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COMPUTATION OF THE REFLECTANCE AND TRANSMITTANCE FOR AN INHOMOGENEOUS LAYERED MEDIUM WITH TURNING POINTS USING THE WKB AND SPPS METHODS

R. Castillo-Pérez*[,1], **A. del C. Cedillo-Díaz**[2], **V. V. Kravchenko**[3] **and H. Oviedo-Galdeano**[1]

*[,1]*SEPI ESIME, National Polytechnic Institute*
Av. IPN s/n, C.P. 07738, Mexico City, Mexico
[2]*SEPI UPIITA, National Polytechnic Institute, Av. IPN 2580, C.P. 07340, Mexico City, Mexico*
[3]*Department of Mathematics, CINVESTAV del IPN, Campus Querétaro, Apartado Postal 1-798, Arteaga # 5, Col. Centro, Santiago de Querétaro, Qro., C.P. 76001 Mexico*
E-mail: rcastillo@ipn.mx

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Abstract. *Electromagnetic wave propagation is currently present in the vast majority of situations which occur in everyday life, whether in mobile communications, HDTV, satellite tracking, broadcasting, etc. Because of this the study of increasingly complex means of propagation of electromagnetic waves has become necessary in order to optimize resources and increase the capabilities of the devices as required by the growing demand for such services.*

Within the electromagnetic wave propagation different parameters are considered that characterize it under various circumstances and of particular importance are the reflectance and transmittance. There are several methods for the analysis of the reflectance and transmittance such as the method of approximation by boundary condition, the plane-wave expansion method (PWE), etc., but this work focuses on the WKB and SPPS methods.

The implementation of the WKB method is relatively simple but is found to be relatively efficient only when working at high frequencies. The SPPS method (Spectral Parameter Powers Series) based on the theory of pseudoanalytic functions, is used to solve this problem through a new representation for solutions of Sturm-Liouville equations and has recently proven to be a powerful tool to solve different boundary value and eigenvalue problems. Moreover, it has a very suitable structure for numerical implementation, which in this case took place in the Matlab software for the evaluation of both conventional and turning points profiles.

The comparison between the two methods allows us to obtain valuable information about their performance which is useful for determining the validity and propriety of their application for solving problems where these parameters are calculated in real-life applications.

1 INTRODUCTION

When a wave traveling from a medium to another is considered, some parameters such as reflectance and transmittance can be identified. They are defined as the ratio of the amplitude of the reflected wave and the transmitted wave with respect to the incident wave amplitude respectively.

The study of these parameters is necessary in many areas of science for increasingly complex media. For example, most modern optical systems could not function without inhomogeneous optical coatings. The telecommunications industry uses various types of layers such as antireflective layers, polarizers and dichroic layers in personal displays, optical filters, inhomogeneous planar waveguides [1], inhomogeneous photonic crystals [2], devices for splitting and combining optical communication channels, and so on. Also the knowledge of the reflectance and transmittance has important applications in ionospheric communications and in the analysis of radiation of antennas [3]. Other areas of study include applications such as environmental studies, precision agriculture, ecology, etc. [4].

There are numerous methods for the analysis of the reflectance and transmittance having different degrees of precision, complexity and efficiency. Some examples of them are the WKB method [5], the method of approximation by boundary condition, the plain wave expansion method PWE [6], the transfer matrix method TMM [7], the variational method [8], the perturbation method [9], differential TMM, etc. (see, e.g., [10, 11, 12, 13, 14]). A recently developed method is the SPPS method (Spectral Parameter Powers Series method [15, 16, 17]).

This work focuses on the WKB [5] and SPPS [18] methods because the WKB method is well known and has been extensively studied, in addition it is relatively easy to implement and efficient especially when working at high frequencies. The SPPS method, which in this particular case is associated with the solutions of a Sturm-Liouville problem, has proven a powerful tool for solving various types of boundary and eigenvalue problems [19, 20, 21] and has a structure which is very suitable for its numerical implementation.

The particular problem discussed in this paper is calculating the reflectance and transmittance of an electromagnetic wave with perpendicular polarization that propagates through an inhomogeneous layered medium with normal incidence. Inhomogeneous media are those for which one or more of their material parameters depend on space and are defined here by means of a refractive index profile, which in this case depends on a single space coordinate. In addition to traditional profiles, profiles with turning points are considered.

We proceeded with the programming of both methods and with the evaluation of different profiles. We used some profiles with known exact solutions (linear, exponential and hyperbolic profiles) as test problems to verify the correct operation of the methods. Then the study of turning points profiles was carried out with both methods.

The organization of this paper is as follows. In Section 2 we introduce the problem to be solved and propose the elements for calculating the reflectance and transmittance. In Section 3 the WKB method is developed and the considerations under which it can be adapted for the analysis of a turning points profile are described. Section 4 introduces the SPPS method and it is shown how the solutions of a Sturm-Liouville equation can be used to construct the solution to our problem. In Section 5 the results of the computations are presented and the performance of the methods is analyzed.

2 WAVE PROPAGATION IN INHOMOGENOUS LAYERED MEDIA

In general the characteristics of wave propagation and dispersion in a non homogeneous medium cannot be described in a simple manner. However, there are special cases where asymptotic methods may apply as, for example, when working with high or low frequency fields [5]. The low frequency approach is applicable whenever the size of a dielectric body is much smaller than a wavelength. The high-frequency approach is useful when the refractive index variation is negligibly small over the distance of one wavelength.

