# Decomposition theory of forms

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PhD Thesis

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# Table of Contents


Chapter 1. Introduction 3

1.1. Notions, notations 4

1.2. Examples, decomposition theorems 6

Chapter 2. The $(\ll_{\mathrm{ac}}, \perp)$-type decomposition 11

2.1. Short-type decomposition of forms 11

2.2. Applications 13

2.2.1. Bounded positive operators 13

2.2.2. Additive set functions 16

2.2.3. Representable functionals 18

Chapter 3. The $(\ll_{\mathrm{cl}}, \perp)$-type decomposition 23

3.1. Lebesgue-type decomposition 23

3.2. Application to bounded charges 29

Chapter 4. The $(\ll_{\mathrm{ad}}, \perp)$-type decomposition 35

4.1. Parallel sum 35

4.2. Hassi - Sebestyén - de Snoo decomposition of forms 38

4.3. Lebesgue decomposition of contents 40

Chapter 5. The uniqueness of short- and Lebesgue-type decompositions 47

5.1. Almost domination and closability 47

5.2. A decomposition of densely defined closable operators 50

5.3. Uniqueness of the Lebesgue-type decomposition 52

Chapter 6. Parallel sum and parallel difference 57

6.1. The equivalence of almost dominated parts 57

6.2. The parallel difference as a minimal solution 59

Chapter 7. Radon–Nikodym-type theorems 63

Chapter 8. Extremal problems for forms 69

8.1. Infimum of forms 69

8.2. Extreme points of form segments 73

Chapter 9. Overview 79

Bibliography 89

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CHAPTER 1

# Introduction

Our aim in the present dissertation is to continue the investigations begun by Seppo Hassi, Zoltán Sebestyén, and Henk de Snoo in their fundamental work [**16**]. Namely, we study the decomposition theory of nonnegative sesquilinear forms.

The thesis is based mainly on the author's papers [**32, 33, 34, 35, 36, 37, 54, 55, 56, 57, 58, 59**]. For the sake of completeness, we include some important results of [**16**].

In this short introductory chapter we recall some basic facts about sesquilinear forms. Furthermore, we present some important examples and decomposition theorems which serve as motivation for this work.

In Chapter 2. we present the $(\ll_{\mathrm{ac}}, \perp)$-type decomposition of forms. The key notion is the short of a form to a linear subspace. This is a generalization of the well-known operator short defined by M. G. Krein [**23**]. A decomposition of a form into a shorted (or absolutely continuous) part and a singular part is called short-type decomposition. As applications, we present some analogous results for bounded positive operators acting on a Hilbert space, for additive set functions on a ring of sets, and for representable positive functionals on a $^*$-algebra.

In Chapter 3. we prove that the $(\ll_{\mathrm{cl}}, \perp)$-type (or Lebesgue-type) decomposition of forms exists. The basic tool in our treatment is the embedding operator between two auxiliary Hilbert spaces associated to the forms in question. As an application of our approach, we also provide a Lebesgue-type decomposition theorem for bounded finitely additive set functions defined on set-rings.

Chapter 4. and Chapter 5. deal mainly with the existence and uniqueness of the so-called $(\ll_{\mathrm{ad}}, \perp)$-type decomposition. These results were proved by Seppo Hassi, Zoltán Sebestyén, and Henk de Snoo in [**16**]. We also show that how this approach can be applied for contents (i.e., for additive set functions defined on set-algebras).

In Chapter 6. we collect some theorems from [**34, 36, 55**]. For example, we identify the parallel difference as the minimal solution of an appropriate equation. We also prove that the almost dominated parts in the mutual Lebesgue decomposition are mutually almost dominated.

The classical Radon-Nikodym theorem can be phrased by means of Hilbert space operators. Namely, by considering the Radon-Nikodym derivative as a (positive self-adjoint) multiplication operator on $L^2(\nu)$. The main result of Chapter 7. is an analogous theorem for forms. We will prove that the quadratic form of the $\mathfrak{w}$-regular part of $\mathfrak{t}$ is derived from a positive self-adjoint operator acting on the Hilbert space associated to the form $\mathfrak{w}$.

It turns out that the Lebesgue-type decomposition is strongly connected with some problems regarding the order structure of forms. Chapter 8. deals with these problems, namely with the characterization of the existence of the infimum of two forms, and the description of the extreme points of form segments.

In the last chapter we make a short overview of our results. Moreover, we show also that how this general theory can be used for applications.

## 1.1. Notions, notations

Let $\mathfrak{X}$ be a complex linear space and let $\mathfrak{t}$ be a nonnegative sesquilinear form (or semi-inner product) on it. That is, $\mathfrak{t}$ is a mapping from the Cartesian product $\mathfrak{X} \times \mathfrak{X}$ to $\mathbb{C}$, which is linear in the first entry, conjugate linear in the second entry, and the corresponding quadratic form $\mathfrak{t}[\,\cdot\,] : \mathfrak{X} \to \mathbb{R}$

$$\forall x \in \mathfrak{X} : \quad \mathfrak{t}[x] := \mathfrak{t}(x,x)$$

is nonnegative. In this thesis all sesquilinear forms are assumed to be nonnegative (unless otherwise stated), hence we write shortly *form*.

The following theorem is the so-called Cauchy-Schwarz inequality for forms (cf. [**61**]).

**Theorem 1.1.** *Let $\mathfrak{t}$ be a form on $\mathfrak{X}$, then for every $x, y \in \mathfrak{X}$ we have*

$$|\mathfrak{t}(x,y)|^2 \leq \mathfrak{t}[x]\mathfrak{t}[y].$$

*Proof.* Let $x, y \in \mathfrak{X}$. For every $\alpha \in \mathbb{R}$ we have

$$0 \leq \mathfrak{t}[x + \alpha y] = \mathfrak{t}[x] + \alpha^2 \mathfrak{t}[y] + 2\alpha \mathfrak{Re}(\mathfrak{t}(x,y)).$$

This is a second degree polynomial in $\alpha$ which has either no root or double root. In any case,

$$\big(\mathfrak{Re}(\mathfrak{t}(x,y))\big)^2 \leq \mathfrak{t}[x]\mathfrak{t}[y].$$

If one chooses a $z \in \mathbb{C}$, $|z| = 1$ such that $|\mathfrak{t}(x,y)| = z\mathfrak{t}(x,y)$ then

$$|\mathfrak{t}(x,y)|^2 = \big(\mathfrak{Re}(z\mathfrak{t}(x,y))\big)^2 = \big(\mathfrak{Re}(\mathfrak{t}(zx,y))\big)^2 \leq \mathfrak{t}[zx]\mathfrak{t}[y] = |z|^2\mathfrak{t}[x]\mathfrak{t}[y] = \mathfrak{t}[x]\mathfrak{t}[y].$$

□

Using the Cauchy-Schwarz inequality it is easy to show that the square root of the quadratic form is a seminorm on $\mathfrak{X}$. Furthermore, it fulfills the parallelogram law

$$\forall x, y \in \mathfrak{X}: \quad \mathfrak{t}[x+y] + \mathfrak{t}[x-y] = 2(\mathfrak{t}[x] + \mathfrak{t}[y])$$

and the polarization identity

$$\forall x, y \in \mathfrak{X}: \quad \mathfrak{t}(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k \mathfrak{t}[x + i^k y].$$

Denote by $\ker \mathfrak{t}$ the kernel of the quadratic form, i.e. $\ker \mathfrak{t} = \{x \in \mathfrak{X} \,|\, \mathfrak{t}[x] = 0\}$. Since the square root of the quadratic form is a seminorm, the set $\ker \mathfrak{t}$ is a linear subspace of $\mathfrak{X}$. The quotient space $\mathfrak{X}/_{\ker \mathfrak{t}}$ with the inner product $(\cdot \,|\, \cdot)_{\mathfrak{t}}$ defined by

$$\big(x + \ker \mathfrak{t} \,|\, y + \ker \mathfrak{t}\big)_{\mathfrak{t}} := \mathfrak{t}(x,y) \quad (x, y \in \mathfrak{X})$$

is an inner product space. The completion $\mathscr{H}_{\mathfrak{t}}$ of this inner product space is called *the Hilbert space associated to* $\mathfrak{t}$.

If $\mathfrak{t}$ and $\mathfrak{w}$ are forms on $\mathfrak{X}$ and $c \geq 0$ is a constant, then the form $\mathfrak{t} + c\mathfrak{w}$ is defined by $(\mathfrak{t} + c\mathfrak{w})[x] := \mathfrak{t}[x] + c\mathfrak{t}[x]$ for all $x \in \mathfrak{X}$. The positive cone of forms (denoted by $\mathcal{F}_+(\mathfrak{X})$) is partially ordered with respect to the ordering

$$\mathfrak{t} \leq \mathfrak{w} \iff \forall x \in \mathfrak{X}: \quad \mathfrak{t}[x] \leq \mathfrak{w}[x].$$

If $\mathfrak{w} \leq \mathfrak{t}$ we say that $\mathfrak{w}$ is *majorized* by $\mathfrak{t}$. The convex set of forms that are majorized by $\mathfrak{t}$ will be denoted by

$$[\mathfrak{o}, \mathfrak{t}] = \big\{\mathfrak{w} \in \mathcal{F}_+(\mathfrak{X}) \,|\, \mathfrak{o} \leq \mathfrak{w} \leq \mathfrak{t}\big\},$$

where $\mathfrak{o}$ denotes the identically zero form. The sequence $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ is said to be nondecreasing (resp., nonincreasing) if $m \leq n$ implies that $\mathfrak{t}_m \leq \mathfrak{t}_n$ (resp., $\mathfrak{t}_n \leq \mathfrak{t}_m$). If $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of forms which is majorized by the form $\mathfrak{w}$ (i.e., $\mathfrak{t}_n \leq \mathfrak{w}$ for all $n \in \mathbb{N}$), then the pointwise supremum $\mathfrak{t}[x] := \sup\limits_{n\in\mathbb{N}} \mathfrak{t}_n[x]$ for all $x \in \mathfrak{X}$ defines a form such that $\mathfrak{t} \leq \mathfrak{w}$. Similarly, if $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ is nonincreasing sequence of forms, then the pointwise infimum $\mathfrak{t}[x] := \inf\limits_{n\in\mathbb{N}} \mathfrak{t}_n[x]$ for all $x \in \mathfrak{X}$ defines a form such that $\mathfrak{o} \leq \mathfrak{t} \leq \mathfrak{t}_1$. To prove these statements it is enough to observe that quadratic form of the limit object defines a seminorm which satisfies the parallelogram identity.

If there exists a constant $c$ such that $\mathfrak{t} \leq c\mathfrak{w}$ then we say that $\mathfrak{t}$ *is dominated by* $\mathfrak{w}$ ($\mathfrak{t} \leq_{\mathrm{d}} \mathfrak{w}$, in symbols). If there exists a nondecreasing sequence $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ of $\mathfrak{w}$-dominated forms such that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ then $\mathfrak{t}$ is called $\mathfrak{w}$-*almost dominated.* We say that the form $\mathfrak{t}$ is $\mathfrak{w}$-*closable*, or *strongly* $\mathfrak{w}$-*absolutely continuous* (denoted by the symbols $\mathfrak{t} \ll_{\mathrm{cl}} \mathfrak{w}$, or $\mathfrak{t} \ll_{\mathrm{s}} \mathfrak{w}$, respectively), if

$$\forall (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}: \quad \big((\mathfrak{t}[x_n - x_m] \to 0) \ \wedge \ (\mathfrak{w}[x_n] \to 0)\big) \implies \mathfrak{t}[x_n] \to 0.$$

We shall prove later that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated precisely when $\mathfrak{w}$-closable.

We say that $\mathfrak{t}$ is $\mathfrak{w}$-*absolutely continuous* ($\mathfrak{t} \ll_{\mathrm{ac}} \mathfrak{w}$) if $\ker \mathfrak{w} \subseteq \ker \mathfrak{t}$, that is to say,

$$\forall x \in \mathfrak{X}: \quad \mathfrak{w}[x] = 0 \implies \mathfrak{t}[x] = 0$$

in analogy with the well-known measure case. Remark that $\mathfrak{w}$-strong absolute continuity implies $\mathfrak{w}$-absolute continuity. To see this consider e.g. constant sequences $(x_n)_{n\in\mathbb{N}}$, $x_n \equiv x \in \ker \mathfrak{w}$ in the definition of $\mathfrak{w}$-strong absolute continuity.

Finally, we say that $\mathfrak{t}$ and $\mathfrak{w}$ are *singular* (in symbols: $\mathfrak{t} \perp \mathfrak{w}$) if the only form which is majorized by both $\mathfrak{t}$ and $\mathfrak{w}$ is the identically zero form, i.e.,

$$\forall \mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}): \quad \big((\mathfrak{s} \le \mathfrak{t}) \ \wedge \ (\mathfrak{s} \le \mathfrak{w})\big) \implies \mathfrak{s} = \mathfrak{o}.$$

The main aim of this dissertation is to prove decomposition theorems in the following fashion: if $\mathfrak{t}$ and $\mathfrak{w}$ are forms, we say that $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ is a $(\ll_\bullet, \perp)$-type decomposition if $\mathfrak{t}_1 \ll_\bullet \mathfrak{w}$ and $\mathfrak{t}_2 \perp \mathfrak{w}$. Here, of course, $\ll_\bullet$ means $\ll_{\mathrm{ac}}$, $\ll_{\mathrm{cl}}$, or $\ll_{\mathrm{ad}}$.

## 1.2. Examples, decomposition theorems

In this section we present some important examples. We mention also some decomposition theorems, which will be investigated later.

**Example 1.2.** Let $\mathfrak{X} = \mathscr{H}$ be a complex Hilbert space with the inner product $(\cdot\,|\,\cdot)$ and denote by $\mathbf{B}_+(\mathscr{H}) = \{A : \ A \in \mathbf{B}(\mathscr{H}), \ 0 \le A\}$ the cone of the bounded positive operators on $\mathscr{H}$. The notion $\le$ always stands for the following relation: $0 \le A$ if $0 \le (Ah, h)$ for all $h \in \mathscr{H}$. We say that $A$ is $B$-absolutely continuous if $A = \sup_{n\in\mathbb{N}} A_n$ for some nondecreasing sequence of $B$-dominated operators. Singularity of $A$ and $B$ means that the greatest lower bound of $A$ and $B$ equals to the identically zero operator.

If $A \in \mathbf{B}_+(\mathscr{H})$ then its induced form will be denoted by $\mathfrak{t}_A$, that is

$$\forall x, y \in \mathfrak{X}: \qquad \mathfrak{t}_A(x, y) := (Ax\,|\,y).$$

If $\mathscr{H}$ is finite dimensional, then $\mathbf{B}_+(\mathscr{H})$ is the cone of positive semidefinite matrices. This special case will play an important role in the next sections.

The following theorem of Ando is the so-called Lebesgue-type decomposition of positive operators. For the details see [**4, 49**].

**Theorem 1.3.** *Let $A$ and $B$ be bounded positive operators on $\mathscr{H}$. Then there is a decomposition of $A$ with respect to $B$ into $B$-absolutely continuous and $B$-singular parts.*

**Example 1.4.** Let $X$ be a non-empty set, let $\mathfrak{A}$ be a set-algebra on it, and let $\mu$ be *content*, i.e., a nonnegative real valued additive set function on $\mathfrak{A}$. Remark that such a function is always bounded because $\mu(A) \leq \mu(X)$ for all $A \in \mathfrak{A}$, and $\mu(X) \in \mathbb{R}$. We say that $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ holds for all $A \in \mathfrak{A}$. If $\mathfrak{A}$ is a $\sigma$-algebra and $\mu$ is $\sigma$ additive, we say that $\mu$ is a *measure*. Of course, any measure is a content as well, and all results in this thesis about contents hold also for measures. The key observation which is needed is the following: if a content $\mu$ is dominated by a measure, then $\mu$ is $\sigma$-additive.

The partially ordered set of contents is a lattice, i.e., for every pair of contents there exist the greatest lower and the least upper bound. Moreover, we can formulate the greatest lower bound of $\mu$ and $\nu$ ($\mu \wedge \nu$, in symbols) with the following infimum

$$\big(\mu \wedge \nu\big)(A) = \inf_{B \in \mathfrak{A}} \big\{\mu(A \setminus B) + \nu(A \cap B)\big\} \quad (A \in \mathfrak{A}).$$

Using König's characterization (see [**21**] and Chapter 3), we say that $\mu$ is strongly absolutely continuous with respect to $\nu$ if $\mu = \sup\limits_{n \in \mathbb{N}} \mu \wedge n\nu$ and $\mu$ is $\nu$-singular if $\mu \wedge \nu = 0$. For a different characterization of strong absolute continuity we refer the reader to [**6**].

Let the complex linear space $\mathfrak{X}$ be the set of $\mathfrak{A}$-simple functions (denoted by $\mathcal{E}$), i.e. the complex linear span of the characteristic functions of the sets in $\mathfrak{A}$

$$\mathfrak{X} := \mathcal{E} = \operatorname{span}_{\mathbb{C}}\{\chi_A : A \in \mathfrak{A}\}.$$

Recall that if $\varphi \in \mathfrak{X}$, i.e., $\varphi$ is an $\mathfrak{A}$-simple function, it is expressible in the form

$$\varphi = \sum_{j=1}^{k} \lambda_j \chi_{A_j},$$

where $\lambda_1, \ldots, \lambda_k$ are non-zero complex numbers and $A_1, \ldots, A_k$ are pairwise disjoint elements of $\mathfrak{A}$. Hereafter we always assume that $A_1, \ldots, A_k$ are pairwise disjoint and nonempty if we write $\varphi = \sum\limits_{j=1}^{k} \lambda_j \chi_{A_j}$.

Let $\mu$ be a content on the algebra $\mathfrak{A}$, and define the form induced by $\mu$

$$\mathfrak{t}_\mu(\varphi, \psi) := \int\limits_X \varphi \cdot \overline{\psi}\, \mathrm{d}\mu \quad (\varphi, \psi \in \mathfrak{X}).$$

The next theorem is the so-called Lebesgue-Darst decomposition theorem. For the details see Chapter 3 or [**9, 51**].

**Theorem 1.5.** *Let $\mu$ and $\nu$ be bounded contents on the algebra $\mathfrak{A}$. Then $\mu$ splits uniquely into $\nu$-strongly absolutely continuous and $\nu$-singular parts.*

**Example 1.6.** We emphasize in advance that the set functions in this example are not assumed to be bounded. In order to distinguish this case to the previous one, we use different notations. Let $T$ be a non-empty set, and let $\mathscr{R}$ be a ring of some subsets of $T$. Let $\alpha$ be a finitely additive nonnegative set function (in this case we shall say that $\alpha$ is a *charge*) on $\mathscr{R}$.

Similarly as in the previous example, the partially ordered set of charges is a lattice, and the greatest lower bound of $\alpha$ and $\beta$ can be written as

$$\big(\alpha \wedge \beta\big)(R) = \inf_{S \in \mathscr{R}} \big\{\alpha(R \setminus S) + \beta(R \cap S)\big\} \qquad (R \in \mathscr{R}).$$

Let us define the form associated to $\alpha$ over the complex linear space $\mathfrak{X} := \mathscr{S}(T, \mathscr{R})$ of $\mathscr{R}$-simple functions by

$$\mathfrak{t}_\alpha(\varphi, \psi) := \int_T \varphi \cdot \overline{\psi}\, \mathrm{d}\alpha.$$

Remark that the right-hand side is just a finite sum again.

**Example 1.7.** Let $\mathscr{A}$ be a complex $^*$-algebra and let $f : \mathscr{A} \to \mathbb{C}$ be a positive linear functional on it (that is, $f(a^*a) \geq 0$ for all $a \in \mathscr{A}$). The form induced by $f$ will be denoted by $\mathfrak{t}_f$

$$\mathfrak{t}_f(a, b) = f(b^*a).$$

The following is the Lebesgue-type decomposition theorem for representable positive functionals. For the details and other interesting results see [**14, 22, 45, 46, 47, 52**].

**Theorem 1.8.** *Let $\mathscr{A}$ be a $^*$-algebra, let $f$ and $g$ be representable positive functionals on $\mathscr{A}$. Then there exists a Lebesgue-type decomposition of $g$ with respect to $f$*

$$g = g_a + g_s.$$

*That is to say, both $g_a$ and $g_s$ are representable functionals such that $g_a$ is a pointwise limit of a nondecreasing sequence of $f$-dominated functionals, and that $g_s$ and $f$ are mutually singular in the order sense.*

**Example 1.9.** Let $S$ be a non-empty set, and let $\mathfrak{E}$ be a complex Banach space (with topological dual $\mathfrak{E}^*$). The dual pairing of $x \in \mathfrak{E}$ and $x^* \in \mathfrak{E}^*$ is denoted by $\langle x, x^* \rangle$. Here the mapping

$$\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E}^* \to \mathbb{C}$$

is linear in its first, conjugate linear in its second variable. The Banach space of bounded linear operators from $\mathfrak{E}$ to $\mathfrak{E}^*$ will be denoted by $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$.

Let $\mathfrak{X}$ be the complex linear space of functions on $S$ with values in $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$ with finite support. We say that the function

$$\mathsf{K} : S \times S \to \mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$$

is a *positive definite operator function*, or shortly a *kernel* on $S$ if

$$\forall f \in \mathfrak{X} : \qquad \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) f(s) \rangle \geq 0.$$

We associate a form with $\mathsf{K}$ by setting

$$\forall f, g \in \mathfrak{X} : \qquad \mathfrak{w}_{\mathsf{K}}(f, g) := \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) g(s) \rangle.$$

The set of kernels will be denoted by $\mathcal{K}_+(\mathfrak{X})$. If $\mathsf{K}$ and $\mathsf{L}$ are kernels, we write $\mathsf{K} \prec \mathsf{L}$ if $\mathfrak{w}_{\mathsf{K}} \leq \mathfrak{w}_{\mathsf{L}}$.

The following is the Lebesgue-type decomposition of positive definite operator functions. For the definitions and other details see [**7**] or Chapter 9.

**Theorem 1.10.** *Let $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ be kernels on $S$. Then $\mathsf{K}$ splits into strongly $\mathsf{L}$-absolutely continuous and $\mathsf{L}$-singular parts.*

CHAPTER 2

# The $(\ll_{\mathrm{ac}}, \perp)$-type decomposition

The main purpose of this chapter is to present an $(\ll_{\mathrm{ac}}, \perp)$-type decomposition for forms. The key notion is the short of a form to a linear subspace. This is a generalization of the well-known operator short defined by M. G. Krein [**23**]. A decomposition of a form into a shorted part and a singular part (with respect to an other form) will be called short-type decomposition. As applications, we present some analogous results for bounded positive operators acting on a Hilbert space; for additive set functions on a ring of sets; and for representable positive functionals on a $*$-algebra. This chapter is based on paper [**33**].

## 2.1. Short-type decomposition of forms

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. The purpose of this section is to show that $\mathfrak{t}$ has a decomposition into a $\mathfrak{w}$-absolutely continuous and a $\mathfrak{w}$-singular part. This type decomposition will be called *short-type decomposition.* The concept of the short of a form, which is introduced in the following lemma, will play an essential role in our further considerations.

**Lemma 2.1.** *Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a linear subspace, and let $\mathfrak{t} \in \mathcal{F}_+(\mathfrak{X})$. Then the following formula defines a form on $\mathfrak{X}$*

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}_{\mathfrak{Y}}[x] := \inf_{y \in \mathfrak{Y}} \mathfrak{t}[x-y].$$

*Furthermore, $\mathfrak{t}_{\mathfrak{Y}}$ is the maximum of the set*

$$\big\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\big|\; (\mathfrak{s} \leq \mathfrak{t}) \;\wedge\; (\mathfrak{Y} \subseteq \ker \mathfrak{s})\big\}.$$

*Proof.* Let $\mathfrak{Y}_{\mathfrak{t}}$ be the following subspace of $\mathscr{H}_{\mathfrak{t}}$

$$\mathfrak{Y}_{\mathfrak{t}} := \big\{y + \ker \mathfrak{t} \;\big|\; y \in \mathfrak{Y}\big\}$$

and consider the orthogonal projection $P$ from $\mathscr{H}_{\mathfrak{t}}$ onto $\overline{\mathfrak{Y}_{\mathfrak{t}}}$ (the closure of $\mathfrak{Y}_{\mathfrak{t}}$). Then for all $x \in \mathfrak{X}$

$$\big\|(I-P)(x+\ker \mathfrak{t})\big\|_{\mathfrak{t}}^2 = \operatorname{dist}^2(x+\ker \mathfrak{t}, \overline{\mathfrak{Y}_{\mathfrak{t}}}) = \inf_{y \in \mathfrak{Y}} \big\|(x-y)+\ker \mathfrak{t}\big\|_{\mathfrak{t}}^2 = \inf_{y \in \mathfrak{Y}} \mathfrak{t}[x-y].$$

Consequently, $\mathfrak{t}_{\mathfrak{Y}}$ is a form, indeed, and $\mathfrak{Y} \subseteq \ker \mathfrak{t}_{\mathfrak{Y}}$. To show the maximality, assume that the quadratic form of $\mathfrak{s}$ vanishes on $\mathfrak{Y}$ and $\mathfrak{s} \le \mathfrak{t}$. According to the triangle inequality we have

$$\mathfrak{s}[x] \le \mathfrak{s}[x-y] \le \mathfrak{t}[x-y]$$

for all $y \in \mathfrak{Y}$, and hence,

$$\mathfrak{s}[x] \le \inf_{y\in\mathfrak{Y}} \mathfrak{t}[x-y] = \mathfrak{t}_{\mathfrak{Y}}[x].$$

□

The form $\mathfrak{t}_{\mathfrak{Y}}$ is called the *short of the form* $\mathfrak{t}$ *to the subspace* $\mathfrak{Y}$.
It follows from the definition that if $\mathfrak{t}$ and $\mathfrak{w}$ are forms and $\mathfrak{Y}$ and $\mathfrak{Z}$ are linear subspaces, then

$$\big((\mathfrak{t} \le \mathfrak{w}) \quad \wedge \quad \mathfrak{Y} \subseteq \mathfrak{Z}\big) \quad \Longrightarrow \quad \mathfrak{t}_{\mathfrak{Z}} \le \mathfrak{w}_{\mathfrak{Y}}.$$

Now, we are in position to state and prove the main result of this chapter.

**Theorem 2.2.** *Let* $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *be forms. Then there exists a* $(\ll_{\mathrm{ac}}, \perp)$*-type decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. Namely,*

$$\mathfrak{t} = \mathfrak{t}_{\ker \mathfrak{w}} + (\mathfrak{t} - \mathfrak{t}_{\ker \mathfrak{w}}),$$

*where the first summand is* $\mathfrak{w}$*-absolutely continuous and the second one is* $\mathfrak{w}$*-singular. Furthermore,* $\mathfrak{t}_{\ker \mathfrak{w}}$ *is the maximum of the set*

$$\big\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\big|\; (\mathfrak{s} \le \mathfrak{t}) \;\wedge\; (\mathfrak{s} \ll_{\mathrm{ac}} \mathfrak{w})\big\}.$$

*Proof.* It follows from the previous lemma that $\mathfrak{t}_{\ker \mathfrak{w}} \ll_{\mathrm{ac}} \mathfrak{w}$, and that $\mathfrak{t}_{\ker \mathfrak{w}}$ is maximal. Let $\mathfrak{s}$ be a form such that $\mathfrak{s} \le \mathfrak{w}$ and $\mathfrak{s} \le \mathfrak{t} - \mathfrak{t}_{\ker \mathfrak{w}}$. Since $\mathfrak{t}_{\ker \mathfrak{w}} \le \mathfrak{t}_{\ker \mathfrak{w}} + \mathfrak{s} \le \mathfrak{t}$ and the quadratic form of $\mathfrak{t}_{\ker \mathfrak{w}} + \mathfrak{s}$ vanishes on $\ker \mathfrak{w}$, the maximality of $\mathfrak{t}_{\ker \mathfrak{w}}$ implies that $\mathfrak{s} = \mathfrak{o}$. □

At the moment we can say the following about the uniqueness of short-type decomposition: if $\mathfrak{t}_{\ker \mathfrak{w}}$ is dominated by $\mathfrak{w}$, then the decomposition is unique. Indeed, let $c$ be a constant such that $\mathfrak{t}_{\ker \mathfrak{w}} \le c\mathfrak{w}$ (we may assume that $c > 1$) and let $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ be an $(\ll_{\mathrm{ac}}, \perp)$-type decomposition. Since $\mathfrak{t}_{\ker \mathfrak{w}}$ is maximal, we have

$$\mathfrak{t}_2 = \mathfrak{t} - \mathfrak{t}_1 \ge \mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1 \ge \frac{1}{c}(\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1) \ge \mathfrak{o} \quad \text{and} \quad \mathfrak{w} \ge \frac{1}{c}\mathfrak{t}_{\ker \mathfrak{w}} \ge \frac{1}{c}(\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1) \ge \mathfrak{o}.$$

Since $\mathfrak{t}_2 \perp \mathfrak{w}$, one concludes that $\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1 = \mathfrak{o}$. We shall see later that the condition $\mathfrak{t}_{\ker \mathfrak{w}} \le_{\mathrm{d}} \mathfrak{w}$ for the uniqueness is not just sufficient, but also necessary.

Finally, observe that $(\mathfrak{t}_{\mathfrak{Y}})_{\mathfrak{Y}} = \mathfrak{t}_{\mathfrak{Y}}$ for each subspace $\mathfrak{Y}$, i.e., shortening to a subspace is an idempotent operation. Furthermore, $\mathfrak{t} \ll_{\mathrm{ac}} \mathfrak{w}$ precisely when $\mathfrak{t}_{\ker \mathfrak{w}} = \mathfrak{t}$.

## 2.2. Applications

In this subsection we apply the previous decomposition theorem for bounded positive operators, for additive set functions, and for representable positive functionals.

**2.2.1. Bounded positive operators.** Let $A \in \mathbf{B}_+(\mathscr{H})$ be a bounded positive operator and consider its induced form $\mathfrak{t}_A$. In view of the Riesz-representation theorem, the correspondence $A \mapsto \mathfrak{t}_A$ defines a bijection between bounded positive operators and bounded nonnegative forms. Consequently, we can define the domination, (strong) absolute continuity, and singularity analogously to the ones defined for forms. We write $A \leq_{\mathrm{d}} B$ if there exists a constant $c$ such that $A \leq cB$. If $Bx = 0$ implies that $Ax = 0$ for all $x \in \mathscr{H}$, we say that $A$ is *$B$-absolutely continuous* ($A \ll_{\mathrm{ac}} B$). The operators $A$ and $B$ are *singular* ($A \perp B$) if 0 is the only positive operator which is dominated by both $A$ and $B$. Finally, $A$ is *$B$-closable*, or $A$ is *strongly $B$-absolutely continuous* ($A \ll_{\mathrm{cl}} B$, in symbols) if for any sequence $(x_n)_{n\in\mathbb{N}} \in \mathscr{H}^{\mathbb{N}}$

$$\big((A(x_n - x_m)\,|\,x_n - x_m) \to 0 \ \wedge\ (Bx_n\,|\,x_n) \to 0\big) \quad\Rightarrow\quad (Ax_n\,|\,x_n) \to 0.$$

Remark that

$$A \ll_{\mathrm{ac}} B \iff \ker B \subseteq \ker A \quad \text{ and } \quad A \perp B \iff \operatorname{ran} A^{1/2} \cap \operatorname{ran} B^{1/2} = \{0\},$$

see [**4**] or [**49**]. It was proved by Krein in [**23**] that if $\mathscr{M}$ is a closed linear subspace of $\mathscr{H}$ and $A \in \mathbf{B}_+(\mathscr{H})$, then the set

$$\big\{S \in \mathbf{B}_+(\mathscr{H}) \ \big|\ (S \leq A) \ \wedge\ (\operatorname{ran} S \subseteq \mathscr{M})\big\}$$

possesses a greatest element. This follows immediately from our previous results, and this is why we say that the form $\mathfrak{t}_{\mathfrak{Y}}$ is the *short of* $\mathfrak{t}$ *to the subspace* $\mathfrak{Y}$. Indeed, let $\mathfrak{t}(x,y) = (Ax\,|\,y)$ and consider the form $\mathfrak{t}_{\mathscr{M}^\perp}$. Since $\mathfrak{t}_{\mathscr{M}^\perp}$ is a bounded form, there exists a unique $S \in \mathbf{B}_+(\mathscr{H})$ such that $\mathfrak{t}_{\mathscr{M}^\perp}(x,y) = (Sx\,|\,y)$ and

$$x \in \mathscr{M}^\perp \Longrightarrow \mathfrak{t}_{\mathscr{M}^\perp}[x] = 0 \Longrightarrow (Sx\,|\,x) = 0 \Longrightarrow \mathscr{M}^\perp \subseteq \ker S \Longrightarrow \operatorname{ran} S \subseteq \mathscr{M}.$$

The maximality of $S$ follows from the maximality of $\mathfrak{t}_{\mathscr{M}^\perp}$. Now, since the map $A \mapsto \mathfrak{t}_A$ is an order preserving positive homogeneous map from $\mathbf{B}_+(\mathscr{H})$ into $\mathcal{F}_+(\mathscr{H})$, the following theorem is an immediate consequence of Theorem 2.2.

**Theorem 2.3.** *Let $A$ and $B$ be bounded positive operators on $\mathscr{H}$. Then there is a decomposition of $A$ with respect to $B$ into $B$-absolutely continuous and $B$-singular parts. Namely,*

$$A = A_{\ll,B} + A_{\perp,B}.$$

*Proof.* Let $A_{\ll,B}$ and $A_{\perp,B}$ be the operators corresponding to $(\mathfrak{t}_A)_{\ker \mathfrak{t}_B}$ and $\mathfrak{t}_A - (\mathfrak{t}_A)_{\ker \mathfrak{t}_B}$, respectively. □

We remark that the short $A_{\mathscr{M}}$ of $A$ to the closed linear subspace $\mathscr{M}$ of the (complex) Hilbert space $\mathscr{H}$ possesses a factorization of the form

$$A_{\mathscr{M}} = A^{1/2} P_{\widetilde{\mathscr{M}}} A^{1/2},$$

where $P_{\widetilde{\mathscr{M}}}$ is defined to be the orthogonal projection onto the closed subspace $\widetilde{\mathscr{M}} := A^{-1/2}\langle \mathscr{M} \rangle$, see Krein [**23**]. This factorization can hold, of course, only if the underlying space is complex. Below we offer an alternative factorization of the operator short that simultaneously treats the real and complex cases. In fact, we show that there exists a complex Hilbert space $\mathscr{H}_A$, associated with the positive operator $A$, such that $A_{\mathscr{M}}$ admits a factorization of the form $J_A(I-P)J_A^*$ where $J_A$ is the canonical continuous embedding of $\mathscr{H}_A$ into $\mathscr{H}$ and $P$ is the orthogonal projection onto an appropriately defined subspace of $\mathscr{H}_A$, associated with $\mathscr{M}$. The construction below is taken from [**38**].

