigg]=0\;.$$

In a similar way, we can also show that

$$[\mathcal{E}(<\underline{x},\underline{\xi}>,\underline{x}\wedge\underline{\xi})]\partial_{\underline{x}}=2\,\mathcal{E}(<\underline{x},\underline{\xi}>,\underline{x}\wedge\underline{\xi})\,\underline{\xi}$$

and

$$\partial_{\underline{\xi}}[\mathcal{E}(<\underline{x},\underline{\xi}>,\underline{x}\wedge\underline{\xi})]=2\underline{x}\,\mathcal{E}(<\underline{x},\underline{\xi}>,\underline{x}\wedge\underline{\xi})\;. \tag{7}$$

It has been used recently by Sommen to introduce Clifford generalizations of the classical Fourier-Borel transform (see [15]).

The Fourier-Bessel kernel is now defined by leaving out the exponential factor $\exp\left(<\underline{x},\underline{\xi}>\right)$ from the Bessel-exponential function.

Definition 1 *The **Fourier-Bessel kernel** takes the form*

$$J(\underline{x}\wedge\underline{\xi})=2^{(m-3)/2}\,\Gamma\left(\frac{m-1}{2}\right)\,|\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\left(J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|)+\frac{\underline{x}\wedge\underline{\xi}}{|\underline{x}\wedge\underline{\xi}|}\,J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\right)\;.$$

*The corresponding **Fourier-Bessel transform** is given by*

$$\mathcal{F}_{\mathrm{bes}}[f](\underline{\xi})=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}J(\underline{x}\wedge\underline{\xi})\;f(\underline{x})\;dV(\underline{x})\;.$$

4 PROPERTIES

4.1 The Fourier-Bessel kernel

We now collect some properties of the Fourier-Bessel kernel.

Property 1 *The Fourier-Bessel kernel* $J(\underline{x} \wedge \underline{\xi})$ *has the following series representation:*

$$J(\underline{x} \wedge \underline{\xi}) = \Gamma\left(\frac{m-1}{2}\right) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\, \Gamma\left(\frac{m-1}{2}+\ell\right)} \left(\frac{|\underline{x} \wedge \underline{\xi}|}{2}\right)^{2\ell} \left(1 + \frac{\underline{x} \wedge \underline{\xi}}{m-1+2\ell}\right) \quad . \tag{8}$$

Proof. This result follows from the series expansion (3) of the Bessel function. □

Property 2 *Similar to the cylindrical Fourier kernel* $\exp(\underline{x} \wedge \underline{\xi}) = \cos(|\underline{x} \wedge \underline{\xi}|) + (\underline{x} \wedge \underline{\xi})$ $\mathrm{sinc}(|\underline{x} \wedge \underline{\xi}|)$*, the Fourier-Bessel kernel takes the form* $K_1 + (\underline{x} \wedge \underline{\xi})\, K_2$ *with* K_1 *and* K_2 *scalar-valued functions of the variable* $|\underline{x} \wedge \underline{\xi}|^2$*. It hence takes the form of a so-called parabivector, i.e. a scalar plus a bivector.*

Property 3 *The Fourier-Bessel kernel* $J(\underline{x} \wedge \underline{\xi})$ *satisfies the Helmholtz equations*

$$(\Delta_{\underline{x}} + |\underline{\xi}|^2)[J(\underline{x} \wedge \underline{\xi})] = 0 \quad \text{and} \quad (\Delta_{\underline{\xi}} + |\underline{x}|^2)[J(\underline{x} \wedge \underline{\xi})] = 0 \ .$$

Proof. Starting from (4) we have consecutively

$$\begin{aligned}
& \partial_{\underline{x}}[\exp(<\underline{x},\underline{\xi}>)\, J(\underline{x} \wedge \underline{\xi})] = 0 \\
\Longleftrightarrow\ & \underline{\xi}\, \exp(<\underline{x},\underline{\xi}>)\, J(\underline{x} \wedge \underline{\xi}) + \exp(<\underline{x},\underline{\xi}>)\, \partial_{\underline{x}}[J(\underline{x} \wedge \underline{\xi})] = 0 \\
\Longleftrightarrow\ & (\underline{\xi} + \partial_{\underline{x}})[J(\underline{x} \wedge \underline{\xi})] = 0 \\
\Longleftrightarrow\ & (\underline{\xi} + \partial_{\underline{x}})^2[J(\underline{x} \wedge \underline{\xi})] = 0 \\
\Longleftrightarrow\ & (-|\underline{\xi}|^2 - 2 <\underline{\xi}, \partial_{\underline{x}}> - \Delta_{\underline{x}})[J(\underline{x} \wedge \underline{\xi})] = 0 \\
\Longleftrightarrow\ & (\Delta_{\underline{x}} + |\underline{\xi}|^2)[J(\underline{x} \wedge \underline{\xi})] = 0 \ ,
\end{aligned}$$