The inhomogeneous media under consideration will have the layered structure shown in Figure 1.

where T is the transmission coefficient and $k_3 = k_0 n_3$. In this case and for non-absorbent media the following energy conservation relation holds

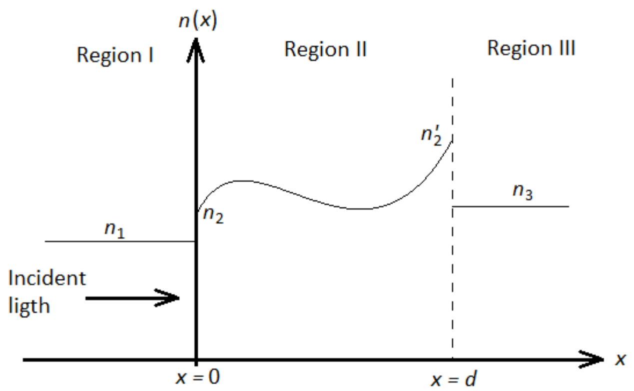

Figure 1. Inhomogeneous layered medium.

The refractive index n has constant values n_1 and n_3 in the regions I and III respectively and is an arbitrary continuous function in the region II. If we suppose an incident wave in region I represented by the scalar function u which stands for a transverse component of the electric field of an s-polarized electromagnetic wave, the following Helmholtz equation is satisfied

$$u''(x) + [q(x) - \beta^2]u(x) = 0 \tag{1}$$

where $q(x) = k_0^2 n^2(x)$, k_0 is the free-space circular wavenumber and $\beta = k_0 \sin\theta$, θ being the angle of incidence (for the sake of simplicity normal incidence is considered, so in what follows $\theta = 0°$, and β vanishes). If the incident wave is supposed to have the form $e^{-ik_1 x}$, where $k_1 = k_0 n_1$, then together with the reflected wave the whole solution for $x < 0$ is

$$u(x) = e^{-ik_1 x} + Re^{ik_1 x}; \qquad x < 0$$

where the constant R is the reflection coefficient whose absolute value is less than 1. The solution corresponding to the transmitted wave in region III has the form

$$u(x) = Te^{-ik_3 x}, \qquad x > d$$

$$|R|^2 + \frac{n_3 |T|^2}{n_1} = 1. \tag{2}$$

The general solution of (1) for $0 < x < d$ is proposed to have the form

$$u = c_1 u_1 + c_2 u_2$$

and consists of two linearly independent solutions u_1 and u_2 in the interval $0 \le x \le d$ such that

$$u_1(0) = 1, \qquad u_1'(0) = 0 \tag{3}$$

$$u_2(0) = 0, \qquad u_2'(0) = 1 \tag{4}$$

and with c_1 and c_2 being arbitrary constants. So, from the continuity and initial conditions the expressions for R and T were found to be [18]

$$R = \frac{-k_1 k_3 u_2(d) - u_1'(d) - ik_3 u_1(d) + ik_1 u_2'(d)}{[u_1'(d) - k_1 k_3 u_2(d)] + i[k_3 u_1(d) + k_1 u_2'(d)]} \tag{5}$$

$$T = \frac{2ik_1[u_1(d)u_2'(d) - u_1'(d)u_2(d)]e^{ik_3 d}}{[u_1'(d) - k_1 k_3 u_2(d)] + i[k_3 u_1(d) + k_1 u_2'(d)]}. \tag{6}$$

These are the formulas for the reflectance and transmittance in an inhomogeneous layered medium.

3 WKB METHOD

3.1 WKB method for profiles without turning points

Initially we work with (1) considering normal incidence

$$\left[\frac{d^2}{dx^2} + q^2(x)\right] u(x) = 0. \tag{7}$$

The following solution is proposed

$$u(x) = e^{\varphi(x)} \tag{8}$$

where $\varphi(x)$ is given by the following expression

$$\varphi(x) = \int \phi(x) dx.$$

It is worth mentioning that $\phi(x)$ will be found later. Now, replacing (8) in (7) we get

$$\frac{d^2}{dx^2} e^{\varphi(x)} + q^2(x) e^{\varphi(x)} = 0. \tag{9}$$

Equation (9) can be expressed as a Riccati equation

$$\phi'(x) + \phi^2(x) + q^2(x) = 0. \tag{10}$$

Knowing that $q(x) = k_0 n(x)$, (10) becomes

$$\phi'(x) + \phi^2(x) + {k_0}^2 n^2(x) = 0\ . \tag{11}$$

Now, $\phi(z)$ can be written as an inverse power series of k_0

$$\phi(x) = \left[\phi_0(x) k_0 + \phi_1(x) + \frac{\phi_2(x)}{k_0} + \frac{\phi_3(x)}{{k_0}^2} + \cdots\right].$$

Taking into account the power series of $\phi^2(x)$, (11) becomes

$$\left[\phi_0^2(x) + n^2(x)\right]k_0^2 + \left[\phi_0'(x) + 2\phi_0(x)\phi_1(x)\right]k_0 + \left[\phi_1'(x) + \phi_1^2(x) + 2\phi_0(x)\phi_2(x)\right] + \cdots = 0$$

where when considering high frequencies, the large value of k_0 allows us to neglect the terms in which it appears in the denominator.

