STATISTICAL INFERENCES OF $R_{s,k} = \Pr(X_{k-s+1:k} > Y)$ FOR GENERAL CLASS OF EXPONENTIATED INVERTED EXPONENTIAL DISTRIBUTION WITH PROGRESSIVELY TYPE-II CENSORED SAMPLES WITH UNIFORMLY DISTRUBUTED RANDOM REMOVALS

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ABSTRACT

The problem of statistical inference of the reliability parameter $\Pr(X_{k-s+1:k} > Y)$ of an s-out-of-$k:G$ system with strength components $X_1, X_2, ..., X_k$ subjected to a common stress Y when X and Y are independent two-parameter general class of exponentiated inverted exponential (GCEIE) progressively type-II right censored data with uniformly random removal random variables, are discussed. We use p-value as a basis for hypothesis testing. There are no exact or approximate inferential procedures for reliability of a multicomponent stress-strength model from the GCEIE based on the progressively type-II right censored data with random or fixed removals available in the literature. Simulation studies and real-world data analyses are given to illustrate the proposed procedures. The size of the test, adjusted and unadjusted power of the test, coverage probability and expected confidence lengths of the confidence interval, and biases of the estimator are also discussed.

DEDICATION

This thesis is dedicated to my mother and father, who taught me that to always persevere in whatever would come my way, no matter what it is, takes patience. It is also dedicated to Jimmy, who has undeniably been the most influencial person in keeping me motivated through countless endeavors over the last 7 years.

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LIST OF ABBREVIATIONS

GCEIED, general class of exponentiated inverted exponential distributions

GCEGIED, general class of exponentiated generalized inverted exponential distributions

$\mathcal{GCEIE}$,general class of exponentiated inverted exponentially distributed

$\mathcal{EIR}$, exponentiated inverted Rayleigh distribution

pmf, probability mass function

MLE, maximum likelihood estimate

CI, confidence interval

UMVUE, uniformly minimum variance unbiased estimator

BBCACI, bootstrap bias-corrected and accelerated confidence interval

MCMC, Markov Chain Monte Carlo

LINEX, linear exponential loss function

HPD, highest posterior density

HPDI, highest posterior density interval

BCI, Bayesian credible interval

DWR, Department of Water Resources

SSE, error sum of squares

BTCI, bootstrap-t confidence interval

C-Method, classical method

GV-Method, generalized variable method

B-Method, Bayesian method

MSE, mean squared error

ER, estimated risk

CP, coverage probability

UP, unadjusted power of a statistical test

AP, adjusted power of a statistical test

UMVU, uniformly minimum variance unbiased

ANOVA, analysis of variance

ANORE, analysis of reciprocals

ANCOVA, analysis of covariance

ANOFRE, analysis of frequency

MANOVA, multivariate analysis of variance

MANCOVA, multivariate analysis of covariance

RAM, read access memory

ln, natural logarithm

GVM, generalized variable method

pdf, probability density function

iid, independent and identically distributed

OOL, offered optical network unit load

LIST OF SYMBOLS

X, random strength

Y, random stress

α, shape parameter of exponentiated inverted exponential distribution

λ, scale parameter of exponentiated inverted exponential distribution

$R_{s:k}$, reliability of a multi-component s-out-of-k: G system, where at least out of s of the k components work (or are good)

$X_{r:n}$, rth order statistic of a simple random sample of size n for the random variable X

$f_X(x)$, probability density function of the random variable X when $X = x$

$F_X(x)$, cumulative distribution function of the random variable X when $X \leq x$

F_{ν_1,ν_2}, Fisher-Snedecore distribution (or simple F distribution) with numerator degrees of freedom $\nu_1$and denominator degrees of freedom ν_2

$\Pr(X \in D)$, Probability of X that belongs to the domain D

$\Pr(X > Y)$, Probability that strength X is greater than stress Y

~, distributed as

$\sum$, single summation

$\prod$, single product

$l(\beta)$, log-likelihood function of β

$\{X\}_{i=1,2,\ldots,I;j=1,2,\ldots,J}$, $I \times J$ matrix

$D(\alpha, \beta, \delta)$, statistical distribution D with a location parameter α, a shape parameter β, and a scale parameter γ

