Harnack Inequalities:

from Harmonic Analysis through Poincare Conjecture to

Matrix Determinant

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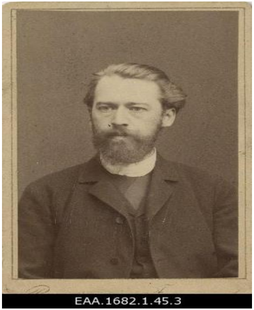
Who is Harnack?

Carl Gustav Axel Harnack (1851-1888),

born in Tartu, Estonia, died in Dresden, Germany.

PhD in 1875 from Felix Klein (...... Klein bottle ......)



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The classical Harnack inequality in Potential Theory

Axel Harnack's book, pp.158, 1887, *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*, Leipzig: V. G. Teubner

In English: *Foundations of the theory of the logarithmic potential and single-valued potential functions in the plane*

in which an inequality of a positive harmonic function was introduced, later generalized to solutions of elliptic or parabolic partial differential equations. Perelman's solution (2003) of the Poincaré conjecture uses a version of the Harnack inequality, found by R. Hamilton (1993), for the Ricci flow.

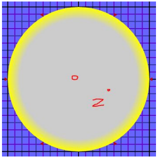
The Harnack inequality in Functional Analysis

Let $f(z)$ be a positive harmonic function on $|z| < 1$ in the plane. Then

$$f(0)\frac{1-|z|}{1+|z|} \le f(z) \le f(0)\frac{1+|z|}{1-|z|}$$

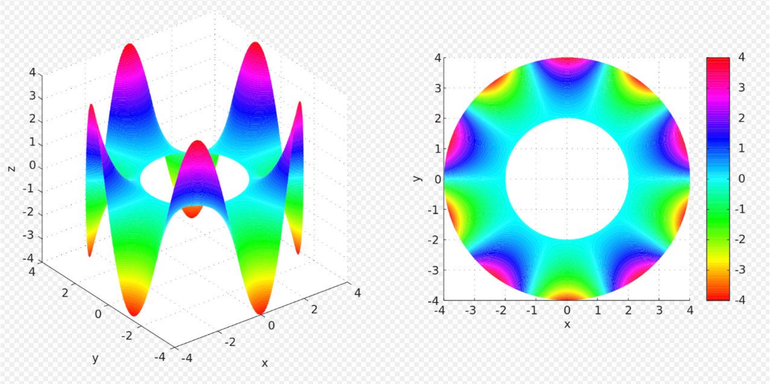
Recall that a harmonic function is a twice continuously differentiable function on an open set (in $\mathbb{R}^n$ or $\mathbb{C}^n$) satisfying the Laplace equation

$$\nabla^2 f = 0$$



Example: Take a point z on $|z| = 0.5$. Then $\frac{1}{3}f(0) \le f(z) \le 3f(0)$.

A harmonic function defined on an annulus



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The Harnack inequality in Functional Analysis

Set $z = re^{i\theta}$, where r is the modulus, θ is the argument of z. Then

$$f(0)\frac{1-r}{1+r} \leq f(z) \leq f(0)\frac{1+r}{1-r}$$

If we scale and translate to an arbitrary disk of radius R with center z_0, then we have for $f(z)$, a positive harmonic function on $|z| < R$,

$$f(z_0)\frac{R-r}{R+r} \leq f(z) \leq f(z_0)\frac{R+r}{R-r}, \quad |z - z_0| < r < R$$

The Harnack inequality in higher dimension

Denote the open ball (in usual topology) centered at x_0 with radius R in the n-dimensional space $\mathbb{R}^n$ by

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

Consider

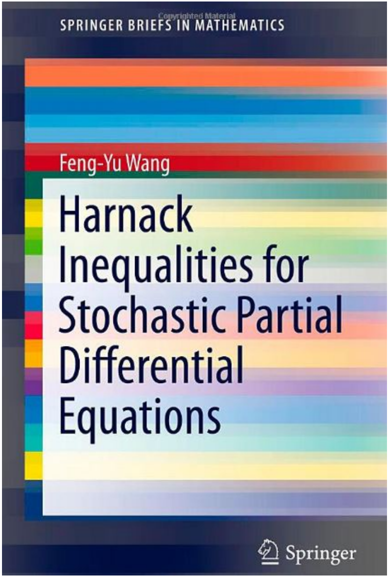
$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r < R\}$$