Let us consider the range space $\operatorname{ran} A$, equipped with the inner product $(\cdot \,|\, \cdot)_A$

$$\forall x, y \in \mathscr{H}: \qquad (Ax \,|\, Ay)_A = (Ax \,|\, y).$$

Note that the operator Schwarz inequality

$$(Ax \,|\, Ax) \le \|A\| (Ax \,|\, x)$$

implies that $(\cdot \,|\, \cdot)_A$ defines an inner product, indeed. Let $\mathscr{H}_A$ stand for the completion of that inner product space. Consider the canonical embedding operator of $\operatorname{ran} A \subseteq \mathscr{H}_A$ into $\mathscr{H}$, defined by

$$\forall x \in \mathscr{H}: \qquad J_A(Ax) := Ax.$$

Then $J_A$ is well defined and continuous due to the operator Schwarz inequality above (namely, by norm bound $\sqrt{\|A\|}$). This mapping has a unique norm preserving extension from $\mathscr{H}_A$ to $\mathscr{H}$ which is denoted by $J_A$ as well. An easy calculation shows that its adjoint $J_A^*$ acts as an operator from $\mathscr{H}$ to $\mathscr{H}_A$ possessing the canonical property

$$\forall x \in \mathscr{H}: \qquad J_A^* x = Ax.$$

This yields the following useful factorization for $A$:

$$A = J_A J_A^*.$$

**Theorem 2.4.** *Let $\mathscr{H}$ be a Hilbert space and let $A \in \mathbf{B}_+(\mathscr{H})$. For a given subspace $\mathscr{M} \subseteq \mathscr{H}$ denote by $P$ the orthogonal projection of $\mathscr{H}_A$ onto the closure of $\{Ax \,|\, x \in \mathscr{M}\}$. Then the short of $A$ to $\mathscr{M}$ equals $J_A(I-P)J_A^*$.*

*Proof.* It is enough to show that the quadratic forms of $J_A(I-P)J_A^*$ and $\mathfrak{t}_{\mathscr{M}^\perp}$ are equal. To verify this let $x \in \mathscr{H}$. Then

$$\begin{aligned}(J_A(I-P)J_A^* x \,|\, x) &= ((I-P)Ax \,|\, (I-P)Ax)_{_A} = \operatorname{dist}^2(Ax, \operatorname{ran} P) \\ &= \inf_{y \in \mathscr{M}} (Ax - Ay \,|\, Ax - Ay)_{_A} = \inf_{y \in \mathscr{M}} (A(x-y) \,|\, x-y) \\ &= \mathfrak{t}_{\mathscr{M}^\perp}[x],\end{aligned}$$

as it is claimed. □

The above construction yields another formula for the quadratic form of the shorted operator:

**Corollary 2.5.** *Let $\mathscr{H}$ be a Hilbert space, $A \in \mathbf{B}_+(\mathscr{H})$ and $\mathscr{M} \subseteq \mathscr{H}$ any closed linear subspace. Then for any $x \in \mathscr{H}$*

$$(J_A(I-P)J_A^* x \,|\, x) = (Ax \,|\, x) - \sup\{|(Ax \,|\, y)|^2 \,|\, y \in \mathscr{M}, (Ay \,|\, y) \le 1\}.$$

*Proof.* For $x \in \mathscr{H}$ we have

$$\begin{aligned}(J_A(I-P)J_A^* x \,|\, x) &= (Ax \,|\, Ax)_{_A} - (P(Ax) \,|\, P(Ax))_{_A} \\ &= (Ax \,|\, x) - \sup\{|(Ax \,|\, Ay)_{_A}|^2 \,|\, y \in \mathscr{M}, (Ay \,|\, Ay)_{_A} \le 1\} \\ &= (Ax \,|\, x) - \sup\{|(Ax \,|\, y)|^2 \,|\, y \in \mathscr{M}, (Ay \,|\, y) \le 1\},\end{aligned}$$

indeed. □

**Corollary 2.6.** *If $A$ and $B$ are bounded positive operators on the Hilbert space $\mathscr{H}$ then the quadratic forms of $A_{\ll,B}$ and $A_{\perp,B}$ can be calculated by the following formulae:*

$$\begin{aligned}(A_{\ll,B} x \,|\, x) &= \inf_{y \in \ker B} (A(x-y) \,|\, x-y), \\ (A_{\perp,B} x \,|\, x) &= \sup\{|(Ax \,|\, y)|^2 \,|\, y \in \ker B, (Ay \,|\, y) \le 1\}.\end{aligned}$$

*Proof.* Since $A_{\ll,B}$ is nothing but the short of $A$ to the closed subspace $\ker B^\perp$, Theorem 2.4 together with the above corollary implies the desired formulae. □

**2.2.2. Additive set functions.** In this section we apply our main theorem for finitely additive nonnegative set functions. We recall first some definitions. Let $T$ be a non-empty set, and let $\mathscr{R}$ be a ring of some subsets of $T$. Let $\alpha$ and $\beta$ be charges on $\mathscr{R}$. We say that the charge $\beta$ is *absolutely continuous with respect to* $\alpha$ (in symbols $\beta \ll_{\rm ac} \alpha$), if $\alpha(R) = 0$ implies $\beta(R) = 0$ for all $R \in \mathscr{R}$. Finally $\beta$ and $\alpha$ are singular if the only charge which is dominated by both $\alpha$ and $\beta$ is the zero charge (or equivalently, $\alpha \wedge \beta = 0$).

Let $\mathscr{S}(T, \mathscr{R})$ be the complex vector space of $\mathscr{R}$-step functions, and for a charge $\alpha$ define the associated form $\mathfrak{t}_\alpha$ as follows:

$$\forall \varphi, \psi \in \mathcal{E}: \quad \mathfrak{t}_\alpha(\varphi, \psi) := \int\limits_T \varphi \cdot \overline{\psi} \, \mathrm{d}\alpha.$$

**Lemma 2.7.** *Let $\alpha$ and $\beta$ be charges on $\mathscr{R}$. Then $\alpha$ is $\beta$-absolutely continuous precisely when $\mathfrak{t}_\alpha$ is $\mathfrak{t}_\beta$-absolutely continuous. Similarly, $\alpha$ and $\beta$ are singular precisely when $\mathfrak{t}_\alpha$ and $\mathfrak{t}_\beta$ are singular.*

*Proof.* First assume that $\alpha \ll_{\rm ac} \beta$, and let $\varphi \in \mathscr{S}(T, \mathscr{R})$ be a step-function such that

$$\mathfrak{t}_\beta[\varphi] = \mathfrak{t}_\beta \left[ \sum_{j=1}^k \lambda_j \chi_{A_j} \right] = \sum_{j=1}^k |\lambda_j|^2 \beta(A_j) = 0.$$

Since the $\lambda_j$'s are non-zero by assumption, it follows that $\beta(A_j) = 0$ for all $1 \le j \le k$. Consequently, $\alpha \ll_{\rm ac} \beta$ implies that

$$0 = \sum_{j=1}^k |\lambda_j|^2 \alpha(A_j) = \mathfrak{t}_\alpha \left[ \sum_{j=1}^k \lambda_j \chi_{A_j} \right] = \mathfrak{t}_\alpha[\varphi].$$

The converse implication is trivial, because if $\beta(R) = 0$ then $\mathfrak{t}_\beta[\chi_R] = 0$ which implies by $\mathfrak{t}_\alpha \ll_{\rm ac} \mathfrak{t}_\beta$ that $\alpha(R) = \mathfrak{t}_\alpha[\chi_R] = 0$. To prove the second statement assume that $\alpha$ and $\beta$ are singular. Let $\mathfrak{t}$ be any form on $\mathscr{S}(T, \mathscr{R})$ such that $\mathfrak{t} \le \mathfrak{t}_\alpha$ and $\mathfrak{t} \le \mathfrak{t}_\beta$. Then for any $E \in \mathscr{R}$ we have

$$\begin{aligned}
0 = (\alpha \wedge \beta)(E) &= \inf\{\mathfrak{t}_\alpha[\chi_{E\cap F}] + \mathfrak{t}_\beta[\chi_{E\setminus F}] \mid F \in \mathscr{R}\} \\
&\ge \inf\{\mathfrak{t}[\chi_{E\cap F}] + \mathfrak{t}[\chi_{E\setminus F}] \mid F \in \mathscr{R}\} \\
&= \inf\left\{ \frac{1}{2}\big( \mathfrak{t}[\chi_{E\cap F} + \chi_{E\setminus F}] + \mathfrak{t}[\chi_{E\setminus F} - \chi_{E\cap F}] \big) \;\Big|\; F \in \mathscr{R} \right\} \\
&\ge \frac{1}{2} \inf\{\mathfrak{t}[\chi_{E\cap F} + \chi_{E\setminus F}] \mid F \in \mathscr{R}\} = \frac{1}{2}\, \mathfrak{t}[\chi_E],
\end{aligned}$$

according to the parallelogram law. Since the square root of the quadratic form of $\mathfrak{t}$ is a seminorm on $\mathscr{S}(T, \mathscr{R})$, it follows from the triangle inequality that $\mathfrak{t} = 0$. The converse implication is obvious because the map $\alpha \mapsto \mathfrak{t}_\alpha$ is order preserving. $\square$

The following lemma plays an essential role in the proof of the short-type decomposition of charges. For every form $\mathfrak{t}$ on $\mathscr{S}(T,\mathscr{R})$ we can associate a nonnegative set function $\vartheta : \mathscr{R} \mapsto \mathbb{R}_+$ to $\mathfrak{t}$ by

$$E \mapsto \mathfrak{t}[\chi_E], \qquad E \in \mathscr{R}, \tag{2.1}$$

which fails to be additive in general. In other words, the natural one-to-one correspondence between additive nonnegative set functions and forms is not surjective. The additivity of $\vartheta$ is characterized by the following lemma.

**Lemma 2.8.** *Let $T$ be a non-empty set, and let $\mathscr{R}$ be a ring of subsets of $T$. For a given form $\mathfrak{t}$ on $\mathscr{S}(T,\mathscr{R})$ the following statements are equivalent:*

(i) *The set function $\vartheta$ defined by correspondence* (2.1) *is additive;*
(ii) $\mathfrak{t}[\varphi] = \mathfrak{t}[|\varphi|]$ *for all $\varphi \in \mathscr{S}(T,\mathscr{R})$.*

*Proof.* If $\vartheta$ is additive, then we have

$$\mathfrak{t}(\varphi,\psi) = \int_T \varphi \cdot \overline{\psi}\, d\vartheta$$

for all $\varphi,\psi \in \mathscr{S}(T,\mathscr{R})$. Hence (i) obviously implies (ii). Conversely, if we assume (ii), then for any pair of disjoint sets $E, F \in \mathscr{R}$ we have

$$\begin{aligned}\vartheta(E)+\vartheta(F) &= \frac{1}{2}\big(\mathfrak{t}[\chi_E+\chi_F]+\mathfrak{t}[\chi_E-\chi_F]\big) = \frac{1}{2}\big(\mathfrak{t}[\chi_{E\cup F}]+\mathfrak{t}[|\chi_E-\chi_F|]\big)\\ &= \frac{1}{2}\big(\mathfrak{t}[\chi_{E\cup F}]+\mathfrak{t}[\chi_{E\cup F}]\big) = \vartheta(E\cup F),\end{aligned}$$

due to the parallelogram law. ☐

The main result of this subsection is the following short-type decomposition of charges. Here we emphasize that, in contrast to the Lebesgue-Darst decomposition [**9**], this decomposition holds for not necessarily bounded set functions as well.

**Theorem 2.9.** *Let $\mathscr{R}$ be a ring of subsets of a non-empty set $T$, and let $\alpha$ and $\beta$ be charges on $\mathscr{R}$. Then there is a decomposition*

$$\alpha = \alpha_{\ll_{\mathrm{ac}},\beta} + \alpha_{\perp,\beta},$$

*where $\alpha_{\ll_{\mathrm{ac}},\beta} \ll_{\mathrm{ac}} \beta$ and $\alpha_{\perp,\beta} \perp \beta$. If $\vartheta$ is a charge such that $\vartheta \le \alpha$ and $\vartheta \ll_{\mathrm{ac}} \beta$, then $\vartheta \le \alpha_{\ll_{\mathrm{ac}},\beta}$.*

*Furthermore, we have the following formula for the absolutely continuous part*

$$\forall R \in \mathscr{R}: \quad \alpha_{\ll_{\mathrm{ac}},\beta}(R) = \inf_{\varphi\in\ker \mathfrak{t}_\beta} \int_R |1-\varphi(t)|^2 \,\mathrm{d}\alpha(t).$$

*Proof.* Let us define the set function $\alpha_{\ll_{\mathrm{ac}},\beta}$ by

$$\forall R \in \mathscr{R}: \quad \alpha_{\ll_{\mathrm{ac}},\beta}(R) := (\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}[\chi_R].$$

It is clear that $\beta(R) = 0$ implies $\alpha_{\ll_{\mathrm{ac}},\beta}(R) = 0$. Our only claim is therefore to prove the additivity of $\alpha_{\ll_{\mathrm{ac}},\beta}$. For this purpose, let $\varphi \in \mathscr{S}(T, \mathscr{R})$. In accordance with the previous lemma, it is enough to show that

$$(\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}[|\varphi|] = (\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}[\varphi].$$

Assume that

$$\varphi = \sum_{i=1}^{k} \lambda_i \cdot \chi_{R_i},$$

where $\{\lambda_i\}_{i=1}^k$ are non-zero complex numbers and $\{R_i\}_{i=1}^k$ are pairwise disjoint elements of $\mathscr{R}$. Define the function $\psi$ as follows

$$\psi := \sum_{i=1}^{k} \frac{|\lambda_i|}{\lambda_i} \cdot \chi_{R_k} + \chi_{T \setminus \cup_{i=1}^k R_i}.$$

Since $|\psi(t)| = 1$ for all $t \in T$, the multiplication with $\psi$ is a bijection on $\mathscr{S}(T, \mathscr{R})$. Furthermore, for every $\eta \in \mathscr{S}(T, \mathscr{R})$ we have that $\eta \in \ker \mathfrak{t}_\beta$ precisely when $\psi \cdot \eta \in \ker \mathfrak{t}_\beta$. (Note that $\psi \notin \mathscr{S}(T, \mathscr{R})$ in general.) As $\mathfrak{t}_\alpha[\zeta] = \mathfrak{t}_\alpha[|\zeta|]$ for all $\zeta \in \mathscr{S}(T, \mathscr{R})$, we have that

$$\begin{aligned}
(\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}[\varphi] &= \inf_{\xi \in \ker \mathfrak{t}_\beta} \mathfrak{t}_\alpha[\varphi - \xi] = \inf_{\xi \in \ker \mathfrak{t}_\beta} \mathfrak{t}_\alpha[|\varphi - \xi|] \\
&= \inf_{\xi \in \ker \mathfrak{t}_\beta} \mathfrak{t}_\alpha[|\psi| \cdot |\varphi - \xi|] = \inf_{\xi \in \ker \mathfrak{t}_\beta} \mathfrak{t}_\alpha[||\varphi| - \psi \cdot \xi|] \\
&= \inf_{\xi \in \ker \mathfrak{t}_\beta} \mathfrak{t}_\alpha[|\varphi| - \psi \cdot \xi] = (\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}[|\varphi|].
\end{aligned}$$

Consequently, $\alpha_{\ll_{\mathrm{ac}},\beta}$ is a charge, which is absolutely continuous with respect to $\beta$. Since $\alpha$ and $\alpha_{\ll_{\mathrm{ac}},\beta}$ are charges, $\alpha_{\perp,\beta} := \alpha - \alpha_{\ll_{\mathrm{ac}},\beta}$ is a charge too, which is derived from $\mathfrak{t}_\alpha - (\mathfrak{t}_\alpha)_{\ker \mathfrak{t}_\beta}$. Hence, $\alpha_{\perp,\beta}$ and $\beta$ are singular. □

If $\mathscr{R}$ is a $\sigma$-algebra, and $\alpha$ and $\beta$ are nonnegative $\sigma$-additive set functions, then the above decomposition coincides with the well-known Lebesgue decomposition. We will discuss this case in the following chapter.

**2.2.3. Representable functionals.** In this subsection we present a short-type decomposition for representable positive functionals, which corresponds to the short type decomposition of their induced forms.

Let $\mathscr{A}$ be a complex *-algebra and let $f : \mathscr{A} \to \mathbb{C}$ be a positive linear functional on it with associated form $\mathfrak{t}_f$.

For positive functionals $f \le g$ means that $\mathfrak{t}_f \le \mathfrak{t}_g$. The positive functional $f$ is called *representable*, if there exists a Hilbert space $\mathscr{H}_f$, a ${}^*$-representation $\pi_f$ of $\mathscr{A}$ into $\mathscr{H}_f$, and a cyclic vector $\xi_f \in \mathscr{H}_f$ such that

$$\forall a \in \mathscr{A}: \quad f(a) = (\pi_f(a)\xi_f \mid \xi_f)_f.$$

Such a triple $(\mathscr{H}_f, \pi_f, \xi_f)$ is provided by the classical GNS-construction (see [**40**] for the details): namely, denote by $N_f$ the set of those elements $a$ such that $f(a^*a) = 0$, and let $\mathscr{H}_f$ stand for the Hilbert space completion of the inner product space

$$\big(\mathscr{A}/_{N_f}, (\cdot \mid \cdot)_f\big); \qquad \forall a, b \in \mathscr{A}: \quad (a + N_f \mid b + N_f)_f := \mathfrak{t}_f(a, b) = f(b^*a).$$

For $a \in \mathscr{A}$ let $\pi_f(a)$ be the left multiplication by $a$:

$$\forall x \in \mathscr{A}: \quad \pi_f(a)(x + N_f) := ax + N_f.$$

The cyclic vector $\xi_f$ is defined as the Riesz-representing vector of the continuous linear functional

$$\mathscr{H}_f \supseteq \mathscr{A}/_{N_f} \to \mathbb{C}; \qquad a + N_f \mapsto f(a).$$

Note also that

$$\pi_f(a)\xi_f = a + N_f.$$

We define the absolute continuity and singularity as for forms. Singularity means that the zero functional is the only representable functional which is dominated by both $f$ and $g$. According to [**45**, Theorem 2], this is equivalent with the singularity of the forms $\mathfrak{t}_f$ and $\mathfrak{t}_g$. We say that $f$ is $g$-absolutely continuous ($f \ll_{\mathrm{ac}} g$), if

$$\forall a \in \mathscr{A}: \quad g(a^*a) = 0 \implies f(a^*a) = 0.$$

A decomposition of $f$ into representable $g$-absolutely continuous and $g$-singular parts is called short-type decomposition.

Now, the short-type decomposition for representable functionals can be stated as follows.

**Theorem 2.10.** *Let $f$ and $g$ be representable positive functionals on the ${}^*$-algebra $\mathscr{A}$. Then $f$ admits a decomposition*

$$f = f_{\ll_{\mathrm{ac}}, g} + f_{\perp, g}$$

*to a sum of representable functionals, where* $f_{\ll_{\mathrm{ac}},g}$ *is* $g$*-absolutely continuous,* $f_{\perp,g}$ *and* $g$ *are singular. Furthermore,* $f_{\ll_{\mathrm{ac}},g}$ *is the greatest among all of the representable functionals* $h$ *such that* $h \leq f$ *and* $h \ll_{\mathrm{ac}} g$*.*

*Proof.* Let $\mathscr{M}$ be the following closed subspace of $\mathscr{H}_f$

$$\mathscr{M} := \overline{\{a + N_f \mid g(a^*a) = 0\}}$$

and let $P$ be the orthogonal projection from $\mathscr{H}_f$ onto $\mathscr{M}$. Then $\mathscr{M}$ and $\mathscr{M}^\perp$ are $\pi_f$-invariant subspaces. Since $\pi_f$ is a $^*$-representation, it is enough to prove that $\mathscr{M}$ is $\pi_f$ invariant. Let $a, x \in \mathscr{A}$ and assume that $g(a^*a) = 0$. Then

$$\pi_f(x)(a + N_f) = xa + N_f \in \mathscr{M}$$

because

$$g(a^*x^*xa) = \|\pi_g(x)(a + N_f)\|_g^2 \leq \|\pi_g(x)\|_g^2 \cdot g(a^*a) = 0.$$

Consequently,

$$\pi_f(x)\langle\mathscr{M}\rangle \subseteq \overline{\pi_f(x)\langle\{a + N_f \mid g(a^*a) = 0\}\rangle} \subseteq \mathscr{M},$$

as it is stated. Now, let us define the functionals

$$f_{\ll_{\mathrm{ac}},g}(a) := (\pi_f(a)(I - P)\xi_f \mid (I - P)\xi_f)_f.$$

$$f_{\perp,g}(a) := (\pi_f(a)P\xi_f \mid P\xi_f)_f.$$

Clearly, $f_{\ll_{\mathrm{ac}},g}$ and $f_{\perp,g}$ are representable positive functionals. On the other hand, since $\mathscr{M}^\perp$ is $\pi_f$-invariant we find that

$$\pi_f(a)(I - P)\xi_f = (I - P)\pi_f(a)(I - P)\xi_f,$$

and using $\pi_f$ invariance of $\mathscr{M}$ one has

$$(I - P)\pi_f(a)P\xi_f = (I - P)P\pi_f(a)P\xi_f = 0,$$

and thus

$$(I - P)\pi_f(a)(I - P)\xi_f = (I - P)\pi_f(a)\xi_f.$$

This gives

$$f_{\ll_{\mathrm{ac}},g}(a^*a) = \|\pi_f(a)(I - P)\xi_f\|_f^2 = \|(I - P)\pi_f(a)\xi_f\|_f^2 = \|(I - P)(a + N_f)\|_f^2 = \mathfrak{t}_{f_{\ll_{\mathrm{ac}},g}}[a].$$

Similarly,

$$f_{\perp,g}(a^*a) = \|P(a + N_f)\|_f^2 = \mathfrak{t}_{f_{\perp,g}}[a].$$

Since $\mathfrak{t}_{f_{\ll_{\mathrm{ac}},g}}$ is $\mathfrak{t}_g$-absolutely continuous, and $\mathfrak{t}_{f_{\perp,g}}$ is $\mathfrak{t}_g$-singular, we infer that $f_{\ll_{\mathrm{ac}},g} \ll_{\mathrm{ac}} g$ and $f_{\perp,g} \perp g$. The maximality of $f_{\ll_{\mathrm{ac}},g}$ follows from the maximality of $\mathfrak{t}_{f_{\ll_{\mathrm{ac}},g}}$. $\square$

CHAPTER 3

# The $(\ll_{\mathrm{cl}}, \perp)$-type decomposition

In this chapter we prove that the $(\ll_{\mathrm{cl}}, \perp)$-type (or *Lebesgue-type*) decomposition of forms exists. This decomposition theorem is a common generalization of several famous decomposition theorems, such as the operator decomposition of T. Ando [**4**], the Lebesgue-Darst decomposition of finitely additive set functions [**9**], and the canonical decomposition of densely defined forms [**43**]. The basic tool in our treatment is the embedding operator between two auxiliary Hilbert spaces associated to the forms in question. As applications of our approach, we also provide the Lebesgue-type decomposition theorems for bounded operators and for bounded finitely additive set functions. This chapter is based on paper [**32**].

The Lebesgue-type decomposition theorem for forms states that for every pair $\mathfrak{t}$ and $\mathfrak{w}$, defined on the complex linear space $\mathfrak{X}$, the form $\mathfrak{t}$ can be decomposed by means of the forms $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ (the so-called regular part) and $\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}$ (the singular part) as

$$\mathfrak{t} = \mathfrak{t}_{\mathrm{reg},\mathfrak{w}} + \mathfrak{t}_{\mathrm{sing},\mathfrak{w}}, \tag{3.1}$$

where $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ is $\mathfrak{w}$-closable and $\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}$ is singular with respect to $\mathfrak{w}$. Our treatment for giving this decomposition is due to the following construction. For a given form $\mathfrak{w}$ consider the auxiliary Hilbert space $\mathscr{H}_{\mathfrak{w}}$. Let denote by $\pi_{\mathfrak{w}}$ the canonical surjection from $\mathfrak{X}$ to $\mathfrak{X}/\ker\mathfrak{w}$, i.e.

$$\pi_{\mathfrak{w}}(x) := x + \ker\mathfrak{w}, \qquad x \in \mathfrak{X}.$$

The embedding operator $J$ from $\mathfrak{X}/\ker(\mathfrak{t}+\mathfrak{w}) \subseteq \mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ into $\mathscr{H}_{\mathfrak{w}}$, defined by

$$\pi_{\mathfrak{t}+\mathfrak{w}}(x) \mapsto \pi_{\mathfrak{w}}(x), \qquad x \in \mathfrak{X}, \tag{3.2}$$

is then a densely defined contraction with respect to the corresponding norms, and $J^{**}$ is the closure of $J$. The orthogonal projection of $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\{\ker J^{**}\}^{\perp}$ is denoted by $P$.

## 3.1. Lebesgue-type decomposition

Let $\mathfrak{X}$ be a complex linear space, and let $\mathfrak{t}$ and $\mathfrak{w}$ be two forms (i.e. semi-inner products) on $\mathfrak{X}$. The purpose in this section is to give the Lebesgue decomposition (3.1) of $\mathfrak{t}$ with respect to $\mathfrak{w}$. We need first the following two lemmas:

**Lemma 3.1.** *Let $J$ be the embedding operator from $\mathfrak{X}/\ker(\mathfrak{t}+\mathfrak{w}) \subseteq \mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ into $\mathscr{H}_{\mathfrak{w}}$, defined by the identification* (3.2). *By setting*

$$\mathfrak{S}(\mathfrak{t},\mathfrak{w}) := \big\{(x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X} \;\big|\; \mathfrak{t}[x_n - x_m] \to 0, \mathfrak{w}[x_n] \to 0\big\}, \tag{3.3}$$

*the kernel of $J^{**}$ can be described by*

$$\ker J^{**} = \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\}. \tag{3.4}$$

*Proof.* Since $J^{**}$ is the closure of $J$, we obtain step by step

$$\begin{aligned}
\ker J^{**} &= \big\{f \in \mathscr{H}_{\mathfrak{t}+\mathfrak{w}} \;\big|\; \exists (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \to f, \pi_{\mathfrak{w}}(x_n) \to 0\big\} \\
&= \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \pi_{\mathfrak{t}+\mathfrak{w}}(x_n - x_m) \to 0, \pi_{\mathfrak{w}}(x_n) \to 0\big\} \\
&= \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, (\mathfrak{t}+\mathfrak{w})[x_n - x_m] \to 0, \mathfrak{w}[x_n] \to 0\big\} \\
&= \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{t}[x_n - x_m] \to 0, \mathfrak{w}[x_n] \to 0\big\} \\
&= \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\},
\end{aligned}$$

as it is claimed. $\square$

**Lemma 3.2.** *Let $P$ stand for the orthogonal projection of $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\{\ker J^{**}\}^{\perp}$, and let us define the mapping $\mathfrak{r} : \mathfrak{X} \to \mathbb{R}_+$ via the following formula:*

$$\mathfrak{r}[x] := \inf\big\{ \lim_{n\to\infty} \mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\}, \qquad x \in \mathfrak{X}. \tag{3.5}$$

*Then for any $x \in \mathfrak{X}$ we have*

$$\big\|P\pi_{\mathfrak{t}+\mathfrak{w}}(x)\big\|^2_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{r}[x] + \mathfrak{w}[x] \quad \textit{and} \quad \big\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\big\|^2_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{t}[x] - \mathfrak{r}[x]. \tag{3.6}$$

*In particular, both $\mathfrak{r}$ and $\mathfrak{t}-\mathfrak{r}$ are (quadratic) forms on $\mathfrak{X}$.*

*Proof.* Since $P$ is the orthogonal projection of $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\{\ker J^{**}\}^{\perp}$, we have for any $x \in \mathfrak{X}$ similarly as before

$$\begin{aligned}
\big\|P\pi_{\mathfrak{t}+\mathfrak{w}}(x)\big\|^2_{\mathfrak{t}+\mathfrak{w}} &= \inf\big\{\|\pi_{\mathfrak{t}+\mathfrak{w}}(x) - y\|^2_{\mathfrak{t}+\mathfrak{w}} \;\big|\; y \in \ker J^{**}\big\} \\
&= \inf\big\{ \lim_{n\to\infty} \|\pi_{\mathfrak{t}+\mathfrak{w}}(x - x_n)\|^2_{\mathfrak{t}+\mathfrak{w}} \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\} \\
&= \inf\big\{ \lim_{n\to\infty} (\mathfrak{t}+\mathfrak{w})[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\} \\
&= \inf\big\{ \lim_{n\to\infty} \mathfrak{t}[x - x_n] + \mathfrak{w}[x] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})\big\} \\
&= \mathfrak{r}[x] + \mathfrak{w}[x].
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{t}[x] + \mathfrak{w}[x] = \left\|\pi_{\mathfrak{t}+\mathfrak{w}}(x)\right\|^2_{\mathfrak{t}+\mathfrak{w}} &= \left\|P\pi_{\mathfrak{t}+\mathfrak{w}}(x)\right\|^2_{\mathfrak{t}+\mathfrak{w}} + \left\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\right\|^2_{\mathfrak{t}+\mathfrak{w}} \\ &= \mathfrak{w}[x] + \mathfrak{r}[x] + \left\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\right\|^2_{\mathfrak{t}+\mathfrak{w}},\end{aligned}$$

which yields the second identity of (3.6). □

We can now formulate the $(\ll_{\mathrm{cl}}, \perp)$-type decomposition theorem.

**Theorem 3.3.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be nonnegative forms on the complex linear space* $\mathfrak{X}$*, and let* $\mathfrak{r}$ *be the form defined by* (3.5)*. Then*

$$\mathfrak{t} = \mathfrak{r} + (\mathfrak{t} - \mathfrak{r})$$

*is a Lebesgue-type decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*:* $\mathfrak{r}$ *is closable with respect to* $\mathfrak{w}$*, and* $\mathfrak{t} - \mathfrak{r}$ *is singular with respect to* $\mathfrak{w}$*. Furthermore,* $\mathfrak{r}$ *is the maximum of all forms majorized by* $\mathfrak{t}$*, which are closable with respect to* $\mathfrak{w}$*.*

*Proof.* In order to show that $\mathfrak{r}$ is closable with respect to $\mathfrak{w}$, consider a sequence $(x_n)_{n\in\mathbb{N}}$ from $\mathfrak{X}$ with $\mathfrak{w}[x_n] \to 0$ and $\mathfrak{r}[x_n - x_m] \to 0$. We should prove that $\mathfrak{r}[x_n] \to 0$. By using formula (3.6) we obtain

$$\mathfrak{r}[x_n - x_m] = \left\|P\pi_{\mathfrak{t}+\mathfrak{w}}(x_n - x_m)\right\|^2_{\mathfrak{t}+\mathfrak{w}} - \mathfrak{w}[x_n - x_m],$$

thus we can conclude that $P(\pi_{\mathfrak{t}+\mathfrak{w}}(x_n))_{n\in\mathbb{N}}$ converges in $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ to a vector $f$. Then the following line of identities

$$\begin{aligned}\|J^{**}f\|^2_{\mathfrak{w}} &= \lim_{n\to\infty} \left\|J^{**}P\pi_{\mathfrak{t}+\mathfrak{w}}(x_n)\right\|^2_{\mathfrak{w}} = \lim_{n\to\infty} \left\|J^{**}\pi_{\mathfrak{t}+\mathfrak{w}}(x_n)\right\|^2_{\mathfrak{w}} \\ &= \lim_{n\to\infty} \left\|J\pi_{\mathfrak{t}+\mathfrak{w}}(x_n)\right\|^2_{\mathfrak{w}} = \lim_{n\to\infty} \|\pi_{\mathfrak{w}}(x_n)\|^2_{\mathfrak{w}} \\ &= \lim_{n\to\infty} \mathfrak{w}[x_n] = 0\end{aligned}$$

implies on the one hand that $f \in \ker J^{**}$. On the other hand,

$$f = \lim_{n\to\infty} P\pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \in \operatorname{ran} P = \{\ker J^{**}\}^{\perp},$$

and therefore $f = 0$. Consequently,

$$\lim_{n\to\infty} \mathfrak{r}[x_n] = \lim_{n\to\infty} \|P\pi_{\mathfrak{t}+\mathfrak{w}}(x_n)\|^2_{\mathfrak{t}+\mathfrak{w}} = 0,$$

indeed.

Our next claim is to show that $\mathfrak{t} - \mathfrak{r}$ and $\mathfrak{w}$ are singular forms with respect to each other. So assume that $\mathfrak{q}$ is a nonnegative form such that $\mathfrak{q} \le \mathfrak{w}$ and $\mathfrak{q} \le \mathfrak{t} - \mathfrak{r}$. Then the second formula of (3.6) gives the following inequalities:

$$\mathfrak{q}[x] \leq \|J^{**}\pi_{\mathfrak{t}+\mathfrak{w}}(x)\|_{\mathfrak{w}}^2 \qquad \text{and} \qquad \mathfrak{q}[x] \leq \|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\|_{\mathfrak{t}+\mathfrak{w}}^2,$$

for any $x \in \mathfrak{X}$. Let $\widetilde{\mathfrak{q}}$ be unique continuous extension of the following (quadratic) form

$$\mathfrak{X}/\ker(\mathfrak{t}+\mathfrak{w}) \ni \pi_{\mathfrak{t}+\mathfrak{w}}(x) \mapsto \mathfrak{q}[x], \qquad x \in \mathfrak{X},$$

to the Hilbert space $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$. Then for any $f \in \mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$ we obtain

$$\begin{aligned}\widetilde{\mathfrak{q}}^{1/2}[f] &\leq \widetilde{\mathfrak{q}}^{1/2}[Pf] + \widetilde{\mathfrak{q}}^{1/2}[(I-P)f] \\ &\leq \|(I-P)Pf\|_{\mathfrak{t}+\mathfrak{w}} + \|J^{**}((I-P)f)\|_{\mathfrak{w}} = 0.\end{aligned}$$

As a consequence, $\mathfrak{q} = 0$, indeed.

It remains only to show the maximality property of $\mathfrak{r}$. So consider a form $\mathfrak{q} \leq \mathfrak{t}$ which is closable with respect to $\mathfrak{w}$, and fix a vector $x \in \mathfrak{X}$. For any sequence $(x_n)_{n\in\mathbb{N}}$ from $\mathfrak{X}$ with $\mathfrak{w}[x_n] \to 0$ and $\mathfrak{t}[x_n - x_m] \to 0$ one obtains

$$\mathfrak{q}^{1/2}[x] \leq \mathfrak{q}^{1/2}[x - x_n] + \mathfrak{q}^{1/2}[x_n] \leq \mathfrak{t}^{1/2}[x - x_n] + \mathfrak{q}^{1/2}[x_n].$$

On the other hand,

$$\mathfrak{q}[x_n - x_m] \leq \mathfrak{t}[x_n - x_m] \to 0 \qquad \text{and} \qquad \mathfrak{w}[x_n] \to 0$$

imply $\mathfrak{q}[x_n] \to 0$. Consequently,

$$\mathfrak{q}^{1/2}[x] \leq \lim_{n\to\infty} \mathfrak{t}^{1/2}[x - x_n]$$

for any sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}, \mathfrak{w})$. This yields just

$$\mathfrak{q}[x] \leq \inf\left\{ \lim_{n\to\infty} \mathfrak{t}[x - x_n] \,\middle|\, (\varphi_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}, \mathfrak{w}) \right\} = \mathfrak{r}[x]$$

for each $x \in \mathfrak{X}$, as desired. □

From now on, we will refer sometimes to $\mathfrak{r}$ (resp., to $\mathfrak{t} - \mathfrak{r}$) as the *regular part* (resp., the *singular part*) of $\mathfrak{t}$ with respect to $\mathfrak{w}$, and we will use the notation $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ (resp., $\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}$).