$\int_a^b ydx$, single definite integral of the function y with respect to x computed from a to b

$\begin{pmatrix} n \\ r \end{pmatrix}$, n choose r ; or n combination r, where r items have been chosen from n items without regard to the order

$\mathrm{P}(A|B)$, conditional probability of the event A given that event B has already occurred

$\widehat{\alpha}$, estimator of the parameter α

A, estimator of the parameter α

$\widehat{\alpha}_{\mathrm{obs}}$, observed (or realized) value (or estimate) of the parameter α

$\prod\prod$, double product

$\sum\sum\sum\sum$, quintuple summation

H_0, null hypothesis

H_a, alternative hypothesis

$\mathbf{X}$, vector of simple random sample from the strength

$\mathbf{Y}$, vector of simple random sample from the stress

$\xrightarrow{D}$, convergence in distribution

$\frac{\partial y}{\partial x}$, dabba y over dabba x; partial derivative of the function y with respect to x

$\iint_D y dx$, double integral of the function y with respect to x computed in the domain D

$\frac{\partial^2 y}{\partial x^2}$, dabba squared y over dabba squared x; second partial derivative of the function y with respect to x

$E(X)$, expected value of the random variable X

χ^2_ν, chi-squared distribution with ν degrees of freedom

$\widehat{\alpha}_{\text{obs}}$, observed (or realized) value (or estimate) of the parameter α

Φ, distribution function of the standard normal distribution

$\text{CI}^{\gamma}_{\eta}$, a $\gamma\%$ confidence interval for the parameter η

$\widehat{\alpha}^*$, bootstrap estimator of the parameter α

z_γ, γth quantile of the standard normal distribution

$H(x)$, Cumulative distribution at the point $X = x$ of the random variable X

$B(\alpha, \beta)$, beta function with parameters α and β

${}_2F_1(\alpha, \beta; \gamma, z)$, hypergeometric function

$\mathcal{E}$, exponentially distributed

$\mathbb{I}_{m\times n}$, $m \times n$ matrix; matrix with m rows and n columns

a^b, a to the power of b; a has been raised to the power of b

$\text{Re}(\xi)$, real part of the complex number ξ

$U(\theta|\mathbf{x})$, posterior distribution of θ given the random sample $\mathbf{x}$

$\pi(\theta)$, prior distribution of θ

$\min\limits_{1 \le i \le n} (X_i)$, first order statistic of the random variable X

$\mathcal{U}_D$, discrete uniformly distributed

$||$, absolute value

$\times$, multiplication

$+$, addition

$-$, subtraction

$/$, division

$>$, greater than; more than

$<$, smaller than; less than

$\geq$, greater than or equals; at least

$\leq$, less than or equals; at most

$=$, equal sign

$\sqrt{}$, square root

$\sum\sum$, double summation

$n \to \infty$, n approaches infinity

$\approx$, approximately

$\mathcal{G}$, gamma distributed

CHAPTER I

INTRODUCTION

The background of exact statistical methods

Exact statistics has a history dated back to Fisher's era when Fisher's exact test (1922) based on the sampling distribution that is conditional on the marginals played a vital role in making inferences of parameters of interest. When statistical inferences are performed, it provides more reliable, accurate, non-misleading results, outperforming procedures based on classical asymptotic and approximate statistical inference methods. The most prominent and major characteristic of exact methods is that statistical inferences are mainly based on exact probability statements that are valid for any sample size. While in exact tests all assumptions of the distribution of the test statistic have to be met, in approximate tests the approximation may be made as close as desired by making the sample size big enough which will result in a significance test that will have a false rejection rate always equal to the significance level of the test. When the sample size is small, the asymptotic and other approximate results may lead to unreliable and misleading conclusions. There are two branches in exact statistics as in approximate or asymptotic statistics: exact parametric procedures where statistical inferences are performed under any parametric distributions and exact nonparametric procedures where any distributional assumptions are not made. Prompted by a conversation he had with Miss Muriel Bristol about whether the tea or milk was added first to her cup, Sir Ronald Aylmer Fisher (1954), The Father of Modern Statistics, for the first time in statistical history devised a comment from her to come up with the idea of " Exact Test" that is to be used in the analysis of contingency tables where sample sizes are small. When the cell counts are small–specifically, if more than twenty percent of the cells, when marginal totals are fixed, have an expected count that is less than five–the χ^2 distribution may not a suitable distributional candidate of the Pearson C^2 or Likelihood Ratio G^2 statistics for testing independence of row and column variables. Such a situation is easily remedied by Fisher's exact test.