since

$$<\underline{\xi}, \partial_{\underline{x}}> [\underline{x} \wedge \underline{\xi}] = 0 \quad \text{and} \quad <\underline{\xi}, \partial_{\underline{x}}> [|\underline{x} \wedge \underline{\xi}|^2] = 0 \ .$$

The other equation is proved similarly. □

Remark 1

1. *The traditional tensorial Fourier kernel* $\exp(-i <\underline{x}, \xi>)$ *also satisfies the Helmholtz equation:*

   $$(\Delta_{\underline{x}} + |\underline{\xi}|^2)[\exp(-i <\underline{x}, \underline{\xi}>)] = 0 \ ,$$

   the cylindrical Fourier kernel $\exp(\underline{x} \wedge \underline{\xi})$ *however not.*

2. *The kernel* $J(\underline{x} \wedge \underline{\xi})$ *also satisfies the refined equation:* $\left(-(\underline{\xi} \wedge \partial_{\underline{x}}) + |\underline{\xi}|^2\right)[J(\underline{x} \wedge \underline{\xi})] = 0$. *Indeed, from the foregoing proof we know that* $(\underline{\xi} + \partial_{\underline{x}})[J(\underline{x} \wedge \underline{\xi})] = 0$. *Let us now decompose* $\underline{x}$ *into components parallel with and orthogonal to* $\underline{\xi}$*:*

$$\underline{x} = \underline{x}_{//} + \underline{x}_{\perp} = \frac{< \underline{x}, \underline{\xi} >}{|\underline{\xi}|^2}\, \underline{\xi} - \frac{\underline{x} \wedge \underline{\xi}}{|\underline{\xi}|^2}\, \underline{\xi} \ .$$

Similarly we can put

$$\partial_{\underline{x}} = \partial_{\underline{x}//} + \partial_{\underline{x}\perp} = \frac{< \partial_{\underline{x}}, \underline{\xi} >}{|\underline{\xi}|^2}\, \underline{\xi} - \frac{\partial_{\underline{x}} \wedge \underline{\xi}}{|\underline{\xi}|^2}\, \underline{\xi} \ .$$

As $\partial_{\underline{x}//}[J(\underline{x} \wedge \underline{\xi})] = 0$*, we thus obtain that*

$$\left(\underline{\xi} - \frac{\underline{\xi}}{|\underline{\xi}|^2}(\underline{\xi} \wedge \partial_{\underline{x}})\right)[J(\underline{x} \wedge \underline{\xi})] = 0$$

or

$$\left(|\underline{\xi}|^2 - (\underline{\xi} \wedge \partial_{\underline{x}})\right)[J(\underline{x} \wedge \underline{\xi})] = 0 \ .$$

Property 4 *The Hermitean conjugate of the Fourier-Bessel kernel takes the form*

$$\left(J(\underline{x} \wedge \underline{\xi})\right)^{\dagger} = J(\underline{\xi} \wedge \underline{x}) = J(\underline{x} \wedge (-\underline{\xi})) \ .$$

Proof. This property follows from $(\underline{x} \wedge \underline{\xi})^{\dagger} = \underline{\xi} \wedge \underline{x}$. □

Property 5 *For* $\underline{x}$ *parallel to* $\underline{\xi}$ *one has* $J(\underline{x} \wedge \underline{\xi}) = 1$.

Proof. If $\underline{x}$ is parallel to $\underline{\xi}$, then $\underline{x} \wedge \underline{\xi} = 0$. Taking into account the series representation (8) the result follows. □

4.2 The Fourier-Bessel transform

First of all, it is observed that in the special case of dimension two, the Clifford-Fourier transform and the cylindrical Fourier transform coincide with the newly introduced Fourier-Bessel transform.