**Corollary 3.4.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$*. The following statements are equivalent:*

(i) $\mathfrak{t}$ *is* $\mathfrak{w}$*-closable;*
(ii) $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = \mathfrak{t}$*;*
(iii) $\ker J^{**} = \{0\}$.

*Proof.* If $\mathfrak{t}$ is $\mathfrak{w}$-closable, then for any $x \in \mathfrak{X}$ and $(x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w})$ we have $\mathfrak{t}[x-x_n] \to \mathfrak{t}[x]$. Therefore,

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] := \inf\big\{ \lim_{n\to\infty} \mathfrak{t}[x-x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w}) \big\} = \mathfrak{t}[x],$$

and thus (i) implies (ii). By assuming (ii), one obtains

$$0 = \mathfrak{t}[x] - \mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] = \big\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\big\|^2_{\mathfrak{t}+\mathfrak{w}}, \qquad x \in \mathfrak{X},$$

thanks to the second formula of (3.6). This means that $I = P$, i.e. $\ker J^{**} = \{0\}$. Finally, if we assume (iii), then clearly $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = \mathfrak{t}$, where $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ is $\mathfrak{w}$-closable thanks to Theorem 3.3. □

The following statement is a direct consequence of the definition of the regular part.

**Proposition 3.5.** *Let $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$, and let $\mathfrak{r}_1$ (resp., $\mathfrak{r}_2$) denote the regular part of $\mathfrak{t}_1$ (resp., of $\mathfrak{t}_2$) with respect to $\mathfrak{w}$.*

(a) *If $\mathfrak{t}_1 \le \mathfrak{t}_2$, then also $\mathfrak{r}_1 \le \mathfrak{r}_2$;*
(b) *If $\mathfrak{t}_1 \le \alpha.\mathfrak{w}$ with some nonnegative constant $\alpha$, then $\mathfrak{t}_1$ is $\mathfrak{w}$-closable.*

*Proof.* Both statements are obvious from the definition of the regular part in (3.5). □

Next we prove an extension of [**42**, Theorem 2], cf. also [**20**, Theorem VI. 1.16]. The proof rests on the following lemma, which may be of interest on its own right.

**Lemma 3.6.** *Let $T$ be a densely defined closable operator between complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. If $P$ stands for the orthogonal projection of $\mathscr{H}$ onto $\{\ker T^{**}\}^\perp$ then for any $x \in \mathscr{H}$ we have*

$$\begin{aligned}\|Px\| &= \inf\big\{ \lim_{n\to\infty} \|x-x_n\| \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} T, (x_n - x_m) \to 0, Tx_n \to 0\big\}\\ &= \inf\big\{ \liminf_{n\to\infty} \|x-x_n\| \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} T, Tx_n \to 0\big\}\\ &= \inf\big\{ \liminf_{n\to\infty} \|x-x_n\| \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom} T, (x_n\,|\,y) \to 0 \textit{ for all } y \in \operatorname{ran} T^*\big\},\end{aligned}$$

*where $(\cdot\,|\,\cdot)$ denotes the inner-product of $\mathscr{H}$.*

*Proof.* Let $A, B$, and $C$ denote the infima expressions above, respectively. For any $x \in \mathscr{H}$ we have of course

$$\|Px\| = \inf\big\{\|x-y\| \;\big|\; y \in \ker T^{**}\big\}.$$

Since for each $y \in \ker T^{**}$ there exists $(y_n)_{n\in\mathbb{N}}$ from $\operatorname{dom} T$ such that $y_n \to y$ and $Ty_n \to 0$, we conclude that $\|Px\| = A$. Inequalities $C \le B \le A$ are obvious, therefore we only have to check $\|Px\| \le C$. First of all we notice that if a sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to a vector $\xi \in \mathscr{H}$, then

$$\limsup_{n\to\infty} \|x_n\| \geq \|\xi\|. \tag{3.7}$$

Indeed, by using the argument of [**29**], we conclude that

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \|\xi - x_n\|^2 &= \|\xi\|^2 + \limsup_{n\to\infty} \|x_n\|^2 - 2 \lim_{n\to\infty} \operatorname{Re}(\xi \,|\, x_n) \\ &= \|\xi\|^2 + \limsup_{n\to\infty} \|x_n\|^2 - 2\operatorname{Re}(\xi \,|\, \xi) = \limsup_{n\to\infty} \|x_n\|^2 - \|\xi\|^2, \end{aligned}$$

which yields inequality (3.7). Consider now a sequence $(x_n)_{n\in\mathbb{N}}$ from $\operatorname{dom} T$ such that $(x_n \,|\, y) \to 0$ for all $y \in \operatorname{dom} T^*$. We may assume boundedness on $(x_n)_{n\in\mathbb{N}}$, and therefore, after twofold choice of appropriate subsequences, we may also suppose that

$$\begin{cases} \liminf\limits_{n\to\infty} \|x - x_n\| = \lim\limits_{n\to\infty} \|x - x_n\|, \\ x_n \to \xi \text{ weakly for some vector } \xi \in \mathscr{H}. \end{cases}$$

We check first that $\xi$ belongs to $\ker T^{**}$ since

$$(\xi \,|\, T^* z) = \lim_{n\to\infty} (x_n \,|\, T^* z) = 0$$

holds for each $z \in \operatorname{dom} T^*$ and therefore we have indeed that $\xi \in \{\operatorname{ran} T^*\}^{\perp} = \ker T^{**}$. Finally, we see that $x - x_n \to x - \xi$ weakly in $\mathscr{H}$, therefore (3.7) gives

$$\lim_{n\to\infty} \|x - x_n\| \geq \|x - \xi\| \geq \inf\big\{ \|x - y\| \;\big|\; y \in \ker T^{**} \big\} = \|Px\|.$$

Consequently, $\|Px\| \leq C$. $\square$

**Theorem 3.7.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. Let $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ stand for the regular part of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Then for each $x \in \mathfrak{X}$*

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] = \inf\big\{ \liminf_{n\to\infty} \mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{w}[x_n] \to 0 \big\}. \tag{3.8}$$

*Proof.* From (3.6) and Lemma 3.6 we conclude that

$$\begin{aligned} \mathfrak{w}[x] + \mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] &= \|P\pi_{\mathfrak{t}+\mathfrak{w}}(x)\|^2_{\mathfrak{t}+\mathfrak{w}} \\ &= \inf\big\{ \liminf_{n\to\infty} \|\pi_{\mathfrak{t}+\mathfrak{w}}(x - x_n)\|^2_{\mathfrak{t}+\mathfrak{w}} \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, J\pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \to 0 \big\} \\ &= \inf\big\{ \liminf_{n\to\infty} \big( \mathfrak{t}[x - x_n] + \mathfrak{w}[x - x_n] \big) \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{w}[x_n] \to 0 \big\} \\ &= \mathfrak{w}[x] + \inf\big\{ \liminf_{n\to\infty} \mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{w}[x_n] \to 0 \big\}, \end{aligned}$$

which gives formula (3.8). $\square$

As an immediate consequence we obtain a generalized version of [**42**, Theorem 2]:

**Corollary 3.8.** *If $\mathfrak{t}$ and $\mathfrak{w}$ are forms on the complex linear space $\mathfrak{X}$, then $\mathfrak{t}$ is $\mathfrak{w}$-closable if and only if for any $x \in \mathfrak{X}$ and for any sequence $(x_n)_{n\in\mathbb{N}}$ from $\mathfrak{X}$ $\mathfrak{w}[x - x_n] \to 0$ implies*

$$\mathfrak{t}[x] \leq \liminf_{n\to\infty} \mathfrak{t}[x_n]$$

*holds for any $x \in \mathfrak{X}$ and for any sequence $(x_n)_{n\in\mathbb{N}}$ from $\mathfrak{X}$ such that $\mathfrak{w}[x - x_n] \to 0$.*

*Proof.* According to Corollary 3.4, $\mathfrak{t}$ is $\mathfrak{w}$-closable if and only if $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$. Since $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} \leq \mathfrak{t}$ holds by definition, the proof can be easily obtained via Theorem 3.7:

$$\begin{aligned}\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] &= \inf\big\{\liminf_{n\to\infty} \mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{w}[x_n] \to 0\big\}\\ &= \inf\big\{\liminf_{n\to\infty} \mathfrak{t}[x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X}, \mathfrak{w}[x - x_n] \to 0\big\},\end{aligned}$$

for all $x \in \mathfrak{X}$. $\square$

## 3.2. Application to bounded charges

The purpose of this section, on the one hand, is to show that notions strong absolute continuity and singularity of bounded charges correspond to the notions closability and singularity of the associated forms, respectively. On the other hand, thanks to this correspondence, we show that the Lebesgue-type decomposition of additive set functions can be derived from that of their associated forms. This result generalizes the well-known Lebesgue decomposition theorem of measures, and the Darst decomposition theorem of contents as well (see [**9, 51**]).

Let $T$ be a non-empty set, and let $\mathscr{R}$ be a ring of subsets of $T$. Let $\mu$ and $\nu$ be *bounded* charges on $\mathscr{R}$, i.e., assume that

$$\sup_{E\in\mathscr{R}} \mu(E) < \infty \qquad \text{and} \qquad \sup_{E\in\mathscr{R}} \nu(E) < \infty.$$

Recall the notions of strong absolute continuity and singularity: $\nu$ is called *strongly absolutely continuous* with respect to $\mu$ (shortly, strongly $\mu$-absolutely continuous) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $E \in \mathscr{R}$ $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$. Similarly as in the previous section for arbitrary charges, $\nu$ is called *singular* with respect to $\mu$ if for $E \in \mathscr{R}$ we have

$$(\mu \wedge \nu)(E) := \inf\{\mu(E \cap F) + \nu(E \setminus F) \mid F \in \mathscr{R}\} = 0. \tag{3.9}$$

Equivalently, $\mu$ and $\nu$ are singular precisely when inequalities $\vartheta \leq \mu$ and $\vartheta \leq \nu$ imply $\vartheta = 0$ for any additive nonnegative set function $\vartheta$.

**Lemma 3.9.** *Let $\mu$ and $\nu$ be bounded additive nonnegative set functions on $\mathscr{R}$, such that $\nu$ is strongly absolutely continuous with respect to $\mu$. Then*

$$\nu(E) = \sup_{n\in\mathbb{N}} (\nu \wedge n.\mu)(E) = \lim_{n\to\infty} (\nu \wedge n.\mu)(E), \qquad E \in \mathscr{R}.$$

*If $\nu$ is both strongly absolutely continuous and singular with respect to $\mu$, then $\nu = 0$.*

*Proof.* Let $E \in \mathscr{R}$ and $\varepsilon > 0$ be fixed; then there is a sequence $(F_n)_{n\in\mathbb{N}}$ from $\mathscr{R}$ such that

$$n\mu(E \cap F_n) + \nu(E \setminus F_n) \le (\nu \wedge n.\mu)(E) + \frac{\varepsilon}{2} \le \sup_{n\in\mathbb{N}} (\nu \wedge n.\mu)(E) + \frac{\varepsilon}{2} =: \alpha + \frac{\varepsilon}{2} \tag{3.10}$$

for all integer $n$. According to the strong $\mu$-absolute continuity of $\nu$, there exists $\delta > 0$ such that $\nu(E') < \frac{\varepsilon}{2}$ for all $E' \in \mathscr{R}$ with $\mu(E') < \delta$. If $k \in \mathbb{N}$ satisfies $\frac{1}{k}(\alpha + \frac{\varepsilon}{2}) < \frac{\varepsilon}{2}$, then $\mu(E \cap F_k) < \delta$, according to (3.10). Therefore,

$$\nu(E) = \nu(E \cap F_k) + \nu(E \setminus F_k) < \frac{\varepsilon}{2} + \alpha + \frac{\varepsilon}{2} =: \sup_{n\in\mathbb{N}} (\nu \wedge n.\mu)(E) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the desired inequality follows. If we assume in addition that $\nu$ is singular with respect to $\mu$, then obviously $\nu \wedge n.\mu = 0$ for all integer $n$. Therefore, by using the first part of the statement, $\nu(E) = 0$ for all $E \in \mathscr{R}$. □

**Theorem 3.10.** *Let $\mu$ and $\nu$ be bounded additive nonnegative set functions on the ring $\mathscr{R}$, and consider their induced forms $\mathfrak{t}_\mu$. Then*

(a) *$\nu$ is strongly $\mu$-absolutely continuous if and only if $\mathfrak{t}_\nu$ is $\mathfrak{t}_\mu$-closable;*
(b) *$\nu$ and $\mu$ are singular if and only if $\mathfrak{t}_\nu$ and $\mathfrak{t}_\mu$ are singular.*

*Proof.* In order to prove the statement (a), assume first that $\nu$ is strongly $\mu$-absolute continuous; in view of Corollary 3.4, it suffices to show $\mathfrak{r} = \mathfrak{t}_\nu$ where $\mathfrak{r}$ stands for the $\mathfrak{t}_\mu$-regular part of $\mathfrak{t}_\nu$. For any integer $k$ let $\mathfrak{t}_{\nu_k}$ denote the form associated to $\nu \wedge k.\mu$. Then we have $\mathfrak{t}_{\nu_k}[\varphi] \le \mathfrak{t}_{\nu_{k+1}}[\varphi] \le (k+1)\,\mathfrak{t}_\mu[\varphi]$ and

$$\lim_{k\to\infty} \mathfrak{t}_{\nu_k}[\varphi] = \mathfrak{t}_\nu[\varphi], \qquad \varphi \in \mathscr{S}(T, \mathscr{R}),$$

via Lemma 3.9. If $\mathfrak{r}_k$ denotes the $\mathfrak{t}_\mu$-regular part of $\mathfrak{t}_{\nu k}$, then $\mathfrak{r}_k = \mathfrak{t}_{\nu_k}$ thanks to Proposition 3.5. Consequently,

$$\mathfrak{t}_{\nu_k}[\varphi] = \mathfrak{r}_k[\varphi] \le \mathfrak{r}[\varphi] \le \mathfrak{t}_\nu[\varphi], \qquad \varphi \in \mathscr{S}(T, \mathscr{R}).$$

By letting $k \to \infty$, this gives $\mathfrak{r} = \mathfrak{t}_\nu$.

Conversely, assume that $\mathfrak{t}_\nu$ is $\mathfrak{t}_\mu$-closable. In order to prove the strong $\mu$-absolute continuity of $\nu$, consider a sequence $(E_n)_{n\in\mathbb{N}}$ from $\mathscr{R}$ with $\mu(E_n) \to 0$; we should prove that $\nu(E_n) \to 0$. Let $J$ stand for the embedding operator from $\mathscr{S}(T, \mathscr{R}) \subseteq \mathscr{H}_{\mathfrak{t}_\mu + \mathfrak{t}_\nu}$ into

$\mathscr{H}_{\mathfrak{t}_\mu}$. For any fixed $f \in \mathscr{H}_{\mathfrak{t}_\mu}$ we have

$$\left|\big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{E_n})\big|J^*f\big)_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}\right|^2 = \left|\big(\pi_{\mathfrak{t}_\mu}(\chi_{E_n})\big|f\big)_{\mathfrak{t}_\mu}\right|^2 \le \mu(E_n)\|f\|^2_{\mathfrak{t}_\mu} \to 0,$$

where the sequence $\big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{E_n})\big)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathscr{H}_{\mathfrak{t}_\mu+\mathfrak{t}_\nu}$, thanks to the boundedness of $\mu$ and $\nu$. In view of Corollary 3.4, ran $J^*$ is dense in $\mathscr{H}_{\mathfrak{t}_\mu+\mathfrak{t}_\nu}$, therefore,

$$\big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{E_n})\big|h\big)_{\mathfrak{t}_\nu+\mathfrak{t}_\mu} \to 0, \qquad \text{for all } h \in \mathscr{H}_{\mathfrak{t}_\mu+\mathfrak{t}_\nu}. \tag{3.11}$$

By setting $F_n := \bigcup_{k=1}^{n} E_k$, we obtain that the sequence $\big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{F_n})\big)_{n\in\mathbb{N}}$ is norm bounded in $\mathscr{H}_{\mathfrak{t}_\mu+\mathfrak{t}_\nu}$, and therefore, it has a weakly convergent subsequence. The corresponding weak limit $\chi \in \mathscr{H}_{\mathfrak{t}_\mu+\mathfrak{t}_\nu}$ then clearly satisfies

$$\big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{E_n})\big|\chi\big)_{\mathfrak{t}_\nu+\mathfrak{t}_\mu} = \mu(E_n)+\nu(E_n)$$

for each integer $n$. By using (3.11), this yields

$$\nu(E_n) \le \big(\pi_{\mathfrak{t}_\nu+\mathfrak{t}_\mu}(\chi_{E_n})\big|\chi\big)_{\mathfrak{t}_\nu+\mathfrak{t}_\mu} \to 0,$$

as it is claimed.

The proof of statement (b) is just the same as in 2.7. ☐

We are now in position to state the Lebesgue decomposition for bounded additive nonnegative set functions.

**Theorem 3.11.** *Let $\mathscr{R}$ be a ring of subsets of a set $T$, and let $\mu$ and $\nu$ be bounded additive nonnegative set functions on $\mathscr{R}$. Then there is a uniquely determined pair $(\nu_a,\nu_s)$ of additive nonnegative set functions on $(T,\mathscr{R})$ with $\nu=\nu_a+\nu_s$ such that $\nu_a$ is strongly absolutely continuous with respect to $\mu$ and that $\nu_s$ is singular with respect to $\mu$.*

*Proof.* Let $\mathfrak{r}$ stand for the $\mathfrak{t}_\mu$-regular part of $\mathfrak{t}_\nu$, i.e.

$$\mathfrak{r}[\varphi] := \inf\big\{\lim_{n\to\infty}\mathfrak{t}_\nu[\varphi-\varphi_n]\;\big|\;(\varphi_n)_{n\in\mathbb{N}}\in\mathfrak{S}(\mathfrak{t}_\nu,\mathfrak{t}_\mu)\big\}, \qquad \varphi\in\mathscr{S}(T,\mathscr{R}).$$

Then $\mathfrak{t}_\nu = \mathfrak{r}+(\mathfrak{t}_\nu-\mathfrak{r})$ is according to the Lebesgue decomposition of the form $\mathfrak{t}_\nu$ with respect to $\mathfrak{t}_\mu$, due to Theorem 3.3. Let us define the nonnegative set function $\nu_a$ on $\mathscr{R}$ by

$$\mathscr{R}\to\mathbb{R}_+, \qquad E\mapsto\mathfrak{r}[\chi_E].$$

If we assume for a moment that $\nu_a$ is additive, then $\nu=\nu_a+(\nu-\nu_a)$ is a Lebesgue decomposition of $\nu$ with respect to $\mu$, thanks to Theorem 3.10. Therefore, our only claim is to prove the additivity of $\nu_a$. So let $\varphi\in\mathscr{S}(T,\mathscr{R})$ be fixed; we should check $\mathfrak{r}[\varphi]=\mathfrak{r}[|\varphi|]$ according to Lemma 2.8. Observe first that for $(\varphi_n)_{n\in\mathbb{N}}\in\mathfrak{S}(\mathfrak{t}_\nu,\mathfrak{t}_\mu)$ we also have $(|\varphi_n|)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}_\nu, \mathfrak{t}_\mu)$, and therefore

$$\mathfrak{r}[|\varphi|] \le \lim_{n\to\infty} \mathfrak{t}_\nu[|\varphi| - |\varphi_n|] \le \lim_{n\to\infty} \mathfrak{t}_\nu[\varphi - \varphi_n],$$

which implies $\mathfrak{r}[\varphi] \ge \mathfrak{r}[|\varphi|]$. In order to obtain the converse inequality, let us consider the following function $\varrho \in \mathscr{S}(T, \mathscr{R})$ defined by

$$\varrho(x) := \begin{cases} \frac{\varphi(x)}{|\varphi(x)|}, & \text{if } \varphi(x) \neq 0, \\ \\ 0, & \text{else.} \end{cases}$$

One easily checks that $(\varrho\cdot\varphi_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}_\nu, \mathfrak{t}_\mu)$ whenever $(\varphi_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}_\nu, \mathfrak{t}_\mu)$, and that $\mathfrak{t}_\nu[\varrho\cdot\psi] \le \mathfrak{t}_\nu[\psi]$ for all $\psi \in \mathscr{S}(T, \mathscr{R})$, according to the additivity of $\nu$. Consequently,

$$\mathfrak{r}[\varphi] \le \lim_{n\to\infty} \mathfrak{t}_\nu[\varphi - \varrho\cdot\varphi_n] = \lim_{n\to\infty} \mathfrak{t}_\nu[(|\varphi| - \varphi_n)\cdot\varrho] \le \lim_{n\to\infty} \mathfrak{t}_\nu[|\varphi| - \varphi_n],$$

which gives $\mathfrak{r}[\varphi] \le \mathfrak{r}[|\varphi|]$. Therefore, $\nu_a$ is additive, as it is claimed.

It remains only to show the uniqueness of the Lebesgue decomposition: assume that there are two additive nonnegative set functions $\nu_1$ and $\nu_2$ such that $\nu_1$ is strongly $\mu$-absolute continuous, $\nu_2$ is singular with respect to $\mu$, and $\nu_1 + \nu_2 = \nu$. Let $\mathfrak{t}_{\nu_1}$ and $\mathfrak{t}_{\nu_2}$ denote their associated forms, respectively. Then $\mathfrak{t}_{\nu_1} + \mathfrak{t}_{\nu_2} = \mathfrak{t}_\nu$ is a Lebesgue-type decomposition of $\mathfrak{t}_\nu$ with respect to $\mathfrak{t}_\mu$. Due to the maximality property of $\mathfrak{r}$, stated in Theorem 3.3, we have $\mathfrak{t}_{\nu_1} \le \mathfrak{r}$. Hence the nonnegative set function $\nu_a - \nu_1 = \nu_2 - (\nu - \nu_a)$ is obviously strongly $\mu$-absolutely continuous and simultaneously $\mu$-singular. Consequently, $\nu_a - \nu_1 = 0$ according to Lemma 3.9. The proof is therefore complete. ☐

The Lebesgue decomposition theorem asserts not only that $\nu$ splits into a strongly absolutely continuous part $\nu_a$ and a singular part $\nu_s$ with respect to $\mu$, but also that $\nu_a$ can be represented in an appropriate fashion as follows:

**Corollary 3.12.** *Let $\mu$ and $\nu$ be additive nonnegative set functions on a ring $\mathscr{R}$. The strongly $\mu$-absolutely continuous part $\nu_a$ of $\nu$ can be calculated by the following formula:*

$$\nu_a(E) = \inf\Big\{ \lim_{n\to\infty} \int_T \big|\chi_E - \varphi_n\big|^2 \, d\nu \,\Big|\, (\varphi_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\nu, \mu) \Big\}, \qquad E \in \mathscr{R},$$

*where $\mathfrak{S}(\nu, \mu)$ denotes the set of all sequences $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathscr{S}(T, \mathscr{R})$ satisfying*

$$\int_T \big|\varphi_n - \varphi_m\big|^2 \, d\nu \to 0 \qquad \textit{and} \qquad \int_T \big|\varphi_n\big|^2 \, d\mu \to 0.$$

*Proof.* Obvious from the proof of Theorem 3.11. ☐

Via Theorem 3.7 another explicit formula for the strongly absolutely continuous part can be given as follows:

**Corollary 3.13.** *Let $\nu, \mu$, and $\nu_a$ just as in Corollary 3.12. Then for $E \in \mathscr{R}$*

$$\nu_a(E) = \inf \Big\{ \liminf_{n\to\infty} \int_T |\varphi_n|^2 \, d\nu \;\Big|\; (\varphi_n)_{n\in\mathbb{N}} \subseteq \mathscr{S}(T, \mathscr{R}), \int_T |\chi_E - \varphi_n|^2 \, d\mu \to 0 \Big\},$$

*Proof.* Obvious from the proof of Theorem 3.11 and from Theorem 3.7. $\square$

CHAPTER 4

# The $(\ll_{\mathrm{ad}}, \perp)$-type decomposition

In this chapter we prove the existence of the $(\ll_{\mathrm{ad}}, \perp)$-type decomposition of forms. This decomposition theorem is a common generalization of those that were mentioned in the introduction, as well. The crucial tool in this treatment is the parallel addition.

## 4.1. Parallel sum

Let $A$ and $B$ be positive semi-definite matrices (or shortly, positive operators) on the finite-dimensional Hilbert space $\mathscr{H}$. The parallel sum $A : B$ of $A$ and $B$ was introduced by Anderson and Duffin [**2**] in study of electrical networks (see also [**1, 3, 12**]). The parallel sum and difference of two nonnegative forms was defined and studied by Hassi, Sebestyén, and de Snoo in [**15**] and [**16**]. In this section we present all their results which are needed in the later chapters.

The properties of the parallel sum are given in the following lemma (cf. [**16**, Proposition 2.2. and Lemma 2.3.].

**Lemma 4.1.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. Then the parallel sum $\mathfrak{t} : \mathfrak{w}$ defined by*

$$(\mathfrak{t} : \mathfrak{w})[x] := \inf_{y \in \mathfrak{X}} \left\{ \mathfrak{w}[y + x] + \mathfrak{t}[y] \right\} \qquad (x \in \mathfrak{X})$$

*is a form. Furthermore, let $\mathfrak{t}$, $\mathfrak{t}_n$, $\mathfrak{w}$, $\mathfrak{w}_n$, and $\mathfrak{s}$ be forms on $\mathfrak{X}$ and let $\lambda$ and $\mu$ be positive numbers. Then*

(*a*) $\mathfrak{t} : \mathfrak{w} = \mathfrak{w} : \mathfrak{t}$,
(*b*) $(\lambda\mathfrak{t}) : (\lambda\mathfrak{w}) = \lambda(\mathfrak{t} : \mathfrak{w})$,
(*c*) $(\mathfrak{t} : \mathfrak{w}) : \mathfrak{s} = \mathfrak{t} : (\mathfrak{w} : \mathfrak{s})$,
(*d*) $\mathfrak{t} : \mathfrak{w} \leq \mathfrak{t}$,
(*e*) $\mathfrak{t} \leq \mathfrak{s}$ *implies* $\mathfrak{t} : \mathfrak{w} \leq \mathfrak{s} : \mathfrak{w}$,
(*f*) $\lambda\mathfrak{t} : \mu\mathfrak{t} = \frac{\mu\lambda}{\mu+\lambda}\mathfrak{t}$, *and*
(*g*) $\mathfrak{t}_n \downarrow \mathfrak{t}$, $\mathfrak{w}_n \downarrow \mathfrak{w}$ *implies* $\mathfrak{t}_n : \mathfrak{w}_n \downarrow \mathfrak{t} : \mathfrak{w}$.

*Proof.* We show first that the map $x \mapsto \sqrt{(\mathfrak{t} : \mathfrak{w})[x]}$ defines a seminorm on $\mathfrak{X}$ and it satisfies the parallelogram law. According to the Jordan-von Neumann theorem this implies that $\mathfrak{t} : \mathfrak{w}$ is a form.

Observe that if $\lambda \neq 0$ then

$$\inf_{y\in\mathfrak{X}} \big\{\mathfrak{w}[y+\lambda x]+\mathfrak{t}[y]\big\} = \inf_{y\in\mathfrak{X}} \big\{\mathfrak{w}[\lambda(y+x)]+\mathfrak{t}[\lambda y]\big\} = |\lambda|^2 \inf_{y\in\mathfrak{X}} \big\{\mathfrak{w}[y+x]+\mathfrak{t}[y]\big\}$$

which shows that $\sqrt{(\mathfrak{t}:\mathfrak{w})[\lambda x]} = |\lambda|\sqrt{(\mathfrak{t}:\mathfrak{w})[x]}$ holds for all $\lambda \in \mathbb{C}$. Now let $x, x' \in \mathfrak{X}$. Then for all $y, y' \in \mathfrak{X}$ we have

$$(\mathfrak{t}:\mathfrak{w})[x+x'] = \inf_{y\in\mathfrak{X}}\{\mathfrak{w}[y+x+x']+\mathfrak{t}[y]\} \leq \mathfrak{w}[y+y'+x+x']+\mathfrak{t}[y+y'],$$

and therefore $(\mathfrak{t}:\mathfrak{w})[x+x']$ is dominated by

$$\begin{aligned}
\mathfrak{w}[y+x]&+2\mathfrak{Re}(\mathfrak{w}(y+x,y'+x'))+\mathfrak{w}[y'+x']+\mathfrak{t}[y]+\mathfrak{Re}(\mathfrak{t}(y,y'))+\mathfrak{t}[y'] \\
&\leq \mathfrak{w}[y+x]+2\sqrt{\mathfrak{w}[y+x]}\sqrt{\mathfrak{w}[y'+x']}+\mathfrak{w}[y'+x']+\mathfrak{t}[y]+2\sqrt{\mathfrak{t}[y]}\sqrt{\mathfrak{t}[y']}+\mathfrak{t}[y'] \\
&\leq \mathfrak{w}[y+x]+\mathfrak{t}[y]+2\sqrt{\mathfrak{w}[y+x]+\mathfrak{t}[y]}\sqrt{\mathfrak{w}[y'+x']+\mathfrak{t}[y']}+\mathfrak{w}[y'+x']+\mathfrak{t}[y'] \\
&= \big(\sqrt{\mathfrak{w}[y+x]+\mathfrak{t}[y]}+\sqrt{\mathfrak{w}[y'+x']+\mathfrak{t}[y']}\big)^2
\end{aligned}$$

for all $y$ and $y'$, hence by taking the infimum we have

$$\sqrt{(\mathfrak{t}:\mathfrak{w})[x+x']} \leq \sqrt{(\mathfrak{t}:\mathfrak{w})[x]}+\sqrt{(\mathfrak{t}:\mathfrak{w})[x']}$$

which shows that $\sqrt{(\mathfrak{t}:\mathfrak{w})[\cdot]}$ is a seminorm on $\mathfrak{X}$. It remains to show that $\sqrt{(\mathfrak{t}:\mathfrak{w})[\cdot]}$ satisfies the parallelogram identity.

Observe first that

$$\begin{aligned}
&2(\mathfrak{w}[y+x]+\mathfrak{t}[y]+\mathfrak{w}[y'+x']+\mathfrak{t}[y']) \\
&\quad= \mathfrak{w}[y+y'+x+x']+\mathfrak{t}[y+y']+\mathfrak{w}[y-y'+x-x']+\mathfrak{t}[y-y']
\end{aligned}$$

holds for all $y$ and $y'$, and hence, taking the infimum on both sides we have the following inequality

$$2((\mathfrak{t}:\mathfrak{w})[x]+(\mathfrak{t}:\mathfrak{w})[x']) \geq (\mathfrak{t}:\mathfrak{w})[x+x']+(\mathfrak{t}:\mathfrak{w})[x-x'].$$

Replacing $y$ and $y'$ by $\frac{y+y'}{2}$ and $\frac{y-y'}{2}$, respectively, we obtain the reverse inequality, and hence,

$$2((\mathfrak{t}:\mathfrak{w})[x]+(\mathfrak{t}:\mathfrak{w})[x']) = (\mathfrak{t}:\mathfrak{w})[x+x']+(\mathfrak{t}:\mathfrak{w})[x-x'].$$

Now, we are going to verify the listed properties of the parallel sum. Since (a), (b), (d), and (e) are immediate consequences of the definition we prove only (c), (f), and (g).

(c) Observe on the one hand that

$$
\begin{aligned}
((\mathfrak{t} : \mathfrak{w}) : \mathfrak{s})[x] &= \inf_{y \in \mathfrak{X}} \{\mathfrak{s}[y + x] + (\mathfrak{t} : \mathfrak{w})[y]\} \\
&= \inf_{y \in \mathfrak{X}} \inf_{z \in \mathfrak{X}} \{\mathfrak{s}[y + x] + \mathfrak{w}[y + z] + \mathfrak{t}[z]\}
\end{aligned}
$$

holds for all $x \in \mathfrak{X}$. On the other hand,

$$
\begin{aligned}
(\mathfrak{t} : (\mathfrak{w} : \mathfrak{s}))[x] &= \inf_{z \in \mathfrak{X}} \{(\mathfrak{w} : \mathfrak{s})[z + x] + \mathfrak{t}[z]\} \\
&= \inf_{z \in \mathfrak{X}} \inf_{y \in \mathfrak{X}} \{\mathfrak{s}[y + z + x] + \mathfrak{w}[y] + \mathfrak{t}[z]\} \\
&= \inf_{z \in \mathfrak{X}} \inf_{y \in \mathfrak{X}} \{\mathfrak{s}[y - z + x] + \mathfrak{w}[y] + \mathfrak{t}[z]\} \\
&= \inf_{z \in \mathfrak{X}} \inf_{y' \in \mathfrak{X}} \{\mathfrak{s}[y' + x] + \mathfrak{w}[y' + z] + \mathfrak{t}[z]\}
\end{aligned}
$$

holds for all $x \in \mathfrak{X}$, thus the comparison of the two expressions gives the required results

(f) Completing squares leads to the following identities

$$
\begin{aligned}
\lambda\mathfrak{t}[y + x] + \mu\mathfrak{t}[y] =& \lambda\mathfrak{t}[x] + 2\lambda\mathfrak{Re}(\mathfrak{t}(y, x)) + (\lambda + \mu)\mathfrak{t}[y] \\
=& \lambda\mathfrak{t}[x] + \mathfrak{t}\big[\sqrt{\lambda + \mu}y + \frac{\lambda}{\sqrt{\lambda + \mu}}x\big] - \frac{\lambda^2}{\lambda + \mu}\mathfrak{t}[x] \\
=& \frac{\lambda\mu}{\lambda + \mu}\mathfrak{t}[x] + \mathfrak{t}\big[\sqrt{\lambda + \mu}y + \frac{\lambda}{\sqrt{\lambda + \mu}}x\big]
\end{aligned}
$$

Since every term is nonnegative, the infimum over all $y$ is attained when $y = \frac{-\lambda}{\lambda+\mu}x$.

(g) The inequality $\lim_{n\to\infty} (\mathfrak{t}_n : \mathfrak{w}_n) \geq \mathfrak{t} : \mathfrak{w}$ is obvious. On the other hand, for every $\varepsilon > 0$ there exists $y_\varepsilon \in \mathfrak{X}$ such that

$$
(\mathfrak{t} : \mathfrak{w})[x] > \mathfrak{w}[y_\varepsilon + x] + \mathfrak{t}[y_\varepsilon] - \varepsilon.
$$

Moreover, for all $n \geq n_{y_\varepsilon,\varepsilon}$ one has

$$
\mathfrak{w}_n[y_\varepsilon + x] + \mathfrak{t}_n[y_\varepsilon] - \varepsilon < \mathfrak{w}[y_\varepsilon + x] + \mathfrak{t}[y_\varepsilon].
$$

These inequalities yield for all $n \geq n_{y_\varepsilon,\varepsilon}$ that

$$
\inf_{y \in \mathfrak{X}} \{\mathfrak{w}_n[y + x] + \mathfrak{t}_n[y + x]\} < (\mathfrak{t} : \mathfrak{w})[x] + 2\varepsilon.
$$

This implies the reverse inequality, and hence $\mathfrak{t}_n : \mathfrak{w}_n \downarrow \mathfrak{t} : \mathfrak{w}$.