Then for any z on the surface of $B_r(x_0)$, i.e., $|z - x_0| = r$, we have

$$f(z_0)\frac{1-\rho}{(1+\rho)^{n-1}} \le f(z) \le f(z_0)\frac{1+\rho}{(1-\rho)^{n-1}}, \quad \rho = \frac{r}{R}$$

(Extensions for general domains; proof by Poisson's formula $\int_{sphere}$)

The Harnack-type inequalities in PDEs



When Pf is measurable for $f \in \mathscr{B}_b(E)$, i.e., the family $\{\mu_x : x \in E\}$ is a transition probability measure, then P is a Markov operator on $\mathscr{B}_b(E)$, i.e., P is contractive and positivity-preserving, and $P1 = 1$.

When a family of stochastic processes $\{(X^x(t))_{t\ge 0} : x \in E\}$ measurable in x is involved, for instance $(X^x(t))_{t\ge 0}$ solves a stochastic differential equation with initial data x, let $\mu_x(t)$ be the distribution of $X^x(t)$. Then we define as in (1.1) a family of Markov operators $(P_t)_{t\ge 0}$. If the family of processes is Markovian, then $(P_t)_{t\ge 0}$ is a semigroup, i.e., $P_{t+s} = P_t P_s$ for $t, s \ge 0$.

In the remainder of the section, we will use coupling by change of measure to establish Harnack-type inequalities and derivative formulas of P.

1.1.1 Harnack Inequalities and Bismut Derivative Formulas

For a Markov operator P on $\mathscr{B}_b(E)$, the Harnack-type inequality considered in this book is of type

$$\Phi(Pf(x)) \le (P\Phi(f)(y))e^{\Psi(x,y)}, \quad x, y \in E, f \in \mathscr{B}_b^+(E), \tag{1.2}$$

where Φ is a nonnegative convex function on $[0, \infty)$ and Ψ is a nonnegative function on E^2. By Jensen's inequality, we may always take $\Psi(x, x) = 0$.

In this book we will mainly consider the following two typical choices of Φ:

(1) **(Harnack inequality with power)** Let $\Phi(r) = r^p$ for some $p > 1$. Then (1.2) reduces to

$$(Pf(x))^p \le (Pf^p(y))e^{\Psi(x,y)}, \quad x, y \in E, f \in \mathscr{B}_b^+(E). \tag{1.3}$$

This inequality, called the Harnack inequality with power p, was first found in [50] for diffusion semigroups with curvature bounded from below.

(2) **(Log-Harnack inequality)** Let $\Phi(r) = e^r$. In this case we may use $\log f$ to replace f, so that (1.2) is equivalent to

$$P\log f(x) \le \log Pf(y) + \Psi(x,y), \quad x, y \in E, f \in \mathscr{B}_b^+(E), f \ge 1. \tag{1.4}$$

Since the inequality does not change by multiplying f by a positive constant, it holds for all uniformly positive functions f. Moreover, using $f + \varepsilon$ to replace f and letting $\varepsilon \downarrow 0$, we may replace the condition $f \ge 1$ by $f \ge 0$. This inequality, called the log-Harnack inequality, was introduced in [56] for reflecting diffusion processes on manifolds with boundary and in [43] for semilinear SPDEs (stochastic partial differential equations) with multiplicative noise.

Theorem 1.1.1 (Harnack inequalities) *If there is a coupling by change of measure (X, Y) for μ_x and μ_y, with changed probability $d\mathbb{Q} := R d\mathbb{P}$, such that $X = Y$, $\mathbb{Q}$-a.s., then*

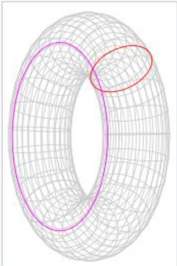
Poincaré Conjecture: A $1M Millennium Prize Problem

Poincaré conjecture (1904-2003):

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

For compact 2-dimensional surfaces without boundary, if every loop can be continuously tightened to a point, then the surface is topologically homeomorphic to a 2-sphere (usually just called a sphere). The Poincaré conjecture, proved by Grigori Perelman, asserts that the same is true for 3-dimensional spaces.

Courtesy: en.wikipedia.org



Neither of the two colored loops on this torus can be continuously tightened to a point. A torus is not homeomorphic to a sphere.