Proposition 1 *In the special case where* $m = 2$*, one has*

$$\mathcal{F}_{bes}[f](\underline{\xi}) = \mathcal{F}_{cyl}[f](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^+}[f](-\underline{\xi}) \ .$$

Proof. Taking into account that (see e.g. [13])

$$J_{1/2}(z) = \left(\frac{\pi}{2}z\right)^{-1/2} \sin(z) \qquad \text{and} \qquad J_{-1/2}(z) = \left(\frac{\pi}{2}z\right)^{-1/2} \cos(z) \ ,$$

we find that for $m = 2$:

$$\begin{aligned} J(\underline{x} \wedge \underline{\xi}) &= \sqrt{\frac{\pi}{2}}\, |\underline{x} \wedge \underline{\xi}|^{1/2} \left(J_{-1/2}(|\underline{x} \wedge \underline{\xi}|) + \frac{\underline{x} \wedge \underline{\xi}}{|\underline{x} \wedge \underline{\xi}|}\, J_{1/2}(|\underline{x} \wedge \underline{\xi}|) \right) \\ &= \cos\left(|\underline{x} \wedge \underline{\xi}|\right) + \frac{\underline{x} \wedge \underline{\xi}}{|\underline{x} \wedge \underline{\xi}|}\, \sin\left(|\underline{x} \wedge \underline{\xi}|\right) \\ &= \exp\left(\underline{x} \wedge \underline{\xi}\right) \ , \end{aligned}$$

which proves the statement. □

Let us now take a look at the operational formulae satisfied by the Fourier-Bessel transform.

Proposition 2 *The Fourier-Bessel transform satisfies*
(i) the linearity property

$$\mathcal{F}_{bes}[f\lambda + g\mu] = \mathcal{F}_{bes}[f]\ \lambda + \mathcal{F}_{bes}[g]\ \mu \qquad ; \qquad \lambda, \mu \in \mathbb{C}_m$$

(ii) the reflection property

$$\mathcal{F}_{bes}[f(-\underline{x})](\underline{\xi}) = \mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi})$$

(iii) Hermitean conjugation

$$\left(\mathcal{F}_{bes}[f(\underline{x})](\underline{\xi})\right)^{\dagger} = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f^{\dagger}(\underline{x})\ J(\underline{\xi} \wedge \underline{x})\ dV(\underline{x})$$

(iv) the change of scale property

$$\mathcal{F}_{bes}[f(a\underline{x})](\underline{\xi}) = \frac{1}{a^m}\ \mathcal{F}_{bes}[f(\underline{x})] \left(\frac{\underline{\xi}}{a}\right) \qquad \text{for } a \in \mathbb{R}_+$$

(v) the differentiation rule

$$\mathcal{F}_{bes}\left[\partial_{\underline{x}}[f(\underline{x})]\right](\underline{\xi}) = -\underline{\xi}\ \mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi})$$

(vi) the multiplication rule

$$\mathcal{F}_{bes}[\underline{x} f(\underline{x})](\underline{\xi}) = -\partial_{\underline{\xi}}\left[\mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi})\right]$$

(vii) the transfer formula

$$\int_{\mathbb{R}^m} \left(\mathcal{F}_{bes}[f](\underline{\xi})\right)^{\dagger} g(\underline{\xi})\ dV(\underline{\xi}) = \int_{\mathbb{R}^m} f^{\dagger}(\underline{\xi})\ \mathcal{F}_{bes}[g](\underline{\xi})\ dV(\underline{\xi})$$

(viii) the rotation rule

$$\begin{aligned} \mathcal{F}_{bes}[f(s\underline{x}\overline{s})](\underline{\xi}) &= \overline{s}\ \mathcal{F}_{bes}[s f(\underline{x})](s\underline{\xi}\overline{s}) \\ &= \overline{s}\ \mathcal{F}_{bes}[s f(\underline{x})\overline{s}](s\underline{\xi}\overline{s}) s \end{aligned}$$

with $s \in \mathrm{Spin}_{\mathbb{R}}(m)$.