□

Remark that parallel addition with these nice properties can be defined also for representable positive functionals [**53**] and for additive set functions [**56**].

Using the concept of parallel sum we can define the almost dominated part of a form with respect to another form. Consider the operator $\mathbf{D}$ which assigns to the pair of forms $(\mathfrak{t}, \mathfrak{w})$ the $\mathfrak{w}$-almost dominated part of $\mathfrak{t}$ by the formula

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} := \sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w}).$$

The following lemma collects some important facts about the operator $\mathbf{D}$. These are elementary consequences of the definition of $\mathbf{D}$ and Lemma 4.1

**Lemma 4.2.** *Let $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{v}$, and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. Then*

(*a*) $(\mathfrak{t} : n\mathfrak{w}) \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$, *for all* $n \in \mathbb{N}$,

(*b*) $\mathbf{D}$ *is monotone in both variables, i.e.,* $\mathfrak{t} \leq \mathfrak{s}$, $\mathfrak{v} \leq \mathfrak{w}$ *implies* $\mathbf{D}_{\mathfrak{v}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{s}$,

(*c*) $\mathbf{D}_{\mathfrak{w}}(\lambda\mathfrak{t}) = \lambda\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *for all* $\lambda \geq 0$,

## 4.2. Hassi - Sebestyén - de Snoo decomposition of forms

In the following theorem we characterize almost dominatedness and singularity in terms of parallel addition.

**Theorem 4.3.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$. Then*

*(a)* $\mathfrak{t} \perp \mathfrak{w} \Leftrightarrow \mathfrak{t} : \mathfrak{w} = 0 \Leftrightarrow \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$,

*(b)* $\mathfrak{t} \ll_{\mathrm{ad}} \mathfrak{w} \Leftrightarrow \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$.

*Proof.* To prove $(a)$ observe that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = 0$ implies $\mathfrak{t} : \mathfrak{w} = 0$ by definition. If $\mathfrak{t} : \mathfrak{w} = 0$, then $\mathfrak{w}$ and $\mathfrak{t}$ are singular, because $0 = \mathfrak{t} : \mathfrak{w} \geq \mathfrak{u} : \mathfrak{u} = \frac{1}{2}\mathfrak{u} \geq 0$ for every form $\mathfrak{u}$ which satisfies $\mathfrak{u} \leq \mathfrak{t}, \mathfrak{w}$. Finally, assume that $\mathfrak{t}$ and $\mathfrak{w}$ are singular, but $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \neq 0$. In this case, there exists $n \in \mathbb{N}$ such that $\mathfrak{t} : n\mathfrak{w} \neq 0$, which is a contradiction, because $0 \neq (\frac{1}{n})\mathfrak{t} : \mathfrak{w} \leq \mathfrak{t}, \mathfrak{w}$.

We are going to prove $(b)$. If $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$ then $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated by definition. For the converse implication observe first that if $\mathfrak{t}$ is $\mathfrak{w}$-dominated, i.e., there exists an $\alpha > 0$ constant such that $\mathfrak{t} \leq \alpha\mathfrak{w}$, then $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$. Indeed, for every $n \in \mathbb{N}$ we have

$$\mathfrak{t} \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t} : n\mathfrak{w} \geq \mathfrak{t} : \left(\frac{n}{\alpha}\right)\mathfrak{t} = \frac{n}{\alpha + n}\mathfrak{t},$$

which implies $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$ by taking supremum in $n$. Now assume that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated, and recall that this guarantees that $\mathfrak{t}$ is a limit of a monotone increasing sequence $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ of $\mathfrak{w}$-dominated forms. According to the previous observation, we have

$$\mathfrak{t}_n = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}_n \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$$

which implies that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$, again by taking supremum. $\square$

Observe that this theorem states that $\mathbf{D}$ is idempotent, i.e., $\mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ for all $\mathfrak{t}$ and $\mathfrak{w}$ (because $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is $\mathfrak{w}$-almost dominated by definition). Observe also that if $\mathfrak{t}$ is both $\mathfrak{w}$-almost dominated and $\mathfrak{w}$-singular, then $\mathfrak{t}$ is the identically zero form. Indeed, according to the previous theorem, $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{o}$.

The following theorem states that the $(\ll_{\mathrm{ad}}, \perp)$-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$ exists for every $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$. This was proved first by Hassi, Sebestyén, and de Snoo in [**16**].

**Theorem 4.4.** *Let* $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *be arbitrary forms on* $\mathfrak{X}$ *and consider the decomposition*

$$\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}).$$

*This decomposition is an* $(\ll_{\mathrm{ad}}, \perp)$*-type decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*, that is,* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \ll_{\mathrm{ad}} \mathfrak{w}$ *and* $(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) \perp \mathfrak{w}$*. Furthermore, this decomposition is extremal in the following sense:*

$$\mathfrak{u} \in \mathcal{F}_+(\mathfrak{X}),\ \mathfrak{u} \leq \mathfrak{t} \textit{ and } \mathfrak{u} \ll_{\mathrm{ad}} \mathfrak{w} \quad \Rightarrow \quad \mathfrak{u} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

*Proof.* First we prove the maximality of $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$. Let $\mathfrak{u}$ be a form such that $\mathfrak{u} \leq \mathfrak{t}$ and $\mathfrak{u} \ll_{\mathrm{ad}} \mathfrak{w}$. According to Theorem 4.3(b) and Lemma 4.1(e) we have

$$\mathfrak{u} = \mathbf{D}_{\mathfrak{w}}\mathfrak{u} = \sup_{n\in\mathbb{N}}(\mathfrak{u} : n\mathfrak{w}) \leq \sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

Using maximality and the fact that the sum of $\mathfrak{w}$-almost dominated forms is $\mathfrak{w}$-almost dominated, one can obtain $\mathbf{D}_{\mathfrak{w}}(\mathfrak{u}+\mathfrak{v}) \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{u}+\mathbf{D}_{\mathfrak{w}}\mathfrak{v}$ for every $\mathfrak{u}, \mathfrak{v} \in \mathcal{F}_+(\mathfrak{X})$. Since $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is $\mathfrak{w}$-almost dominated by definition, it is enough to prove that $\mathfrak{w}$ and $\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ are singular, or equivalently, $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = 0$. Combining Theorem 4.3 with the following line the singularity of $\mathfrak{w}$ and $\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is proved

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\big(\mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})\big) \geq \mathbf{D}_{\mathfrak{w}}(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

□

Here we emphasize the following important consequence of maximality, which was used in the proof:

$$\mathbf{D}_{\mathfrak{w}}(\mathfrak{u} + \mathfrak{v}) \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{u} + \mathbf{D}_{\mathfrak{w}}\mathfrak{v}.$$

Remark also that this type decomposition is not unique in general. We shall see later that this decomposition is unique precisely when $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq c\mathfrak{w}$ for some $c \geq 0$.

## 4.3. Lebesgue decomposition of contents

In this section we present a new approach for the Lebesgue decomposition of finitely additive measures (or contents, for short). Using the main result of this chapter we show that the Lebesgue decomposition of contents exists, and corresponds to the Lebesgue decomposition of their induced forms.

There are many authors who studied the decomposition of additive set functions defined on set algebras, or on lattices of sets (see e.g. [**6, 8, 9, 21, 28**]. A Lebesgue decomposition of additive set functions is constructed and characterized first by R. B. Darst [**9**]. We shall prove that the content $\mu$ is strongly absolutely continuous (resp., singular) with respect to the content $\nu$ precisely when the induced form $\mathfrak{t}_\mu$ is almost dominated (resp., singular) by the induced form $\mathfrak{t}_\nu$.

Let $\mathfrak{A}$ be an algebra of subsets of a set $X$, and let $\mu$, $\mu_n$ $(n \in \mathbb{N})$, and $\nu$ be contents on it. We say that $\mu$ is dominated by $\nu$ (or $\mu$ is $\nu$-dominated) if there exists a $c > 0$ such that $\mu \leq c\nu$. If $\mu_n \leq \mu_{n+1} \leq \nu$ for every $n \in \mathbb{N}$, then the set function defined by the pointwise limit

$$\mu(A) := \sup_{n \in \mathbb{N}} \mu_n(A) \qquad (A \in \mathfrak{A})$$

is a content, and $\mu \leq \nu$. If $\mu$ is a pointwise limit of a nondecreasing sequence of $\nu$-dominated sequence then $\mu$ is called almost dominated by $\nu$ ($\mu \ll_{\mathrm{ad}} \nu$ in symbols). We say that $\mu$ is strongly absolutely continuous with respect to $\nu$ (and write $\mu \ll_{\mathrm{s}} \nu$), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \varepsilon$, whenever $A \in \mathfrak{A}$ and $\nu(A) < \delta$. Note that if $\vartheta \leq \mu$ and $\mu \ll_{\mathrm{s}} \nu$ then $\vartheta \ll_{\mathrm{s}} \nu$. Remark that if $\mu$ and $\nu$ are measures on the $\sigma$-algebra $\mathfrak{A}$ then $\mu \ll_{\mathrm{s}} \nu$ is equivalent with the usual notion of absolute continuity (denoted by $\mu \ll_{\mathrm{ac}} \nu$), that is $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathfrak{A}$. For contents we have $\mu \ll_{\mathrm{s}} \nu \Rightarrow \mu \ll_{\mathrm{ac}} \nu$.

We say that $\mu$ is singular with respect to $\nu$ (or $\mu$ and $\nu$ are singular, $\mu \perp \nu$), if $\mu \wedge \nu = 0$. For measures this is equivalent with the existence of a measurable subset $P$ such that $\mu(A) = \mu(A \cap P)$ and $\nu(A) = \nu(A \setminus P)$ for all $A \in \mathfrak{A}$.

The following two results was proved by König in [**21**].

**Theorem 4.5.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the following statements are equivalent.*

*(i)* $\mu \ll_{\mathrm{s}} \nu$.

*(ii)* $\lim\limits_{n \to +\infty} \big(\mu \wedge n\nu\big) = \sup\limits_{n \in \mathbb{N}} \big(\mu \wedge n\nu\big) = \mu$.

*Proof.* To prove *(ii)* $\Rightarrow$ *(i)* assume that $\mu \not\ll_{\mathrm{s}} \nu$. Then there exists $\varepsilon > 0$ and $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{A}$ such that $\nu(A_n) \to 0$ and $\mu(A_n) > \varepsilon$ for every $n \in \mathbb{N}$. Now fix $N \in \mathbb{N}$ such that

$$\big(\mu \wedge n\nu\big)(X) > \mu(X) - \frac{\varepsilon}{2} \quad (n \geq N).$$

Since $\big(\mu \wedge n\nu\big)(X) \leq \mu(X \setminus A_k) + n\nu(A_k)$ by definition, we have

$$\mu(X \setminus A_k) + n\nu(A_k) > \mu(X) - \frac{\varepsilon}{2} \quad (n \geq N).$$

Hence, if $k$ is big enough, we have

$$\mu(A_k) < n\nu(A_k) + \frac{\varepsilon}{2} < \varepsilon.$$

This is contradiction. For the converse implication see Lemma 3.9. $\square$

**Corollary 4.6.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the set function*

$$\mu_{\mathrm{r},\nu}(A) := \sup_{n\in\mathbb{N}} \big(\mu \wedge n\nu\big)(A) \quad (A \in \mathfrak{A})$$

*is a content, and $\mu_{\mathrm{r},\nu} \ll_{\mathrm{s}} \nu$.*

*Proof.* Since the sequence $(\mu \wedge n\nu)_{n\in\mathbb{N}}$ is monotonically nondecreasing and majorized by $\mu$, then $\mu_{\mathrm{r},\nu}$ is a content, and $\mu_{\mathrm{r},\nu} \leq \mu$. Let $k \in \mathbb{N}$ be fixed. Clearly, $\mu_{\mathrm{r},\nu} \wedge k\nu \leq \mu \wedge k\nu$ holds, and

$$\mu \wedge k\nu \leq \sup_{n\in\mathbb{N}} \big(\mu \wedge n\nu\big) = \mu_{\mathrm{r},\nu}.$$

Consequently, $\mu_{\mathrm{r},\nu} \wedge k\nu = \mu \wedge k\nu$, and

$$\sup_{n\in\mathbb{N}} \big(\mu_{\mathrm{r},\nu} \wedge n\nu\big) = \sup_{n\in\mathbb{N}} \big(\mu \wedge n\nu\big) = \mu_{\mathrm{r},\nu}.$$

By the previous theorem we obtain that $\mu_{\mathrm{r},\nu} \ll_{\mathrm{s}} \nu$. $\square$

Now, we are going to investigate the connection between the Lebesgue decomposition of forms and the Lebesgue decomposition of contents. Let $X$ be a set, let $\mathfrak{A}$ be an algebra on it, and let $\mu$ and $\nu$ be contents on $\mathfrak{A}$. Let the complex linear space $\mathfrak{X}$ be the complex linear span of the characteristic functions of the sets in $\mathfrak{A}$, i.e.,

$$\mathfrak{X} := \operatorname{span}_{\mathbb{C}}\{\chi_A \,|\, A \in \mathfrak{A}\}.$$

Let $\mu$ be a content on the algebra $\mathfrak{A}$, and consider its induced form

$$\mathfrak{t}_\mu(\varphi, \psi) := \int_X \varphi \cdot \overline{\psi}\, \mathrm{d}\mu \quad (\varphi, \psi \in \mathfrak{X}).$$

In the next lemma we collect some simple but useful property of the above defined assignment $\mu \mapsto \mathfrak{t}_\mu$.

**Lemma 4.7.** *Let $\mu$, $\mu_n$, and $\nu$ be contents on the algebra $\mathfrak{A}$. Then*

*(a)* $\mu = \nu$ *if and only if* $\mathfrak{t}_\mu = \mathfrak{t}_\nu$,
*(b)* $\mu \leq \nu$ *if and only if* $\mathfrak{t}_\mu \leq \mathfrak{t}_\mu$,
*(c)* $\mathfrak{t}_{c\mu} = c\mathfrak{t}_\mu$ *for all* $c \geq 0$,
*(d)* $\mu$ *is dominated by* $\nu$ *if and only if* $\mathfrak{t}_\mu$ *is dominated by* $\mathfrak{t}_\nu$,
*(e)* $\mathfrak{t}_{\mu+\nu} = \mathfrak{t}_\mu + \mathfrak{t}_\nu$, *and*
*(f)* *if* $\mu_n \uparrow \mu$, *then* $\mathfrak{t}_{\mu_n} \uparrow \mathfrak{t}_\mu$.

*Proof.* The proof is an easy computation based on the equalities $\mathfrak{t}_\mu[\chi_A] = \mu(A)$ and

$$\mathfrak{t}_\mu\left[\sum_{j=1}^k \lambda_j \chi_{A_j}\right] = \sum_{j=1}^k |\lambda_j|^2 \mu(A_j).$$

□

Recall that if $\mu$ and $\nu$ are contents on the algebra $\mathfrak{A}$, then $\mu_{\mathrm{r},\nu}$ denotes the content $\sup_{n\in\mathbb{N}}\big(\mu \wedge n\nu\big)$. In the following theorem we describe the almost dominated part of an induced form with respect to an other one.

**Theorem 4.8.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the $\mathfrak{t}_\nu$-almost dominated part of $\mathfrak{t}_\mu$ is always induced by a content and belongs to $\mu_{\mathrm{r},\nu}$, i.e.,* $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu = \mathfrak{t}_{\mu_{\mathrm{r},\nu}}$.

*Proof.* We prove first that $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu$ and $\mathfrak{t}_{\mu_{\mathrm{r},\nu}}$ coincide on the characteristic functions. Let $A \in \mathfrak{A}$ be fixed, and estimate $\big(\mathfrak{t}_\mu : \mathfrak{t}_{n^2\nu}\big)[\chi_A]$ from above

$$\begin{aligned}
\big(\mathfrak{t}_\mu : \mathfrak{t}_{n^2\nu}\big)[\chi_A] &= \inf_{g\in\mathfrak{X}} \big\{\mathfrak{t}_\mu[\chi_A - g] + \mathfrak{t}_{n^2\nu}[g]\big\} \\
&\leq \inf_{E\in\mathfrak{A}} \big\{\mathfrak{t}_\mu[\chi_{A\setminus E}] + \mathfrak{t}_{n^2\nu}[\chi_E]\big\} \\
&= \inf_{E\in\mathfrak{A}} \big\{\mu(A\setminus E) + n^2\nu(E)\big\} \\
&= \big(\mu \wedge n^2\nu\big)(A) \\
&= \mathfrak{t}_{\mu\wedge n^2\nu}[\chi_A].
\end{aligned}$$

The converse inequality follows from Lemma 4.7 $(b)$ and from Lemma 4.1

$$\begin{aligned}
\big(\mathfrak{t}_\mu : \mathfrak{t}_{n^2\nu}\big)[\chi_A] &= \big(\mathfrak{t}_\mu : n\mathfrak{t}_{n\nu}\big)[\chi_A] \\
&\geq \big(\mathfrak{t}_{\mu\wedge n\nu} : n\mathfrak{t}_{\mu\wedge n\nu}\big)[\chi_A] \\
&= \tfrac{n}{n+1}\mathfrak{t}_{\mu\wedge n\nu}[\chi_A] \\
&= \tfrac{n}{n+1}\big(\mu \wedge n\nu\big)(A).
\end{aligned}$$

Taking supremum in $n$, we obtain that

$$\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu[\chi_A] = \sup_{n\in\mathbb{N}}(\mathfrak{t}_\mu : n^2\mathfrak{t}_\nu)[\chi_A] = \sup_{n\in\mathbb{N}}\big(\mu \wedge n^2\nu\big)(A) = \mu_{\mathrm{r},\nu}(A) = \mathfrak{t}_{\mu_{\mathrm{r},\nu}}[\chi_A].$$

In the second step we show that $\mathfrak{t}_\mu : n\mathfrak{t}_\nu \le \mathfrak{t}_{\mu\wedge n\nu}$. Let $\varepsilon > 0$ and $\varphi \in \mathfrak{X}$ be fixed, and assume that $\varphi = \sum\limits_{j=1}^{k}\lambda_j\chi_{A_j}$. For any $j \in \{1,\dots,k\}$ there exist disjoint sets $A'_j$ and $A''_j$ in $\mathfrak{A}$ such that $A'_j \cup A''_j = A_j$ and

$$\big(\mu \wedge n\nu\big)(A_j) \le \mu(A'_j) + n\nu(A''_j) \le \big(\mu \wedge n\nu\big)(A_j) + \tfrac{\varepsilon}{k|\lambda_j|^2}.$$

Then

$$\begin{aligned}
\big(\mathfrak{t}_\mu : n\mathfrak{t}_\nu\big)\left[\sum_{j=1}^{k}\lambda_j\chi_{A_j}\right] &= \inf_{g\in\mathfrak{X}}\left\{\mathfrak{t}_\mu\left[\sum_{j=1}^{k}\lambda_j\chi_{A_j} - g\right] + n\mathfrak{t}_\nu[g]\right\}\\
&\le \mathfrak{t}_\mu\left[\sum_{j=1}^{k}\lambda_j\chi_{A'_j}\right] + n\mathfrak{t}_\nu\left[\sum_{j=1}^{k}\lambda_j\chi_{A''_j}\right]\\
&= \sum_{j=1}^{k}|\lambda_j|^2\mu(A'_j) + \sum_{j=1}^{k}|\lambda_j|^2 n\nu(A''_j)\\
&\le \sum_{j=1}^{k}|\lambda_j|^2\Big((\mu\wedge n\nu)(A_j) + \tfrac{\varepsilon}{|\lambda_j|^2 k}\Big)\\
&\le \sum_{j=1}^{k}|\lambda_j|^2\big(\mu\wedge n\nu\big)(A_j) + \varepsilon\\
&= \mathfrak{t}_{\mu\wedge n\nu}\left[\sum_{j=1}^{k}\lambda_j\chi_{A_j}\right] + \varepsilon.
\end{aligned}$$

Using Lemma 4.7 $(f)$ it follows that $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu \le \mathfrak{t}_{\mu_{\mathrm{r},\nu}}$. Finally, since $\mathfrak{t}_{\mu_{\mathrm{r},\nu}} - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu$ is a form, the square root of its quadratic form is a seminorm. Consequently, using the triangle inequality we have that

$$\begin{aligned}
\left((\mathfrak{t}_{\mu_{\mathrm{r},\nu}} - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu)\Big[\sum_{j=1}^{k}\lambda_j\chi_{A_j}\Big]\right)^{1/2} &\le \sum_{j=1}^{k}\Big((\mathfrak{t}_{\mu_{\mathrm{r},\nu}} - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu)[\lambda_j\chi_{A_j}]\Big)^{1/2}\\
&= \sum_{j=1}^{k}|\lambda_j|\Big((\mathfrak{t}_{\mu_{\mathrm{r},\nu}} - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu)[\chi_{A_j}]\Big)^{1/2}\\
&= \sum_{j=1}^{k}|\lambda_j|\Big(\mathfrak{t}_{\mu_{\mathrm{r},\nu}}[\chi_{A_j}] - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu[\chi_{A_j}]\Big)^{1/2} = 0
\end{aligned}$$

holds for all $\varphi = \sum\limits_{j=1}^{k}\lambda_j\chi_{A_j} \in \mathfrak{X}$. Hence, we proved that $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu = \mathfrak{t}_{\mu_{\mathrm{r},\nu}}$.

$\square$

**Remark 4.9.** We can prove in the same way as in the second step that if $\mu$ and $\nu$ are contents on the algebra $\mathfrak{A}$ then $\mathfrak{t}_\mu : \mathfrak{t}_\nu \leq \mathfrak{t}_{\mu\wedge\nu}$.

Using the above theorem we can characterize strong absolute continuity of contents by means of their induced forms.

**Lemma 4.10.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the following statements are equivalent.*

(*i*) $\mu \ll_{\mathrm{s}} \nu$.

(*ii*) $\mathfrak{t}_\mu \ll_{\mathrm{ad}} \mathfrak{t}_\nu$.

*Proof.* If $\mu \ll_{\mathrm{s}} \nu$ then $\mu = \sup_{n\in\mathbb{N}} \big(\mu \wedge n\nu\big) = \mu_{\mathrm{r},\nu}$. According to Theorem 4.8 and Theorem 4.4 we have $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu = \mathfrak{t}_{\mu_{\mathrm{r},\nu}} = \mathfrak{t}_\mu$, i.e., $\mathfrak{t}_\mu \ll_{\mathrm{ad}} \mathfrak{t}_\nu$. The converse implication is similar. If $\mathfrak{t}_\mu \ll_{\mathrm{ad}} \mathfrak{t}_\nu$ then $\mathfrak{t}_{\mu_{\mathrm{r},\nu}} = \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu = \mathfrak{t}_\mu$. Consequently, $\mu = \mu_{\mathrm{r},\nu}$, i.e., $\mu \ll_{\mathrm{s}} \nu$. □

As a consequence, observe immediately that $\mu_{\mathrm{r},\nu}$ is the maximum of all contents which are majorized by $\mu$ and strongly absolutely continuous with respect to $\nu$.

In the following lemma we describe singularity of contents via the induced forms. Recall that $\mu$ is singular with respect to $\nu$ if $\mu \wedge \nu = 0$, and $\mathfrak{t}_\mu$ is singular with respect to $\mathfrak{t}_\nu$ if $\mathfrak{s} \leq \mathfrak{t}_\mu$ and $\mathfrak{s} \leq \mathfrak{t}_\nu$ imply that $\mathfrak{s} = 0$.

**Lemma 4.11.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the following statements are equivalent.*

(*i*) *$\mu$ is singular with respect to $\nu$.*

(*ii*) *$\mathfrak{t}_\mu$ is singular with respect to $\mathfrak{t}_\nu$.*

*Proof.* First assume that $\mu \wedge \nu = 0$. In this case $\mathfrak{t}_{\mu\wedge\nu} = 0$. From Remark 4.9 it follows that $\mathfrak{t}_\mu : \mathfrak{t}_\nu = 0$, therefore, $\mathfrak{t}_\mu$ is singular with respect to $\mathfrak{t}_\nu$ by Theorem 4.3 (*a*). On the other hand, if $\mathfrak{t}_\mu$ is singular with respect to $\mathfrak{t}_\nu$, then $\mathfrak{t}_\mu : \mathfrak{t}_\nu = 0$ according to Theorem 4.3 (*a*). Hence, we have $0 = \mathfrak{t}_\mu : \mathfrak{t}_\nu \geq \mathfrak{t}_{\mu\wedge\nu} : \mathfrak{t}_{\mu\wedge\nu} = \frac{1}{2}\mathfrak{t}_{\mu\wedge\nu}$, i.e., $\mu \wedge \nu = 0$. □

Now, we prove the Lebesgue decomposition theorem for contents via their induced forms.

**Theorem 4.12.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then $\mu$ admits a Lebesgue decomposition with respect to $\nu$, namely*

$$\mu = \mu_{\mathrm{r},\nu} + (\mu - \mu_{\mathrm{r},\nu})$$

*Furthermore, the Lebesgue decomposition is unique.*

*Proof.* By Theorem 4.4 it follows that the decomposition

$$\mathfrak{t}_\mu = \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu + \big(\mathfrak{t}_\mu - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu\big)$$

is a Lebesgue decomposition, i.e., $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu$ is almost dominated by $\mathfrak{t}_\nu$ and $\mathfrak{t}_\mu - \mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu$ is singular with respect to $\mathfrak{t}_\nu$. We know from Theorem 4.8 that $\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_\mu$ is an induced form and belongs to $\mu_{\mathrm{r},\nu}$. Since $\mu_{\mathrm{r},\nu} \leq \mu$, so $\mu - \mu_{\mathrm{r},\nu}$ is a content and it is clear that $\mathfrak{t}_\mu - \mathfrak{t}_{\mu_{\mathrm{r},\nu}} = \mathfrak{t}_{\mu - \mu_{\mathrm{r},\nu}}$. It follows from Lemma 4.10 and Lemma 4.11 that $\mu_{\mathrm{r},\nu} \ll_{\mathrm{ac}} \nu$ and $\mu_{\mathrm{s},\nu} := \mu - \mu_{\mathrm{r},\nu}$ is singular with respect to $\nu$, i.e., the Lebesgue decomposition exists. Assume that the decomposition is not unique. Let $\mu = \mu_1 + \mu_2$ be a decomposition, and consider the induced forms. We show that $\mathfrak{t}_{\mu_1} + \mathfrak{t}_{\mu_2}$ is not a Lebesgue decomposition for $\mathfrak{t}_\mu$, unless $\mu_1 = \mu_{\mathrm{r},\nu}$.

Since $\mathfrak{t}_{\mu_{\mathrm{r},\nu}}$ is the maximum of all forms majorized by $\mathfrak{t}_\mu$, which are almost dominated by $\mathfrak{t}_\nu$, we obtain that $\mathfrak{t}_{\mu_{\mathrm{r},\nu}} - \mathfrak{t}_{\mu_1}$ is a form, which is induced by $\mu_{\mathrm{r},\nu} - \mu_1$. Recall that $\mu_{\mathrm{r},\nu} - \mu_1$ is strongly absolutely continuous with respect to $\nu$, hence we have that

$$\mathbf{D}_{\mathfrak{t}_\nu}\mathfrak{t}_{\mu_2} = \mathbf{D}_{\mathfrak{t}_\nu}\big(\mathfrak{t}_{\mu_{\mathrm{r},\nu}-\mu_1} + \mathfrak{t}_{\mu_{\mathrm{s},\nu}}\big) \geq \mathbf{D}_{\mathfrak{t}_\nu}\big(\mathfrak{t}_{\mu_{\mathrm{r},\nu}-\mu_1}\big) + \mathbf{D}_{\mathfrak{t}_\nu}\big(\mathfrak{t}_{\mu_{\mathrm{s},\nu}}\big) = \mathbf{D}_{\mathfrak{t}_\nu}\big(\mathfrak{t}_{\mu_{\mathrm{r},\nu}-\mu_1}\big),$$

which is equal to $\mathfrak{t}_{\mu_{\mathrm{r},\nu}-\mu_1}$. Therefore, $\mathfrak{t}_{\mu_2}$ is not $\mathfrak{t}_\nu$-singular unless $\mu_{\mathrm{r},\nu} = \mu_1$. $\square$

CHAPTER 5

# The uniqueness of short- and Lebesgue-type decompositions

In this chapter we present (without any significant modification) some results of the fundamental paper of Hassi, Sebestyén and de Snoo [**16**]. Namely, we give a necessary and sufficient condition for the uniqueness of the decomposition theorems. First we show that the $(\ll_{\mathrm{cl}}, \perp)$-type decomposition coincides with the $(\ll_{\mathrm{ad}}, \perp)$-type decomposition. In fact, we are going to prove that

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

After that, we will investigate densely defined closable operators. These results will help us to characterize the uniqueness of the different type of decompositions. This part is included because of the sake of completeness. For the references and other remarks see the original paper. As an application, we will close this chapter with the characterization of closed range operators.

## 5.1. Almost domination and closability

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$, and recall that $J$ is the embedding operator

$$\mathfrak{X}/\ker(\mathfrak{t}+\mathfrak{w}) \subseteq \mathscr{H}_{\mathfrak{t}+\mathfrak{w}} \hookrightarrow \mathscr{H}_{\mathfrak{w}},$$

defined by

$$\pi_{\mathfrak{t}+\mathfrak{w}}(x) \mapsto \pi_{\mathfrak{w}}(x), \qquad x \in \mathfrak{X}.$$

Recall also that the kernel of $J^{**}$ can be described by

$$\ker J^{**} = \big\{ \lim_{n\to\infty} \pi_{\mathfrak{t}+\mathfrak{w}}(x_n) \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t},\mathfrak{w}) \big\}.$$

where

$$\mathfrak{S}(\mathfrak{t},\mathfrak{w}) := \big\{ (x_n)_{n\in\mathbb{N}} \subseteq \mathfrak{X} \;\big|\; \mathfrak{t}[x_n - x_m] \to 0, \mathfrak{w}[x_n] \to 0 \big\}.$$

According to (3.6) we know that

$$\big\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x)\big\|^2_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{t}[x] - \mathfrak{r}[x] = \mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x].$$

The next theorem gives a characterization for the singular and regular part (see Theorem 3.4. and Theorem 3.5. in [**16**].

**Theorem 5.1.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$*, and consider the* $(\ll_{\mathrm{cl}}, \perp)$*-type decomposition of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. Then the* $\mathfrak{w}$*-singular part of* $\mathfrak{t}$ *can be written as*

$$\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x] = \mathfrak{t}[x] + \inf_{y\in\mathfrak{X}} \big\{\mathfrak{w}[x+y] - \inf_{z\in\mathfrak{X}}\{\mathfrak{t}[z] + \mathfrak{w}[z-y]\}\big\}, \tag{5.1}$$

*and the regular part as*

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] = \sup_{y\in\mathfrak{X}} \big\{ \inf_{z\in\mathfrak{X}}\{\mathfrak{w}[z+y] + \mathfrak{t}[z]\} - \mathfrak{w}[x+y]\big\}. \tag{5.2}$$

*Proof.* Since $J^*(\operatorname{ran} J)$ is dense in $\mathscr{H}_{\mathfrak{w}} \ominus \ker J^{**}$, it follows that

$$\begin{aligned}
\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x] &= \Big((I-P)(x+\ker(\mathfrak{t}+\mathfrak{w}) \,|\, (I-P)(x+\ker(\mathfrak{t}+\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} \\
&= \inf_{y\in\operatorname{dom} J} \left\{ \Big(x+\ker(\mathfrak{t}+\mathfrak{w}) + J^*(y+\ker\mathfrak{w}) \,|\, x+\ker(\mathfrak{t}+\mathfrak{w}) + J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} \right\} \\
&= \inf_{y\in\operatorname{dom} J} \left\{ \mathfrak{t}[x] + \mathfrak{w}[x] + \mathfrak{w}(x,y) + \mathfrak{w}(y,x) + \Big(J^*(y+\ker\mathfrak{w}) \,|\, J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} \right\} \\
&= \mathfrak{t}[x] + \inf_{y\in\operatorname{dom} J} \left\{ \mathfrak{w}[x+y] - \mathfrak{w}[y] + \Big(J^*(y+\ker\mathfrak{w}) \,|\, J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} \right\}.
\end{aligned}$$

On the other hand, since $\operatorname{dom} J$ is dense in $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$, we have

$$\begin{aligned}
0 &= \inf_{z\in\operatorname{dom} J} \left\{ \Big(z+\ker(\mathfrak{t}+\mathfrak{w}) + J^*(y+\ker\mathfrak{w}) \,|\, z+\ker(\mathfrak{t}+\mathfrak{w}) + J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} \right\} \\
&= \Big(J^*(y+\ker\mathfrak{w}) \,|\, J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} + \inf_{z\in\operatorname{dom} J} \{\mathfrak{t}[z] + \mathfrak{w}[z] + \mathfrak{w}(y,z) + \mathfrak{w}(z,y)\} \\
&= -\,\mathfrak{w}[y] + \Big(J^*(y+\ker\mathfrak{w}) \,|\, J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} + \inf_{z\in\mathfrak{X}}\{\mathfrak{t}[z] + \mathfrak{w}[z+y]\},
\end{aligned}$$

and hence,

$$\Big(J^*(y+\ker\mathfrak{w}) \,|\, J^*(y+\ker\mathfrak{w})\Big)_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{w}[y] - \inf_{z\in\mathfrak{X}}\{\mathfrak{t}[z] + \mathfrak{w}[z+y]\}$$

Combining these two equations we have the desired equality

$$\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x] = \mathfrak{t}[x] + \inf_{y\in\mathfrak{X}} \big\{\mathfrak{w}[x+y] - \inf_{z\in\mathfrak{X}}\{\mathfrak{t}[z] + \mathfrak{w}[z-y]\}\big\}.$$

Since $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = \mathfrak{t} - \mathfrak{t}_{\mathrm{sing},\mathfrak{w}}$, we have the following formula for the $\mathfrak{w}$-regular part of $\mathfrak{t}$

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] = \sup_{y\in\mathfrak{X}} \big\{ \inf_{z\in\mathfrak{X}}\{\mathfrak{w}[z+y] + \mathfrak{t}[z]\} - \mathfrak{w}[x+y]\big\}$$

□

Now, define the parallel difference of two forms. This notion was introduced and studied first by Hassi, Sebestyén, and de Snoo in [**15**]. Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$, and define the *parallel difference* $\mathfrak{t} \div \mathfrak{w}$ of $\mathfrak{t}$ and $\mathfrak{w}$ as a mapping from $\mathfrak{X}$ to $\mathbb{R} \cup \{\infty\}$ by

$$(\mathfrak{t} \div \mathfrak{w})[x] = \sup_{y \in \mathfrak{X}} \{\mathfrak{t}[x+y] - \mathfrak{w}[y]\} \quad (x \in \mathfrak{X}). \tag{5.3}$$

We say that the parallel difference $\mathfrak{t} \div \mathfrak{w}$ exists, if $\mathfrak{t}$ and $\mathfrak{w}$ are forms, and (5.3) is the quadratic form of a form. It is clear that if $\mathfrak{w}_1 \leq \mathfrak{w}_2$ then $\mathfrak{t} \div \mathfrak{w}_1 \geq \mathfrak{t} \div \mathfrak{w}_2$. Now observe that $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ can be written as

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w} \tag{5.4}$$

according to (5.2). The following Lemma (see Proposition 2.7. in [**16**]) plays an important role in this thesis. The operator version of this result is due to Eriksson and Leutwiler, see [**11**, Lemmas 2.6,2.7]

**Lemma 5.2.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$ *Then we have the following line of identities*

$$\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : \mathfrak{w} = \mathfrak{t} : \mathfrak{w} \tag{5.5}$$

*Moreover, for any form* $\mathfrak{s}$*,*

$$\mathfrak{t} : \mathfrak{w} \leq \mathfrak{s} : \mathfrak{w} \qquad \Leftrightarrow \qquad \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{s}. \tag{5.6}$$

*Proof.* Observe first that $\mathfrak{t} : \mathfrak{w}$ is dominated by $\mathfrak{w}$, and hence $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) = \mathfrak{t} : \mathfrak{w}$. Since the parallel sum is commutative, associative, and monotone in both variables we have

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) = \sup_{n \in \mathbb{N}}((\mathfrak{t} : \mathfrak{w}) : n\mathfrak{w}) = \sup_{n \in \mathbb{N}}((\mathfrak{t} : n\mathfrak{w}) : \mathfrak{w}) \leq (\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : \mathfrak{w} \leq \mathfrak{t} : \mathfrak{w}.$$

Now we are going to prove the second statement. If $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{s}$ then

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{s} : \mathfrak{w} = \mathfrak{s} : \mathfrak{w}$$

holds according to the first part. To prove the converse implication we use induction. The case $n = 1$ is clear, assume it holds for $n$, and observe that

$$\begin{aligned} \left(\frac{1}{n+1}\mathfrak{t}\right) : \mathfrak{w} &= \left(\frac{1}{n}\mathfrak{t} : \mathfrak{w}\right) : \mathfrak{t} \leq \left(\frac{1}{n}\mathfrak{s} : \mathfrak{w}\right) : \mathfrak{t} \\ &\leq \left(\frac{1}{n}\mathfrak{s}\right) : (\mathfrak{w} : \mathfrak{t}) \leq \left(\frac{1}{n}\mathfrak{s}\right) : (\mathfrak{w} : \mathfrak{s}) = \leq \left(\frac{1}{n+1}\mathfrak{s}\right) : \mathfrak{w} \end{aligned}$$

which completes the proof. $\square$

The following theorem states that the $(\ll_{\mathrm{cl}}, \perp)$-type decomposition of forms coincides with the $(\ll_{\mathrm{ad}}, \perp)$-type decomposition.