Perelman's Harnack inequality in his solution to Poincaré Conjecture

Perelman resolved the Poincaré conjecture in 2003...

Perelman's solution uses a version of the Harnack inequality for the Ricci flow, found by R. Hamilton (1993), which is an extension of a result of P. Li and S.-T. Yau (1986).

Li-Yau -> Hamilton -> Perelman

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A MATRIX HARNACK ESTIMATE FOR THE HEAT EQUATION

RICHARD S. HAMILTON

In the important paper[LY] by Peter Li and S.-T. Yau, they show how the classical Harnack principle for the heat equation on a manifold can be derived from a differential inequality. In particular, they show that for any positive solution $f > 0$ of the heat equation

$$\frac{\partial f}{\partial t} = \Delta f$$

on a compact Riemannian manifold of dimension m solving the equation for $t > 0$, if the manifold has weakly positive Ricci curvature $R_{ij} \geq 0$ then for any vector field V on M

$$\frac{\partial f}{\partial t} + \frac{1}{2t} f + 2Df(V) + f|V|^2 \geq 0;$$

a similar result holds with an error term if the Ricci curvature is bounded below. The quadratic version in V given here is equivalent to the more complicated formula in their paper by choosing the optimal V. We shall show in this paper that the Harnack estimate of Li and Yau is the trace of a full matrix inequality.

Main Theorem. *If M is a compact Riemannian manifold and $f > 0$ is a positive solution to the heat equation on M*

$$\frac{\partial f}{\partial t} = \Delta f$$

for $t > 0$, then for any vector field V_i on M we have

$$D_i D_j f + \frac{1}{2t} f g_{ij} + D_i f \cdot V_j + D_j f \cdot V_i + f V_i V_j \geq 0$$

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9 Differential Harnack inequality for solutions of the conjugate heat equation

9.1 Proposition. *Let $g_{ij}(t)$ be a solution to the Ricci flow $(g_{ij})_t = -2R_{ij}, 0 \leq t \leq T$, and let $u = (4\pi(T-t))^{-\frac{n}{2}} e^{-f}$ satisfy the conjugate heat equation $\square^* u = -u_t - \triangle u + Ru = 0$. Then $v = [(T-t)(2\triangle f - |\nabla f|^2 + R) + f - n]u$ satisfies*

$$\square^* v = -2(T-t)|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2(T-t)} g_{ij}|^2 \qquad (9.1)$$

Proof. Routine computation.

Clearly, this proposition immediately implies the monotonicity formula (3.4); its advantage over (3.4) shows up when one has to work locally.

9.2 Corollary. *Under the same assumptions, on a closed manifold M, or whenever the application of the maximum principle can be justified, min v/u is nondecreasing in t.*

9.3 Corollary. *Under the same assumptions, if u tends to a δ-function as $t \to T$, then $v \leq 0$ for all $t < T$.*

Proof. If h satisfies the ordinary heat equation $h_t = \triangle h$ with respect to the evolving metric $g_{ij}(t)$, then we have $\frac{d}{dt}\int hu = 0$ and $\frac{d}{dt}\int hv \geq 0$. Thus we only need to check that for everywhere positive h the limit of $\int hv$ as $t \to T$ is nonpositive. But it is easy to see, that this limit is in fact zero.

9.4 Corollary. *Under assumptions of the previous corollary, for any smooth curve $\gamma(t)$ in M holds*

$$-\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot\gamma(t)|^2) - \frac{1}{2(T-t)} f(\gamma(t), t) \qquad (9.2)$$

Proof. From the evolution equation $f_t = -\triangle f + |\nabla f|^2 - R + \frac{n}{2(T-t)}$ and $v \leq 0$ we get $f_t + \frac{1}{2}R - \frac{1}{2}|\nabla f|^2 - \frac{f}{2(T-t)} \geq 0$. On the other hand, $-\frac{d}{dt} f(\gamma(t), t) = -f_t - < \nabla f, \dot\gamma(t) > \leq -f_t + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\dot\gamma|^2$. Summing these two inequalities, we get (9.2).