Inspired by the Fisher's original treatment of hypothesis testing statistics (1954), Weerahandi searched for an extreme region, a unbiased subset of sample space formed by minimal sufficient statistics having observed sample points on its boundary, to generalize the existing p-values to come up with exact solutions for different problems arise in hypothesis testing. For exact tests, readers are referred to Fisher (1922), Weerhandi (1995, 2005), Metha and Patel (1997), and many others.

As extensions (not alternatives) to conventional inference methods, generalized p-values (Tsui and Weerahandi 1989) and generalized confidence intervals (Weerahandi 1993) based on exact probability statements are introduced to remedy and overcome drawbacks of other conventional exact and approximate inference methods. Conventional methods alone do not always provide exact solutions to:

1. Problems involving nuisance parameters such as that of comparing the means of two exponential distributions and making inferences of the second moments of a random

variable whose underlying distribution is normal,

2. Problems of making inferences of complicated functions of parameters of underlying distributions such as Offered Optical Network Unit Load (OOL) in Data Transmission,

3. Problems of making inferences in the face of small samples, especially that are found in biomedical researches.

Practitioners often resort to asymptotic results in search of approximate solutions in the face of all the above mentioned problems. This newly developed promising approach, generalized variable method, provides exact solutions for such drastic, difficult, intrigue problems.

The generalized p-value and confidence interval have been widely applied to variety of practical settings where standard and conventional solutions do not exist for confidence interval estimation and hypothesis testing: Weerahandi (1995, 2004), Weerahandi and Berger (1999), Gamage and Weerahandi (1998), Ananda and Weerahandi (1997), Ananda (1995, 1998, and 1999), Gunasekera and Ananda (2009), Tian and Wu (2007), Krishnamoorthy and Lu (2003), and Zhou and Mathew (1994).

Exponentiated inverted family of distributions

Two-parameter gamma and two-parameter Weibull are the most popular distributions for analyzing any lifetime data. Gamma has a long history and it has several desirable properties, see Johnson, Kotz, and Balakrishnan (1994) for the different properties of the two-parameter gamma distribution. It has wide variety of applications in different fields other than being a lifetime distribution. (see, Alexander 1962, Jackson 1963, Klinken 1961, and Masuyama and Kuroiwa 1952). The two parameters of a gamma distribution represent the scale and shape of the distribution, and because of them, the distribution has quite a bit of flexibility to analyze any positive real data. It has increasing as well as decreasing failure rates depending on the shape parameter, which gives an extra edge over exponential distribution, which has only constant failure rate. Since sum of independent and identically distributed (*iid*) gamma random variables has a gamma distribution, it has also a nice physical interpretation. If a system has one component and n-spare parts, and if the component and each spare parts have *iid* gamma lifetime distributions, then the lifetime distribution of the system also follows a gamma distribution. Another interesting property of the family of gamma distributions is that it has likelihood ratio ordering, with respect to the shape parameter, when the scale parameter remains constant. It naturally implies the ordering in hazard rate as well as in distribution. But one major disadvantage of the gamma distribution is that the distribution function or survival function cannot be expressed in a closed form if the shape parameter is not an integer. Since it is in terms of an incomplete gamma function, one needs to obtain the distribution function, survival function or the failure rate by numerical integration. This makes gamma distribution little bit unpopular compared to the Weibull distribution, which has a nice distribution function, survival function and hazard function. Weibull distribution was originally proposed by Weibull (1939), a Swedish physicist, and he used it to represent the distribution of the breaking strength of materials. Weibull distribution also has the scale and shape parameters. In recent years the Weibull distribution becoming very popular to analyze lifetime data mainly because in presence of censoring it is much easier to handle, at least numerically, compared to a gamma distribution. It also has increasing and decreasing failure rates depending on the shape parameter. Physically it represents a series system, because the minimum of i.i.d. Weibull distributions also follows a Weibull distribution. Several applications of the Weibull distribution can be found in Plait (1962) and Johnson (1968) although some of the negative points of the Weibull distribution can be found in Gorski (1968). One of the disadvantages can be pointed out that the asymptotic convergence to normality for the distribution of the maximum likelihood estimators is very slow (Bain, 1976). Therefore most of the asymptotic inferences (for example asymptotic unbiasedness or asymptotic confidence interval) may not be very accurate unless the sample size is very large. Some ramifications of this problem can be found in Bain (1976). It also does not enjoy any ordering properties like gamma distribution.