**Theorem 5.3.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$. *Then* $\mathfrak{t}$ *is* $\mathfrak{w}$*-closable precisely when* $\mathfrak{t}$ *is* $\mathfrak{w}$*-almost dominated. In fact,* $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$*, and hence.*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w}. \tag{5.7}$$

*Proof.* First assume that $\mathfrak{t}$ is $\mathfrak{w}$-closable, i.e., $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$. According to Lemma 5.2, Corollary 3.4, and Proposition 3.5, we have

$$\mathfrak{t} = \mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w} = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w} = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t})_{\mathrm{reg},\mathfrak{w}} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}.$$

According to Theorem 4.3 $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}$ means that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated.

For the converse inequality assume that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated, i.e., there exists a nondecreasing sequence of $\mathfrak{w}$-dominated forms $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ such that $\sup\limits_{n\in\mathbb{N}} \mathfrak{t}_n = \mathfrak{t}$. Using Corollary 3.4 and Proposition 3.5 again we have the following line of inequalities

$$\mathfrak{t}_n = (\mathfrak{t}_n)_{\mathrm{reg},\mathfrak{w}} \leq \mathfrak{t}_{\mathrm{reg},\mathfrak{w}} \leq \mathfrak{t}.$$

Taking the supremum in $n$ we obtain that $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}} = \mathfrak{t}$, i.e., $\mathfrak{t}$ is $\mathfrak{w}$-closable. □

**Corollary 5.4.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$. *Then*

$$(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : (\mathfrak{w} + \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathfrak{o}.$$

*Proof.* According to (3.6) and the previous theorem we have the following for every $x \in \mathfrak{X}$

$$\begin{aligned}\Big((\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) : (\mathfrak{w} + \mathbf{D}_{\mathfrak{w}}\mathfrak{t})\Big)[x] &= \inf_{y\in\mathfrak{X}} \left\{\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x-y)\|^2_{\mathfrak{t}+\mathfrak{w}} + \|P\pi_{\mathfrak{t}+\mathfrak{w}}(y)\|^2_{\mathfrak{t}+\mathfrak{w}}\right\} \\ &= \inf_{y\in\mathfrak{X}} \left\{\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x-y)\|^2_{\mathfrak{t}+\mathfrak{w}} + \|P\pi_{\mathfrak{t}+\mathfrak{w}}(-y)\|^2_{\mathfrak{t}+\mathfrak{w}}\right\} \\ &= \inf_{y\in\mathfrak{X}} \left\{\|(I-P)\pi_{\mathfrak{t}+\mathfrak{w}}(x) - \pi_{\mathfrak{t}+\mathfrak{w}}(y)\|^2_{\mathfrak{t}+\mathfrak{w}}\right\}\end{aligned}$$

which is equals to zero because $\{\pi_{\mathfrak{t}+\mathfrak{w}}(y) \,|\, y \in \mathfrak{X}\}$ is dense $\mathscr{H}_{\mathfrak{t}+\mathfrak{w}}$. □

## 5.2. A decomposition of densely defined closable operators

This section contains the results of Section 4.1 and Section 4.2 of [**16**]. For singular operators and relations, and for a canonical decomposition of general linear relations see [**17**]. The first two lemmas can be found in [**16**] as Lemma 4.1 and Lemma 4.2.

**Lemma 5.5.** *Let* $T$ *be a linear relation from a Hilbert space* $\mathscr{H}$ *to a Hilbert space* $\mathscr{K}$. *Then the following statements are equivalent:*

(i) $T^{**}$ *is a bounded linear operator;*
(ii) $\operatorname{ran} T^{**} \subseteq \operatorname{dom} T^{*}$*;*
(iii) $\operatorname{dom} T^{*} = \mathscr{K}$.

*Proof.* First we prove $(i) \Leftrightarrow (iii)$. It is known that $T^{**}$ is a bounded operator precisely when $\operatorname{dom} T^{**}$ is closed and $\operatorname{mul} T^{**} = \{0\}$. On the one hand, $\operatorname{dom} T^{**}$ is closed if and only if $\operatorname{dom} T^{*}$ is closed (see e.g. [**44**]). On the other hand, $\operatorname{mul} T^{**} = \{0\}$ means that $\operatorname{dom} T^{*}$ is dense, hence $\operatorname{dom} T^{*} = \mathscr{K}$. Now, recall that $T^{*} = JT^{\perp}$, where $J\{f, f'\} = \{f', -f\}$. Hence identities

$$\mathscr{H} \times \mathscr{K} = \overline{T} \oplus T^{\perp} = T^{**} \oplus JT^{*},$$

lead to

$$\mathscr{H} = \operatorname{dom} T^{**} + \operatorname{ran} T^{*} \qquad\qquad \mathscr{K} = \operatorname{ran} T^{**} + \operatorname{dom} T^{*}, \tag{5.8}$$

which implies $(i) \Leftrightarrow (iii)$. □

Similarly, the bounded invertibility of $T^{**}$ can be characterized as follows.

**Lemma 5.6.** *Let $T$ be a linear relation from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$. Then the following statements are equivalent*

(i) $T^{**}$ has a bounded inverse;
(ii) $\operatorname{dom} T^{**} \subseteq \operatorname{ran} T^{*}$;
(iii) $\operatorname{ran} T^{*} = \mathscr{H}$.

A closable operator $T$ is bounded if and only if $T^{**}$ is bounded. Consequently, Lemma 5.5 says that $T$ is not bounded only if $\operatorname{dom} T^{*} \neq \mathscr{K}$. The following theorem is Proposition 4.3 in [**16**].

**Theorem 5.7.** *Let $T$ be a densely defined closable operator from $\mathscr{H}$ to $\mathscr{K}$. Let $v \in \mathscr{K}$ and let $P_v$ be the orthogonal projection from $\mathscr{K}$ onto* $\operatorname{span}\{v\}$*. Then $T$ has the following orthogonal decomposition*

$$T = A + B \tag{5.9}$$

*where the densely defined operators $A$ and $B$ are defined by*

$$A = (I - P_v)T, \qquad\qquad B = P_v T. \tag{5.10}$$

*Here $A$ is closable and*

(i) *if $v \in \operatorname{dom} T^{*}$, then $B^{**} \in \mathbf{B}(\mathscr{H}, \mathscr{K})$;*
(ii) *if $v \in \mathscr{H} \setminus \operatorname{dom} T^{*}$, then $B$ is a singular operator, i.e., $\operatorname{ran} B \subseteq \operatorname{mul} B^{**}$.*

*In case $(i)$ one has $B^{**}h = (h \,|\, T^{*}v)_{\mathscr{H}} v$ and in case $(ii)$ $B^{**} = \mathscr{H} \times \operatorname{span}\{v\}$.*

*Proof.* The decomposition $T = A + B$ is clearly an orthogonal decomposition. Since $T$ is densely defined and closable, the adjoint $T^{*}$ is a closed densely defined operator. Let

$v \in \mathscr{K}$, $\|v\|_{\mathscr{K}} = 1$. It follows from the definition of $A$ that

(5.11)

$$A^* = T^*(I - P_v) = \big\{\{f, g\} \in \mathscr{H} \times \mathscr{K} \mid f - (f \mid v)_{\mathscr{K}} v \in \operatorname{dom} T^*, g = T^*(f - (f \mid v)_{\mathscr{K}} v)\big\}.$$

In particular,

$$\operatorname{dom} A^* = \operatorname{span}\{v\} \oplus \big(\operatorname{dom} T^* \cap \operatorname{span}\{v\}^{\perp}\big). \tag{5.12}$$

Since span is one dimensional $\operatorname{dom} T^* \cap \operatorname{span}\{v\}^{\perp}$ is dense in $\operatorname{span}\{v\}^{\perp}$ (see [**39**]), and hence $\operatorname{dom} A^*$ is dense in $\mathscr{K}$. Consequently, $\operatorname{mul} A^{**} = (\operatorname{dom} A^*)^{\perp} = \{0\}$, which means that $A$ is a closable operator. On the other hand,

$$B^* = T^* P_v = \big\{\{f, g\} \in \mathscr{H} \times \mathscr{K} \mid \{(f \mid v)_{\mathscr{K}} v, g\} \in T^*\big\}, \tag{5.13}$$

□

which shows that if $v \in \operatorname{dom} T^*$ then $\operatorname{dom} B^* = \mathscr{K}$. Lemma 5.5 implies that $B$ is a densely defined bounded operator and $B^{**} \in \mathbf{B}(\mathscr{H}, \mathscr{K})$. Finally, observe that in this case the closure of $B$ is given by $B^{**} h = (h \mid T^* v)_{\mathscr{H}} v$ for all $h \in \mathscr{H}$.

If $v \in \mathscr{K} \setminus \operatorname{dom} T^*$ then (5.13) shows that $\{f, g\} \in B^*$ precisely when $(f \mid v)_{\mathscr{K}} = 0$ and $g = 0$. Hence $B^*$ is given by

$$B^* = \big\{\{f, 0\} \in \mathscr{H} \times \mathscr{K} \mid (f \mid v)_{\mathscr{K}} = 0\big\} = \operatorname{span}\{v\}^{\perp} \times \{0\}. \tag{5.14}$$

Consequently, $\operatorname{mul} B^{**} = (\operatorname{dom} B^*)^{\perp} = \operatorname{span}\{v\} \supseteq \operatorname{ran} B$, i.e., $B$ is a singular operator. In this case the formula for the closure of $B$ is obtained by taking adjoints in (5.14).

## 5.3. Uniqueness of the Lebesgue-type decomposition

The following theorem was motivated by the uniqueness result of Ando [**4**], and can be found in [**16**] as Theorem 4.4.

**Theorem 5.8.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$ and let $\mathfrak{t}$ be almost dominated by $\mathfrak{w}$. Then the following statements are equivalent*

(i) $\mathfrak{t}$ *is not dominated by* $\mathfrak{w}$*;*

(ii) $\mathfrak{t}$ *ha a decomposition* $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ *where the non-zero form* $\mathfrak{t}_1$ *is almost dominated by* $\mathfrak{w}$ *and the non-zero form* $\mathfrak{t}_2$ *is singular with respect to* $\mathfrak{w}$.

*Proof.* Define the linear relation $T$ on $\mathscr{H}_{\mathfrak{w}} \times \mathscr{H}_{\mathfrak{t}}$ by

$$T = \big\{\{x + \ker \mathfrak{w}, x + \ker \mathfrak{t}\} \in \mathscr{H}_{\mathfrak{w}} \times \mathscr{H}_{\mathfrak{t}} \mid x \in \mathfrak{X}\big\}. \tag{5.15}$$

Clearly, $T$ is densely defined and, $\mathfrak{t}$ is $\mathfrak{w}$ closable precisely when $T$ is the graph of a closable operator. Furthermore, $\mathfrak{t}$ is dominated by $\mathfrak{w}$ precisely when $T$ is a bounded operator.

First we prove $(i) \Rightarrow (ii)$. If $\mathfrak{t}$ is not dominated by $\mathfrak{w}$, then $T$ is not bounded. Hence, $\operatorname{dom} T^* \neq \mathscr{H}_{\mathfrak{t}}$ according to Lemma 5.5, and one can choose a unit vector $v \in \mathscr{H}_{\mathfrak{t}} \setminus \operatorname{dom} T^*$. Let $P_v$ be the orthogonal projection from $\mathscr{H}_{\mathfrak{t}}$ onto $\operatorname{span}\{v\}$. Then the decomposition of $T$ (see Theorem 5.7) leads to the decomposition of the form $\mathfrak{t}$

$$\mathfrak{t}[x] = \mathfrak{t}_1[x] + \mathfrak{t}_2[x] \qquad (x \in \mathfrak{X}) \tag{5.16}$$

where $\mathfrak{t}_1$ is defined by

$$\mathfrak{t}_1[x] = \|(I - P_v)T(y + \ker \mathfrak{w})\|_{\mathfrak{t}}^2 = \|(I - P_v)(x + \ker \mathfrak{t})\|_{\mathfrak{t}}^2, \qquad (x \in \mathfrak{X}) \tag{5.17}$$

and $\mathfrak{t}_2$ is defined by

$$\mathfrak{t}_2[x] = \|P_v T(x + \ker \mathfrak{w})\|_{\mathfrak{t}}^2 = \|(T(x + \ker \mathfrak{w}\,|v\,)_{\mathfrak{t}} v\|_{\mathfrak{t}}^2, \qquad (x \in \mathfrak{X}) \tag{5.18}$$

It follows from Theorem 5.7 that the form $\mathfrak{t}_1$ is $\mathfrak{w}$-closable and hence $\mathfrak{t}_1$ is $\mathfrak{w}$-almost dominated by. Since $v \in \mathscr{H}_{\mathfrak{t}} \setminus \operatorname{dom} T^*$, $\mathfrak{t}_2$ is non-trivial. Furthermore, $P_v T$ is singular and $(P_v T)^{**} = \mathscr{H}_{\mathfrak{w}} \times \operatorname{span}\{v\}$. In particular, $\operatorname{dom} P_v T \subseteq \ker(P_v T)^{**} = \mathscr{H}_{\mathfrak{w}}$ and therefore

$$\inf_{y + \ker \mathfrak{w} \in \mathscr{H}_{\mathfrak{w}}} \left\{ \|(y - x) + \ker \mathfrak{w}\|_{\mathscr{H}_{\mathfrak{w}}}^2 + \|P_v T(x + \ker \mathfrak{w})\|_{\mathscr{H}_{\mathfrak{t}}}^2 \right\} = 0 \tag{5.19}$$

holds for all $x \in \mathfrak{X}$, or equivalently,

$$\inf_{y \in \mathfrak{X}} \{\mathfrak{w}[y - x] + \mathfrak{t}_2[y]\} = 0. \tag{5.20}$$

This is equivalent with the singularity of $\mathfrak{w}$ and $\mathfrak{t}_2$. For the converse implication observe that if $\mathfrak{t}$ is dominated by $\mathfrak{w}$, then also $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are dominated by $\mathfrak{w}$. In this case, $\mathfrak{t}_2$ is both $\mathfrak{w}$-almost dominated and $\mathfrak{w}$-singular, and hence, $\mathfrak{t}_2$ must be $\mathfrak{o}$, which is contradiction. □

Now we are ready to state and prove the main result of this chapter (see Theorem 4.6 in [**16**]).

**Theorem 5.9.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$. The Lebesgue-type (i.e., $(\ll_{\mathrm{cl}}, \perp)$ or $(\ll_{\mathrm{ad}}, \perp)$-type) decomposition is unique if and only if $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ (or equivalently, $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$) is dominated by $\mathfrak{w}$.*

*Proof.* Assume first that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is dominated by $\mathfrak{w}$, and let $\mathfrak{t}_1 + \mathfrak{t}_2$ be a Lebesgue-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. According to the maximality of $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$, we have on the one hand

$$\mathfrak{o} \le \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 \le \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \le_{\mathrm{d}} \mathfrak{w},$$

which implies $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 \ll_{\mathrm{ad}} \mathfrak{w}$. On the other hand

$$\mathfrak{t}_2 = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 \geq \mathfrak{o}$$

which implies $(\mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1) \perp \mathfrak{w}$. Consequently, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 = \mathfrak{o}$. To prove the converse implication assume that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is not dominated by $\mathfrak{w}$. Then there is a non-trivial $\mathfrak{w}$-Lebesgue decomposition of $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2.$$

(That is, $\mathfrak{t}_1 \ll_{\mathrm{ad}} \mathfrak{w}$, $\mathfrak{t}_2 \perp \mathfrak{w}$.) In this case $\mathfrak{t}$ can be written as

$$\mathfrak{t} = [\mathfrak{t}_1 + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})] + [\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})],$$

which is clearly a $\mathfrak{w}$-Lebesgue decomposition of $\mathfrak{t}$. Indeed, $\mathfrak{t}_1 + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ is $\mathfrak{w}$-almost dominated, being the sum of two such forms, and $\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ is $\mathfrak{w}$ singular because it is just the $\mathfrak{w}$-singular part of $\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ with respect to $\mathfrak{w}$. It remains only to show that this decomposition differs from the decomposition $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$. Assume indirectly that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{t}_1 + \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}),$$

which leads to

$$\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}_2 + \mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} - \mathfrak{t}_1 = \mathfrak{t}_2.$$

This implies that $\mathfrak{t}_2$ is simultaneously $\mathfrak{w}$-almost dominated and (by assumption) singular, consequently, $\mathfrak{t}_2 = \mathfrak{o}$ which is a contradiction. □

Using the previous theorem, we can characterize also the uniqueness of the short-type decomposition.

**Theorem 5.10.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$. The $(\ll_{\mathrm{ac}}, \perp)$-type) decomposition is unique if and only if $\mathfrak{t}_{\ker \mathfrak{w}}$ is dominated by $\mathfrak{w}$.*

*Proof.* Assume first that $\mathfrak{t}_{\ker \mathfrak{w}}$ is dominated by $\mathfrak{w}$, and let $c$ be a constant such that $\mathfrak{t}_{\ker \mathfrak{w}} \leq c\mathfrak{w}$ (we may assume that $c > 1$). Let $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$ be an $(\ll, \perp)$-type decomposition. Since $\mathfrak{t}_{\ker \mathfrak{w}}$ is maximal, we have

$$\mathfrak{t}_2 = \mathfrak{t} - \mathfrak{t}_1 \geq \mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1 \geq \frac{1}{c}(\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1) \geq \mathfrak{o} \quad \text{and} \quad \mathfrak{w} \geq \frac{1}{c}\mathfrak{t}_{\ker \mathfrak{w}} \geq \frac{1}{c}(\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1) \geq \mathfrak{o}.$$

Since $\mathfrak{t}_2 \perp \mathfrak{w}$, one concludes that $\mathfrak{t}_{\ker \mathfrak{w}} - \mathfrak{t}_1 = \mathfrak{o}$. For the converse implication recall that if $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated (or equivalently, $\mathfrak{w}$-closable), then it is $\mathfrak{w}$-absolutely continuous. Consequently, every $(\ll_{\mathrm{ad}}, \perp)$-type decomposition is a $(\ll, \perp)$-type decomposition as well. If the $(\ll, \perp)$-type decomposition is unique, then $\mathfrak{t}_{\ker \mathfrak{w}} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$, and $\mathfrak{t}_{\ker \mathfrak{w}} \leq_{\mathrm{d}} \mathfrak{w}$ according to the previous theorem. □

We close this chapter with an interesting application. Namely, the following theorem is a characterization of closed range operators. We know from Theorem 2.3 that if $A$ is a bounded positive operator on the Hilbert space $\mathscr{H}$ then $A$ splits into the sum of a $B$-absolutely continuous and a $B$-singular part. The following theorem states that if $\operatorname{ran} B$ is closed, then the decomposition is unique.

**Theorem 5.11.** *Let $B$ be a bounded positive operator with closed range. Then for every $A \in \mathbf{B}_+(\mathscr{H})$*

$$A = A_{\ll,B} + A_{\perp,B}.$$

*is the unique decomposition of $A$ into $B$-absolutely continuous and $B$-singular parts.*

*Proof.* If $\operatorname{ran} B$ is closed, the inclusion $\ker B \subseteq \ker A_{\ll,B}$ implies that $\operatorname{ran} A_{\ll,B} \subseteq \operatorname{ran} B$. Furthermore, if $\operatorname{ran} B$ is closed, then the following two sets are identical according to the well-known theorem of Douglas [**10**]

$$\big\{S \in \mathbf{B}_+(\mathscr{H}) \bigm| (S \leq A) \ \wedge \ (\operatorname{ran} S \subseteq \operatorname{ran} B)\big\} = \big\{S \in \mathbf{B}_+(\mathscr{H}) \bigm| (S \leq A) \ \wedge \ (S \leq_{\mathrm{d}} B)\big\}.$$

Consequently, the statement follows from Theorem 2.3 and the previous theorem. □

Observe also that if $\operatorname{ran} B$ is closed, then $A_{\ll,B}$ coincides with $\mathbf{D}_B A$ in the sense of Ando [**4**], and therefore it is strongly absolutely continuous (or closable) with respect to $B$. Furthermore, according to [**49**, Theorem 7] we have the following characterization of closed range positive operators.

**Theorem 5.12.** *Let $B$ be a bounded positive operator. Then the following are equivalent*

$(i)$ $\operatorname{ran} B$ *is closed,*

$(ii)$ $\forall A \in \mathbf{B}_+(\mathscr{H}): \quad A_{\ll,B} \leq_{\mathrm{d}} B$,

$(iii)$ $\forall A \in \mathbf{B}_+(\mathscr{H}): \quad \mathbf{D}_B A \leq_{\mathrm{d}} B$.

*If any of $(i)-(iii)$ fulfills, then $\mathbf{D}_B A = A_{\ll,B}$ for all $A \in \mathbf{B}_+(\mathscr{H})$.*

CHAPTER 6

# Parallel sum and parallel difference

The aim of this short chapter is to investigate the inverse operation of parallel addition, the so called *parallel subtraction.* In the first section we present an interesting application of identity

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w}.$$

Namely, we prove that the almost dominated parts $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$ are mutually almost dominated for every $\mathfrak{t}$ and $\mathfrak{w}$. In the second section we give a necessary and sufficient condition for the solvability of the equation $\mathfrak{t} : \mathfrak{x} = \mathfrak{s}$ (with unknown $\mathfrak{x}$).

## 6.1. The equivalence of almost dominated parts

All results of this section can be found in [**55**]. The operator version of Lemma 6.1 is due to Pekarev and Šmul'jan (Theorem 3.7 in [**25**]).

**Lemma 6.1.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$*. Then we have*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t} : \mathfrak{w}.$$

*Proof.* Since $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$, $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{w}$, and the parallel addition is monotone, we have

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t} : \mathfrak{w}$$

To show the converse inequality, let $f$ and $g \in \mathfrak{X}$ be fixed ($h := f + g$), and estimate $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}[f] + \mathbf{D}_{\mathfrak{t}}\mathfrak{w}[g]$ using the formula $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{w}$ and the parallelogram identity.

$$\begin{aligned}
\mathbf{D}_{\mathfrak{w}}\mathfrak{t}[f] + \mathbf{D}_{\mathfrak{t}}\mathfrak{w}[g] &= \sup_{u\in\mathfrak{X}} \big\{(\mathfrak{t} : \mathfrak{w})[f+u] - \mathfrak{w}[u]\big\} + \sup_{v\in\mathfrak{X}} \big\{(\mathfrak{t} : \mathfrak{w})[g+v] - \mathfrak{t}[v]\big\} \\
&\geq \sup_{u,v\in\mathfrak{X}} \big\{(\mathfrak{t} : \mathfrak{w})[f+u] + (\mathfrak{t} : \mathfrak{w})[g+v] - \mathfrak{w}[u] - \mathfrak{t}[v]\big\} \\
&\geq \sup_{\substack{u,v\in\mathfrak{X}\\ u+v=h}} \big\{\tfrac{1}{2}(\mathfrak{t} : \mathfrak{w})[2h] - \mathfrak{w}[u] - \mathfrak{t}[v]\big\} \\
&= \tfrac{1}{2}(\mathfrak{t} : \mathfrak{w})[2h] - \inf_{\substack{u,v\in\mathfrak{X}\\ u+v=h}} \big\{\mathfrak{w}[u] + \mathfrak{t}[v]\big\} \\
&= 2(\mathfrak{t} : \mathfrak{w})[h] - (\mathfrak{t} : \mathfrak{w})[h] \\
&= (\mathfrak{t} : \mathfrak{w})[h].
\end{aligned}$$

Hence, by taking the infimum over $f$ and $g$ with the restriction $f+g=h$, it follows that

$$(\mathfrak{t}:\mathfrak{w})[h] \leq \inf_{\substack{f,g\in\mathfrak{X}\\ f+g=h}} \left\{\mathbf{D}_{\mathfrak{w}}\mathfrak{t}[f]+\mathbf{D}_{\mathfrak{t}}\mathfrak{w}[g]\right\} = (\mathbf{D}_{\mathfrak{w}}\mathfrak{t}:\mathbf{D}_{\mathfrak{t}}\mathfrak{w})[h] \quad (h\in\mathfrak{X}).$$

Consequently, $\mathfrak{t}:\mathfrak{w}=\mathbf{D}_{\mathfrak{w}}\mathfrak{t}:\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$. ☐

An analogous theorem for representable functionals can be found in [**52**].

**Theorem 6.2.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$. *Then we have*

$$(\mathbf{D}_{\mathfrak{t}}\mathfrak{w}:\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) \div \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{w},$$

*i.e.,* $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$ *is almost dominated by* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$. *And by symmetry,* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *is almost dominated by* $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$.

*Proof.* Since $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}\leq\mathfrak{t}$, it follows from Theorem 6.1 and the definition of parallel subtraction that

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}:\mathbf{D}_{\mathfrak{t}}\mathfrak{w}) \div \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = (\mathfrak{t}:\mathfrak{w}) \div \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \geq (\mathfrak{t}:\mathfrak{w}) \div \mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{w}.$$

On the other hand,

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t}:\mathbf{D}_{\mathfrak{t}}\mathfrak{w}) \div \mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathbf{D}_{\mathfrak{w}}\mathfrak{t}}\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}.$$

☐

**Corollary 6.3.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$. *Then*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathbf{D}_{\mathfrak{t}}\mathfrak{w}}\mathfrak{t}. \tag{6.1}$$

*Proof.* Since $\mathfrak{u}_1\leq\mathfrak{u}_2$ implies $\mathbf{D}_{\mathfrak{u}_1}\leq\mathbf{D}_{\mathfrak{u}_2}$ by definition, we have on the one hand that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \geq \mathbf{D}_{\mathbf{D}_{\mathfrak{t}}\mathfrak{w}}\mathfrak{t}.$$

On the other hand, using (5.7), Lemma 5.2, and the properties of parallel subtraction we have

$$\mathbf{D}_{\mathbf{D}_{\mathfrak{t}}\mathfrak{w}}\mathfrak{t} = (\mathfrak{t}:\mathbf{D}_{\mathfrak{t}}\mathfrak{w}) \div \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = (\mathfrak{t}:\mathfrak{w}) \div \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \geq (\mathfrak{t}:\mathfrak{w}) \div \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

☐

Since almost domination and closability are equivalent concepts, Theorem 3.10 and Theorem 3.11 give the following corollary for bounded charges (see also [**58**]).

**Corollary 6.4.** *Let* $\mathscr{R}$ *be a ring of subsets of a non-empty set* $T$, *and let* $\mu$ *and* $\nu$ *be bounded additive non-negative set functions on* $\mathscr{R}$. *Consider the* $(\ll_{\mathrm{cl}},\perp)$*-type decompositions with respect to each other. Then the regular parts* $\mu_{a,\nu}$ *and* $\nu_{a,\mu}$ *are equivalent* *charges. That is, $\mu_{a,\nu}$ is strongly $\nu_{a,\mu}$-absolutely continuous, and $\nu_{a,\mu}$ is strongly $\mu_{a,\nu}$-absolutely continuous.*

## 6.2. The parallel difference as a minimal solution

In this section we identify $\mathfrak{w} \div \mathfrak{t}$ as the minimal solution of an appropriate equation. First we need some definitions and technical lemmas.

The first notion which is needed is the parallel difference of two forms. Recall that if $\mathfrak{t}$ and $\mathfrak{w}$ are (not necessary nonnegative) sesquilinear forms on the complex linear space $\mathfrak{X}$, then the *parallel difference* $\mathfrak{t} \div \mathfrak{w}$ of $\mathfrak{t}$ and $\mathfrak{w}$ is a mapping from $\mathfrak{X}$ to $\mathbb{R} \cup \{\infty\}$,which is defined by

$$(\mathfrak{t} \div \mathfrak{w})[x] = \sup_{y \in \mathfrak{X}} \{\mathfrak{t}[x+y] - \mathfrak{w}[y]\} \quad (x \in \mathfrak{X}). \tag{6.2}$$

If $\mathfrak{t}$ and $\mathfrak{w}$ are nonnegative, we say that the parallel difference $\mathfrak{t} \div \mathfrak{w}$ exists, if $\mathfrak{t} \div \mathfrak{w}$ is a nonnegative form. The second notion which is needed is the *complement of forms.* We define the complement of $\mathfrak{w}$ with respect to $\mathfrak{t}$ (or shortly, the $\mathfrak{t}$-complement of $\mathfrak{w}$) as a mapping from $\mathfrak{X}$ to $\mathbb{R} \cup \{+\infty\}$ by the following supremum

$$\mathfrak{w}_{\mathfrak{t}}[x] := \sup_{y \in \mathfrak{X}} \{\mathfrak{t}(x,y) + \mathfrak{t}(y,x) - \mathfrak{w}[y]\} \quad (x \in \mathfrak{X}). \tag{6.3}$$

Remark that the supremum need not be finite, i.e. $\mathfrak{w}_{\mathfrak{t}}$ is not a form in general. We say that the complement $\mathfrak{w}_{\mathfrak{t}}$ exists, if the formula above defines a form. Observe that $\mathfrak{w} \le \mathfrak{v}$ implies $\mathfrak{v}_{\mathfrak{t}} \le \mathfrak{w}_{\mathfrak{t}}$.

The following lemma provides a useful formula for the $\mathfrak{t}$-complement of $\mathfrak{w}$ using the concept of the parallel difference.

**Lemma 6.5.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be sesquilinear forms on the complex linear space $\mathfrak{X}$. Then*

$$\mathfrak{w}_{\mathfrak{t}} = \mathfrak{t} + (\mathfrak{t} - \mathfrak{w}) \div \mathfrak{t}. \tag{6.4}$$

*Proof.* The assertion follows immediately from the definitions (5.3) and (6.3). Completing squares leads to

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{t}}[x] &= \sup_{y \in \mathfrak{X}} \{\mathfrak{t}(x,y) + \mathfrak{t}(y,x) - \mathfrak{w}[y]\} \\
&= \sup_{y \in \mathfrak{X}} \{\mathfrak{t}[x] + \mathfrak{t}[y] - \mathfrak{t}[x-y] - \mathfrak{w}[y]\} \\
&= \mathfrak{t}[x] + \sup_{y \in \mathfrak{X}} \{(\mathfrak{t} - \mathfrak{w})[y] - \mathfrak{t}[x-y]\} \\
&= \mathfrak{t}[x] + (\mathfrak{t} - \mathfrak{w}) \div \mathfrak{t}[x].
\end{aligned}$$

□

**Corollary 6.6.** *Let* $\mathfrak{t}$ *and* $\mathfrak{s}$ *be sesquilinear forms on the complex linear space* $\mathfrak{X}$*. Then*

$$(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}-\mathfrak{t}=(\mathfrak{t}-\mathfrak{s})_{\mathfrak{s}}+\mathfrak{s}=\mathfrak{s}\div\mathfrak{t} \tag{6.5}$$

*Furthermore, if* $\mathfrak{t}$ *and* $\mathfrak{s}$ *are forms, then*

$$(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}\geq\mathfrak{t}. \tag{6.6}$$

*Proof.* The identity $(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}-\mathfrak{t}=\mathfrak{s}\div\mathfrak{t}$ follows immediately from (6.4). For the other equality observe that

$$\mathfrak{t}(x,h)+\mathfrak{t}(h,x)-(\mathfrak{t}-\mathfrak{s})[h]-\mathfrak{t}[x]=\mathfrak{s}(x,h-x)+\mathfrak{s}(h-x,x)-(\mathfrak{t}-\mathfrak{s})[h-x]+\mathfrak{s}[x].$$

Hence, we can obtain that

$$\begin{aligned}(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}[x]-\mathfrak{t}[x]&=\sup_{h\in\mathfrak{X}}\{\mathfrak{t}(x,h)+\mathfrak{t}(h,x)-(\mathfrak{t}-\mathfrak{s})[h]-\mathfrak{t}[x]\}\\&=\sup_{h\in\mathfrak{X}}\{\mathfrak{s}(x,h-x)+\mathfrak{s}(h-x,x)-(\mathfrak{t}-\mathfrak{s})[h-x]+\mathfrak{s}[x]\}\\&=\sup_{g\in\mathfrak{X}}\{\mathfrak{s}(x,g)+\mathfrak{s}(g,x)-(\mathfrak{t}-\mathfrak{s})[g]+\mathfrak{s}[x]\}\\&=(\mathfrak{t}-\mathfrak{s})_{\mathfrak{s}}[x]+\mathfrak{s}[x]\end{aligned}$$

holds for all $x\in\mathfrak{X}$. □

The previous results imply immediately the following generalization of Theorem 3. [**24**] for forms. It is not connected with the subject of this section, we present it for its own interest.