Ky Fan's Harnack type inequality for Operators

Theorem (Fan 1988)

Let F be an operator-valued analytic function on the open unit disk $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that for any $z \in \mathfrak{D}$, $F(z)$ is an operator on a complex Hilbert space $\mathbb{H}$ with $\mathrm{Re}\, F(z) > 0$ and $F(0) = I$. Then

$$\frac{1-|z|}{1+|z|} I \leq \mathrm{Re}\, F(z) \leq \frac{1+|z|}{1-|z|} I$$

Proof. For each x in $\mathbb{H}$ with $\|x\| = 1$, define the complex-valued $f_x(z) = \langle F(z)x, x \rangle$. Use the classical Harnack inequality. □

Note: There is an analog for $\mathrm{Im}\, F(z)$.

Tung's Harnack type inequality for matrices

Theorem (Tung 1964)

Let Z be an $n \times n$ complex matrix with singular values r_k that satisfy $0 \le r_k < 1$, $k = 1, 2, \ldots, n$ (i.e., Z is a strict contraction). Let Z^* denote the conjugate transpose of Z and I be the $n \times n$ identity matrix. Then for any $n \times n$ unitary matrix U

$$\prod_{k=1}^{n} \frac{1 - r_k}{1 + r_k} \le \frac{\det(I - Z^* Z)}{|\det(I - UZ)|^2} \le \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k} \qquad (1)$$

Proof. Consider $f(U) = \det((I - ZU^*)(I - UZ^*))$ for fixed strict contraction Z and use the method of Lagrange multipliers. □

Marcus and Hua

Marcus (1965) gave another proof and pointed out that Tung's inequality is equivalent to

$$\prod_{k=1}^{n}(1-r_k) \leq |\det(I-A)| \leq \prod_{k=1}^{n}(1+r_k) \tag{2}$$

for any $n \times n$ matrix A with the same singular values as the contractive matrix Z.

L.-K. Hua (1965) gave a proof of (2) using an inequality he had previously obtained in 1955: For strict contractions A, B,

$$\begin{pmatrix} (I-A^*A)^{-1} & (I-B^*A)^{-1} \\ (I-A^*B)^{-1} & (I-B^*B)^{-1} \end{pmatrix} \geq 0$$

Remarks

(i). In the book by Marshall, Olkin and Arnold, Tung's theorem is cited in which the condition that A be contractive is missing.

(ii). Inequalities (1) and (2) are not equivalent for general matrices. The right-hand side inequality in (2) is true for all $n \times n$ matrices A; that is,

$$|\det(I - A)| \leq \prod_{k=1}^{n} (1 + r_k)$$

Restatement of Tung's Theorem and equality case

Theorem (Left inequality)

Let Z be an $n \times n$ positive semidefinite matrix with eigenvalues $r_1, r_2, \ldots, r_n$. Let U be an $n \times n$ unitary matrix such that $I - UZ$ is nonsingular. Then

$$\prod_{k=1}^{n} \frac{|1 - r_k|}{1 + r_k} \le \frac{|\det(I - Z^2)|}{|\det(I - UZ)|^2} \tag{3}$$

with equality if and only if Z has an eigenvalue 1 or UZ has eigenvalues $-r_1, -r_2, \ldots, -r_n$. If both Z and $I - Z$ are nonsingular, the strict inequality holds for $U \neq -I$.

Restatement of Tung's Theorem and equality case

Theorem (Right inequality)

Let Z be an $n \times n$ positive semidefinite matrix with eigenvalues $r_1, r_2, \ldots, r_n$. Let U be an $n \times n$ unitary matrix such that $I - UZ$ is nonsingular. If $0 \le r_k < 1$, $k = 1, 2, \ldots, n$, then

$$\frac{\det(I - Z^2)}{|\det(I - UZ)|^2} \le \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k} \tag{4}$$

with equality if and only if UZ has eigenvalues $r_1, r_2, \ldots, r_n$. If Z is nonsingular, then the strict inequality in (4) holds if $U \ne I$.

Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be real vectors whose components are arranged in nonincreasing order:

$$x_1 \geq x_2 \geq \cdots \geq x_n, \quad y_1 \geq y_2 \geq \cdots \geq y_n$$

If

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n$$

we say that x is weakly majorizaed by y, written $x \prec_w y$. If the last inequality becomes equality, then x is majorized by y, denoted $x \prec y$.