Proof.
(i)-(iv) Straightforward.
(v) First, by means of the Clifford-Stokes theorem (see e.g. [1]) we obtain

$$\begin{aligned}
\mathcal{F}_{bes}\left[\partial_{\underline{x}}[f(\underline{x})]\right](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{x} \wedge \underline{\xi})\, \partial_{\underline{x}}[f(\underline{x})]\, dV(\underline{x}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{\partial\mathbb{R}^m} J(\underline{x} \wedge \underline{\xi})\, d\sigma_{\underline{x}}\, f(\underline{x}) \\
&\quad - \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} [J(\underline{x} \wedge \underline{\xi})]\partial_{\underline{x}}\, f(\underline{x})\, dV(\underline{x}) \\
&= -\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} [J(\underline{x} \wedge \underline{\xi})]\partial_{\underline{x}}\, f(\underline{x})\, dV(\underline{x}) \quad .
\end{aligned}$$

Furthermore inserting in the above equation the following formula

$$[J(\underline{x} \wedge \underline{\xi})]\partial_{\underline{x}} = J(\underline{x} \wedge \underline{\xi})\, \underline{\xi} = \underline{\xi}\, J(\underline{\xi} \wedge \underline{x}) = \underline{\xi}\, J(\underline{x} \wedge (-\underline{\xi})) \quad ,$$

we find

$$\mathcal{F}_{bes}\big[\partial_{\underline{x}}[f(\underline{x})]\big](\underline{\xi}) = -\underline{\xi}\, \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J\big(\underline{x} \wedge (-\underline{\xi})\big)\, f(\underline{x})\, dV(\underline{x}) = -\underline{\xi}\, \mathcal{F}_{bes}[f(\underline{x})](-\underline{\xi}) \quad .$$

(vi) From (7) we derive

$$J(\underline{x} \wedge \underline{\xi})\, \underline{x} = -\partial_{\underline{\xi}}[J\big(\underline{x} \wedge (-\underline{\xi})\big)] \quad ,$$

which yields

$$\begin{aligned}
\mathcal{F}_{bes}[\underline{x}\, f(\underline{x})](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{x} \wedge \underline{\xi})\, \underline{x}\, f(\underline{x})\, dV(\underline{x}) \\
&= -\partial_{\underline{\xi}} \left[\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J\big(\underline{x} \wedge (-\underline{\xi})\big)\, f(\underline{x})\, dV(\underline{x}) \right] \\
&= -\partial_{\underline{\xi}} \big[\mathcal{F}_{bes}[f(\underline{x})]\big](-\underline{\xi}) \quad .
\end{aligned}$$

(vii) Using property (iii) and changing the order of integration, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(\mathcal{F}_{bes}[f](\underline{\xi})\big)^{\dagger}\, g(\underline{\xi})\, dV(\underline{\xi}) \\
&\qquad = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} f^{\dagger}(\underline{x})\, J(\underline{\xi} \wedge \underline{x})\, dV(\underline{x}) \right) g(\underline{\xi})\, dV(\underline{\xi}) \\
&\qquad = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f^{\dagger}(\underline{x}) \left(\int_{\mathbb{R}^m} J(\underline{\xi} \wedge \underline{x})\, g(\underline{\xi})\, dV(\underline{\xi}) \right) dV(\underline{x}) \\
&\qquad = \int_{\mathbb{R}^m} f^{\dagger}(\underline{x})\, \mathcal{F}_{bes}[g](\underline{x})\, dV(\underline{x}) \quad .
\end{aligned}$$

(viii) By means of the substitution $\underline{u} = s\underline{x}\overline{s}$ for which it holds that $dV(\underline{u}) = dV(\underline{x})$ and $\underline{x} = \overline{s}\underline{u}s$, we obtain

$$\begin{aligned}\mathcal{F}_{bes}[f(s\underline{x}\overline{s})](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{x} \wedge \underline{\xi})\, f(s\underline{x}\overline{s})\, dV(\underline{x}) \\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\overline{s}\underline{u}s \wedge \underline{\xi})\, f(\underline{u})\, dV(\underline{u}) \\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J\big(\overline{s}(\underline{u} \wedge s\underline{\xi}\overline{s})s\big)\, f(\underline{u})\, dV(\underline{u}) \ .\end{aligned}$$

Moreover, as $(\overline{s}(\underline{x} \wedge \underline{t})s)^\ell = \overline{s}(\underline{x} \wedge \underline{t})^\ell s$ for $\ell \in \mathbb{N}$, we arrive at

$$\begin{aligned}\mathcal{F}_{bes}[f(s\underline{x}\overline{s})](\underline{\xi}) &= \overline{s}\, \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{u} \wedge s\underline{\xi}\overline{s})\ s\ f(\underline{u})\, dV(\underline{u}) \\ &= \overline{s}\, \mathcal{F}_{bes}[s\ f(\underline{x})](s\underline{\xi}\overline{s}) \ .\end{aligned}$$