**Theorem 6.7.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$*. Then*

$$\mathbf{D}_{\mathfrak{t}}\mathfrak{w}=(\mathfrak{t}-\mathfrak{t}:\mathfrak{w})_{\mathfrak{t}:\mathfrak{w}}+\mathfrak{t}:\mathfrak{w}. \tag{6.7}$$

*Proof.* The equality $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}=(\mathfrak{t}-\mathfrak{t}:\mathfrak{w})_{\mathfrak{t}:\mathfrak{w}}+\mathfrak{t}:\mathfrak{w}$ follows immediately from (6.5), because

$$(\mathfrak{t}-\mathfrak{t}:\mathfrak{w})_{\mathfrak{t}:\mathfrak{w}}+\mathfrak{t}:\mathfrak{w}=(((\mathfrak{t}:\mathfrak{w})\div\mathfrak{t})-\mathfrak{t}:\mathfrak{w})+\mathfrak{t}:\mathfrak{w}=\mathbf{D}_{\mathfrak{t}}\mathfrak{w}.$$

□

**Lemma 6.8.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$ *such that* $0\leq\mathfrak{t}\leq\mathfrak{w}$*. Then*

$$\mathfrak{t}=\mathfrak{w}_{\mathfrak{t}}+(\mathfrak{w}-\mathfrak{t}):\mathfrak{t}. \tag{6.8}$$

*Hence,* $\mathfrak{w}_{\mathfrak{t}}$ *is a form which is majorized by* $\mathfrak{t}$*. In particular,*

$$\mathfrak{t}=(\mathfrak{w}+\mathfrak{t})_{\mathfrak{t}}+\mathfrak{w}:\mathfrak{t} \tag{6.9}$$

*Proof.* Since $\mathfrak{t}[x-y] = \mathfrak{t}(x-y, x-y) = \mathfrak{t}[x] + \mathfrak{t}[y] - \mathfrak{t}(x,y) - \mathfrak{t}(y,x)$ holds for all $x$ and $y$ in $\mathfrak{X}$, it follows that

$$\begin{aligned}
\mathfrak{w}_{\mathfrak{t}}[x] &= \sup_{y\in\mathfrak{X}} \{\mathfrak{t}(x,y) + \mathfrak{t}(y,x) - \mathfrak{w}[y]\} \\
&= \sup_{y\in\mathfrak{X}} \{-\mathfrak{t}[x-y] + \mathfrak{t}[x] + \mathfrak{t}[y] - \mathfrak{w}[y]\} \\
&= \mathfrak{t}[x] + \sup_{y\in\mathfrak{X}} \{(\mathfrak{t}-\mathfrak{w})[y] - \mathfrak{t}[x-y]\} \\
&= \mathfrak{t}[x] - \inf_{y\in\mathfrak{X}} \{(\mathfrak{w}-\mathfrak{t})[y] + \mathfrak{t}[x-y]\} \\
&= \mathfrak{t}[x] - \big((\mathfrak{w}-\mathfrak{t}) : \mathfrak{t}\big)[x].
\end{aligned}$$

The sesquilinear form $(\mathfrak{w}-\mathfrak{t}) : \mathfrak{t}$ is nonnegative, and majorized by $\mathfrak{t}$, hence $0 \le \mathfrak{w}_{\mathfrak{t}}$ and $\mathfrak{w}_{\mathfrak{t}} \le \mathfrak{t}$. □

For the operator version of the following result see Theorem 3.2 in [**25**].

**Theorem 6.9.** *Let $\mathfrak{t}$ and $\mathfrak{s}$ be forms on the complex linear space $\mathfrak{X}$ such that $\mathfrak{t} \ge \mathfrak{s} \ge 0$. Then the following statements are equivalent.*

- (*i*) *There exists a form $\mathfrak{w}$ such that $\mathfrak{t} : \mathfrak{w} = \mathfrak{s}$.*
- (*ii*) *$(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}$ is a (sesquilinear) form on $\mathfrak{X}$ satisfying $\big((\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}\big)_{\mathfrak{t}} = \mathfrak{t}-\mathfrak{s}$.*

*Furthermore, if the equation $\mathfrak{t} : \mathfrak{w} = \mathfrak{s}$ is solvable for $\mathfrak{w}$, then there exists a minimal solution, namely $\mathfrak{s} \div \mathfrak{t}$.*

*Proof.* First assume that there exists $\mathfrak{w} \ge 0$ such that $\mathfrak{t} : \mathfrak{w} = \mathfrak{s}$. Recall that

$$\mathbf{D}_{\mathfrak{t}}\mathfrak{w} = (\mathfrak{w} : \mathfrak{t}) \div \mathfrak{t} \le \mathfrak{w},$$

hence (6.5) implies that

$$\mathfrak{t} + \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t} + (\mathfrak{w} : \mathfrak{t}) \div \mathfrak{t} = \mathfrak{t} + \mathfrak{s} \div \mathfrak{t} = (\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}.$$

Hence, in particular $(\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}$ and $\mathfrak{s} \div \mathfrak{t}$ are nonnegative forms on $\mathfrak{X}$. Since $\mathfrak{t} + \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \ge 0$ and $(\mathfrak{t} + \mathbf{D}_{\mathfrak{t}}\mathfrak{w}) - \mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \ge 0$, it follows from (5.5) and (6.8) that

$$\big((\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}\big)_{\mathfrak{t}} = (\mathfrak{t} + \mathbf{D}_{\mathfrak{t}}\mathfrak{w})_{\mathfrak{t}} = \mathfrak{t} - \big((\mathfrak{t} + \mathbf{D}_{\mathfrak{t}}\mathfrak{w}) - \mathfrak{t}\big) : \mathfrak{t} = \mathfrak{t} - \mathbf{D}_{\mathfrak{t}}\mathfrak{w} : \mathfrak{t} = \mathfrak{t} - \mathfrak{t} : \mathfrak{w} = \mathfrak{t} - \mathfrak{s}.$$

For the converse implication, assume that $\big((\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}\big)_{\mathfrak{t}} = \mathfrak{t}-\mathfrak{s}$. Then, by (6.6) and (6.9) it follows that the equation $\mathfrak{t} : \mathfrak{z} = \mathfrak{s}$ is solvable, and $\mathfrak{w} := (\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}} - \mathfrak{t}$ is a solution, because

$$\begin{aligned}
\mathfrak{s} &= \mathfrak{t} - (\mathfrak{t}-\mathfrak{s}) = \mathfrak{t} - \big((\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}}\big)_{\mathfrak{t}} = \mathfrak{t} - \big(\mathfrak{t} + (\mathfrak{t}-\mathfrak{s})_{\mathfrak{t}} - \mathfrak{t}\big)_{\mathfrak{t}} \\
&= \mathfrak{t} - (\mathfrak{t}+\mathfrak{w})_{\mathfrak{t}} = \mathfrak{t} - (\mathfrak{t} - \mathfrak{t} : \mathfrak{w}) = \mathfrak{t} : \mathfrak{w}.
\end{aligned}$$

So, the equivalence of $(i)$ and $(ii)$ is proved. Now, assume that there exists $\mathfrak{w}$, such that $\mathfrak{t} : \mathfrak{w} = \mathfrak{s}$. In this case $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} = (\mathfrak{t} : \mathfrak{w}) \div \mathfrak{t} = \mathfrak{s} \div \mathfrak{t}$, i.e. the parallel difference $\mathfrak{s} \div \mathfrak{t}$ exists and $0 \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{s} \div \mathfrak{t} \leq \mathfrak{w}$. On the other hand, it follows from (5.5) that

$$\mathfrak{t} : (\mathfrak{s} \div \mathfrak{t}) = \mathfrak{t} : \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t} : \mathfrak{w} = \mathfrak{s},$$

i.e. $\mathfrak{s} \div \mathfrak{t}$ is a solution. Since $\mathfrak{w}$ was an arbitrary solution, $\mathfrak{s} \div \mathfrak{t}$ is minimal. This completes the proof. $\square$

CHAPTER 7

# Radon–Nikodym-type theorems

The Lebesgue decomposition theorem and the Radon-Nikodym theorem are cornerstones of the classical measure theory. These theorems were generalized in several settings and several ways (see e.g. [**14, 18, 31, 41, 60**]). For example, Radon–Nikodym-type questions for forms were investigated independently by Zs. Tarcsay, from a different point of view. For more information and applications we refer the reader to [**50**].

The main purpose of this chapter (which is based on paper [**37**]) is to formulate and prove Radon-Nikodym type results for forms. Consider first a simple version of the classical theorem.

**Theorem 7.1.** *Let $\mu$ and $\nu$ be real valued finite measures defined on a common $\sigma$-algebra $\mathfrak{A}$ of subsets of a set $X$. If $\mu$ is absolutely continuous with respect to $\nu$ then there exists a nonnegative $\nu$-integrable function $f$ such that*

$$\mu(A) = \int_A f \,\mathrm{d}\nu \qquad (\forall A \in \mathfrak{A}).$$

*The function $f$ is unique up to a set of $\nu$ measure zero.*

By means of positive operators the previous theorem can be phrased as follows: if $\mu$ is absolutely continuous with respect to $\nu$, we can compute the $\|\cdot\|_{L^2(\mu)}$ norm of every $\mathfrak{A}$-simple function via a (not necessarily bounded) multiplication operator, which is positive and self-adjoint. Indeed,

$$\|\chi_A\|^2_{L^2(\mu)} = \mu(A) = \int_A f \,\mathrm{d}\nu = \int_X f\chi_A \,\mathrm{d}\nu = \int_X |f^{1/2}\chi_A|^2 \,\mathrm{d}\nu = \|f^{1/2}\chi_A\|^2_{L^2(\nu)}$$

holds for every $A \in \mathfrak{A}$, and similarly, for every $\mathfrak{A}$-simple function. Our main purpose is to prove an analogous theorem in the context of forms. Namely, for every pair of forms $\mathfrak{t}$ and $\mathfrak{w}$, the Lebesgue decomposition exists, and the almost dominated part can be computed via a (not necessarily bounded) positive self-adjoint operator acting on a Hilbert space associated to the form $\mathfrak{w}$.

Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$, and consider the Hilbert spaces $\mathscr{H}_{\mathfrak{w}}$ and $\mathscr{H}_{\mathfrak{w}+\mathfrak{t}}$. The kernel of $\mathfrak{w}$ and $\mathfrak{w}+\mathfrak{t}$ will be denoted by $\mathfrak{L}$ and $\mathfrak{K}$, respectively, while the identity operator on these spaces will be denoted by $I_{\mathfrak{w}}$ and $I_{\mathfrak{w}+\mathfrak{t}}$, respectively. Recall that since

$$\mathfrak{K} = \ker \mathfrak{w} \cap \ker \mathfrak{t} \subseteq \ker \mathfrak{w} = \mathfrak{L},$$

the linear operator $J$, defined by

$$J(x + \mathfrak{K}) := x + \mathfrak{L} \quad (x \in \mathfrak{X}),$$

is a densely defined contraction from $\mathscr{H}_{\mathfrak{w}+\mathfrak{t}}$ to $\mathscr{H}_{\mathfrak{w}}$. Consequently, its closure (i.e., its second adjoint) $J^{**}$ is a contraction with $\operatorname{dom} J^{**} = \mathscr{H}_{\mathfrak{w}+\mathfrak{t}}$ and $\operatorname{ran} J^{**}$ is dense in $\mathscr{H}_{\mathfrak{w}}$. Moreover, $J^*$ is an injective contraction with $\operatorname{dom} J^* = \mathscr{H}_{\mathfrak{w}}$ and $\operatorname{ran} J^*$ is dense in $\mathscr{H}_{\mathfrak{w}+\mathfrak{t}} \ominus \ker J^{**}$. Consider the operator $J^{**}J^*$, which is an everywhere defined positive self-adjoint operator on $\mathscr{H}_{\mathfrak{w}}$. Since $\ker J^* = \ker J^{**}J^*$, the operator $J^{**}J^*$ is injective and its (not necessarily bounded) inverse is also a densely defined positive self-adjoint operator. The positive self-adjoint square root of these operators will be denoted by $(J^{**}J^*)^{1/2}$ and $(J^{**}J^*)^{-1/2}$, respectively.

As was mentioned in the introduction, the classical Radon-Nikodym theorem (see Theorem 7.1) can be phrased by means of Hilbert space operators. This is done by considering the Radon-Nikodym derivative as a (positive self-adjoint) multiplication operator on $L^2(\nu)$. In the following theorem we present an analogous result. Namely, the quadratic form of the $\mathfrak{w}$-regular part of $\mathfrak{t}$ is derived from a positive self-adjoint operator acting on the Hilbert space associated to the form $\mathfrak{w}$.

Before stating the main result of this chapter, recall that the form induced by the orthogonal projection $I - P := Q$ from $\mathscr{H}_{\mathfrak{w}+\mathfrak{t}}$ onto $\{\ker J^{**}\}$

$$\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x] := \|Q(x + \mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} \quad (x \in \mathfrak{X}), \tag{7.1}$$

is singular with respect to $\mathfrak{w}$.

**Theorem 7.2.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. Then*

$$\mathfrak{t}[x] = \|[(J^{**}J^*)^{-1} - I_{\mathfrak{w}}]^{1/2}(x + \mathfrak{L})\|^2_{\mathfrak{w}} + \|Q(x + \mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} \quad (x \in \mathfrak{X}) \tag{7.2}$$

*is a Lebesgue-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$.*

*Proof.* Let $x$ and $y$ be arbitrary elements of $\mathfrak{X}$, and consider

$$\|(I_{\mathfrak{w}+\mathfrak{t}} - Q)(x + \mathfrak{K}) + J^*(y + \mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}}.$$

According to the equalities

$$\big(J^*(y + \mathfrak{L}) \,|\, Q(x + \mathfrak{K})\big)_{\mathfrak{w}+\mathfrak{t}} = \big(y + \mathfrak{L} \,|\, J^{**}Q(x + \mathfrak{K})\big)_{\mathfrak{w}} = 0,$$

and

$$\big(x+\mathfrak{K} \,|\, Q(x+\mathfrak{K})\big)_{\mathfrak{w}+\mathfrak{t}} = \|Q(x+\mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}}$$

one obtains that

$$\begin{aligned}
&\|(I_{\mathfrak{w}+\mathfrak{t}}-Q)(x+\mathfrak{K})+J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} = \\
&= \|x+\mathfrak{K}+J^*(y+\mathfrak{L})-Q(x+\mathfrak{K})]\|^2_{\mathfrak{w}+\mathfrak{t}} = \\
&= \|x+\mathfrak{K}\|^2_{\mathfrak{w}+\mathfrak{t}} + \|J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} + \|Q(x+\mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} + 2\mathfrak{Re}\big(J^*(y+\mathfrak{L}) \,|\, Q(x+\mathfrak{K})\big)_{\mathfrak{w}+\mathfrak{t}} + \\
&\qquad + 2\mathfrak{Re}\big(x+\mathfrak{K} \,|\, J^*(y+\mathfrak{L})\big)_{\mathfrak{w}+\mathfrak{t}} - 2\mathfrak{Re}\big(x+\mathfrak{K} \,|\, Q(x+\mathfrak{K})\big)_{\mathfrak{w}+\mathfrak{t}} = \\
&= \|x+\mathfrak{K}\|^2_{\mathfrak{w}+\mathfrak{t}} + \|J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} - \|Q(x+\mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} + 2\mathfrak{Re}\big(x+\mathfrak{K} \,|\, J^*(y+\mathfrak{L})\big)_{\mathfrak{w}+\mathfrak{t}} = \\
&= \mathfrak{w}[x] + \mathfrak{t}[x] + \|(J^{**}J^*)^{1/2}(y+\mathfrak{L})\|^2_{\mathfrak{w}} - \|Q(x+\mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} + \mathfrak{w}(g,h) + \mathfrak{w}(h,g),
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}
\|J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} &= \big(J^*(y+\mathfrak{L}) \,|\, J^*(y+\mathfrak{L})\big)_{\mathfrak{w}+\mathfrak{t}} = \big(J^{**}J^*(y+\mathfrak{L}) \,|\, y+\mathfrak{L}\big)_{\mathfrak{w}} = \\
&= \big((J^{**}J^*)^{1/2}(y+\mathfrak{L}) \,|\, (J^{**}J^*)^{1/2}(y+\mathfrak{L})\big)_{\mathfrak{w}} = \|(J^{**}J^*)^{1/2}(y+\mathfrak{L})\|^2_{\mathfrak{w}}
\end{aligned}$$

and

$$\big(J^*(y+\mathfrak{L}) \,|\, x+\mathfrak{K}\big)_{\mathfrak{w}+\mathfrak{t}} = \big(y+\mathfrak{L} \,|\, J^{**}(x+\mathfrak{K})\big)_{\mathfrak{w}} = \big(y+\mathfrak{L} \,|\, x+\mathfrak{L}\big)_{\mathfrak{w}} = \mathfrak{w}(y,x).$$

There is another observation which is needed. Namely, completing squares gives

$$\begin{aligned}
&\|(J^{**}J^*)^{1/2}(y+\mathfrak{L})\|^2_{\mathfrak{w}} + \mathfrak{w}[x] + \mathfrak{w}(x,y) + \mathfrak{w}(y,x) = \\
&\quad = \|(J^{**}J^*)^{-1/2}(x+\mathfrak{L}) + (J^{**}J^*)^{1/2}(y+\mathfrak{L})\|^2_{\mathfrak{w}} - \|[(J^{**}J^*)^{-1} - I_{\mathfrak{w}}]^{1/2}(x+\mathfrak{L})\|^2_{\mathfrak{w}}.
\end{aligned}$$

We are now in the position to verify (7.2). Observe first that

$$\inf_{y\in\mathfrak{X}} \|(I_{\mathfrak{w}+\mathfrak{t}}-Q)(x+\mathfrak{K})+J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} = 0,$$

because $\operatorname{ran} J^*$ is dense in $(\ker J^{**})^{\perp}$. On the other hand, since $\operatorname{ran} J^{**}J^* \subseteq \operatorname{ran}(J^{**}J^*)^{1/2}$ and $\operatorname{ran} J^{**}J^*$ is dense in $\mathscr{H}_{\mathfrak{w}}$, one obtains

$$\inf_{y\in\mathfrak{X}} \|(J^{**}J^*)^{-1/2}(x+\mathfrak{L}) + (J^{**}J^*)^{1/2}(y+\mathfrak{L})\|^2_{\mathfrak{w}} = 0,$$

and hence

$$\begin{aligned}
&\inf_{y\in\mathfrak{X}} \|(I_{\mathfrak{w}+\mathfrak{t}}-Q)(x+\mathfrak{K})+J^*(y+\mathfrak{L})\|^2_{\mathfrak{w}+\mathfrak{t}} = \\
&= \mathfrak{t}[x] - \|Q(x+\mathfrak{K})\|^2_{\mathfrak{w}+\mathfrak{t}} - \|[(J^{**}J^*)^{-1} - I_{\mathfrak{w}}]^{1/2}(x+\mathfrak{L})\|^2_{\mathfrak{w}}.
\end{aligned}$$

Since $\mathfrak{t}_{\mathrm{sing},\mathfrak{w}}$ is singular with respect to $\mathfrak{w}$, it remains to show that the form $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$, defined by

$$\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}[x] := \mathfrak{t}[x] - \mathfrak{t}_{\mathrm{sing},\mathfrak{w}}[x] = \|[(J^{**}J^*)^{-1} - I_{\mathfrak{w}}]^{1/2}(x+\mathfrak{L})\|_{\mathfrak{w}}^2 \quad (x \in \mathfrak{X}), \tag{7.3}$$

is $\mathfrak{w}$-closable, or equivalently, $\mathfrak{w}$-almost dominated. From now on we will use the notation

$$T := [(J^{**}J^*)^{-1} - I_{\mathfrak{w}}]^{1/2}.$$

Due to the spectral theorem, there exists a unique resolution of the identity $E$ on the Borel subsets of the real line, such that

$$T = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}E(\lambda).$$

For a fixed $x \in \mathfrak{X}$, the Borel measure $\omega \mapsto \big(E(\omega)(x+\mathfrak{L}), x+\mathfrak{L}\big)_{\mathfrak{w}}$ will be denoted by $E_x$. Consider the *bounded* positive self-adjoint operators $T_n$ on $\mathscr{H}_{\mathfrak{w}}$, defined by

$$T_n := \int_{-\infty}^{+\infty} \min\{n, \lambda\} \, \mathrm{d}E(\lambda),$$

and the forms

$$\mathfrak{t}_n[x] := \big(T_n^2(x+\mathfrak{L}), x+\mathfrak{L}\big)_{\mathfrak{w}} = \|T_n(x+\mathfrak{L})\|_{\mathfrak{w}}^2. \tag{7.4}$$

Since the inequality

$$\mathfrak{t}_n[x] = \int_{-\infty}^{+\infty} |\min\{n,\lambda\}|^2 \, \mathrm{d}E_x(\lambda) \leq \int_{-\infty}^{+\infty} |\lambda|^2 \, \mathrm{d}E_x(\lambda) = \mathfrak{t}_{\mathrm{r},\mathfrak{w}}[x]$$

holds for all $x \in \mathfrak{X}$, we have $\mathfrak{t}_n \leq \mathfrak{t}_{\mathrm{r},\mathfrak{w}}$. On the other hand,

$$\mathfrak{t}_n[x] = \|T_n(x+\mathfrak{L})\|_{\mathfrak{w}}^2 \leq \|T_n\|_{\mathbf{B}(\mathscr{H}_{\mathfrak{w}})}^2 \|x+\mathfrak{L}\|_{\mathfrak{w}}^2 = \|T_n\|_{\mathbf{B}(\mathscr{H}_{\mathfrak{w}})}^2 \mathfrak{w}[x],$$

i.e. the forms $\mathfrak{t}_n$ are majorized by $\mathfrak{t}$, dominated by $\mathfrak{w}$ and clearly

$$\sup_{n\in\mathbb{N}} \mathfrak{t}_n = \mathfrak{t}_{\mathrm{reg},\mathfrak{w}} \tag{7.5}$$

which means by definition that $\mathfrak{t}_{\mathrm{reg},\mathfrak{w}}$ is almost dominated by $\mathfrak{w}$. □

We close this section with a weaker Radon–Nikodym-type result. The classical Radon–Nikodym theorem (stated in Theorem 7.1) can be view also as follows

$$(1\,|\,\chi_F)_{L^2(\mu)} = \mu(F) = \int_F f \, \mathrm{d}\nu = \int_T f \cdot \chi_F \, \mathrm{d}\nu = (f\,|\,\chi_F)_{L^2(\nu)} = \lim_{n\to\infty} (f_n|\chi_F)_{L^2(\nu)} \tag{7.6}$$

where $f_n$'s are step functions. So that, the following theorem is a natural generalization.

**Lemma 7.3.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$ and assume that $\mathfrak{t} \leq c \cdot \mathfrak{w}$ for some $c > 0$. Then for every $y \in \mathfrak{X}$ there exists a unique vector $\xi_y$ in $\mathscr{H}_{\mathfrak{w}}$ such that*

$$\forall x \in \mathfrak{X} : \qquad \mathfrak{t}(x, y) = (x + \ker \mathfrak{w} \mid \xi_y)_{\mathfrak{w}}.$$

*Proof.* Let $y$ be an arbitrary but fixed element of $\mathfrak{X}$ and define the linear functional $\Phi_y$ as follows

$$\Phi_y : \mathfrak{X}/_{\ker \mathfrak{w}} \to \mathbb{C}; \qquad x + \ker \mathfrak{w} \mapsto (x + \ker \mathfrak{t} \mid y + \ker \mathfrak{t})_{\mathfrak{t}}.$$

According to the Cauchy-Schwarz inequality and the assumption it is clear that $\Phi_y$ is a bounded linear functional. Indeed,

$$|\Phi_y(x + \ker \mathfrak{w})|^2 \leq \|x + \ker \mathfrak{t}\|_{\mathfrak{t}}^2 \cdot \|y + \ker \mathfrak{t}\|_{\mathfrak{t}}^2 \leq c^2 \cdot \|x + \ker \mathfrak{w}\|_{\mathfrak{w}}^2 \cdot \|y + \ker \mathfrak{w}\|_{\mathfrak{w}}^2.$$

Consequently, due to the Riesz representation theorem there exists a unique vector $\xi_y$ in $\mathscr{H}_{\mathfrak{w}}$ such that

$$\forall x \in \mathfrak{X} : \qquad \mathfrak{t}(x, y) = (x + \ker \mathfrak{t} \mid y + \ker \mathfrak{t})_{\mathfrak{t}} = \Phi_y(x + \ker \mathfrak{t}) = (x + \ker \mathfrak{w} \mid \xi_y)_{\mathfrak{w}}.$$

□

**Theorem 7.4.** *Let $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ be forms on $\mathfrak{X}$ and let $\mathfrak{t}$ be almost dominated by $\mathfrak{w}$. Then for every $y \in \mathfrak{X}$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ such that*

$$\forall x \in \mathfrak{X} : \qquad \mathfrak{t}(x, y) = \lim_{n\to+\infty} \mathfrak{w}(x, y_n).$$

*Proof.* Fix an arbitrary $y \in \mathfrak{X}$. Since $\mathfrak{t}$ is almost dominated by $\mathfrak{w}$, there exists a suitable sequence $(\mathfrak{t}_n)_{n\in\mathbb{N}}$ of $\mathfrak{w}$-dominated forms and a sequence $(\xi_{y,n})_{n\in\mathbb{N}}$ of representant vectors such that

$$\lim_{n\to+\infty} \mathfrak{t}_n = \mathfrak{t} \qquad \text{and} \qquad (\forall x \in \mathfrak{X})\ (\forall n \in \mathbb{N}) : \quad \mathfrak{t}_n(x, y) = (x + \ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}}.$$

As $\mathfrak{t}_n \leq \mathfrak{t}$, we can apply the Cauchy–Schwarz inequality on the form $\mathfrak{t} - \mathfrak{t}_n$ that gives

$$\left|\big(\mathfrak{t} - \mathfrak{t}_n\big)(x, y)\right|^2 \leq \big(\mathfrak{t} - \mathfrak{t}_n\big)[x]\big(\mathfrak{t} - \mathfrak{t}_n\big)[y] \to 0, \qquad n \to +\infty,$$

whence we infer that

$$\mathfrak{t}(x, y) = \lim_{n\to+\infty} t_n(x, y) = \lim_{n\to+\infty} (x + \ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}}.$$

Since $\mathfrak{X}/_{\ker \mathfrak{w}}$ is dense in $\mathcal{H}_{\mathfrak{w}}$ we can choose a sequence $(y_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ such that

$$\left\|\xi_{y,n} - (y_n + \ker \mathfrak{w})\right\|_{\mathfrak{w}} \to 0.$$

According to the Cauchy–Schwarz inequality, this implies that

$$\left|(x + \ker \mathfrak{w} \mid \xi_{y,n})_{\mathfrak{w}} - (x + \ker \mathfrak{w} \mid y_n + \ker \mathfrak{w})_{\mathfrak{w}}\right| \to 0$$

and thus

$$\forall x \in \mathfrak{X} : \qquad \mathfrak{t}(x, y) = \lim_{n\to+\infty} \mathfrak{w}(x, y_n).$$

$\square$

CHAPTER 8

# Extremal problems for forms

The aim of this chapter is to investigate some extremal problems that are in close relation with the Lebesgue decomposition.

## 8.1. Infimum of forms

In this section (which is based on [**54**]) we present a generalization of Ando's theorem for nonnegative forms. It is known that $\mathbf{B}_{\mathrm{sa}}(\mathscr{H})$ is a so-called antilattice with the order induced by the cone of positive operators [**19**]. That is, for self-adjoint operators $A$ and $B$ the infimum $A \wedge B \in \mathbf{B}_{\mathrm{sa}}(\mathscr{H})$ exists precisely when $A$ and $B$ are comparable.

The situation is completely different when consider the positive cone $\mathbf{B}_+(\mathscr{H})$ instead of $\mathbf{B}_{\mathrm{sa}}(\mathscr{H})$. The problem whether the infimum $A \wedge B \in \mathbf{B}_+(\mathscr{H})$ exists for two positive operators has been studied by several authors in mathematical physics. Particularly, Moreland and Gudder have solved it when the space is finite dimensional [**13**]. In the general case a necessary and sufficient condition was given by Ando [**5**]. He used the concepts of the parallel sum and the generalized short of positive operators. These notions were introduced by Anderson, Duffin and Trapp for the matrix case (see [**1, 3, 2**]). The infinite dimensional case was further studied by Ando [**4**], Fillmore and Williams [**12**] and Pekarev and Šmul'jan [**25**].

Next we briefly review some basic definitions. For more details the reader is referred to [**5**] and [**25**].

Let $\mathscr{H}$ be a complex Hilbert space and denote by $\mathbf{B}_+(\mathscr{H})$ the cone of bounded positive operators on $\mathscr{H}$. We say that the positive operators $A$ and $B$ are comparable if $B \leq A$ or $A \leq B$. The parallel sum $A : B$ of two positive operators $A$ and $B$ is defined by the quadratic form

$$\big((A : B)h \,|\, h\big) = \inf_{g \in \mathscr{H}} \big\{(Ag \,|\, g) + (B(h-g) \,|\, h-g)\big\} \quad (h \in \mathscr{H}). \tag{8.1}$$

The notion of generalized short of operators was introduced by Ando in [**4**] by means of the strong limit

$$[B]A := \lim_{n\to\infty} \big(nB : A\big). \tag{8.2}$$

Note that $0 \le nB : A \le (n+1)B : A \le A$ and $[B]A = \sup_{n\in\mathbb{N}} \big(nB : A\big)$.

Now, we are ready to present of the result of Ando (Theorem 6. in [**5**]):

**Theorem 8.1.** *Given $A, B \in \mathbf{B}_+(\mathscr{H})$, the infimum $A \wedge B$ exists in $\mathbf{B}_+(\mathscr{H})$ if and only if $[A]B$ and $[B]A$ are comparable. In this case*

$$A \wedge B = \min\{[A]B, [B]A\}.$$

The following construction (cf. [**27**]) gives an opportunity to consider forms in terms of positive operators. Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative form on the complex linear space $\mathfrak{X}$ such that $\mathfrak{w} \le \mathfrak{t}$ and consider their auxiliary Hilbert spaces $\mathscr{H}_\mathfrak{t}$ and $\mathscr{H}_\mathfrak{w}$, respectively.

Let us define the operator $J_{\mathfrak{t},\mathfrak{w}}$ from $\mathfrak{X}/_{\ker\mathfrak{t}} \subseteq \mathscr{H}_\mathfrak{t}$ to $\mathscr{H}_\mathfrak{w}$ by

$$J_{\mathfrak{t},\mathfrak{w}}(x + \ker\mathfrak{t}) = x + \ker\mathfrak{w} \quad (x \in \mathfrak{X}). \tag{8.3}$$

Since $\ker\mathfrak{t} \subseteq \ker\mathfrak{w}$ we have that the operator $J_{\mathfrak{t},\mathfrak{w}}$ is a densely defined contraction, hence its closure (i.e. the second adjoint of $J_{\mathfrak{t},\mathfrak{w}}$) is a contraction with $\operatorname{dom} J_{\mathfrak{t},\mathfrak{w}}^{**} = \mathscr{H}_\mathfrak{t}$ such that $\operatorname{ran} J_{\mathfrak{t},\mathfrak{w}}^{**}$ is dense in $\mathscr{H}_\mathfrak{w}$. Now, consider the bounded operator $J_{\mathfrak{t},\mathfrak{w}}^{*}J_{\mathfrak{t},\mathfrak{w}}^{**}$ on $\mathscr{H}_\mathfrak{t}$ and observe that

$$\begin{aligned}
\big(J_{\mathfrak{t},\mathfrak{w}}^{*}J_{\mathfrak{t},\mathfrak{w}}^{**}(x+\ker\mathfrak{t}), x+\ker\mathfrak{t}\big)_\mathfrak{t} &= \big(J_{\mathfrak{t},\mathfrak{w}}^{**}(x+\ker\mathfrak{t}), J_{\mathfrak{t},\mathfrak{w}}^{**}(x+\ker\mathfrak{t})\big)_\mathfrak{w}\\
&= \big(x+\ker\mathfrak{w}, x+\ker\mathfrak{w}\big)_\mathfrak{w}\\
&= \mathfrak{w}[x].
\end{aligned}$$

Recall that if $\mathscr{H}$ is a Hilbert space then there is a bijective isometric correspondence between the operators in $\mathbf{B}(\mathscr{H})$ and the bounded sesquilinear forms on $\mathscr{H}$, given by $A \mapsto \mathfrak{t}_A$, where $A \in \mathbf{B}(\mathscr{H})$ and

$$\mathfrak{t}_A(x,y) = \big(Ax, y\big) \quad (x, y \in \mathscr{H}). \tag{8.4}$$

Let us denote by $\mathcal{F}(\mathfrak{t})$ the set of nonnegative forms on $\mathfrak{X}$ which are majorized by $\mathfrak{t}$ and let $\mathcal{E}(\mathfrak{t})$ be the set of those positive operators on $\mathscr{H}_\mathfrak{t}$ which are majorized by the identity. The mapping defined by the above construction will be denoted by $\mathrm{i}_\mathfrak{t}$, i.e.

$$\mathrm{i}_\mathfrak{t} : \mathcal{F}(\mathfrak{t}) \to \mathcal{E}(\mathfrak{t});\ \mathfrak{w} \mapsto \mathrm{i}_\mathfrak{t}(\mathfrak{w}) = J_{\mathfrak{t},\mathfrak{w}}^{*}J_{\mathfrak{t},\mathfrak{w}}^{**}. \tag{8.5}$$

It easily follows from the definition that $\mathrm{i}_{\mathfrak{t}}(0) = 0$ and $\mathrm{i}_{\mathfrak{t}}(\mathfrak{t}) = I$ where $0$ and $I$ denote the zero and the unit elements in $\mathbf{B}(\mathscr{H}_{\mathfrak{t}})$, respectively.

The following lemma plays a key role in the proof of our main result.

**Lemma 8.2.** *Let* $\mathrm{i}_{\mathfrak{t}}$ *be the above defined map. Then*

(a) $\mathrm{i}_{\mathfrak{t}}$ *is bijective.*
(b) *If* $\mathfrak{u}$ *and* $\mathfrak{w} \in \mathcal{F}(\mathfrak{t})$, *then* $\mathfrak{u} \leq \mathfrak{w}$ *if and only if* $\mathrm{i}_{\mathfrak{t}}(\mathfrak{u}) \leq \mathrm{i}_{\mathfrak{t}}(\mathfrak{w})$.
(c) *If* $(\mathfrak{u}_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}(\mathfrak{t})$ *such that* $\mathfrak{u}_n \uparrow \mathfrak{u}$, *then* $\mathfrak{u} \in \mathcal{F}(\mathfrak{t})$ *and* $\mathrm{i}_{\mathfrak{t}}(\mathfrak{u}) = \sup_{n\in\mathbb{N}} \mathrm{i}_{\mathfrak{t}}(\mathfrak{u}_n)$.
(d) *If* $\mathfrak{u}, \mathfrak{w}$ *and* $\mathfrak{u} + \mathfrak{w} \in \mathcal{F}(\mathfrak{t})$, *then* $\mathrm{i}_{\mathfrak{t}}(\mathfrak{u}) + \mathrm{i}_{\mathfrak{t}}(\mathfrak{w}) = \mathrm{i}_{\mathfrak{t}}(\mathfrak{u} + \mathfrak{w})$.
(e) *If* $\mathfrak{u}$ *and* $\mathfrak{w} \in \mathcal{F}(\mathfrak{t})$, *then* $\mathfrak{u} : \mathfrak{w} \in \mathcal{F}(\mathfrak{t})$ *and* $\mathrm{i}_{\mathfrak{t}}(\mathfrak{u} : \mathfrak{w}) = \mathrm{i}_{\mathfrak{t}}(\mathfrak{u}) : \mathrm{i}_{\mathfrak{t}}(\mathfrak{w})$.
(f) *If* $\mathfrak{u}$ *and* $\mathfrak{w} \in \mathcal{F}(\mathfrak{t})$, *then* $\mathbf{D}_{\mathfrak{u}}\mathfrak{w} \in \mathcal{F}(\mathfrak{t})$ *and* $\mathrm{i}_{\mathfrak{t}}(\mathbf{D}_{\mathfrak{u}}\mathfrak{w}) = [\mathrm{i}_{\mathfrak{t}}(\mathfrak{u})]\mathrm{i}_{\mathfrak{t}}(\mathfrak{w})$.