$$x \prec_{\mathrm{w}} y, \quad x \prec y$$

Replacing $\sum$ by $\prod$, we have log-majorization:

$$x \prec_{\mathrm{wlog}} y, \quad x \prec_{\log} y$$

Lemma

Lemma

Let $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *be nonnegative vectors and assume that* y *is not a permutation of* x *(i.e., the multisets* $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_n\}$ *are not equal). Denote* $\tilde{z} = (1 + z_1, 1 + z_2, \ldots, 1 + z_n)$. *We have:*

$$\text{If} \quad x \prec_{\log} y, \quad \text{then} \quad \tilde{x} \prec_{\text{wlog}} \tilde{y}$$

and

$$\prod_{k=1}^{n} (1 + x_k) < \prod_{k=1}^{n} (1 + y_k). \tag{5}$$

Proof. $f(t) = \ln(1 + e^t)$ is strictly increasing & convex on $(0, \infty)$. □

Lemma

If all $x_i, y_i \in [0,1)$, x is not a permutation of y, and $x \prec_{\log} y$, then

$$\prod_{k=1}^{n}(1-x_k) > \prod_{k=1}^{n}(1-y_k). \tag{6}$$

Proof. $-\ln(1-e^t)$ is strictly increasing and convex on $(-\infty, 0)$. □

Proof of the equality case of the theorems

Proof of the Theorems. Use Majorization Theory.

Only show the equality cases. For (3), if Z has a singular (eigen-) value 1, then both sides vanish. If UZ has eigenvalues $-r_1, -r_2, \ldots, -r_n$, then $\det(I - UZ) = \prod_{k=1}^{n}(1 + r_k)$. Equality is readily seen. Conversely, suppose equality occurs in (3). We further assume that no r_k $(k = 1, 2, \ldots, n)$ equals 1. Since $|\det(I - Z^2)| = \prod_{k=1}^{n} |1 - r_k|(1 + r_k)$, we have

$$|\det(I - UZ)| = \prod_{k=1}^{n}(1 + r_k). \tag{7}$$

Moreover, by Weyl majorization inequality

$$|\lambda(UZ)| \prec_{\log} \sigma(UZ) = \sigma(Z) = \lambda(Z),$$

where $\lambda(X)$ and $\sigma(X)$ denote the vectors of the eigenvalues and singular values of matrix X, respectively.

With $\lambda_k(X)$ denoting the eigenvalues of the $n \times n$ matrix X, $k = 1, 2, \dots, n$, by the lemma, we have

$$0 < |\det(I - UZ)| = \prod_{k=1}^{n} |1 - \lambda_k(UZ)| \le \prod_{k=1}^{n} (1 + |\lambda_k(UZ)|) \le \prod_{k=1}^{n} (1 + r_k).$$

Thus, (7) yields $|1 - \lambda_k(UZ)| = 1 + |\lambda_k(UZ)|$ for all k, which implies $\lambda_k(UZ) \le 0$ for $k = 1, 2, \dots, n$, i.e., all eigenvalues of $-UZ$ are nonnegative. If $|\lambda(UZ)| = \lambda(-UZ)$ is not a permutation of $\lambda(Z)$, then, by strict inequality (5), we have $\prod_{k=1}^{n}(1 + |\lambda_k(UZ)|) < \prod_{k=1}^{n}(1 + \lambda_k(Z)) = \prod_{k=1}^{n}(1 + r_k)$, a contradiction to (7). It follows that UZ has the eigenvalues $-r_1, -r_2, \dots, -r_n$.

For the equality in (4), it occurs if and only if $\prod_{k=1}^{n}(1-r_k) = |\det(I-UZ)|$. Note that $|\lambda(UZ)| \prec_{\log} \sigma(Z)$ and

$$\prod_{k=1}^{n} |1-\lambda_k(UZ)| \geq \prod_{k=1}^{n}(1-|\lambda_k(UZ)|) \geq \prod_{k=1}^{n}(1-\sigma_k(Z)) = \prod_{k=1}^{n}(1-r_k). \quad (8)$$

The first equality in (8) occurs if and only if all $\lambda_k(UZ)$ are in $[0,1)$; the second equality occurs if and only if $\lambda(UZ)$ is a permutation of $\sigma(Z)$, i.e., $\mathrm{Spec}(UZ) = \mathrm{Spec}(Z)$.