Note that we also have the more symmetric form

$$\begin{aligned}\mathcal{F}_{bes}[f(s\underline{x}\overline{s})](\underline{\xi}) &= \overline{s}\, \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} J(\underline{u} \wedge s\underline{\xi}\overline{s})\ s\ f(\underline{u})\, \overline{s}\, dV(\underline{u})\ s \\ &= \overline{s}\, \mathcal{F}_{bes}[sf(\underline{x})\overline{s}](s\underline{\xi}\overline{s})\ s \ . \qquad \square\end{aligned}$$

5 FOURIER-BESSEL SPECTRUM OF THE L_2-BASIS CONSISTING OF GENERALIZED CLIFFORD-HERMITE FUNCTIONS

Now we will calculate the Fourier-Bessel spectrum of the L_2-basis (2) consisting of generalized Clifford-Hermite functions. The calculation method is based on the Funk-Hecke theorem in space (see Theorem 1).

5.1 The Fourier-Bessel spectrum of $\phi_{2p,k,j}$

Let us first calculate the Fourier-Bessel transform of a general basis element $\phi_{2p,k,j}$ which is given, up to constants, by

$$H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \ .$$

From Corollary 1 it is clear that we have to make a distinction between k even and k odd.

5.1.1 k even

Expressing the generalized Clifford-Hermite polynomial of even degree in terms of the classical Laguerre polynomial on the real line, introducing spherical co-ordinates

$$\underline{x} = r\underline{\omega} \ , \ \ \underline{\xi} = \rho\underline{\eta} \ , \ \ r = |\underline{x}| \ , \ \ \rho = |\underline{\xi}| \ , \ \ \underline{\omega}, \underline{\eta} \in S^{m-1} \ ,$$

using formula (1) and denoting $t_{\underline{\eta}} =< \underline{\omega}, \underline{\eta} >$, we find consecutively

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
=\ & 2^p\ p!\ \mathcal{F}_{bes}\left[L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ P_k(\underline{x})\right](\underline{\xi}) \\
=\ & \frac{2^p\ p!\ 2^{(m-3)/2}\ \Gamma\ \left(\frac{m-1}{2}\right)}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} |\underline{x}\wedge\underline{\xi}|^{(3-m)/2}\Bigg(J_{(m-3)/2}(|\underline{x}\wedge\underline{\xi}|) + \frac{\underline{x}\wedge\underline{\xi}}{|\underline{x}\wedge\underline{\xi}|} \\
& J_{(m-1)/2}(|\underline{x}\wedge\underline{\xi}|)\Bigg)\ L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ P_k(\underline{x})\ dV(\underline{x}) \\
=\ & \frac{2^p\ p!\ 2^{(m-3)/2}\ \Gamma\ \left(\frac{m-1}{2}\right)}{(2\pi)^{m/2}} \Bigg\{\int_{\mathbb{R}^m} L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ r^k\ \left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)^{(3-m)/2} \\
& J_{(m-3)/2}\left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)\ P_k(\underline{\omega})\ dV(\underline{x}) \\
-\ & \rho\int_{\mathbb{R}^m} L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ r^{k+1}\ \left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)^{(1-m)/2}\ t_{\underline{\eta}} \\
& J_{(m-1)/2}\left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)\ P_k(\underline{\omega})\ dV(\underline{x}) \\
-\ & \underline{\xi}\int_{\mathbb{R}^m} L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ r^{k+1}\ \left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)^{(1-m)/2} \\
& J_{(m-1)/2}\left(r\rho\sqrt{1-t_{\underline{\eta}}^2}\right)\ \underline{\omega}\ P_k(\underline{\omega})\ dV(\underline{x})\Bigg\}\ . \qquad (9)
\end{aligned}$$