In this paper we consider a two-parameter exponentiated inverted exponential distribution and study some of its properties. The two parameters of an exponentiated inverted exponential distribution represent the shape and the scale parameter like a gamma distribution or a Weibull distribution. It also has the increasing or decreasing failure rate depending of the shape parameter. The density function varies significantly depending of the shape parameter. It is observed that it wide variety of properties which are quite similar to those of a gamma distribution but it has an explicit expression of the distribution function or the survival function like a Weibull distribution. It has also likelihood ratio ordering with respect to the shape parameter, when the scale parameter is kept constant., and for fixed scale and shape parameters there is a stochastic ordering among distributions.

Reliability of a multicomponent system

We treat the problem of testing, and estimating and constructing the confidence intervals of, the *reliability parameter* $R_{s,k} = \Pr(\text{at least } s \text{ of the } (X_1, X_2, ..., X_k) \text{ exceed } Y) = \Pr(X_{k-s+1:k} > Y)$ in the *multicomponent stress-strength model,* developed by Bhattacharyya and Johnson (1974), when k statistically independent and identical strength components $X_1, X_2, ..., X_k$ of a system that have a common probability density function (*pdf*) $f_X(x)$ and $X_{k-s+1:k}$ is the $(k-s+1)$th order statistics of $(X_1, X_2, ..., X_k)$a common stress Y experienced by the system that has a *pdf* $f_Y(y)$. The system functions when s $(1 \leq s \leq k)$ or more of the components simultaneously survive. This system is referred to as an s-out-of-$k : G$ (or s-out-of-$k : F$) system because a k-component system works (or is *good*) if and only if at least s of the k components work (or are *good*), and the system is referred to as s-out-of-$k : F$ because the k-component system *fails* if and only if at least s of the k components *fail.* Based on these two definitions, a s-out-of-$k : G$ system is equivalent to an $(k-s+1)$-out-of-$k : F$ system. In the reliability context, the multicomponent stress-strength model can be described as an assessment of reliability of an s-out-of-$k : G$ system. Its practical application range from communication and industrial systems to logistic and military systems. For example, in suspension bridges, the deck is supported by a series of vertical cables hung from the towers. Suppose a suspension bridge consisting of k number of vertical cable pairs. The bridge will only survive if minimum s number of vertical cable through the deck are not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion, etc. As another example, a V-8 engine of an automobile it may be possible to drive the car if only four cylinders are firing. However, if less than four cylinders fire, then the automobile cannot be driven. Thus, the functioning of the engine may be represented by a 4-out-of-8 : G system. Other examples include an electrical power station containing eight generating units produces the right amount of electricity only if at least 6 units are working; the demand of the electricity of a district is fulfilled only if 6-out-of-8 wind roses are operating at all times; a communication system for a navy can be successful only if 6 transmitters out of 10 are operational to cover a district; a semi-trailer pulled by a truck can be driven safely as long as 6-out-of-8 tires are in good conditions. For an extensive reviews of s-out-of-k and related systems, see Kuo and Zuo (2003).

Reliability of a multicomponent system based on general class of exponentiated inverted exponential distributions

We consider the case where f_X and f_Y are from the *general class of exponentiated inverted exponential distributions* (GCEIEDs) (Mudholkar et. al 1995), which is the counter part of the *general class of exponentiated generalized inverted experiential distributions* (GCEGIEDs) (Cordeiro et. al 2013). The former is derived by raising the cumulative distribution function (*cdf*) of an arbitrary parental (or underlying or baseline) *general class of inverted exponential distributions* to an additional non-negative parameter, say α; which is solely responsible for the skewness, kurtosis, and the tails of the resulting GCEIEDs; and the latter is derived by raising the complement of the survival function of the underlaying *general class of inverted exponential distributions* that has been raised to a shape parameter, say α, to an additional non-negative parameter, say β. Also, instead of using the complete data, we observe progressively type-II censored samples with uniformly distributed random removals for stress and strength from GCEIEDs.