Equating the coefficients of each power of k_0 to zero, we obtain an infinite number of equations, but we only took the first three

$$\phi_0^2(x) + n^2(x) = 0 \tag{12}$$

$$\phi_0'(x) + 2\phi_0(x)\phi_1(x) = 0 \tag{13}$$

$$\phi_1'(x) + \phi_1^2(x) + 2\phi_0(x)\phi_2(x) = 0. \tag{14}$$

From (12) we can get the value of $\phi_0(x)$

$$\phi_0(x) = \pm in(x). \tag{15}$$

In order to find $\phi_1(x)$, using $\frac{d}{dx}(\ln u) = \frac{\frac{du}{dx}}{u}$ equation (14) turns out to be

$$\phi_1 = -\frac{\phi_0'}{2\phi_0} = -\frac{1}{2}\frac{d\ln\phi_0}{dx} = \frac{d\ln{\phi_0}^{-1/2}}{dx}. \tag{16}$$

Now, from (8) for

$$\phi(x) \approx \phi_0 k_0 + \phi_1 + \frac{\phi_2}{k_0} + \frac{\phi_3}{{k_0}^2}$$

we get

$$u(x) \approx e^{\int[\phi_0 k_0 + \phi_1 + \frac{\phi_2}{k_0} + \frac{\phi_3}{{k_0}^2}]dx}.$$

Neglecting all the terms which have k_0 in the denominator and replacing the values of (15) and (16) we get

$$u(x) = \frac{1}{\sqrt{\phi_0}} \cdot e^{\pm i\int k_0 n(x)dx}. \tag{17}$$

Developing the previous expression we can state it in terms of $q(x)$

$$u(x) = \frac{1}{q(x)^{1/2}}\left[a \cdot e^{-i\int q(x)dx} + b \cdot e^{+i\int q(x)dx}\right]. \tag{18}$$

The constants a and b can be found using the initial conditions. Then the solution consists of a wave u_1 traveling in the $+z$ direction and a wave u_2 traveling in the direction of $-z$

$$u_1(x) = a \cdot \frac{1}{q^{1/2}} e^{-i\int q dx} \tag{19}$$

$$u_2(x) = b \cdot \frac{1}{q^{1/2}} e^{+i\int q dx}. \tag{20}$$

After obtaining the solutions (19) and (20) that are solutions of the quadratic equation with inhomogeneous parameters (7), we proceed to find the value of the constants, which together with the above equations will constitute the complete solution v which will allow us to satisfy the initial conditions (3) and (4).

In order to do this we propose two linearly independent solutions given by

$$v_1(x) = a_1 u_1(x) + a_2 u_2(x) \tag{21}$$

and

$$v_2(x) = b_1 u_1(0) + b_2 u_2(x). \tag{22}$$

The constants for the conditions (3) were found to be

$$a_1 = \frac{1}{2} q^{1/2}(0)\left(1 - \frac{1}{2i} q^{-2}(0) \cdot q'(0)\right) \tag{23}$$

and

$$a_2 = q^{1/2}(0) - \frac{1}{2} q^{1/2}(0) \left(1 - \frac{1}{2i} q^{-2}(0) \cdot q'(0)\right). \tag{24}$$

For the conditions (4) we found

$$b_1 = -\frac{1}{2iq^{1/2}(0)} \tag{25}$$

and also

$$b_2 = +\frac{1}{2iq^{1/2}(0)}. \tag{26}$$

Solutions (21) and (22) can be used for calculating the reflectance and transmitance using (5) and (6).

3.2 WKB method for profiles with turning points

The solutions found with the WKB method (19) and (20) in Section 3.1 for the Helmholtz equation (7) can be used to find the reflectance and transmittance of profiles that do not have a zero crossing singularity, i.e., when the function $q^2(x)$ (which depends on the refractive index) does not pass through zero. In the case of profiles having such zero crossing, the point where they cross zero (here denoted by x_0) is called the turning point. (Figure 2).

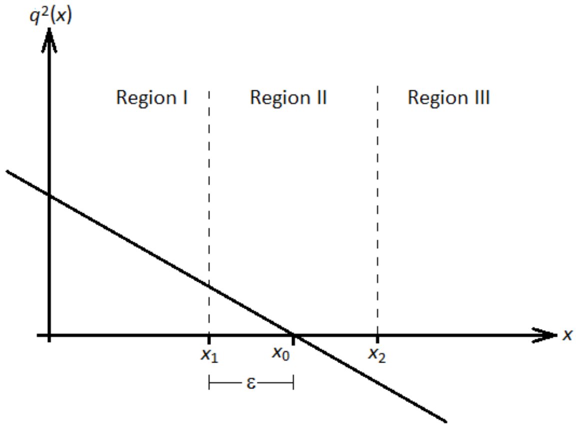

Figure 2. Profile with turning points.

In order to study the above problem, the inhomogeneous region of the medium is in its turn divided into three regions. Let x_1 delimit the region II by the left hand side and x_2 delimit the region II by the right hand side. This region is a ε-neighburhood of the turning point x_0. In region I and region III, the solutions found in Section 3.1, (19) and (20) are valid, but in region II, the one where the turning points lie, a different solution should be calculated.

In order to find the solution in region II [5] one can go back to the Helmholtz equation (7) for which $q^2(x)$ can then be expanded around x_0 using the Taylor series. In this case only the first term will be used

$$q^2(x) = -a(x - x_0), \tag{27}$$

where a is the slope at x_0:

$$a = -\left.\frac{d(q^2)}{dx}\right|_{x=x_0}.$$

Then (7) becomes

$$\left[\frac{d^2}{dx^2} - a(x - x_0)\right] u(x) = 0. \tag{28}$$

To see what happens around the turning point, one must find the solution of (28). This is possible, using the solutions in the Region I (21) and (22) in order to calculate the solutions in region II that we will denote by $\widetilde{v_1}(x)$ and $\widetilde{v_2}(x)$. In addition, it is necessary to connect the solutions found for the Region I with the solutions in the Region II, for which new initial conditions are introduced (Cauchy problem). The following continuity conditions are to be imposed

$$\widetilde{v_1}(x_1) = v_1(x_1), \qquad \widetilde{v_1}'(x_1) = v_1'(x_1) \tag{29}$$

$$\widetilde{v_2}(x_1) = v_2(x_1), \qquad \widetilde{v_2}'(x_1) = v_2'(x_1). \tag{30}$$