Furthermore, as k is even, Corollary 1 implies that both the second and third integral vanish. Moreover, applying the Funk-Hecke theorem in space and expressing the Legendre polynomial in terms of the Gegenbauer polynomial

$$P_{k,m}(t) = \frac{k!\ (m-3)!}{(k+m-3)!}\ C_k^{(m-2)/2}(t)\ \ ,$$

we obtain

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = \frac{2^p\ p!\ 2^{(m-3)/2}\ \Gamma\left(\frac{m-1}{2}\right)}{(2\pi)^{m/2}} \\
&\quad \frac{k!\ (m-3)!}{(k+m-3)!}\ A_{m-1}\ P_k(\underline{\eta})\ \left(\int_0^{+\infty} L_p^{m/2+k-1}(r^2)\ \exp\left(-\frac{r^2}{2}\right)\ r^{k+m-1}\ dr\right) \\
&\quad \left(\int_{-1}^{1}(r\rho\sqrt{1-t^2})^{(3-m)/2} J_{(m-3)/2}(r\rho\sqrt{1-t^2})\ (1-t^2)^{(m-3)/2}\ C_k^{(m-2)/2}(t)\ dt\right)\ . \qquad (10)
\end{aligned}$$

By means of the substitution $t = \cos(x)$, the second integral is turned into

$$\int_{-1}^{1} (r\rho\sqrt{1-t^2})^{(3-m)/2} \; J_{(m-3)/2}(r\rho\sqrt{1-t^2}) \; (1-t^2)^{(m-3)/2} \; C_k^{(m-2)/2}(t) \; dt$$
$$= (r\rho)^{(3-m)/2} \int_0^{\pi} J_{(m-3)/2}(r\rho \sin(x)) \; C_k^{(m-2)/2}(\cos(x)) \; \sin(x)^{(m-1)/2} \; dx \; . \quad (11)$$

Taking into account the integral formula (see [10], p. 832, 7.333, formula 1 with $\theta = \frac{\pi}{2}$):

$$\int_0^{\pi} \big(\sin(x)\big)^{\nu+1} \; C_n^{\nu+1/2}(\cos(x)) \; J_\nu\big(a \sin(x)\big) \; dx$$
$$= (-1)^{n/2} \left(\frac{2\pi}{a}\right)^{1/2} C_n^{\nu+1/2}(0) \; J_{\nu+1/2+n}(a) \qquad , \qquad n = 0, 2, 4, \ldots \quad (12)$$

equation (11) becomes

$$\int_{-1}^{1} (r\rho\sqrt{1-t^2})^{(3-m)/2} \; J_{(m-3)/2}(r\rho\sqrt{1-t^2}) \; (1-t^2)^{(m-3)/2} \; C_k^{(m-2)/2}(t) \; dt$$
$$= \; (r\rho)^{(3-m)/2} \; (-1)^{k/2} \left(\frac{2\pi}{r\rho}\right)^{1/2} C_k^{(m-2)/2}(0) \; J_{(m-2)/2+k}(r\rho)$$
$$= \; \frac{\sqrt{2\pi} \; \Gamma\left(\frac{m-2+k}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right) \; \Gamma\left(\frac{k}{2}+1\right)} \; (r\rho)^{1-m/2} \; J_{(m-2)/2+k}(r\rho) \; ,$$

where we have also used the fact that (see e.g. [13])

$$C_n^{\lambda}(0) = (-1)^{n/2} \frac{\Gamma\left(\lambda + \frac{n}{2}\right)}{\Gamma(\lambda) \; \Gamma\left(\frac{n}{2}+1\right)}$$

for n even.
Hence, the Fourier-Bessel spectrum (10) becomes

$$\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x}) \; P_k(\underline{x}) \; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = 2^p \; p! \; \frac{k! \; (m-3)!}{(k+m-3)!} \; \frac{\Gamma\left(\frac{m-2+k}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right) \; \Gamma\left(\frac{k}{2}+1\right)}$$
$$\rho^{1-m/2-k} \; P_k(\underline{\xi}) \int_0^{+\infty} r^{k+m/2} \; L_p^{m/2+k-1}(r^2) \; \exp\left(-\frac{r^2}{2}\right) J_{(m-2)/2+k}(r\rho) \; dr \; .$$

Substituting r^2 for y in the integral formula (see e.g. [13], p. 244)

$$\int_0^{+\infty} \exp\left(-\frac{y}{2}\right) y^{\alpha/2} \; L_n^{(\alpha)}(y) \; J_\alpha(\sqrt{xy}) \; dy = 2 \; (-1)^n \; x^{\alpha/2} \; \exp\left(-\frac{x}{2}\right) L_n^{(\alpha)}(x)$$

yields

$$\int_0^{+\infty} \exp\left(-\frac{r^2}{2}\right) r^{\alpha+1} \; L_n^{(\alpha)}(r^2) \; J_\alpha(r\sqrt{x}) \; dr = (-1)^n \; x^{\alpha/2} \; \exp\left(-\frac{x}{2}\right) L_n^{(\alpha)}(x) \; . \quad (13)$$