In this research, we consider the reliability parameter $R_{s,k} = \Pr(\text{at least } s \text{ of the } (X_1, X_2, ..., X_k) \text{ exceed } Y)$ or $R_{s,k} = \Pr(X_{k-s+1:k} > Y)$, where $X_{k-s+1:k}$ is $(k-s+1)$th order statistic of the remaining $(k-s+1)$ strength components X_s through X_k, is more than or equal to Y. That is, the highest order statistic $X_{k-s+1:k}$ of the $(k-s+1)$ strength components X_s through X_k must at least more than or equal to Y for the system to function. For instance, in a 13-out-of-18 : G system, there are $(k-s+1 = 18-13+1 = 6)$ strength components, that is, $X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{18}$. When they are ordered, that is $X_{1:18}, X_{2:18}, X_{3:18}, X_{4:18}, X_{5:18}, X_{6:18}$,its highest ordered value,that is $X_{6:18}$, must be greater than or equal to Y for the survival of the system, that is, $R_{13,18} = \Pr(\text{at least 13 of the } (X_1, X_2, ..., X_{18}) \text{ exceed } Y) = \Pr(X_{6:18} > Y)$.

Suppose $X_1, X_2, ..., X_k$ is a simple random sample from the *general class of exponentiated inverted exponential distribution* with a shape parameter $\alpha_1$and a scale parameter λ, For brevity, we shall also say that $X_j \sim \mathcal{GCEIE}\,(\alpha_1, \lambda), j = 1, 2, ..., k$, with its common survival function *(sf)* $S_{X_j}(x_j) = [(G(1/x_j)]^{\alpha_1}$,where $G(1/x_j) = 1 - \exp\{\lambda Q(1/x_j)\}$ is the *cdf* of the underlying general class of inverted exponential distribution whose *pdf* $f_{X_j}(x_j) = \alpha_1 \lambda Q^{'}(1/x_j) \exp\{-\lambda Q(1/x_j)\}[1 - \exp\{\lambda Q(1/x_j)\}]^{\alpha_1 - 1}$ and *cdf* $F_{X_j}(x_j) = 1 - [1 - \exp\{\lambda Q(1/x_j)\}]^{\alpha_1}$;$x_j > 0, \alpha_1, \lambda > 0$; for $j = 1, 2, ..., k$,and prime ($\prime$) being the first derivative with respect to *(w.r.t.)* x_i. Also, suppose Y be a random variable that is distributed as $\mathcal{GCEIE}\,(\alpha_2, \lambda)$ with its *pdf* $f_Y(y) =\alpha_2 \lambda Q^{'}(1/y)$ $\exp\{-\lambda Q(1/y)\}[1 - \exp\{\lambda Q(1/y)\}]^{\alpha_2 - 1}$ and *cdf* $F_Y(y) = 1 - [1 - \exp\{\lambda Q(1/y)\}]^{\alpha_2}$; $y > 0, \alpha_2, \lambda > 0$. This class (or family) of distributions includes *exponentiated inverted exponential, exponentiated inverted Rayleigh, and exponentiated inverted Pareto distributions* when $Q(1/z) = 1/z$, $Q(1/z) = 1/z^2$, and $Q(1/z) = \ln(1 + 1/z)$, respectively. Mudholkar et al. (1995) introduced the exponentiated Weibull distribution and since then, a number of authors have proposed and generalized many standard distributions based on the exponentiated distributions; to name few: Lemonte and Cordeiro (2011), Ghitany et al. (2014), Silva et al. (2010), and Gupta and Kundu (2001), Krishna and Kumar (2013), and references therin. The reliability in a multi-component stress-strength model, based on $X_{ij} \sim \mathcal{GCEIE}\,(\alpha_1, \lambda)$ and $Y \sim \mathcal{GCEIE}\,(\alpha_2, \lambda)$, is then given by