Now assume that Z is nonsingular and suppose that equality holds in (4). Then UZ has eigenvalues $r_1, r_2, \ldots, r_n$. Moreover, the singular values of UZ are $r_1, r_2, \ldots, r_n$. Let $P = UZ$. Then the eigenvalues of P are just the singular values of P. So P is positive definite. It follows that $U = PZ^{-1}$ has only positive eigenvalues. Since U is unitary, U has to be the identity matrix. The case for (3) is similar. $\square$

Extension of Tung's Theorem on Harnack inequality

Theorem (Lin and Z. 2017)

Let Z_i, $i = 1, 2, \ldots, m$, be $n \times n$ positive semidefinite matrices. Suppose that the eigenvalues of Z_i are r_{ik} satisfying $0 \le r_{ik} < 1$, $k = 1, 2, \ldots, n$. Then for any $n \times n$ unitary matrix U and positive scalars w_i, $i = 1, 2, \ldots, m$, $\sum_{i=1}^m w_i = 1$, we have

$$\prod_{k=1}^{n}\prod_{i=1}^{m}\left(\frac{1-r_{ik}}{1+r_{ik}}\right)^{w_i} \le \frac{\det(I-(\sum_{i=1}^m w_i Z_i)^2)}{|\det(I-U\sum_{i=1}^m w_i Z_i)|^2} \le \prod_{k=1}^{n}\prod_{i=1}^{m}\left(\frac{1+r_{ik}}{1-r_{ik}}\right)^{w_i}. \quad (9)$$

Equality on the left-hand side occurs if and only if all Z_i are equal to Z, say, and Z has an eigenvalue 1 or Spec(UZ)=Spec($-Z$) (in which $U = -I$ if Z is nonsingular); Equality on the right-hand side occurs if and only if all Z_i are equal to Z, say, and Spec(UZ)=Spec(Z) (in which $U = I$ if Z is nonsingular).

Outline of the Proof

Fact: for $n \times n$ Hermitian A and B, if $\lambda_k(A+B) = \lambda_k(A) + \lambda_k(B)$ for all k, then A and B are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonal in the same order.

Fan's majorization $\lambda(H+S) \prec \lambda(H) + \lambda(S)$ for $n \times n$ Hermitian matrices H and S and Lewent's inequality for $x_i \in [0,1)$,

$$\frac{1+\sum_{i=1}^n \alpha_i x_i}{1-\sum_{i=1}^n \alpha_i x_i} \leq \prod_{i=1}^n \left(\frac{1+x_i}{1-x_i}\right)^{\alpha_i},$$

where $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$. Equality holds iff $x_1 = x_2 = \cdots = x_n$.

Let $r_{ik}^{\downarrow}$ be the kth largest eigenvalue of Z_i, $i = 1, 2, \ldots, m$, and s_k be the kth largest eigenvalue of $W := \sum_{i=1}^m w_i Z_i$, $k = 1, 2, \ldots, n$.

$$\lambda(W) \prec \sum_{i=1}^m w_i \lambda(Z_i), \text{ i.e., } \sum_{k=1}^{\ell} s_k \leq \sum_{k=1}^{\ell} \sum_{i=1}^m w_i r_{ik}^{\downarrow}, \qquad \ell = 1, 2, \ldots, n.$$

(Note that the components of $\lambda(\cdot)$ are in nonincreasing order.)

Now the convexity and the monotonicity of the function $f(t) = \ln \frac{1+t}{1-t}$, $0 \leq t < 1$, imply

$$\sum_{k=1}^{n} \ln \frac{1+s_k}{1-s_k} \leq \sum_{k=1}^{n} \ln \frac{1+\sum_{i=1}^{m} w_i r_{ik}^{\downarrow}}{1-\sum_{i=1}^{m} w_i r_{ik}^{\downarrow}},$$

where equality holds if and only if $s_k = \sum_{i=1}^{m} w_i r_{ik}^{\downarrow}$ for all k; that is, $\lambda(W) = \sum_{i=1}^{m} w_i \lambda(Z_i)$. It follows that all Z_i are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonals in the same order (nonincreasing, say).

Applying the exponential function to both sides and using Lewent's inequality yield

$$\begin{aligned}\prod_{k=1}^{n}\frac{1+s_k}{1-s_k} &\le \prod_{k=1}^{n}\frac{1+\sum_{i=1}^{m} w_i r_{ik}^{\downarrow}}{1-\sum_{i=1}^{m} w_i r_{ik}^{\downarrow}} \\ &\le \prod_{k=1}^{n}\prod_{i=1}^{m}\left(\frac{1+r_{ik}^{\downarrow}}{1-r_{ik}^{\downarrow}}\right)^{w_i} \\ &= \prod_{k=1}^{n}\prod_{i=1}^{m}\left(\frac{1+r_{ik}}{1-r_{ik}}\right)^{w_i},\end{aligned} \tag{10}$$

in which equality occurs in the second inequality if and only if $r_{1k} = r_{2k} = \cdots = r_{mk}$ for $k = 1, 2, \ldots, n$. Thus both equalities in (10) hold if and only if $Z_1 = Z_2 = \cdots = Z_m$.