Using the above result, we finally arrive at

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\; P_k(\underline{x})\; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&= 2^p\, p!\, \frac{k!\,(m-3)!}{(k+m-3)!}\, \frac{\Gamma\left(\frac{m-2+k}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right)\,\Gamma\left(\frac{k}{2}+1\right)}\, P_k(\underline{\xi})\,(-1)^p\, \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) L_p^{(m-2+2k)/2}(|\underline{\xi}|^2) \\
&= (-1)^p \frac{k!\,(m-3)!}{(k+m-3)!} \frac{\Gamma\left(\frac{m-2+k}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right)\,\Gamma\left(\frac{k}{2}+1\right)}\, H_{2p,k}(\sqrt{2}\underline{\xi})\, P_k(\underline{\xi})\, \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \\
&= \frac{(-1)^{k/2}\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{-k+1}{2}\right)\Gamma\left(\frac{k+m-1}{2}\right)}\,(-1)^p\, H_{2p,k}(\sqrt{2}\underline{\xi})\, P_k(\underline{\xi})\, \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) ,
\end{aligned}$$

where in the last line we have used the formulae (see e.g. [13]):

$$\Gamma(2z) = \pi^{-1/2}\, 2^{2z-1}\,\Gamma(z)\,\Gamma\left(\frac{1}{2}+z\right) \quad \text{and} \quad \Gamma\left(\frac{1}{2}+z\right)\,\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)} .$$

5.1.2 k odd

In case where k is odd, the first integral in (9) vanishes.
By means of the Funk-Hecke theorem in space we obtain

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\; P_k(\underline{x})\; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&= \frac{2^p\, p!\, 2^{(m-3)/2}\Gamma\left(\frac{m-1}{2}\right)}{(2\pi)^{m/2}}\, A_{m-1}\,\rho\, P_k(\underline{\eta}) \left(\int_0^{+\infty} L_p^{m/2+k-1}(r^2)\, r^{k+m}\, \exp\left(-\frac{r^2}{2}\right) dr\right) \\
&\left(\int_{-1}^{1} (r\rho\sqrt{1-t^2})^{(1-m)/2}\, J_{(m-1)/2}(r\rho\sqrt{1-t^2})\,(1-t^2)^{(m-3)/2}\left(P_{k+1,m}(t) - tP_{k,m}(t)\right) dt\right) .
\end{aligned}$$

Next, taking into account the Gegenbauer recurrence relation (see e.g. [13])

$$(k+2\lambda)\, tC_k^\lambda(t) - (k+1)\, C_{k+1}^\lambda(t) = 2\lambda\,(1-t^2)\, C_{k-1}^{\lambda+1}(t) ,$$

we have that

$$P_{k+1,m}(t) - tP_{k,m}(t) = -\frac{k!\,(m-2)!}{(k+m-2)!}\,(1-t^2)\, C_{k-1}^{m/2}(t) ,$$

which in its turn yields

$$\begin{aligned}
&\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\; P_k(\underline{x})\; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = -\frac{2^p\, p!\, 2^{(m-3)/2}\,\Gamma\left(\frac{m-1}{2}\right)}{(2\pi)^{m/2}}\, \frac{k!\,(m-2)!}{(k+m-2)!} \\
&A_{m-1}\,\rho\, P_k(\underline{\eta}) \left(\int_0^{+\infty} L_p^{m/2+k-1}(r^2)\, r^{k+m}\, \exp\left(-\frac{r^2}{2}\right) dr\right) \\
&\left(\int_{-1}^{1} (r\rho\sqrt{1-t^2})^{(1-m)/2}\, J_{(m-1)/2}(r\rho\sqrt{1-t^2})\,(1-t^2)^{(m-1)/2}\, C_{k-1}^{m/2}(t)\, dt\right) .
\end{aligned}$$

In a similar way as before, by applying the integral formulae (12) and (13), we find the following Fourier-Bessel image in case of k odd:

$$\mathcal{F}_{bes}\left[H_{2p,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{(k-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(-\frac{k}{2}\right)\Gamma\left(\frac{k+m}{2}\right)}\ (-1)^p\ H_{2p,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi})\ .$$

5.2 The Fourier-Bessel spectrum of $\phi_{2p+1,k,j}$

Seen the calculations of the Fourier-Bessel spectrum of the basis function $\phi_{2p+1,k,j}$ given, up to constants, by

$$H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)$$

are very similar to the ones of the previous subsection, we restrict ourselves to giving the results.