$$
\begin{aligned}
R_{s,k} &= \Pr(\text{at least } s \text{ of the } (X_1, X_2, ..., X_k) \text{ exceed } Y), \\
&= \Pr(X_{k-s+1:k} > Y), \\
&= \sum_{i=s}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \int_{-\infty}^{\infty} (1 - F_X(y))^{\,i} (F_X(y))^{k-i} dF_Y(y), \\
&= \sum_{i=s}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \alpha_2 \int_0^1 u^{\alpha_1 i + \alpha_2 - 1} (1 - u^{\alpha_1})^{k-i} du, \ \text{where } u = 1 - e^{-\lambda Q(1/y)} \\
&= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} k-i \\ j \end{pmatrix} (-1)^j \alpha_2 \int_0^1 u^{\alpha_1 (i+j) + \alpha_2 - 1} du, \\
&= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} k-i \\ j \end{pmatrix} \frac{(-1)^j \alpha_2}{\alpha_1 (i+j) + \alpha_2} \qquad (1.1)
\end{aligned}
$$

Exponetiated inverted exponentially distributed progressively Type-II right censored data with unifromly distributed random removals

Let X denote strength of a component that is statistically distributed with $\mathcal{GCEIE}\;(\alpha, \lambda)$ whose density is given by $f_X(x) = \alpha\lambda Q^{'}(1/x) \exp\{-\lambda Q(1/x)\}\times$ $[1 - \exp\{\lambda Q(1/x)]^{\alpha - 1}$. Consider that $X_{1:m:n} \leq X_{2:m:n} \leq\cdots\leq X_{m:m:n}$ is the corresponding progressively type II right censored sample, with censoring scheme $\mathbf{R} = \mathbf{r} = (r_1, r_2, ..., r_m)$; where m denote the number of failures observed before termination from n items that are on test, and $r_1, r_2, ..., r_m$ denote the corresponding numbers of units randomly removed (withdrawn) from the test. Furthermore, let $x_{1:m:n} \leq x_{2:m:n} \leq \cdots \leq x_{m:m:n}$ be the observed ordered lifetimes. Let r_i denote the number of units removed at the time of the ith failure, $0 \leq r_i \leq n - m - \sum_{j=1}^{i-1} r_j, i = 2, 3, ..., m-1$, with $0 \leq r_1 \leq n - m$ and $r_m = n - m - \sum_{j=1}^{m-1} r_j$, where r_i's are non-pre-specified integers and m are pre-specified integers. Note that if $r_1, r_2, ..., r_{m-1} = 0$, so that $r_m = n - m$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $r_1 = r_2 = ... = r_m = 0$, so that $m = n$, progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case).

Since the joint density of $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ is given by

$$\prod_{i=1}^{m} A_i f(x_{i:m:n})[1 - \int_0^{x_i} f(x_{i:m:n}) dx_{i:m:n}]^{r_i}, \text{ where } A_i = n - \sum_{j=1}^{i-1} (r_j + 1) \tag{1.2}$$

So, the conditional likelihood function is given by

$$\begin{aligned} L(\alpha|R &= r) = (\alpha\lambda)^m \prod_{i=1}^{m} A_i Q^{'}(1/x_i) \exp\{-\lambda Q(1/x_{i:m:n})\} \times \\ & [1 - \exp\{\lambda Q(1/x_{i:m:n})]^{\alpha(1+r_i)\,-\,1}, \end{aligned} \tag{1.3}$$

where A_i is as defined in (1.2).

Now, suppose that the number of units removed at each failure time $R_i (i = 1, 2, m-1)$ follows a discrete uniform distribution; for brevity, we shall also say $R_i \sim \mathcal{U}_D(0, n-m-\sum_{j=1}^{i-1} r_j)$; with probability mass function (pmf)

$$\begin{aligned} P(R_i &= r_i | R_{i-1} = r_{i-1}, R_{i-2} = r_{i-2}, ..., R_1 = r_1) = \frac{1}{n-m-\sum_{j=1}^{i-1} r_j + 1}, \\ & i = 2, 3, ..., m-1, \end{aligned} \tag{1.4}$$

and

$$P(R_1 = r_1) = \frac{1}{n-m+1}.$$

Suppose further that $R_i (i = 1, 2, ..., m-1)$ is independent of $x_{i:m:n}$, then the unconditional likelihood function can be expressed as

$$L(\alpha) = L(\alpha | R = r) P(R = r),$$

where $P(R = r) = \prod_{i=1}^{m-1} P(R_i = r_i | R_{i-1} = r_{i-1}, R_{i-2} = r_{i-2}, ..., R_1 = r_1)$.