*Proof.*

(a) If $\mathfrak{u}, \mathfrak{w} \in \mathcal{F}(\mathfrak{t})$, such that $\mathfrak{u} \neq \mathfrak{w}$, then there exists $x \in \mathfrak{X}$ such that $\mathfrak{u}[x] \neq \mathfrak{w}[x]$. Hence the equalities

$$\mathfrak{u}[x] = \big(J_{\mathfrak{t},\mathfrak{u}}^{*} J_{\mathfrak{t},\mathfrak{u}}^{**}(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}}$$

and

$$\mathfrak{w}[x] = \big(J_{\mathfrak{t},\mathfrak{w}}^{*} J_{\mathfrak{t},\mathfrak{w}}^{**}(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}}$$

imply that $\mathrm{i}_{\mathfrak{t}}(\mathfrak{u}) = J_{\mathfrak{t},\mathfrak{u}}^{*} J_{\mathfrak{t},\mathfrak{u}}^{**} \neq J_{\mathfrak{t},\mathfrak{w}}^{*} J_{\mathfrak{t},\mathfrak{w}}^{**} = \mathrm{i}_{\mathfrak{t}}(\mathfrak{w})$. Consequently the mapping $\mathrm{i}_{\mathfrak{t}}$ is injective. On the other hand let $A$ be an arbitrary element of $\mathcal{E}(\mathfrak{t})$. Then $\big(A(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}} \leq \big(I(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}} = \mathfrak{t}[x]$ thus it is clear that the sesquilinear form $\mathfrak{w}$ defined by $\mathfrak{w}(x, y) := \big(A(x + \ker \mathfrak{t}) \,|\, y + \ker \mathfrak{t}\big)_{\mathfrak{t}}$ is in $\mathcal{F}(\mathfrak{t})$. Moreover,

$$\big(\mathrm{i}_{\mathfrak{t}}(\mathfrak{w})(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}} = \big(J_{\mathfrak{t},\mathfrak{w}}^{*} J_{\mathfrak{t},\mathfrak{w}}^{**}(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t}\big)_{\mathfrak{t}} = \mathfrak{w}[x]$$

and

$$\big(A(x + \ker \mathfrak{t}), x + \ker \mathfrak{t}\big)_{\mathfrak{t}} = \mathfrak{w}[x]$$

imply that $\mathrm{i}_{\mathfrak{t}}(\mathfrak{w}) = A$. Hence $\mathrm{i}_{\mathfrak{t}}$ is surjective.

(b) The equivalence follows from the equalities $\mathfrak{u}[x] = \big(J_{\mathfrak{t},\mathfrak{u}}^{*} J_{\mathfrak{t},\mathfrak{u}}^{**}(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t})\big)_{\mathfrak{t}}$ and $\mathfrak{w}[x] = \big(J_{\mathfrak{t},\mathfrak{w}}^{*} J_{\mathfrak{t},\mathfrak{w}}^{**}(x + \ker \mathfrak{t}) \,|\, x + \ker \mathfrak{t})\big)_{\mathfrak{t}}$ for all $x \in \mathfrak{X}$.

(c) Recall that if $(\mathfrak{u}_n)_{n\in\mathbb{N}}$ is a monotonically nondecreasing sequence of nonnegative forms which is bounded from above by a nonnegative form $\mathfrak{t}$, then the pointwise limit $\sup_{n\in\mathbb{N}} \mathfrak{u}_n[x] = \mathfrak{u}[x]$ exists and defines a nonnegative form, such that $\mathfrak{u} \leq \mathfrak{t}$. Using $(b)$ the sequence $\big(\mathrm{i}_\mathfrak{t}(\mathfrak{u}_n)\big)_{n\in\mathbb{N}}$ is monotonically nondecreasing and bounded from above by $\mathrm{i}_\mathfrak{t}(\mathfrak{t})$. The equality $\mathrm{i}_\mathfrak{t}(\mathfrak{u}) = \sup_{n\in\mathbb{N}} \mathrm{i}_\mathfrak{t}(\mathfrak{u}_n)$ follows again from $(b)$.

(d) Combining $(\mathfrak{u}+\mathfrak{w})[x] = \mathfrak{u}[x]+\mathfrak{w}[x]$ with the equality

$$\mathfrak{s}[x] = \big(J^*_{\mathfrak{t},\mathfrak{s}} J^{**}_{\mathfrak{t},\mathfrak{s}}(x+\ker\mathfrak{t}) \,|\, x+\ker\mathfrak{t}\big)_\mathfrak{t}$$

and replacing $\mathfrak{s}$ respectively with $\mathfrak{u}$, $\mathfrak{w}$, and $\mathfrak{u}+\mathfrak{w}$ the required equality follows.

(e) Recall that $(\mathfrak{u}:\mathfrak{w}) \leq \mathfrak{u},\mathfrak{w}$ and that $(\mathfrak{u}:\mathfrak{w})[x] = \inf_{g\in\mathfrak{X}}\{\mathfrak{w}[g+x]+\mathfrak{u}[g]\}$. For $A, B \in \mathcal{E}(\mathfrak{t})$, the parallel sum $A : B$ is given by the formula

$$\big((A:B)(x+\ker\mathfrak{t}) \,|\, x+\ker\mathfrak{t}\big)_\mathfrak{t} =$$
$$= \inf_{y+\ker\mathfrak{t}\in\mathscr{H}_\mathfrak{t}} \Big\{\big(B(x+y+\ker\mathfrak{t}) \,|\, x+y+\ker\mathfrak{t}\big)_\mathfrak{t} + \big(A(y+\ker\mathfrak{t}) \,|\, y+\ker\mathfrak{t}\big)_\mathfrak{t}\Big\}.$$

Replacing $A$ and $B$ respectively by $\mathrm{i}_\mathfrak{t}(\mathfrak{u})$ and $\mathrm{i}_\mathfrak{t}(\mathfrak{w})$, and using that

$$\mathfrak{w}[g+x] = \big(\mathrm{i}_\mathfrak{t}(\mathfrak{w})(g+x+\ker\mathfrak{t}) \,|\, g+x+\ker\mathfrak{t}\big)_\mathfrak{t}$$

and

$$\mathfrak{u}[g] = \big(\mathrm{i}_\mathfrak{t}(\mathfrak{u})(g+\ker\mathfrak{t}) \,|\, g+\ker\mathfrak{t}\big)_\mathfrak{t}$$

the equality $\mathrm{i}_\mathfrak{t}(\mathfrak{u}:\mathfrak{w}) = \mathrm{i}_\mathfrak{t}(\mathfrak{u}) : \mathrm{i}_\mathfrak{t}(\mathfrak{w})$ follows.

(f) Since $(\mathfrak{w} : n\mathfrak{u}) \leq \mathfrak{w} \leq \mathfrak{t}$, so $\mathbf{D}_\mathfrak{u}\mathfrak{w} = \sup_{n\in\mathbb{N}}(\mathfrak{w} : n\mathfrak{u})$ is an element $\mathcal{F}(\mathfrak{t})$. Furthermore, $\mathrm{i}_\mathfrak{t}$ is isotone and preserves the parallel sum, hence

$$\mathrm{i}_\mathfrak{t}(\mathbf{D}_\mathfrak{u}\mathfrak{w}) = \mathrm{i}_\mathfrak{t}(\sup_{n\in\mathbb{N}}(\mathfrak{w}:n\mathfrak{u})) = \sup_{n\in\mathbb{N}}(\mathrm{i}_\mathfrak{t}(\mathfrak{w}) : n\mathrm{i}_\mathfrak{t}(\mathfrak{u})) = [\mathrm{i}_\mathfrak{t}(\mathfrak{u})]\mathrm{i}_\mathfrak{t}(\mathfrak{w})$$

holds.

□

Now we are ready to prove the main result of this section. Namely, we give a necessary and sufficient condition for the existence of the infimum of two given nonnegative forms.

**Theorem 8.3.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be nonnegative forms on the complex linear space $\mathfrak{X}$. Then the following statements are equivalent.*

(i) $\mathbf{D}_\mathfrak{t}\mathfrak{w} \leq \mathbf{D}_\mathfrak{w}\mathfrak{t}$ *or* $\mathbf{D}_\mathfrak{w}\mathfrak{t} \leq \mathbf{D}_\mathfrak{t}\mathfrak{w}$.

(ii) $\mathbf{D}_\mathfrak{t}\mathfrak{w} \leq \mathfrak{t}$ *or* $\mathbf{D}_\mathfrak{w}\mathfrak{t} \leq \mathfrak{w}$.

(iii) *The infimum* $\mathfrak{t}\wedge\mathfrak{w}$ *exists.*

*Proof.* Assume that $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ (resp. $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$). From the inequality $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$ (resp. $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{w}$) it follows that $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$ (resp. $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{w}$).

On the contrary assume that $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t}$. The inequality $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{w}$ implies that $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \ll_{\mathrm{ad}} \mathfrak{w}$. On the other hand, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is the maximum of all forms which are majorized by $\mathfrak{t}$ and almost dominated with respect to $\mathfrak{w}$, hence $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$. Similarly, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{w}$ implies that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$. So the equivalence of (i) and (ii) is proved.

Assuming (i), the infimum of $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$ exists and equals to the smaller. We shall prove that $\mathfrak{t} \wedge \mathfrak{w}$ exists and equals to $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \wedge \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$. Let $\mathfrak{s}$ be a nonnegative form such that $\mathfrak{s} \leq \mathfrak{t}$ and $\mathfrak{s} \leq \mathfrak{w}$. As in the proof of (i)$\Rightarrow$(ii), we have that $\mathfrak{s} \leq \mathfrak{w}$ implies $\mathfrak{s} \ll_{\mathrm{ad}} \mathfrak{w}$. Thus the inequality $\mathfrak{s} \leq \mathfrak{t}$ and the maximality of $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ gives $\mathfrak{s} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$. Similarly, we have that $\mathfrak{s} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$, and hence $\mathfrak{s} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \wedge \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$. On the other hand, from the inequalities $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{t}$ and $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{w}$ it follows that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \wedge \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t}, \mathfrak{w}$. This shows that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \wedge \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t} \wedge \mathfrak{w}$ holds.

Finally, we prove that the existence of $\mathfrak{t} \wedge \mathfrak{w}$ implies that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ and $\mathbf{D}_{\mathfrak{t}}\mathfrak{w}$ are comparable. Consider the sets $\mathcal{F}(\mathfrak{t}+\mathfrak{w})$, and $\mathcal{E}(\mathfrak{t}+\mathfrak{w})$ and the mapping $\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}$. Recall that $\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}$ is defined by

$$\mathrm{i}_{\mathfrak{t}+\mathfrak{w}} : \mathcal{F}(\mathfrak{t}+\mathfrak{w}) \to \mathcal{E}(\mathfrak{t}+\mathfrak{w}); \ \mathfrak{u} \mapsto \mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{u}) = J^*_{\mathfrak{t}+\mathfrak{w},\mathfrak{u}} J^{**}_{\mathfrak{t}+\mathfrak{w},\mathfrak{u}}.$$

Recall also that $\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}$ is bijective, order preserving, and preserves the parallel sum. Consequently, if the infimum of $\mathfrak{t}$ and $\mathfrak{w}$ in $\mathcal{F}_+(\mathfrak{t}+\mathfrak{w})$ exists, then $\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{t})$ and $\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{w})$ have a greatest lower bound in $B_+(\mathscr{H}_{\mathfrak{t}+\mathfrak{w}})$. Using Theorem 8.1 this is equivalent to

$$[\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{t})]\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{w}) \leq [\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{w})]\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{t}) \text{ or } [\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{w})]\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{t}) \leq [\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{t})]\mathrm{i}_{\mathfrak{t}+\mathfrak{w}}(\mathfrak{w}).$$

Now Lemma 8.2 $(f)$ completes the proof. □

**Remark 8.4.** Observe that Ando's theorem was used in the proof above only in the special case when $A, B \in \mathbf{B}_+(\mathscr{H})$ and $A + B = I$. (See [**5**], Theorem 2.)

## 8.2. Extreme points of form segments

Our next purpose is to describe the extreme points of $[0, \mathfrak{t}]$. These results were motivated by Pekarev's paper [**26**] (see also [**11, 31**]). First we recall the notion of closability. If $\mathfrak{t}$ and $\mathfrak{w}$ are forms, $\mathfrak{t}$ is called $\mathfrak{w}$-*closable* if

$$\mathfrak{w}[x_n] \to 0 \quad \text{and} \quad \mathfrak{t}[x_n - x_m] \to 0 \quad \text{imply} \quad \mathfrak{t}[x_n] \to 0$$

for any sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}$. Recall also the nontrivial fact that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated if and only if $\mathfrak{t}$ is $\mathfrak{w}$-closable.

To prove the main result of this section we need also the following lemma.

**Lemma 8.5.** *Let $\mathfrak{t}, \mathfrak{w}, \mathfrak{h} \in \mathcal{F}_+(\mathfrak{X})$ be forms and assume that $\mathfrak{t}$ is $\mathfrak{w}$-dominated. Then the following statements are equivalent*

(i) $\mathfrak{w} \perp \mathfrak{h}$,
(ii) $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}+\mathfrak{h}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$.

*Proof.* Since $\mathfrak{w} \perp \mathfrak{h}$ is equivalent with $\mathbf{D}_{\mathfrak{w}}\mathfrak{h} = 0$, implication $(ii) \Rightarrow (i)$ follows from

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}+\mathfrak{h}) \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + \mathbf{D}_{\mathfrak{w}}\mathfrak{h} \geq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

To prove $(i) \Rightarrow (ii)$ observe first that $\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{w}}(\mathfrak{t}+\mathfrak{h}) \leq \mathfrak{t}+\mathfrak{h}$, consequently, $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}+\mathfrak{h}) = \mathfrak{t}+\mathfrak{k}$ with some $\mathfrak{k} \in \mathcal{F}_+(\mathfrak{X})$ $(0 \leq \mathfrak{k} \leq \mathfrak{h})$. Assume indirectly that $\mathfrak{k} \neq 0$. Since $\mathfrak{w} \perp \mathfrak{h}$ and $\mathfrak{k} \leq \mathfrak{h}$, we have $\mathfrak{w} \perp \mathfrak{k}$. Consequently, $\mathfrak{k}$ is not $\mathfrak{w}$-closable, because the only form which is simultaneously $\mathfrak{w}$-singular and $\mathfrak{w}$-closable is the identically zero form. In this case, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}$ such that $\mathfrak{k}[x_n - x_m] \to 0$ and $\mathfrak{w}[x_n] \to 0$, but $\mathfrak{k}[x_n] \nrightarrow 0$. Since $\mathfrak{t}$ is $\mathfrak{w}$-dominated, $\mathfrak{t}[x_n] \to 0$ holds for this sequence, and thus $\mathfrak{t}[x_n - x_m] \to 0$, which is contradiction. Indeed, $\mathfrak{t}+\mathfrak{k}$ is $\mathfrak{w}$-closable by assumption, but

$$(\mathfrak{t}+\mathfrak{k})[x_n - x_m] \to 0, \qquad \mathfrak{w}[x_n] \to 0 \qquad (\mathfrak{t}+\mathfrak{k})[x_n] \nrightarrow 0.$$

□

Assume that $\mathfrak{t}$ and $\mathfrak{w}$ are forms such that $\mathfrak{w} \leq \mathfrak{t}$. We say that $\mathfrak{w}$ is a *disjoint part* of $\mathfrak{t}$ if $\mathfrak{w}$ and $\mathfrak{t}-\mathfrak{w}$ are singular. The following theorem states that the extreme points of $[0, \mathfrak{t}]$ are precisely the disjoint parts of $\mathfrak{t}$.

For the operator version of the following theorem we refer the reader to [**11**].

**Theorem 8.6.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms, and assume that $\mathfrak{w} \leq \mathfrak{t}$. Then the following statements are equivalent.*

(i) $\mathfrak{w}$ is an extreme point of $[0, \mathfrak{t}]$,
(ii) $\mathfrak{w}$ is a disjoint part of $\mathfrak{t}$,
(iii) $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{w}$.
(iv) $(\lambda\mathfrak{u}) : (\mu\mathfrak{t}) = \frac{\lambda\mu}{\lambda+\mu}\mathfrak{u}$ for all $\lambda, \mu > 0$.
(v) $(\lambda\mathfrak{u}) : \mathfrak{t} = \mathfrak{u} : (\lambda\mathfrak{t})$ for all $\lambda > 0$.

*Proof.* To prove $(i) \Rightarrow (ii)$ assume that $\mathfrak{w}$ is not singular with respect to $\mathfrak{t} - \mathfrak{w}$. In this case, according to Theorem 4.3 (a), $\mathfrak{w} : (\mathfrak{t} - \mathfrak{w}) \neq 0$, and hence,

$$\frac{1}{2}\Big(\mathfrak{w} - \big(\mathfrak{w} : (\mathfrak{t} - \mathfrak{w})\big)\Big) + \frac{1}{2}\Big(\mathfrak{w} + \big(\mathfrak{w} : (\mathfrak{t} - \mathfrak{w})\big)\Big) = \mathfrak{w}$$

is a nontrivial convex combination in $[0, \mathfrak{t}]$. Implication $(ii) \Rightarrow (iii)$ follows directly from Theorem 4.3(b) and Lemma 8.5

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}(\mathfrak{w} + (\mathfrak{t} - \mathfrak{w})) = \mathbf{D}_{\mathfrak{w}}\mathfrak{w} = \mathfrak{w}.$$

To prove $(iii) \Rightarrow (i)$, assume that $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \mathfrak{w}$, but $\mathfrak{w}$ is not an extreme point of $[0, \mathfrak{t}]$. In this case, there exists $\mathfrak{h}, \mathfrak{k} \in [0, \mathfrak{t}]$ $(\mathfrak{h} \neq \mathfrak{k})$ and $\lambda \in (0, 1)$ such that $\mathfrak{w} = \lambda\mathfrak{h} + (1 - \lambda)\mathfrak{k}$. According to Proposition 4.1, we have

$$\mathfrak{t} : n\mathfrak{w} \geq \mathfrak{h} : n\lambda\mathfrak{h} = \frac{n\lambda}{n\lambda + 1}\mathfrak{h} \quad \Rightarrow \quad \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \geq \mathfrak{h}$$

and similarly,

$$\mathfrak{t} : n\mathfrak{w} \geq \mathfrak{k} : n(1 - \lambda)\mathfrak{k} = \frac{n(1 - \lambda)}{n(1 - \lambda) + 1}\mathfrak{k} \quad \Rightarrow \quad \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} \geq \mathfrak{k}.$$

These imply that $\mathfrak{w} = \mathfrak{h} = \mathfrak{k}$, because if there exists $a \in \mathfrak{X}$ such that $\mathfrak{w}[x] > \mathfrak{h}[x]$ or $\mathfrak{w}[x] > \mathfrak{k}[x]$ then

$$\mathfrak{w}[x] = \lambda\mathfrak{w}[x] + (1 - \lambda)\mathfrak{w}[x] > \lambda\mathfrak{h}[x] + (1 - \lambda)\mathfrak{k}[x] = \mathfrak{w}[x],$$

which is contradiction. Now we prove the implications $(iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (iii)$. Assume that $\mathfrak{w}$ is a $\mathfrak{t}$-quasi unit, and observe that

$$(\lambda\mathfrak{w}) : (\mu\mathfrak{t}) = (\lambda\mathfrak{w}) : \big(\mathbf{D}_{\lambda\mathfrak{w}}(\mu\mathfrak{t})\big) = (\lambda\mathfrak{w}) : (\mu\mathbf{D}_{\mathfrak{w}}\mathfrak{t}) = (\lambda\mathfrak{w}) : (\mu\mathfrak{w}) = \frac{\lambda\mu}{\lambda + \mu}\mathfrak{w}.$$

according to the properties of the parallel sum and the following equalities

$$\mathfrak{t} : \mathfrak{w} = \mathbf{D}_{\mathfrak{w}}(\mathfrak{t} : \mathfrak{w}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} : \mathfrak{w}.$$

Assuming $(iv)$ it is clear that

$$(\lambda\mathfrak{w}) : \mathfrak{t} = \frac{\lambda}{1 + \lambda}\mathfrak{w} = \mathfrak{w} : (\lambda\mathfrak{t}).$$

Finally, since $\mathfrak{w} \leq \mathfrak{t}$, property $(v)$ implies that

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} = \sup_{n \in \mathbb{N}}\big(\mathfrak{t} : (n\mathfrak{w})\big) = \sup_{n \in \mathbb{N}}\big((n\mathfrak{t}) : \mathfrak{w}\big) = \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{w}.$$

□

Remark that if $\mathfrak{w}$ is an extreme point of $[0,\mathfrak{t}]$, then the equivalence $(i) \Leftrightarrow (ii)$ asserts that $\mathfrak{t}-\mathfrak{w}$ belongs also to $\mathrm{ex}[0,\mathfrak{t}]$.

The following observation will be very useful, hence we state it separately. This is just the translation of [**11**, Theorem 4.2, Corollary 4.3] to the language of forms.

**Lemma 8.7.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$. Then the following statements are equivalent.*

(i) $\mathfrak{t}$ *is a $\mathfrak{w}$-quasi-unit.*
(ii) $\mathfrak{t} : \mathfrak{w} = \frac{1}{2}\mathfrak{t}$.

*Proof.* If $\mathfrak{t}$ is a $\mathfrak{w}$-quasi-unit then $\mathfrak{t} : \mathfrak{w} = \mathfrak{t} : \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t} : \mathfrak{t} = \frac{1}{2}\mathfrak{t}$ according to (5.5). To prove the converse implication, assume that $\mathfrak{t} : \mathfrak{w} = \frac{1}{2}\mathfrak{t}$. Define the sequence $\lambda_1 = 1$, $\lambda_k := \lambda_{k-1}(\lambda_{k-1}+2)$ for $k \geq 2$. Since $\lambda_k > 2^{k-1}$, it is enough to show that

$$(\lambda_k\mathfrak{t}) : \mathfrak{w} = \frac{\lambda_k}{1+\lambda_k}\mathfrak{t}$$

holds for all $k \in \mathbb{N}$. If this is the case, taking the supremum in $k \in \mathbb{N}$ we obtain $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t}$.

If $k = 1$ then $\mathfrak{t} : \mathfrak{w} = \frac{1}{2}\mathfrak{t}$ holds by assumption. Now assume that it holds for some $k \in \mathbb{N}$. Using the hypothesis twice, we obtain that

$$\begin{aligned}\frac{1}{1+\lambda_k}\mathfrak{t} = \mathfrak{t} : \frac{1}{\lambda_k}\mathfrak{w} &= \left(\frac{1+\lambda_k}{\lambda_k}\big((\lambda_k\mathfrak{t}) : \mathfrak{w}\big)\right) : \frac{1}{\lambda_k}\mathfrak{w} = \\ &= ((1+\lambda_k)\mathfrak{t}) : \left(\frac{1+\lambda_k}{\lambda_k}\mathfrak{w} : \frac{1}{\lambda_k}\mathfrak{w}\right) = \big((1+\lambda_k)\mathfrak{t}\big) : \frac{1+\lambda_k}{\lambda_k(\lambda_k+2)}\mathfrak{w} = \\ &= (1+\lambda_k)\left(\mathfrak{t} : \frac{1}{\lambda_{k+1}}\mathfrak{w}\right) = \frac{1+\lambda_k}{\lambda_{k+1}}\big((\lambda_{k+1}\mathfrak{t}) : \mathfrak{w}\big)\end{aligned}$$

and hence,

$$(\lambda_{k+1}\mathfrak{t}) : \mathfrak{w} = \frac{\lambda_{k+1}}{(1+\lambda_k)^2}\mathfrak{t} = \frac{\lambda_{k+1}}{1+\lambda_{k+1}}\mathfrak{t}.$$

□

Observe that if $\mathfrak{t}$ and $\mathfrak{w}$ are forms on $\mathfrak{X}$, then the form $\mathfrak{z} \in [0,\mathfrak{t}])$ becomes a solution of

$$(\mathfrak{t}-\mathfrak{z}) : (\mathfrak{w}+\mathfrak{z}) = 0 \tag{8.6}$$

precisely when $\mathfrak{w}+\mathfrak{z}$ is an extreme point of the interval $[0,\mathfrak{t}+\mathfrak{w}]$. Indeed, the expression $(\mathfrak{t}-\mathfrak{z}) : (\mathfrak{w}+\mathfrak{z})$ can be written as $(\mathfrak{t}+\mathfrak{w}-(\mathfrak{w}+\mathfrak{z})) : (\mathfrak{w}+\mathfrak{z}) = 0$, i.e. $\mathfrak{w}+\mathfrak{z}$ is a disjoint part of $\mathfrak{t}+\mathfrak{w}$, or equivalently, $\mathfrak{w}+\mathfrak{z} \in \mathrm{ex}[0,\mathfrak{t}+\mathfrak{w}]$.

Remark that $\mathrm{ex}[\mathfrak{t},\mathfrak{w}] = \{\mathfrak{t}+\mathfrak{u} : \ \mathfrak{u} \in \mathrm{ex}[0,\mathfrak{w}-\mathfrak{t}]\}$. Obviously $\mathrm{ex}[\mathfrak{t},\mathfrak{w}]$ contains all points of $\mathrm{ex}[0,\mathfrak{w}]$ in $[\mathfrak{t},\mathfrak{w}]$. The converse is not true if $\mathfrak{t} \notin \mathrm{ex}[0,\mathfrak{w}]$.

**Theorem 8.8.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. Then the following statements are equivalent* (*i*) $\mathfrak{t}$ *is an extreme point of* $[0, \mathfrak{t}+\mathfrak{w}]$.

(*ii*) $\mathrm{ex}[\mathfrak{t}, \mathfrak{t}+\mathfrak{w}] \subseteq \mathrm{ex}[0, \mathfrak{t}+\mathfrak{w}]$.

*Proof.* The implication $(ii) \Rightarrow (i)$ is trivial. Conversely, assume that $\mathfrak{t}$ is an extreme point of $[0, \mathfrak{t}+\mathfrak{w}]$, or equivalently $\mathfrak{t} : \mathfrak{w} = 0$. If $\mathfrak{u} \in \mathrm{ex}[\mathfrak{t}, \mathfrak{t}+\mathfrak{w}]$, then there exists $\mathfrak{v} \in \mathrm{ex}[0, \mathfrak{w}]$ such that $\mathfrak{u} = \mathfrak{t} + \mathfrak{v}$. Recall that $\mathfrak{v}$ is an extreme point of $[0, \mathfrak{w}]$ exactly when $\mathbf{D}_{\mathfrak{v}}(\mathfrak{w} - \mathfrak{v}) = 0$ and $\mathbf{D}_{\mathfrak{v}}\mathfrak{w} = \mathfrak{v}$. Remark also that $\mathbf{D}_{\mathfrak{w}}(\mathfrak{t} + \mathfrak{v}) = \mathbf{D}_{\mathfrak{w}}\mathfrak{v}$, because $\mathfrak{v}$ is dominated by $\mathfrak{w}$, and $\mathfrak{t}$ is singular with respect to $\mathfrak{w}$. Now, we need to show that $\mathfrak{t} + \mathfrak{v} \in \mathrm{ex}[0, \mathfrak{t} + \mathfrak{w}]$, i.e. $(\mathfrak{t} + \mathfrak{v}) : (\mathfrak{t} + \mathfrak{w} - (\mathfrak{t} + \mathfrak{v})) = 0$. Or equivalently,

$$\mathbf{D}_{\mathfrak{t}+\mathfrak{v}}(\mathfrak{w} - \mathfrak{v}) = 0.$$

Applying Corollary 6.3 and the previous remarks we have that

$$\mathbf{D}_{\mathfrak{t}+\mathfrak{v}}(\mathfrak{w} - \mathfrak{v}) = \mathbf{D}_{\mathbf{D}_{\mathfrak{w}-\mathfrak{v}}(\mathfrak{t}+\mathfrak{v})}(\mathfrak{w} - \mathfrak{v}) \leq \mathbf{D}_{\mathbf{D}_{\mathfrak{w}}(\mathfrak{t}+\mathfrak{v})}(\mathfrak{w} - \mathfrak{v}) = \mathbf{D}_{\mathbf{D}_{\mathfrak{w}}\mathfrak{v}}(\mathfrak{w} - \mathfrak{v}) = \mathbf{D}_{\mathfrak{v}}(\mathfrak{w} - \mathfrak{v}),$$

and hence, the proof of the equivalence $(ii) \Leftrightarrow (i)$ is complete. □

**Remark 8.9.** Replacing $\mathfrak{w}$ with $\mathfrak{w} - \mathfrak{t}$ we have the following statement:

$$\mathfrak{t} \in \mathrm{ex}[0, \mathfrak{w}] \quad \Leftrightarrow \quad \mathrm{ex}[\mathfrak{t}, \mathfrak{w}] \subseteq \mathrm{ex}[0, \mathfrak{w}].$$

The followings are immediate consequence of the previous theorems

**Corollary 8.10.** *Let* $A, B \in \mathbf{B}_+(\mathscr{H})$ *be bounded positive operators on the Hilbert space* $\mathscr{H}$ *such that* $A \leq B$. *Then the following statements are equivalent:*

(*i*) *$A$ is an extreme point of the convex set* $[0, B]$.

(*ii*) *The almost dominated part of $B$ with respect $A$ is equal to $A$.*

(*iii*) $A : (B - A) = 0$.

(*iv*) $\mathrm{ex}[A, B] \subseteq \mathrm{ex}[0, B]$.

**Corollary 8.11.** *Let $\mu$ and $\nu$ be contents on the algebra $\mathfrak{A}$. Then the following statements are equivalent:*

(*i*) *$\mu$ is an extreme point of the convex set* $[0, \nu]$.

(*ii*) *The almost dominated part of $\nu$ with respect $\mu$ is equal to $\mu$.*

(*iii*) $\mu \wedge (\nu - \mu) = 0$.

(*iv*) $\mathrm{ex}[\mu, \nu] \subseteq \mathrm{ex}[0, \nu]$.

Finally, we are going to prove that the set of $\mathfrak{w}$-quasi-units (denoted by $Q(\mathfrak{w})$) is a lattice in contrast with the set $\mathcal{F}_+(\mathfrak{X})$. For the operator version of this result see Theorem 4.2 $(iii) \Leftrightarrow (i)$ and theorem 5.4 in [**11**].

**Theorem 8.12.** *Let $\mathfrak{w}$ be a form, and consider the set $Q(\mathfrak{w})$ of $\mathfrak{w}$-quasi-units. Then the partially ordered set $(Q(\mathfrak{w}), \leq)$ is a lattice. Namely, if $\mathfrak{s}$ and $\mathfrak{t}$ are $\mathfrak{w}$-quasi units, then the greatest lower bound $\mathfrak{s} \curlywedge \mathfrak{t}$ and the least upper bound $\mathfrak{s} \curlyvee \mathfrak{t}$ in $Q(\mathfrak{w})$ exist, and*

$$\mathfrak{s} \curlywedge \mathfrak{t} = 2(\mathfrak{s} : \mathfrak{t}), \qquad\qquad \mathfrak{s} \curlyvee \mathfrak{t} = \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}.$$

*Furthermore,* $\mathfrak{s} \curlywedge \mathfrak{t} = \mathfrak{s} \wedge \mathfrak{t} = \mathbf{D}_{\mathfrak{s}}\mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{s}$.

*Proof.* First observe that $2(\mathfrak{s} : \mathfrak{t})$ is a $\mathfrak{w}$-quasi-unit. Indeed,

$$[2(\mathfrak{s} : \mathfrak{t})] : \mathfrak{w} = [2(\mathfrak{s} : \mathfrak{t})] : [2(\mathfrak{w} : \mathfrak{w})] = 2[(\mathfrak{s} : \mathfrak{w}) : (\mathfrak{t} : \mathfrak{w})] = 2[(\frac{1}{2}\mathfrak{s}) : (\frac{1}{2}\mathfrak{t})] = \mathfrak{s} : \mathfrak{t}.$$

Now let $\mathfrak{u}$ be a form such that $\mathfrak{u} \leq \mathfrak{s}$ and $\mathfrak{u} \leq \mathfrak{t}$. According to

$$2(\mathfrak{s} : \mathfrak{t}) \geq 2(\mathfrak{u} : \mathfrak{u}) = \mathfrak{u},$$

it is enough to show that $2(\mathfrak{s} : \mathfrak{t}) \leq \mathfrak{s}$ and $2(\mathfrak{s} : \mathfrak{t}) \leq \mathfrak{t}$. This follows immediately from the previous lemma, because $2(\mathfrak{s} : \mathfrak{t}) \leq 2(\mathfrak{w} : \mathfrak{t}) = \mathfrak{t}$ and $2(\mathfrak{s} : \mathfrak{t}) \leq 2(\mathfrak{s} : \mathfrak{w}) = \mathfrak{s}$. This shows also that $2(\mathfrak{s} : \mathfrak{t})$ is the infimum of $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathcal{F}_+(\mathfrak{X})$. Now we show that $2(\mathfrak{s} : \mathfrak{t}) = \mathbf{D}_{\mathfrak{s}}\mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{s}$. Since $\mathbf{D}_{\mathfrak{t}}\mathfrak{s} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w} = \mathfrak{t}$ we have $\mathbf{D}_{\mathfrak{t}}\mathfrak{s} = \mathbf{D}_{\mathfrak{s}}(\mathbf{D}_{\mathfrak{t}}\mathfrak{s}) \leq \mathbf{D}_{\mathfrak{s}}\mathfrak{t}$. And by symmetry, $\mathbf{D}_{\mathfrak{s}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{s}$. On the other hand, $\mathbf{D}_{\mathfrak{s}}\mathfrak{t} \leq \mathfrak{t}$ and $\mathbf{D}_{\mathfrak{t}}\mathfrak{s} \leq \mathfrak{s}$ imply $\mathbf{D}_{\mathfrak{s}}\mathfrak{t} : \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t} : \mathfrak{s}$. Since $\mathbf{D}_{\mathfrak{s}}\mathfrak{t} = \mathbf{D}_{\mathfrak{t}}\mathfrak{s}$ we conclude that $\frac{1}{2}\mathbf{D}_{\mathfrak{s}}\mathfrak{t} = \frac{1}{2}\mathbf{D}_{\mathfrak{t}}\mathfrak{s} \leq \mathfrak{t} : \mathfrak{s}$. For the converse inequality observe that $2(\mathfrak{s} : \mathfrak{t}) \leq 2(\mathfrak{s} : \mathfrak{w}) = \mathfrak{s}$, and hence $2(\mathfrak{s} : \mathfrak{t}) = \mathbf{D}_{\mathfrak{t}}(2(\mathfrak{s} : \mathfrak{t})) \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$.