It is possible to solve this system using the mathematical package Matlab, which in its turn uses an internal process based on Maple. The command used to obtain the solutions is *dsolve* and it allows the solution of a quadratic equation with initial conditions to be found. The command used in Matlab to solve (28) for the initial conditions (29) was

$$dsolve('D2v \ = +a*(x-x0)*v', 'v(x1) \ = \ v1x1', 'Dv(x1) = dv1x1', 'x').$$

The resulting expression after applying the *simplify* command is:

$$\widetilde{v_1} = \frac{Bi(\zeta_2)\cdot\left(v_1'(x_1)\cdot Ai\ (\zeta_1) - \ av_1(x_1)\cdot Ai'(\zeta_1)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}\right)}{a\big(Ai(\zeta_1)\cdot Bi'(\zeta_1) - \ Ai'(\zeta_1)\cdot Bi(\zeta_1)\big)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}}$$
$$- \ \frac{Ai(\zeta_2)\cdot\left(v_1'(x_1)\cdot Bi(\zeta_1) - \ av_1(x_1)\cdot Bi'(\zeta_1)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}\right)}{a\big(Ai(\zeta_1)\cdot Bi'(\zeta_1) - \ Ai'(\zeta_1)\cdot Bi(\zeta_1)\big)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}} \tag{31}$$

where

$$\zeta_1 = ax_1\left(-\frac{1}{a}\right)^{\frac{2}{3}} - \ ax_0\left(-\frac{1}{a}\right)^{\frac{2}{3}},$$

$$\zeta_2 = ax\left(-\frac{1}{a}\right)^{\frac{2}{3}} - \ ax_0\left(-\frac{1}{a}\right)^{\frac{2}{3}}$$

and for the initial conditions (30) the solution is the following expression

$$\widetilde{v_2} = \frac{Bi(\zeta_2)\cdot\left(v_2'(x_1)\cdot Ai\ (\zeta_1) - \ av_2(x_1)\cdot Ai'(\zeta_1)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}\right)}{a\big(Ai(\zeta_1)\cdot Bi'(\zeta_1) - \ Ai'(\zeta_1)\cdot Bi(\zeta_1)\big)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}}$$

$$-\quad \frac{Ai(\zeta_2)\cdot\left(v_2'(x_1)\cdot Bi(\zeta_1)-\ av_2(x_1)\cdot Bi'(\zeta_1)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}\right)}{a\big(Ai(\zeta_1)\cdot Bi'(\zeta_1)-\ Ai'(\zeta_1)\cdot Bi(\zeta_1)\big)\cdot\left(-\frac{1}{a}\right)^{\frac{2}{3}}}. \tag{32}$$

For Region III we take up the solutions found with the WKB method (19) and (20) and proceed to find the constants that will complement the proposed final solutions

$$\widetilde{\widetilde{v_1}}(x) = a_1\, u_1(x) + a_2 u_2(x) \tag{33}$$

$$\widetilde{\widetilde{v_2}}(x) = b_1\, u_1(x) + b_2 u_2(x) \tag{34}$$

for which with similar initial conditions to those used in Region II were imposed that lead to the following constants

$$a_1 = \frac{u_2(x_2)\cdot\widetilde{v_1}'(x_2) - \widetilde{v_1}(x_2)\cdot u_2{}'(x_2)}{u_1{}'(x_2)\cdot u_2(x_2) - u_1(x_2)\cdot u_2{}'(x_2)} \tag{35}$$

$$a_2 = \frac{\widetilde{v_1}(x_2) - a_1\cdot u_1(x_2)}{u_2(x_2)} \tag{36}$$

$$b_1 = \frac{u_2(x_2)\cdot\widetilde{v_2}'(x_2) - \widetilde{v_2}(x_2)\cdot u_2{}'(x_2)}{u_1{}'(x_2)\cdot u_2(x_2) - u_1(x_2)\cdot u_2{}'(x_2)} \tag{37}$$

$$b_2 = \frac{\widetilde{v_2}'(x_2) - b_1\cdot u_1(x_2)}{u_2(x_2)}. \tag{38}$$

Equations (33) and (34) are the final solutions needed to find the reflectance and transmittance for a turning points profile.

In Section 4 we will work with these profiles using the SPPS method. This will later allow us to stablish a comparison of the performance of both methods.

4 SPPS METHOD

An application of the theory of pseudoanalytic functions corresponds to the theory of linear ordinary differential equations of second order. One of them is the Sturm-Liouville equation, which is of fundamental importance because of the many situations in mathematical physics where it arises, and that has the following form

$$(pv')' + qv = \beta^2 rv \tag{39}$$

for which p, q, r and v are complex-valued functions of an independent real variable $x \in [0, d]$ and β is an arbitrary complex constant. The coefficients p, q and r depend on the considered problem and are proposed so that there is a particular solution v_0 (which is also a complex-valued function of a real variable x) of the homogeneous equation

$$(pv_0')' + qv_0 = 0 \tag{40}$$

such that the functions $\boldsymbol{v_0^2 r}$ and $\boldsymbol{1/(v_0^2 p)}$ are continuous in the interval $[\boldsymbol{0}, \boldsymbol{d}]$.