It is evident that $P(R = r)$ does not depend on the parameters λ and α, and hence the MLE of θ can be obtained by the conditional likelihood function given in (1.3) directly.

Assuming that λ is given (or known), the maximum likelihood estimate of α can be derived by solving the equation:

$$\frac{d}{d\alpha}\ln L(\alpha|R=r)=\frac{m}{\lambda}-\sum_{i=1}^{m}(1+r_i)\ln[1-\exp\{-\lambda Q(1/x_{i:m:n})\}]=0.$$

Hence, we show that the MLE $\widehat{\alpha}$ of α is given by

$$\widehat{\alpha}=\frac{m}{\sum\limits_{i=1}^{m}(1+r_i)\ln[1-\exp\{-\lambda Q(1/x_{i:m:n})\}]}. \tag{1.5}$$

Now, let $Y_{i:m:n}=(1+r_i)\ln[1-\exp\{-\lambda Q(1/x_{i:m:n})\}], i=1,2,...,m$. It is easy to show that $Y_{1:m:n}\leq Y_{2:m:n}\leq\cdots\leq Y_{m:m:n}$ is a progressively type II censored sample from the exponential distribution with mean $(1/\alpha)$. For a fixed set of $\mathbf{R}=\mathbf{r}=(r_1,r_2,...,r_m)$, let us consider the following scaled (generalized) spacings

$$\begin{aligned}
W_1 &= nY_{1:m:n}\\
W_2 &= (n-r_1-1)(Y_{2:m:n}-Y_{1:m:n})\\
&\;\;\vdots
\end{aligned}$$

$$W_i = (n - \sum_{j=1}^{i-1} r_j - (i-1))(Y_{i:m:n} - Y_{i-1:m:n})$$

$$.$$

$$.$$

$$.$$

$$W_m = (n - \sum_{j=1}^{m-1} r_j - (m-1))(Y_{m:m:n} - Y_{m-1:m:n})$$

Balaksrihnan and Aggarwala (2000) proved that the progressively type II right censored spacings $W_1, W_2, ..., W_m$ are all independent and identically distributed as exponential with the mean $(1/\theta)$, that is, $W_i \sim \mathcal{E}(1/\theta) = \mathcal{G}(1, 1/\theta)$, where $\mathcal{E}(\beta)$ is an exponential distribution with a mean (or scale parameter) β, and $\mathcal{G}(\gamma, \delta)$ is a gamma distribution with a shape parameter γ and a scale parameter δ. Then, $W = \sum_{i=1}^{m} W_i = (1 + r_i) \ln[1 - \exp\{-\lambda Q(1/x_{i:m:n})\}] \sim \mathcal{G}(m, 1/\alpha)$. Since we can write the denominator of $\widehat{\theta}$ as the sum of m independent generalized spacings, we can find that, conditionally on a fixed set of $\mathbf{R} = \mathbf{r} = (r_1, r_2, ..., r_m)$, $U = 2\alpha W = 2m\alpha/\widehat{\alpha}$ has a chi-square distribution with $2m$ degrees of freedom., that is, $U|\mathbf{R} \sim \chi^2_{2m}$. In addition, because conditional distribution of U is independent of $\mathbf{R} = \mathbf{r}$, it must follow that the marginal distribution of U is also a chi-square distribution with $2m$ degrees of freedom, that is, $U \sim \chi^2_{2m}$.