To prove that the least upper bound of $\mathfrak{s}$ and $\mathfrak{t}$ in $Q(\mathfrak{w})$ is $\mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}$ take an $\mathfrak{u} \in Q(\mathfrak{w})$ such that $\mathfrak{u} \geq \mathfrak{s}$ and $\mathfrak{u} \geq \mathfrak{t}$ and observe that $\mathfrak{u} = \mathbf{D}_{\mathfrak{u}}\mathfrak{w} = \mathbf{D}_{2\mathfrak{u}}\mathfrak{w} \geq \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}$. Since $\mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w} \geq \mathbf{D}_{\mathfrak{s}}\mathfrak{w} = \mathfrak{s}$ and $\mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w} \geq \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \geq \mathfrak{t}$ it is enough to show that $\mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}$ is a $\mathfrak{w}$-quasi unit, or equivalently, a disjoint part. This follows immediately from Corollary 5.4, because

$$(\mathfrak{w} - \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}) : \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w} \leq (\mathfrak{w} - \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}) : (\mathfrak{w} + \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}) = \mathfrak{o},$$

which means that $\mathfrak{w} - \mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}$ and $\mathbf{D}_{\mathfrak{s}+\mathfrak{t}}\mathfrak{w}$ are singular according to Theorem 4.3(a). $\square$

CHAPTER 9

# Overview

In this closing chapter we make a short overview of our results. Moreover, we show also that how this general theory can be used for applications.

## Notions, notations

Let $\mathfrak{X}$ be a complex linear space and let $\mathfrak{t}$ be a nonnegative sesquilinear form (or shortly just *form*) on it. That is, $\mathfrak{t}$ is a mapping from $\mathfrak{X} \times \mathfrak{X}$ to $\mathbb{C}$, which is linear in the first argument, antilinear in the second argument, and the corresponding quadratic form

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}[x] := \mathfrak{t}(x, x)$$

is nonnegative. A crucial fact is that a form is uniquely determined via its quadratic form due to the polarization formula

$$\forall x, y \in \mathfrak{X}: \quad \mathfrak{t}(x, y) = \frac{1}{4} \sum_{k=0}^{3} i^k \mathfrak{t}[x + i^k y].$$

The set of forms is denoted by $\mathcal{F}_+(\mathfrak{X})$. For $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ we write $\mathfrak{t} \leq \mathfrak{w}$ if $\mathfrak{t}[x] \leq \mathfrak{w}[x]$ for all $x \in \mathfrak{X}$. *Domination* means that there exists a constant $c$ such that $\mathfrak{t} \leq c \cdot \mathfrak{w}$. Using the ordering we can define singularity and almost domination. The forms $\mathfrak{t}$ and $\mathfrak{w}$ are *singular* ($\mathfrak{t} \perp \mathfrak{w}$) if for every form $\mathfrak{s}$ the inequalities $\mathfrak{s} \leq \mathfrak{t}$ and $\mathfrak{s} \leq \mathfrak{w}$ imply that $\mathfrak{s} = \mathfrak{o}$ (i.e., $\mathfrak{s}$ is the identically zero form).

We say that $\mathfrak{t}$ is *almost dominated by* $\mathfrak{w}$ (in symbols: $\mathfrak{t} \ll_{\mathrm{ad}} \mathfrak{w}$) if there exists a monotonically nondecreasing sequence of forms $\mathfrak{t}_n$, each dominated by $\mathfrak{w}$, such that $\mathfrak{t} = \sup_{n\in\mathbb{N}} \mathfrak{t}_n$ (pointwise supremum). The form $\mathfrak{t}$ is $\mathfrak{w}$-*closable* ($\mathfrak{t} \ll_{\mathrm{cl}} \mathfrak{w}$, in symbols), if

$$\forall (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}: \quad \big((\mathfrak{t}[x_n - x_m] \to 0) \ \wedge \ (\mathfrak{w}[x_n] \to 0)\big) \implies \mathfrak{t}[x_n] \to 0.$$

It is a crucial thing that $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated precisely when $\mathfrak{t}$ is $\mathfrak{w}$-closable. The form $\mathfrak{t}$ is called *absolutely continuous with respect to* $\mathfrak{w}$ (or $\mathfrak{t}$ is $\mathfrak{w}$-absolutely continuous, in symbols: $\mathfrak{t} \ll_{\mathrm{ac}} \mathfrak{w}$), if $\mathfrak{w}[x] = 0$ implies $\mathfrak{t}[x] = 0$ for all $x \in \mathfrak{X}$. If $\mathfrak{t} \in \mathcal{F}_+(\mathfrak{X})$ then the square root of its quadratic form defines a seminorm on $\mathfrak{X}$. Hence the set

$$\ker \mathfrak{t} := \big\{ x \in \mathfrak{X} \ \big| \ \mathfrak{t}[x] = 0 \big\}$$

is a linear subspace of $\mathfrak{X}$. The Hilbert space $\mathscr{H}_{\mathfrak{t}}$ denotes the completion of the inner product space $\mathfrak{X}/_{\ker\mathfrak{t}}$ equipped with the natural inner product

$$\forall x, y \in \mathfrak{X}: \quad (x + \ker\mathfrak{t} \mid y + \ker\mathfrak{t})_{\mathfrak{t}} := \mathfrak{t}(x, y).$$

Observe that $\mathfrak{t}$ is $\mathfrak{w}$-closable if and only if the canonical embedding (which assigns the coset $x + \ker\mathfrak{t}$ to $x + \ker\mathfrak{w}$) from $\mathscr{H}_{\mathfrak{w}}$ to $\mathscr{H}_{\mathfrak{t}}$ is well-defined. Strong absolute continuity means that this embedding is a closable operator.

## Decomposition theorems

We establish two decomposition theorems. The first one is the so-called short-type decomposition, which is a decomposition of $\mathfrak{t}$ into absolutely continuous and singular parts. The key notion is the *short* of a form to a linear subspace of $\mathfrak{X}$, which is a generalization of the well known concept of operator short.

If $\mathfrak{t}$ and $\mathfrak{w}$ are forms on $\mathfrak{X}$, then the short of $\mathfrak{t}$ to the subspace $\ker\mathfrak{w}$ is

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}_{\ker\mathfrak{w}}[x] := \inf_{y\in\ker\mathfrak{w}} \mathfrak{t}[x - y].$$

The short-type decomposition theorem is stated as follows.

**Theorem 9.1.** *Let $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ be forms on $\mathfrak{X}$. Then there exists a short-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Namely,*

$$\mathfrak{t} = \mathfrak{t}_{\ker\mathfrak{w}} + (\mathfrak{t} - \mathfrak{t}_{\ker\mathfrak{w}}),$$

*where the first summand is $\mathfrak{w}$-absolutely continuous and the second one is $\mathfrak{w}$-singular. Furthermore, $\mathfrak{t}_{\ker\mathfrak{w}}$ is the largest element of the set*

$$\big\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\big|\; (\mathfrak{s} \le \mathfrak{t}) \;\wedge\; (\mathfrak{s} \ll_{\mathrm{ac}} \mathfrak{w})\big\}.$$

*The decomposition is unique precisely when $\mathfrak{t}_{\ker\mathfrak{w}}$ is dominated by $\mathfrak{w}$.*

A decomposition of $\mathfrak{t}$ into a $\mathfrak{w}$-almost dominated (or equivalently, $\mathfrak{w}$-closable) and $\mathfrak{w}$-singular parts is called Lebesgue-type decomposition. This is a generalization of the well-known operator decomposition of T. Ando. In order to establish the existence of such a decomposition, we need to introduce the notion of parallel sum. The parallel sum $\mathfrak{t} : \mathfrak{w}$ is determined by the formula

$$\forall x \in \mathfrak{X}: \quad (\mathfrak{t} : \mathfrak{w})[x] := \inf_{y\in\mathfrak{X}} \big\{\mathfrak{t}[x - y] + \mathfrak{w}[y]\big\}.$$

We can define also the operator $\mathbf{D}_{\mathfrak{w}} : \mathcal{F}_+(\mathfrak{X}) \to \mathcal{F}_+(\mathfrak{X})$ as follows

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} := \sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w}).$$

The form $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is the so-called almost dominated part of $\mathfrak{t}$ with respect to $\mathfrak{w}$, as the following fundamental theorem states.

**Theorem 9.2.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$. Then the decomposition*

$$\mathfrak{t} = \mathbf{D}_{\mathfrak{w}}\mathfrak{t} + (\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$$

*is a Lebesgue-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. That is, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is almost dominated by $\mathfrak{w}$, $(\mathfrak{t} - \mathbf{D}_{\mathfrak{w}}\mathfrak{t})$ is $\mathfrak{w}$-singular. Furthermore, $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is the largest element of the set*

$$\big\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\big|\; (\mathfrak{s} \le \mathfrak{t}) \;\wedge\; (\mathfrak{s} \ll_{\mathrm{ad}} \mathfrak{w})\big\}.$$

*The decomposition is unique precisely when $\mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ is dominated by $\mathfrak{w}$.*

Moreover, for the almost dominated part we have the following two formulae

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t})[x] = \inf\big\{\lim_{n\to+\infty}\mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} : (\mathfrak{t}[x_n - x_m] \to 0) \;\wedge\; (\mathfrak{w}[x_n] \to 0)\big\}$$

and

$$(\mathbf{D}_{\mathfrak{w}}\mathfrak{t})[x] = \inf\big\{\liminf_{n\to+\infty}\mathfrak{t}[x - x_n] \;\big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} :\; \mathfrak{w}[x_n] \to 0\big\}.$$

It turns out that if $\mathfrak{t}$ and $\mathfrak{w}$ are forms, then the almost dominated parts have an interesting property.

**Theorem 9.3.** *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on $\mathfrak{X}$, and consider their Lebesgue-type decompositions with respect to each other. Then the almost dominated parts are mutually almost dominated, i.e.,*

$$\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \qquad \textit{and} \qquad \mathbf{D}_{\mathfrak{t}}\mathfrak{w} \ll_{\mathrm{ad}} \mathbf{D}_{\mathfrak{w}}\mathfrak{t}.$$

## Radon-Nikodym theorems

Similarly as for measures, the regular part can be characterized in an appropriate fashion. The next result is the Radon-Nikodym theorem for forms. To see the analogy consider the following line of equalities:

$$\|\chi_F\|^2_{L^2(\mu)} = \mu(F) = \int_F f \,\mathrm{d}\nu = \int_T f \cdot \chi_F \,\mathrm{d}\nu = \|\sqrt{f} \cdot \chi_F\|^2_{L^2(\nu)}$$

**Theorem 9.4.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$. *The following statements are equivalent:*

(i) $\mathfrak{t}$ *is* $\mathfrak{w}$*-closable,*

(ii) *There is a positive selfadjoint (in general, unbounded) operator* $T$ *in* $\mathscr{H}_{\mathfrak{w}}$ *such that* $\mathfrak{X}/_{\ker\mathfrak{w}} \subseteq \operatorname{dom} T^{1/2}$ *and*

$$\forall x \in \mathfrak{X} : \quad \mathfrak{t}[x] = \|T^{1/2}(x + \ker\mathfrak{w})\|_{\mathfrak{w}}^2.$$

A weaker result, which is according to

$$(1|\chi_F)_{L^2(\mu)} = \mu(F) = \int_F f \, \mathrm{d}\nu = \int_T f \cdot \chi_F \, \mathrm{d}\nu = (f|\chi_F)_{L^2(\nu)}$$

an other possible generalization of the classical Radon-Nikodym theorem is stated as follows.

**Theorem 9.5.** *Let* $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *be forms on* $\mathfrak{X}$ *and let* $\mathfrak{t}$ *be almost dominated by* $\mathfrak{w}$. *Then for every* $y \in \mathfrak{X}$ *there exists a sequence* $(y_n)_{n\in\mathbb{N}}$ *in* $\mathfrak{X}$ *such that*

$$\forall x \in \mathfrak{X} : \qquad \mathfrak{t}(x, y) = \lim_{n\to+\infty} \mathfrak{w}(x, y_n).$$

## Extremal questions

It turns out that the Lebesgue-type decomposition is in close relation with some problems regarding the order structure of forms. The first natural question is whether the infimum (i.e., the greatest lower bound) $\mathfrak{t}\wedge\mathfrak{w}$ of $\mathfrak{t}$ and $\mathfrak{w}$ exists in $\mathcal{F}_+(\mathfrak{X})$. Recall that the infimum of $\mathfrak{t}$ and $\mathfrak{w}$ exists if there is a form denoted by $\mathfrak{t}\wedge\mathfrak{w}$, for which $\mathfrak{t}\wedge\mathfrak{w} \leq \mathfrak{t}$, $\mathfrak{t}\wedge\mathfrak{w} \leq \mathfrak{w}$, and the inequalities $\mathfrak{u} \leq \mathfrak{t}$ and $\mathfrak{u} \leq \mathfrak{w}$ imply that $\mathfrak{u} \leq \mathfrak{t}\wedge\mathfrak{w}$.

**Theorem 9.6.** *Let* $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ *be forms on* $\mathfrak{X}$. *Then the following statements are equivalent.*

(i) $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathbf{D}_{\mathfrak{w}}\mathfrak{t}$ *or* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathbf{D}_{\mathfrak{t}}\mathfrak{w}$.

(ii) $\mathbf{D}_{\mathfrak{t}}\mathfrak{w} \leq \mathfrak{t}$ *or* $\mathbf{D}_{\mathfrak{w}}\mathfrak{t} \leq \mathfrak{w}$.

(iii) *The infimum* $\mathfrak{t}\wedge\mathfrak{w}$ *exists.*

The Lebesgue-type decomposition turns up again by examining the extreme points of the convex set $[0, \mathfrak{t}]$. Here the segment $[\mathfrak{t}_1, \mathfrak{t}_2]$ for $\mathfrak{t}_1 \leq \mathfrak{t}_2$ is defined to be the convex set

$$[\mathfrak{t}_1, \mathfrak{t}_2] = \big\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \;\big|\; \mathfrak{t}_1 \leq \mathfrak{s} \leq \mathfrak{t}_2\big\}.$$

The following theorems characterize the extreme points of form segments.

**Theorem 9.7.** *Let* $\mathfrak{u}$ *and* $\mathfrak{t}$ *be forms on* $\mathfrak{X}$*, such that* $\mathfrak{u} \le \mathfrak{t}$*. The following statements are equivalent*

(*i*) $\mathfrak{u}$ *and* $\mathfrak{t} - \mathfrak{u}$ *are singular,*

(*ii*) $\mathbf{D}_{\mathfrak{u}}\mathfrak{t} = \mathfrak{u}$,

(*iii*) $\mathfrak{u}$ *is an extreme point of the convex set* $[0, \mathfrak{t}]$.

Furthermore, we have the following characterization.

**Theorem 9.8.** *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on* $\mathfrak{X}$*. Then the following statements are equivalent*

(*i*) $\mathfrak{t}$ *is an extreme point of* $[0, \mathfrak{t} + \mathfrak{w}]$,

(*ii*) $\mathrm{ex}[\mathfrak{t}, \mathfrak{t} + \mathfrak{w}] \subseteq \mathrm{ex}[0, \mathfrak{t} + \mathfrak{w}]$.

*Replacing* $\mathfrak{w}$ *with* $\mathfrak{w} - \mathfrak{t}$ *(if* $\mathfrak{t} \le \mathfrak{w}$*) we have*

$$\mathfrak{t} \in \mathrm{ex}[0, \mathfrak{w}] \quad \Leftrightarrow \quad \mathrm{ex}[\mathfrak{t}, \mathfrak{w}] \subseteq \mathrm{ex}[0, \mathfrak{w}].$$

## 9.1. Applications

In this section we carry over the previous theorems for positive definite operator functions. Szymański in [**48**] presented a general dilation theory governed by forms. We will see (after making some generalities) that the absolutely continuous part in Theorem 9.1 (and the almost dominated part in Theorem 9.2) is the largest dilatable part in some sense. Finally, we describe some order properties of kernels. Throughout this section we will use the notations of [**16**, Section 7], which is our main reference. Recall again that almost domination and strong absolute continuity (or closability) are equivalent concepts for forms.

Let $S$ be a non-empty set, and let $\mathfrak{E}$ be a complex Banach space (with topological dual $\mathfrak{E}^*$). The dual pairing of $x \in \mathfrak{E}$ and $x^* \in \mathfrak{E}^*$ is denoted by $\langle x, x^* \rangle$. Here the mapping

$$\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E}^* \to \mathbb{C}$$

is linear in its first, conjugate linear in its second variable. The Banach space of bounded linear operators from $\mathfrak{E}$ to $\mathfrak{E}^*$ will be denoted by $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$.

Let $\mathfrak{X}$ be the complex linear space of functions on $S$ with values in $\mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$ with finite support. We say that the function

$$\mathsf{K} : S \times S \to \mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$$

is a *positive definite operator function*, or shortly a *kernel* on $S$ if

$$\forall f \in \mathfrak{X} : \qquad \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) f(s) \rangle \ge 0.$$

We associate a form with $\mathsf{K}$ by setting

$$\forall f, g \in \mathfrak{X}: \qquad \mathfrak{w}_{\mathsf{K}}(f,g) := \sum_{s,t \in S} \langle f(t), \mathsf{K}(s,t) g(s) \rangle.$$

The set of kernels will be denoted by $\mathcal{K}_+(\mathfrak{X})$. If $\mathsf{K}$ and $\mathsf{L}$ are kernels, we write $\mathsf{K} \prec \mathsf{L}$ if $\mathfrak{w}_{\mathsf{K}} \leq \mathfrak{w}_{\mathsf{L}}$.

The following lemma states that the order structures of forms and of kernels are the same. This statement was proved by Hassi, Sebestyén, and de Snoo in [**16**, Lemma 7.1]). An analogous result in context of bounded positive operators can be found in [**7**, (2.2) Theorem].

We emphasize here that this is always the crucial question when we want to apply our general results.

If we take two objects, consider its induced forms, and make the decomposition theorems, it is not clear that whether the shorted or closable part is induced by an object. However, the following lemma guarantees that this is the case when the objects are positive definite operator functions.

**Lemma 9.9.** *Let $\mathsf{K} \in \mathcal{K}_+(\mathfrak{X})$ be a kernel on $S$ with associated form $\mathfrak{w}_{\mathsf{K}}$ and let $\mathfrak{w}$ be a form on $\mathfrak{X}$. Then the following statements are equivalent*

(*i*) $\mathfrak{w} \leq \mathfrak{w}_{\mathsf{K}}$,
(*ii*) $\mathfrak{w} = \mathfrak{w}_{\mathsf{L}}$ *for a unique kernel* $\mathsf{L} \prec \mathsf{K}$.

*Proof.* Implication $(ii) \Rightarrow (i)$ follows from the definitions. To prove the converse implication define for each $s \in S$ and $x \in \mathfrak{E}$ the function

$$h_{s,x} \in \mathfrak{X}; \qquad \forall u \in S: \quad h_{s,x}(u) := \delta_s(u) x$$

where $\delta_s$ is the Dirac function concentrated to $s$. Now, define $\mathsf{L}$ pointwise as follows. For each $s, t \in S$

$$\forall x, y \in \mathfrak{E}: \qquad \langle x, \mathsf{L}(s,t) y \rangle := \mathfrak{w}(h_{t,x}, h_{s,y})$$

It follows from the nonnegativity of $\mathfrak{w}[\cdot]$ that

$$\sum_{s,t \in S} \langle f(t), \mathsf{L}(s,t) f(s) \rangle$$

is nonnegative for all $f \in \mathfrak{X}$. The only thing we need is to show that $L(s,t) \in \mathbf{B}(\mathfrak{E}, \mathfrak{E}^*)$. According to the Cauchy-Schwarz inequality, we have for all $x, y \in \mathfrak{E}$ that

$$
\begin{aligned}
&|\langle x, L(s,t)y\rangle|^2 = |\mathfrak{w}(h_{t,x}, h_{s,y})|^2 \leq \mathfrak{w}[h_{t,x}] \cdot \mathfrak{w}[h_{s,y}] \leq \mathfrak{w}_{\mathsf{K}}[h_{t,x}] \cdot \mathfrak{w}_{\mathsf{K}}[h_{s,y}] \\
&\quad = \langle x, K(t,t)x\rangle \cdot \langle y, K(s,s)y\rangle \leq \|K(t,t)\|_{\mathbf{B}(\mathfrak{E},\mathfrak{E}^*)} \cdot \|K(s,s)\|_{\mathbf{B}(\mathfrak{E},\mathfrak{E}^*)} \cdot \|x\|_{\mathfrak{E}}^2 \cdot \|y\|_{\mathfrak{E}}^2.
\end{aligned}
$$

□

We emphasize here that the preceding is the key observation of this section. Most of the results gathered below are immediate consequences of this lemma, and the theorems listed in the previous sections of this overview.

Now, we can define domination, almost domination, singularity, closability, and (strong) absolute continuity of kernels via their associated forms. We say that $\mathsf{K}$ *is* $\mathsf{L}$*-almost dominated;* $\mathsf{L}$*-closable; (strongly)-*$\mathsf{L}$*-absolutely continuous* if $\mathfrak{w}_{\mathsf{K}}$ is $\mathfrak{w}_{\mathsf{L}}$-almost dominated; $\mathfrak{w}_{\mathsf{L}}$-closable; (strongly)-$\mathfrak{w}_{\mathsf{L}}$-absolutely continuous, respectively. $\mathsf{K}$ *and* $\mathsf{L}$ *are singular* if $\mathfrak{w}_{\mathsf{K}}$ and $\mathfrak{w}_{\mathsf{L}}$ are singular.

Before stating the short-type and Lebesgue-type decomposition of kernels, we mention a result of W. Szymański (reduced to our less general setting). For the details we refer the reader to [**48**, (3.5) Theorem].

**Theorem 9.10.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$ *with associated forms* $\mathfrak{w}_{\mathsf{K}}$ *and* $\mathfrak{w}_{\mathsf{L}}$*. Then*

(a) $\mathsf{K}$ *is absolutely continuous with respect to* $\mathsf{L}$ *(i.e.,* $\ker \mathfrak{w}_{\mathsf{L}} \subseteq \ker \mathfrak{w}_{\mathsf{K}}$*) if and only if there exists a Hilbert space* $\mathcal{H}$ *and a linear mapping* $T : \mathfrak{X}/_{\ker \mathfrak{w}_{\mathsf{L}}} \to \mathcal{H}$ *such that*

$$
\langle y, K(s,t)x\rangle = \big(T(h_{t,y} + \ker \mathfrak{w}_{\mathsf{L}}) \bigm| T(h_{s,x} + \ker \mathfrak{w}_{\mathsf{L}})\big)_{\mathcal{H}},
$$

(b) $\mathsf{K}$ *is strongly absolutely continuous with respect to* $\mathsf{L}$ *(i.e.,* $\mathfrak{w}_{\mathsf{K}}$ *is strongly* $\mathfrak{w}_{\mathsf{L}}$*-absolutely continuous) if and only if there exists a Hilbert space* $\mathcal{H}$ *and a closed linear mapping* $T : \mathfrak{X}/_{\ker \mathfrak{w}_{\mathsf{L}}} \to \mathcal{H}$ *such that*

$$
\langle y, K(s,t)x\rangle = \big(T(h_{t,y} + \ker \mathfrak{w}_{\mathsf{L}}) \bigm| T(h_{s,x} + \ker \mathfrak{w}_{\mathsf{L}})\big)_{\mathcal{H}}.
$$

The operator $T$ is called the *dilation* of $\mathsf{K}$ and the auxiliary space $\mathcal{H}$ is called the dilation space.

In view of the previous theorem, the following two decomposition theorems can be stated as follows. For every pair of kernels $\mathsf{K}$ and $\mathsf{L}$ there is a maximal part of $\mathsf{K}$ which has a (closed) dilation with respect to $\mathsf{L}$. These are straightforward consequences of Theorem 9.1 and Theorem 9.2.

**Theorem 9.11.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$. *Then there exists a short-type decomposition of* $\mathsf{K}$ *with respect to* $\mathsf{L}$*, i.e., the first summand is* $\mathsf{L}$*-absolutely continuous and the second one is* $\mathsf{L}$*-singular. Namely*

$$\mathsf{K} = \mathsf{K}_{\mathrm{ac},\mathsf{L}} + \mathsf{K}_{\mathrm{s},\mathsf{L}},$$

*where*

$$\sum_{s,t\in S} \langle f(t), \mathsf{K}_{\mathrm{ac},\mathsf{L}}(s,t)f(s)\rangle = \inf_{g\in\ker \mathfrak{w}_{\mathsf{L}}} \sum_{s,t\in S} \langle f(t) - g(t), \mathsf{K}(s,t)(f(s) - g(s))\rangle.$$

*The decomposition is unique precisely when* $\mathsf{K}_{\mathrm{ac},\mathsf{L}}$ *is dominated by* $\mathsf{L}$.

**Theorem 9.12.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$. *Then the decomposition*

$$\mathsf{K} = \mathbf{D}_{\mathsf{L}}\mathsf{K} + (\mathsf{K} - \mathbf{D}_{\mathsf{L}}\mathsf{K}),$$

*is a Lebesgue-type decomposition of* $\mathsf{K}$ *with respect to* $\mathsf{L}$. *That is,* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is strongly* $\mathsf{L}$*-absolutely continuous,* $(\mathsf{K}-\mathbf{D}_{\mathsf{L}}\mathsf{K})$ *is* $\mathsf{L}$*-singular. The almost dominated part* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is defined via*

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}} = \mathbf{D}_{\mathfrak{w}_{\mathsf{L}}}\mathfrak{w}_{\mathsf{K}},$$

*and hence*

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}}[f] = \inf\big\{ \lim_{n\to+\infty} \mathfrak{w}_{\mathsf{K}}[f - g_n] \;\big|\; (g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} : (\mathfrak{w}_{\mathsf{K}}[g_n - g_m] \to 0) \;\wedge\; (\mathfrak{w}_{\mathsf{L}}[g_n] \to 0)\big\}$$

*and*

$$\mathfrak{w}_{\mathbf{D}_{\mathsf{L}}\mathsf{K}}[f] = \inf\big\{ \liminf_{n\to+\infty} \mathfrak{w}_{\mathsf{K}}[f - g_n] \;\big|\; (g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} :\; \mathfrak{w}_{\mathsf{L}}[x_n] \to 0\big\}.$$

*The decomposition is unique precisely when* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is dominated by* $\mathsf{L}$.

Due to Theorem 9.5 we have the following Radon–Nikodym-type result for kernels.

**Corollary 9.13.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$ *and assume that* $\mathsf{K}$ *is almost dominated by* $\mathsf{L}$. *Then for every* $g \in \mathfrak{X}$ *there exists a sequence* $(g_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}$ *such that*

$$\forall f \in \mathfrak{X} : \quad \sum_{s,t\in S} \langle f(t), \mathsf{K}(s,t)g(s)\rangle = \lim_{n\to+\infty} \sum_{s,t\in S} \langle f(t), \mathsf{L}(s,t)g_n(s)\rangle.$$

The following statements are immediate consequences of Theorem 9.3 and Theorem 9.6.

**Corollary 9.14.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$*, then* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *is* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$*-almost dominated. And by symmetry,* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$ *is* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$*-almost dominated.*

**Corollary 9.15.** *Let* $\mathsf{K}$ *and* $\mathsf{L}$ *be kernels on* $S$*. Then the infimum* $\mathsf{K} \wedge \mathsf{L}$ *of* $\mathsf{K}$ *and* $\mathsf{L}$ *exists precisely when* $\mathbf{D}_{\mathsf{K}}\mathsf{L}$ *and* $\mathbf{D}_{\mathsf{L}}\mathsf{K}$ *are comparable.*

Finally, we have the following characterizations according to Theorem 9.7 and Theorem 9.8.

**Corollary 9.16.** *Let* $\mathsf{J}, \mathsf{K} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$*, such that* $\mathsf{J} \prec \mathsf{K}$*. The following statements are equivalent.*

$(i)$ $\mathsf{J}$ *and* $\mathsf{K} - \mathsf{J}$ *are singular.*

$(ii)$ $\mathbf{D}_{\mathsf{J}}\mathsf{K} = \mathsf{J}$.

$(iii)$ $\mathsf{J}$ *is an extreme point of the convex set* $[0, \mathsf{K}] = \big\{\mathsf{U} \in \mathcal{K}_+(\mathfrak{X}) \bigm| 0 \prec \mathsf{U} \prec \mathsf{K}\big\}$.

In view of Theorem 9.10 the previous corollary says that the extreme points of the convex set $[0, \mathsf{K}]$ are precisely those kernels that have closed dilation.

**Corollary 9.17.** *Let* $\mathsf{K}, \mathsf{L} \in \mathcal{K}_+(\mathfrak{X})$ *be kernels on* $S$*. Then the following statements are equivalent*

$(i)$ $\mathsf{K}$ *is an extreme point of* $[0, \mathsf{K} + \mathsf{L}]$.

$(ii)$ $\mathrm{ex}[\mathsf{K}, \mathsf{K} + \mathsf{L}] \subseteq \mathrm{ex}[0, \mathsf{K} + \mathsf{L}]$.

*Replacing* $\mathsf{L}$ *with* $\mathsf{L} - \mathsf{K}$ *(if* $\mathsf{K} \prec \mathsf{L}$*) we have*

$$\mathsf{K} \in \mathrm{ex}[0, \mathsf{L}] \quad \Leftrightarrow \quad \mathrm{ex}[\mathsf{K}, \mathsf{L}] \subseteq \mathrm{ex}[0, \mathsf{L}].$$

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# Summary


The Lebesgue decomposition theorem and the Radon-Nikodym theorem are cornerstones of the classical measure theory. These theorems were generalized in several settings and several ways. In 2009, Seppo Hassi, Zoltán Sebestyén, and Henk de Snoo proved a Lebesgue-type decomposition theorem for nonnegative sesquilinear forms (or shortly, for *forms*). The present dissertation contains the author's contributions to this general decomposition theory.

The first part deals with several important decomposition theorems of forms. The reason is that there are a lot of objects in analysis that induce sesquilinear forms in a very natural way. For example, bounded positive operators, finite measures on $\sigma$-algebras, positive definite kernels on Banach spaces, positive linear functionals on (Banach) $^*$-algebras, and so on.

We establish two different kind of decomposition theorems. The first one is the so-called short-type decomposition, which is a decomposition of a form into absolutely continuous and singular parts with respect to an other form. The key notion here is the so called *short of a form* to a linear subspace of the underlying vector space. This is a generalization of the classical notion of operator short defined by Krein.

The second decomposition result is the Lebesgue-type decomposition, i.e., a decomposition of a form into almost dominated (or equivalently, closable) and singular parts. This is a common generalization of the well-known Ando-decomposition of bounded positive operators, the canonical decomposition of densely defined quadratic forms proved by Simon, and the Lebesgue–Darst decomposition of finitely additive nonnegative set functions.

We also show that the regular part in the Lebesgue-type decomposition theorem can be described in an appropriate fashion (similarly as for measures). Namely, regularity is characterized by means of densely defined closable operators.

We close our investigations with two extremal problems in which the regular part plays an important role. The first one is the characterization of whether the infimum of two given forms exists (with respect to the ordering). The second problem is to describe the extreme points of form segments.

## Magyar nyelvű összefoglalás

A Lebesgue felbontási tétel és a Radon–Nikodym tétel a klasszikus mérték- és integrálelmélet két sarokköve. Ezen tételeknek számtalan általánosítása ismert a matematika különböző területein. Seppo Hassi, Sebestyén Zoltán, és Henk de Snoo nevéhez fűződik a Lebesgue felbontási tétel általánosítása nemnegatív sesquilineáris formákra (röviden: *formákra*). Ezt az általánosítást, illetve ennek következményeit és alkalmazásait mutatja be a disszertáció.

A matematika számos területén találkozhatunk olyan tételekkel, amelyek bizonyos objektumok reguláris, illetve szinguláris részekre való felbonthatóságát garantálják. Természetes módon adódik a kérdés, hogy megadható-e ezen tételeknek egységes tárgyalása, azaz megfogalmazhatóak-e olyan tételek, amelyeknek ezek az analóg eredmények mind speciális esetei. Az első részben ezt a kérdést válaszoljuk meg.

Két különböző típusú felbontási tételt igazolunk. Az első az úgynevezett *short-típusú felbontás*, amely Mark Grigorievich Krein híres eredményének általánosítása. Bevezetjük formák lineáris altérre való shortjának fogalmát, majd az általános eredményt felhasználva igazolunk egy-egy analóg felbontási tételt operátorokra, halmazfüggvényekre, illetve reprezentálható funkcionálokra.

A másik eredmény az úgynevezett *Lebesgue-típusú felbontási tétel*, amely közös általánosítása számtalan nevezetes tételnek. Többek között Tsuyoshi Ando pozitív operátorokra vonatkozó tételének, Barry Simon sűrűn definiált formák felbontására vonatkozó tételének, és a végesen additív mértékek felbonthatóságát garantáló Lebesgue-Darst tételnek.

A Radon-Nikodym tétel mintájára megmutatjuk, hogy a Lebesgue-típusú felbontásban szereplő reguláris rész mindig egyfajta kanonikus alakba írható. Nevezetesen, bebizonyítjuk, hogy az abszolút folytonosság jellemezhető sűrűn definiált lezárható operátorok segítségével.

A disszertáció második felében olyan extremális kérdéseket válaszolunk meg, amelyekben a Lebesgue-típusú felbontás kulcsszerepet játszik. Az egyik probléma annak eldöntése, hogy két adott formának létezik-e infimuma a kvadratikus alakok pontonkénti rendezésére nézve, a másik pedig egy úgynevezett szegmens extremális pontjainak meghatározása. Bebizonyítjuk továbbá, hogy az ilyen formaszegmensek extremális pontjai hálót alkotnak.

## [38]ADATLAP
### a doktori értekezés nyilvánosságra hozatalához

**I. A doktori értekezés adatai**

A szerző neve: Titkos Tamás
MTMT-azonosító: 10029504
A doktori értekezés címe és alcíme: Decomposition theory of forms
DOI-azonosító[39]: 10.15476/ELTE.2016.004
A doktori iskola neve: ELTE Matematika Doktori Iskola
A doktori iskolán belüli doktori program neve: Elméleti matematika doktori program
A témavezető neve és tudományos fokozata: Dr. Sebestyén Zoltán, az MTA doktora
A témavezető munkahelye: ELTE Alkalmazott Analízis és Számításmatematikai tanszék.

**II. Nyilatkozatok**

1. A doktori értekezés szerzőjeként[40]
a) hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom a Természettudományi Kar Tudományszervezési és Egyetemközi Kapcsolatok Osztályának ügyintézőjét Bíró Évát, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.

**2.** A doktori értekezés szerzőjeként kijelentem, hogy
a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;
b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.

**3.** A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.

Kelt: 2016. 01. 11.

Titkos Tamás
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a doktori értekezés szerzőjének aláírása

[38]Beiktatta az Egyetemi Doktori Szabályzat módosításárólszóló CXXXIX/2014. (VI. 30.) Szen. sz. határozat. Hatályos: 2014. VII.1. napjától.
[39]A kari hivatal ügyintézője tölti ki.
[40]A megfelelő szöveg aláhúzandó.
[41]A doktori értekezés benyújtásával egyidejűleg be kell adni a tudományági doktori tanácshoz a szabadalmi, illetőleg oltalmi bejelentést tanúsító okiratot és a nyilvánosságra hozatal elhalasztása iránti kérelmet.
[42]A doktori értekezés benyújtásával egyidejűleg be kell nyújtani a minősített adatravonatkozó közokiratot.
[43]A doktori értekezés benyújtásával egyidejűleg be kell nyújtani a mű kiadásáról szóló kiadói szerződést.