The general solution for (39) takes the form [18]

$$v = c_1 v_1 + c_2 v_2 \tag{41}$$

where $c_1\ and\ c_2$ are arbitrary complex constants, v_1 and v_2 are defined in the following way

$$v_1 = v_0 \sum_{even\, n=0}^{\infty} \beta^n \tilde{X}^{(n)} \tag{42}$$

$$v_2 = v_0 \sum_{odd\, n=1}^{\infty} \beta^n X^{(n)} \tag{43}$$

where $\tilde{X}^{(n)}$ and $X^{(n)}$ are defined as

$$\tilde{X}^{(0)} \equiv 1, \qquad\qquad X^{(0)} \equiv 1, \tag{44}$$

and for $n \in \mathbb{N}$,

$$\tilde{X}^{(n)}(x) = \begin{cases} \int_0^x \tilde{X}^{(n-1)}(s) v_0^2(s) r(s) ds, & n\ odd, \\ \int_0^x \tilde{X}^{(n-1)}(s) \dfrac{1}{v_0^2(s)p(s)} ds, & n\ even, \end{cases} \tag{45}$$

$$X^{(n)}(x) = \begin{cases} \int_0^x X^{(n-1)}(s) \dfrac{1}{v_0^2(s)p(s)} ds, & n\ odd, \\ \int_0^x X^{(n-1)}(s) v_0^2(s) r(s) ds, & n\ even. \end{cases} \tag{46}$$

With the above recursive formulas, the solution v can be found. The solution consisting of the equations (41)-(43) is a power series in β that is really attractive for the numerical solution of spectral problems, initial value and boundary value problems. Both series in (42) and (43), which are called Spectral Parameter Power Series (SPPS), converge uniformly on [0, d] and as shown in [16] it is easy to obtain a rough but useful estimate of the rest of the series. This estimate provides a simple tool to calculate the number of powers N which guarantees an a priori established accuracy.

The required non vanishing particular solution v_0 of (42) and (43), at least in the case of a regular equation with real-valued coefficients, always exists and and can be easily constructed as follows [18]: take any two linearly independent solutions $v_{0,1}$ y $v_{0,2}$ of (40), then their zeros do not coincide (otherwise their Wronskian is zero and are not linearly independent) and then v_0 can be chosen as $v_0 = v_{0,1} + iv_{0,2}$.

Then v_0 can be constructed in a manner similar to the solutions v_1 and v_2 considering a special case of the already presented result when $q \equiv 0$ and $\beta = 1$, which was already known to Weyl [22]. That is, let $1/p$ and r be continuous on [0, d]. The general solution of equation $(pv')' = rv$ in (0, d) has the form (41) where c_1 and c_2 are arbitrary constantsand v_1 and v_2 are defined by equations (42)-(46) with $v_0 \equiv \beta = 1$.

As a special case another important situation is considered. Very often in electromagnetic theory (see for example [8]) it is necessary to solve the equation

$$-\frac{d^2v(x)}{dx^2} + k^2 q(x) v(x) = 0 \tag{47}$$

for different values of the complex constant k^2. It can be seen that the above equation is practically the same as (7). Its general solution can be represented as follows

$$v = c_1 v_3 + c_2 v_4 \tag{48}$$

where

$$v_3 = \sum_{even\, n=0}^{\infty} k^n \widetilde{W}^{(n)}$$

$$v_4 = \sum_{odd\, n=1}^{\infty} k^{n-1} W^{(n)}$$

with $\widetilde{W}^{(n)}$ and $W^{(n)}$ defined by

$$\widetilde{W}^{(0)} \equiv 1, \qquad\qquad W^{(0)} \equiv 1, \tag{49}$$

and for $n \epsilon \mathbb{N}$,

$$\widetilde{W}^{(n)}\,(x) = \begin{cases} \int_0^x \widetilde{W}^{(n-1)}(s)q(s)ds, & n\, odd, \\ \int_0^x \widetilde{W}^{(n-1)}(s)ds, & n\ even, \end{cases} \tag{50}$$

$$W^{(n)}(x) = \begin{cases} \int_0^x W^{(n-1)}(s)ds, & n\, odd, \\ \int_0^x W^{(n-1)}(s)q(s)ds, & n\ even. \end{cases} \tag{51}$$

and for c_1 and c_2 two arbitrary complex constants. Thus, once $\widetilde{W}^{(n)}$ and $W^{(n)}$ are calculated up to a certain power *N*, an approximate solution of (47) is simply a polynomial on k with its calculated coefficients $\widetilde{W}^{(n)}$ and $W^{(n)}$. This observation is also valid for the case of solution (41), (42) and (43) of equation (39). Furthermore this property is very useful for the numerical solution of the corresponding spectral problems, which reduces to finding the zeros of the polynomials with respect to k and β respectively [18].

5 CALCULATIONS

5.1 Calculation for profiles without turning points

Initially, to ensure that the programs developed for the WKB and SPPS methods were functioning correctly, we studied some well known profiles having analytic solutions based on Bessel functions [23, 24]. Such profiles are the linear, exponential and hyperbolic profiles.

For example for the exponential profile its refractive index profile is described by

$$n(x) = n_2 \cdot \exp\left[\frac{x}{d} \ln\left(\frac{n_2{}'}{n_2}\right)\right] \ . \tag{52}$$

The following data were used in our programs:

- Incidence angle $\theta_i = 0°$, that is, normal incidence
- Size of the non homogeneous medium $d = 1 \times 10^{-6}$ m
- Number of points in the evaluated interval $[0\ d]$: 1000
- Refractive index of medium in region I, $n_1 = 1$
- Refractive index of medium in region III, $n_3 = 1.5$
- Refractive index of the boundary of medium 1, $n_2 = 1.4$

- Refractive index of the boundary of medium 3, $n_2{}' = 2.1$
- Wavelengths, approximately from 2 μm to 100 μm
- Frequencies, approximately from 3 THz to 150 THz.