By (4), we have

$$\frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U\sum_{i=1}^m w_i Z_i)|^2} \le \prod_{k=1}^n \left(\frac{1+s_k}{1-s_k}\right). \tag{11}$$

Combining (10) and (11) gives the second inequality of (9).

Note that the inequalities in (10) reverse by taking reciprocals, which implies

$$\prod_{k=1}^n \frac{1-s_k}{1+s_k} \ge \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1-r_{ik}}{1+r_{ik}}\right)^{w_i}. \tag{12}$$

Then by (3), we have

$$\frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U\sum_{i=1}^m w_i Z_i)|^2} \ge \prod_{k=1}^n \left(\frac{1-s_k}{1+s_k}\right). \tag{13}$$

Combining (12) and (13) yields the first inequality of (9).

If either equality holds in (9), then all Z_i are equal to Z, say. The conclusions are immediate from Theorem 3. □

Corollary

Corollary

Let Z_i, $i = 1, 2, \ldots, m$, be $n \times n$ complex matrices with singular values r_{ik} such that $0 \le r_{ik} < 1$, $k = 1, 2, \ldots, n$. Then for any $n \times n$ unitary matrix U

$$\frac{\det(I - \sum_{i=1}^{m} w_i Z_i^* Z_i)}{|\det(I - U \sum_{i=1}^{m} w_i |Z_i|)|^2} \le \prod_{k=1}^{n} \prod_{i=1}^{m} \left(\frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i},$$

where $w_i > 0$, $i = 1, 2, \ldots, m$, such that $\sum_{i=1}^{m} w_i = 1$. Equality occurs if and only if all Z_i have the same absolute value, say Z, and Spec(UZ)=Spec(Z) (in which $U = I$ if Z is nonsingular).

Proof. With $(\sum_{i=1}^{m} w_i |Z_i|)^2 \le \sum_{i=1}^{m} w_i |Z_i|^2$. □

An example

In view of the inequality in the corollary, it is tempting to have the lower bound inequality

$$\prod_{k=1}^{n}\prod_{i=1}^{m}\left(\frac{1-r_{ik}}{1+r_{ik}}\right)^{w_i} \le \frac{\det(I-\sum_{i=1}^{m} w_i Z_i^* Z_i)}{|\det(I-U\sum_{i=1}^{m} w_i|Z_i|)|^2}.$$

However, this is not true. Set $m = n = 2$, $w_1 = w_2 = 1/2$ and take

$$Z_1 = \begin{pmatrix} 0.34 & -0.15 \\ -0.15 & 0.07 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0.02 & -0.01 \\ -0.01 & 0.01 \end{pmatrix},$$

$$U = \begin{pmatrix} -0.60 & 0.80 \\ 0.80 & 0.60 \end{pmatrix}.$$

One may check that the left hand side is 0.6281, while the right hand side is 0.6250.

A question

Replace Z with U^*Z and leave the singular values unchanged in the Theorem. Giving an upper bound and lower bound, in terms of the singular values of individual matrices, for the quantity $\frac{\det(I-\sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I-\sum_{i=1}^m w_i Z_i)|^2}$, where Z_i, $i=1,2,\ldots,m$, are general contractive matrices. We would guess

$$\prod_{k=1}^n \prod_{i=1}^m \left(\frac{1-r_{ik}}{1+r_{ik}}\right)^{w_i} \le \frac{\det(I-\sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I-\sum_{i=1}^m w_i Z_i)|^2} \le \prod_{k=1}^n \prod_{i=1}^m \left(\frac{1+r_{ik}}{1-r_{ik}}\right)^{w_i}. \quad (14)$$

The first inequality in (14) is untrue in general as it is disproved by substituting Z_1 and Z_2 in (14) with $U|Z_1|$ and $U|Z_2|$, respectively, in the previous example. However, simulation seems to support the second inequality which is unconfirmed yet.

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