5.2.1 k even

$$\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{k/2}\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(\frac{k+m-1}{2}\right)}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi})\ .$$

5.2.2 k odd

$$\mathcal{F}_{bes}\left[H_{2p+1,k}(\sqrt{2}\underline{x})\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= \frac{(-1)^{(k+1)/2}\sqrt{\pi}\,\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{-k}{2}\right)\Gamma\left(\frac{k+m}{2}\right)}\ (-1)^p\ H_{2p+1,k}(\sqrt{2}\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ P_k(\underline{\xi})\ .$$

6 CONCLUSION

In this paper we introduced a new multi-dimensional integral transform within the Clifford analysis framework, the so-called Fourier-Bessel transform. We have shown that in the two-dimensional case it coincides with the Clifford-Fourier and cylindrical Fourier transforms introduced earlier and that it satisfies similar operational formulae to those of the classical tensorial Fourier transform. Moreover, in the last section we have proved that the L_2-basis elements consisting of generalized Clifford-Hermite functions are eigenfunctions of the Fourier-Bessel transform, which is also the case for the Clifford-Fourier transform. The fact that these L_2-basis elements are simultaneous eigenfunctions of the Clifford-Fourier and Fourier-Bessel transform, will allow us in [7] to express in the even dimensional case the Clifford-Fourier transform in terms of the Fourier-Bessel transform. The latter will then lead to a closed form of the Clifford-Fourier kernel in case of even dimension. Note that, apart from the special two-dimensional case, we had not succeeded yet in obtaining such a closed form.

REFERENCES

[1] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*. Pitman Publishers, Boston-London-Melbourne, 1982.

[2] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Fourier Transform. *J. Fourier Anal. Appl.*, **11 (6)**, 669-681, 2005.

[3] F. Brackx, N. De Schepper and F. Sommen, The Two-Dimensional Clifford-Fourier Transform. *J. Math. Imaging Vision*, **26 (1-2)**, 5-18, 2006.

[4] F. Brackx, N. De Schepper and F. Sommen, The Fourier Transform in Clifford Analysis. *Advances in Imaging and Electron Physics*, **156**, 55-203, 2008.

[5] F. Brackx, N. De Schepper and F. Sommen, The Cylindrical Fourier Transform. In: (digital) Proceedings of AGACSE2008.

[6] F. Brackx, N. De Schepper and F. Sommen, The Cylindrical Fourier Spectrum of an L2-basis consisting of Generalized Clifford-Hermite Functions. In: AIP Conference Proceedings Volume 1048, Numerical Analysis and Applied Mathematics: International Conference of Numerical Analysis and Applied Mathematics, Kos (Greece), 16-20 September 2008, Theodore E. Simos, George Psihoyios and Ch. Tsitouras (eds.), 686-690.

[7] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Fourier integral kernel in the even dimensional case, submitted.

[8] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor–Valued Functions*. Kluwer Academic Publishers, Dordrecht, 1992.

[9] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*. Cambridge University Press, Cambridge, 1991.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Academic Press, New York - London - Toronto - Sydney - San Francisco, 1980.

[11] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*. Wiley, Chichester, 1997.

[12] H. Hochstadt, *The functions of mathematical physics*. In ”Pure and Applied Mathematics,” Volume 23, Wiley-Interscience, New York, 1971.

[13] W. Magnus, F. Oberhettinger and R.-P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[14] F. Sommen, Special Functions in Clifford analysis and Axial Symmetry. *J. Math. Anal. Appl.*, **130(1)**, 110–133, 1988.

[15] F. Sommen, Fourier-Borel Transforms in Clifford Analysis and the Dual Fischer Decomposition. In: AIP Conference Proceedings Volume 1048, Numerical Analysis and Applied Mathematics: International Conference of Numerical Analysis and Applied Mathematics, Kos (Greece), 16-20 September 2008, Theodore E. Simos, George Psihoyios and Ch. Tsitouras (eds.), 695-696.