The program used to implement the SPPS method performed the calculations using 31 formal powers, approximating the functions needed to calculate the recursive integrals by splines of order two and using 500 segments for integrating. From the parameters (p, q and r) in (39) one can obtain a particular solution v_0 for (40). The same parameters are used to calculate the formal powers (45)-(46), with which two linearly independent solutions are found according to formulas (42) and (43) which in turn are used to find the solution (41) that will be a solution of (39).

The obtained results are shown in the following figures.

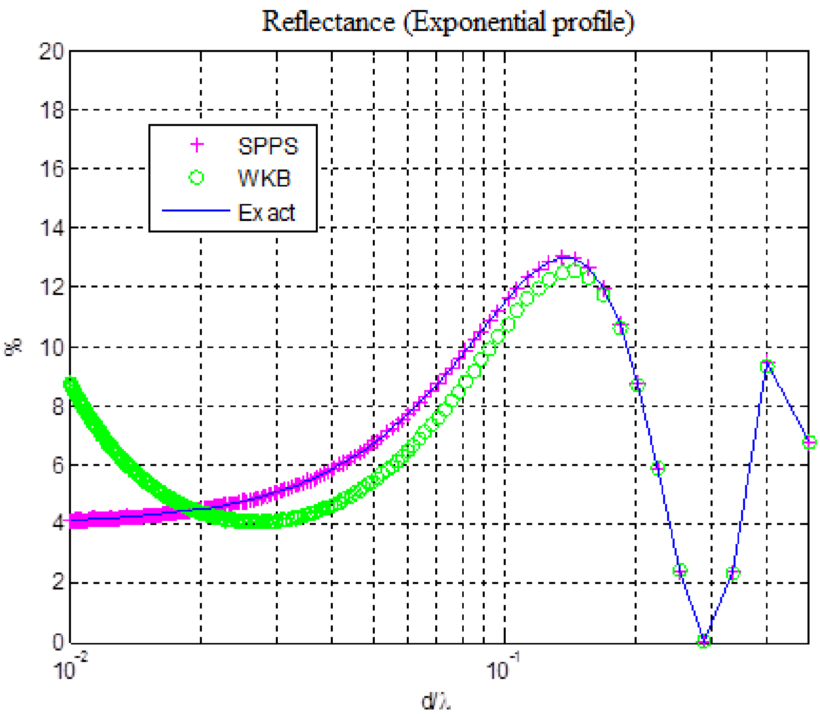

Figure 3. Reflectance for the exponential profile.

The elapsed time for the program which implemented the WKB method was approximately of 1.5 seconds, using a laptop computer with an Intel Core i7 2.8 GHz processor, with 8 GB of RAM. The SPPS method for its execution needed a time of around 11 minutes.

Another comparison which was performed was estimating the absolute error of the two methods in comparison with the exact solutions for each of the profiles. Some of the resulting graphs are shown in Figures 5 and 6.

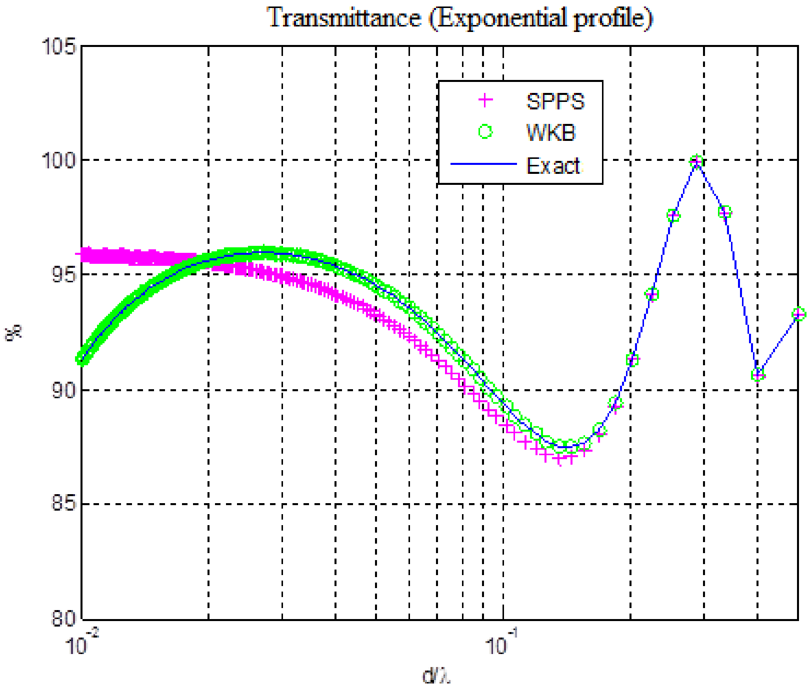

Figure 4. Transmittance for the exponential profile.

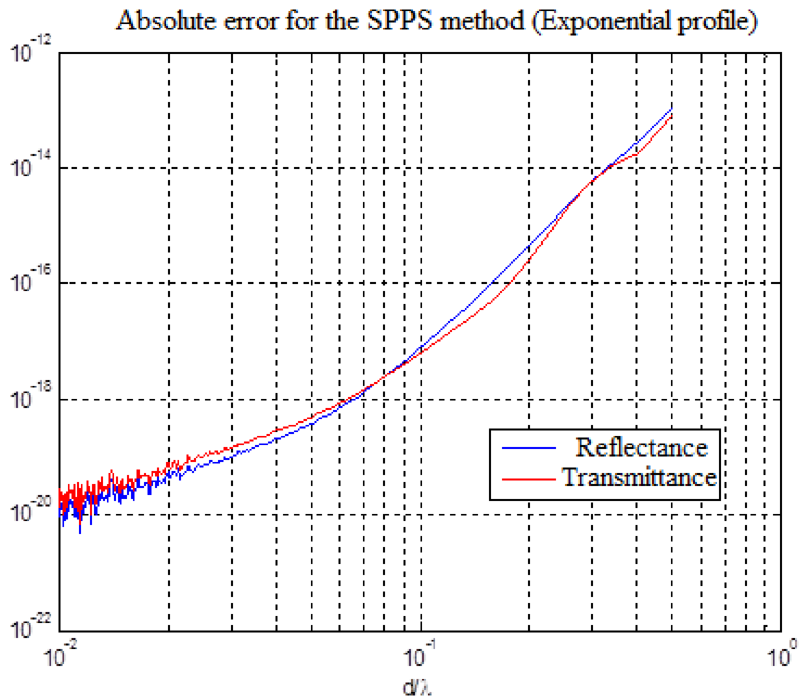

Figure 5. Absolute error for the exponential profile using the SPPS method.

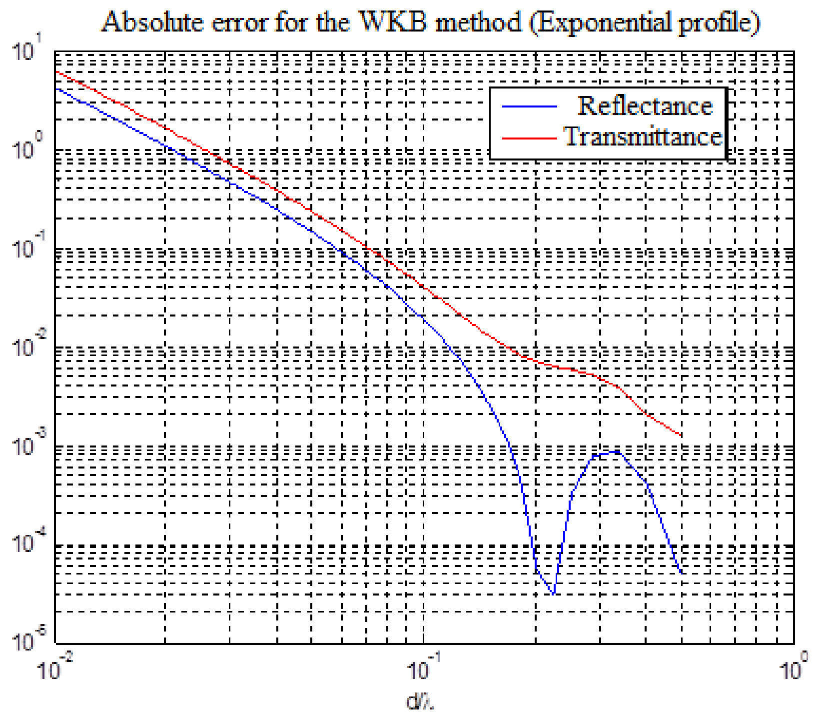

Figure 6. Absolute error for the exponential profile using the WKB method.

5.2 Calculation for profiles with turning points

In order to study profiles with turning points we proceeded to enter into the program that implements the WKB solution a profile $n(x)$ with a zero crossing. It is shown in Figure 7.

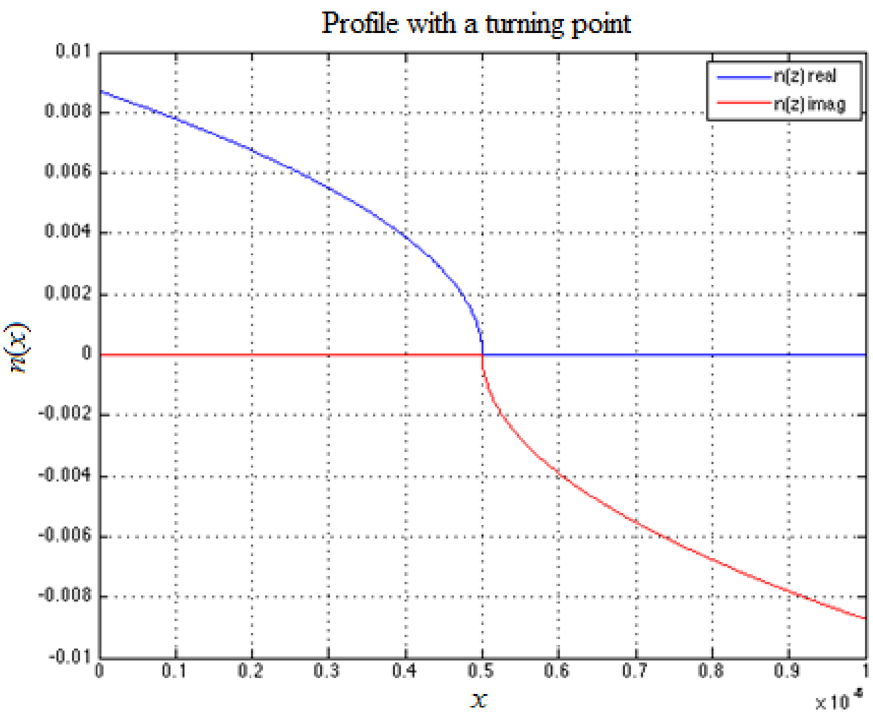

Figure 7. Profile for $n(x)$ with turning points with $\boldsymbol{x_0 = 0.5 \times 10^{-6}}$.

The percentage of the reflectance and transmittance was obtained with the modified WKB method for a profile with turning points replacing (33) and (34) in the expressions (5) and (6) and their values are shown in Figure 8.

The same profile with turning points was tested with the use of the SPPS method. The results are shown in Figure 9.

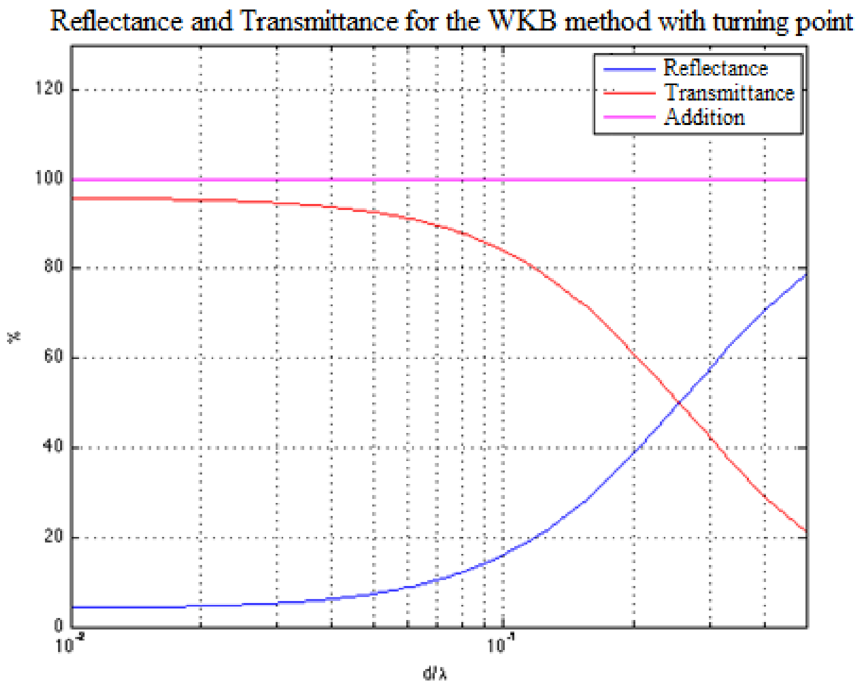

Figure 8. Reflectance and transmittance calculated with the modified WKB method for turning points.

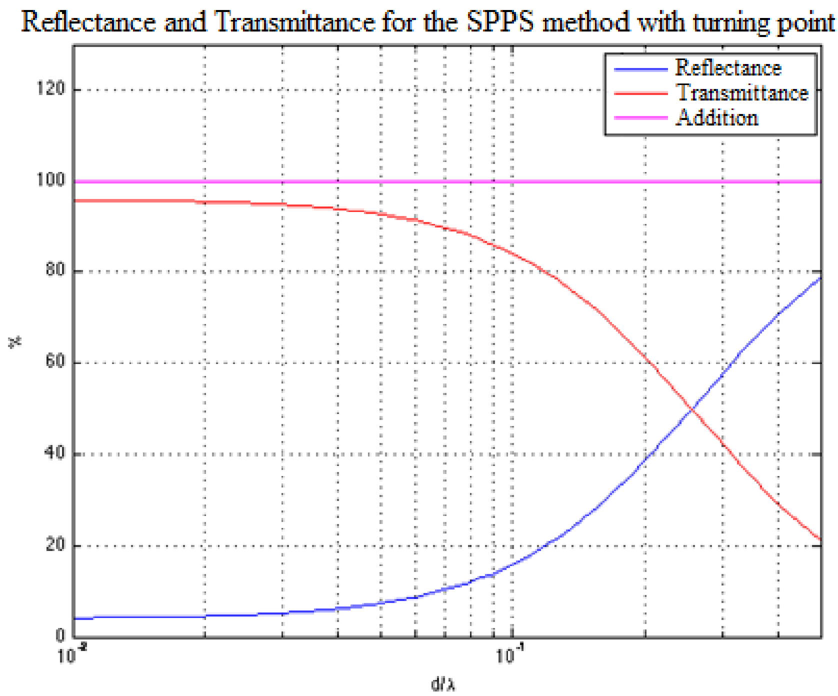

Figure 9. Reflectance and transmittance calculated with the modified SPPS method for turning points.

It is worth noting that for the SPPS method no changes at all are needed for its implementation (which implies another advantage of it). The WKB method should be specifically adjusted in order to work with profiles with turning points in which $q^2(x)$ is arbitrary.

6 CONCLUSIONS

Up to date there was no comparative analysis between the SPPS method and the WKB method, and it was presented in this paper. The numerical implementation of both methods does not represent any difficulty.

For the first time different profiles with turning points were analyzed using the SPPS method. Note that the SPPS method has very few limitations in terms of the profiles it can evaluate compared with the WKB method which requires a more thorough work. As an example, unlike the WKB method, no modifications are required in the case of the profile with turning points for the SPPS method.

Talking about computation time, the WKB method is much faster (2 seconds) compared with the SPPS method (11 minutes). The SPPS method's precision was much higher because in comparison with the exact solutions, in the worst case an accuracy of 10^{-13} was obtained. In contrast, we found that the WKB method in the best case could only deliver an accuracy of 10^{-4}.

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