A number of authors have proposed and developed various inferential techniques for the reliability in multicomponent stress-strength system using various underlying distributions for complete data as well as censored data; see Hanagal (1999), Eryilmaz (2010), Rao, et al. (2015). For a comprehensive discussion on different stress-strength models, along with more theories and examples, the reader is referred to the monograph of Kotz et al. (2003). In these studies, maximum likelihood estimator (MLE), moment estimator, and asymptotic confidence interval were obtained, but the generalized variable method due to Tusi and Weerahandi (1989) was not taken into consideration. On the other hand , inferences for the reliability in multicomponent stress-strength system using data with fixed removal as well as data with random removals have not been discussed in the literature. In addition, Bayesian- and generalized-variable-method based inferences for the data with fixed removals as well as the data with random removals have not been discussed in th literature.

The main purpose of this thesis is to develop various approaches to obtain confidence interval for, and to perform hypothesis testing of, the reliability parameter $R_{s,k}$, where the strength and stress variables are independent and belong to the family of exponentiated inverted exponential distributions. Toward this, we mainly develop methods based on the concept of classical procedures, generalized variable procedures as well as Bayesian procedures. Maximum likelihood estimation is one of the most popular methods for estimating the parameters of continuous distributions because of its attractive properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality. Under *The classical method* Section, we discuss the MLEs of the parameters of the $\mathcal{GCEIE}$ distribution and their asymptotic properties to derive the MLE of the reliability function $R_{s,k}$. In the next section, *The Bayesian method,* we deal with the problem of estimating the parameters α_1 and α_2, and the reliability function $R_{s,k}$ under various loss functions. The prior distribution for the parameters of the model has been taken as a natural conjugate prior. The loss functions and Bayes estimates for a parameter δ are under those loss functions are as follows: under the squared error (SE) loss function $L_{SE}(\delta,\widehat{\delta})=(\delta-\widehat{\delta})^2$, weighted squared error (WSE) loss function $L_{WSE}(\delta,\widehat{\delta})=w(\delta)(\delta-\widehat{\delta})^2$, linear (L) loss function $L_L(\delta,\widehat{\delta})=k_0(\delta-\widehat{\delta})$ if $\delta\geq\widehat{\delta}$ and $k_1(\widehat{\delta}-\delta)$ if $\delta<\widehat{\delta}$, absolute error (AE) loss function $L_{AE}(\delta,\widehat{\delta})=\left|\delta-\widehat{\delta}\right|$, linear exponential (LINEX) loss function $L_{LINEX}(\delta,\widehat{\delta})=k\left\{\exp\left[c\left(\delta-\widehat{\delta}\right)\right]-c(\delta-\widehat{\delta})-1\right\}$, percentage (P) loss function $L_P(\delta,\widehat{\delta})=(\delta-\widehat{\delta})^2/\delta$, and 0-1 (ZO) loss function $L_{ZO}(\delta,\widehat{\delta})=0$ if $\left|\delta-\widehat{\delta}\right|<c$ and 1 if $\left|\delta-\widehat{\delta}\right|\geq c$, Bayes estimates of δ are, respectively, $\widehat{\delta}_B^{SE}=E_{\pi(\delta|\mathbf{x})}(\delta)$, $\widehat{\delta}_B^{WSE}=E_{\pi(\delta|\mathbf{x}}\left[\delta w(\delta)\right]/E_{\pi(\delta|\mathbf{x})}\left[w(\delta)\right]$, $\widehat{\delta}_B^{L}=[k_0/(k_0+k_1)]$th-Fractile of the $\pi(\delta|\mathbf{x})$, $\widehat{\delta}_B^{AE}=Median_{\pi(\delta|\mathbf{x})}(\delta)$, $\widehat{\delta}_B^{LINEX}=(-1/c)\ln\{E_{\pi(\delta|\mathbf{x})}[\exp(-c\delta)]\}$, and $\widehat{\delta}_B^{ZO}=Mode_{\pi(\delta|\mathbf{x})}(\delta)$, where k and c are shape and scale parameters of the LINEX loss function, respectively and $\delta-\widehat{\delta}$ denotes the scalar estimation error in using $\widehat{\delta}$ to estimate δ. Note that In Bayesian statistics, a maximum a posteriori probability (MAP) estimate is a mode of the posterior distribution. The MAP can be used to obtain a point estimate of an unobserved quantity on the basis of empirical data. It is closely related to Fisher's method of maximum likelihood (ML) estimation, but employs an augmented optimization objective which incorporates a prior distribution over the quantity one wants to estimate. MAP estimation can therefore be seen as a regularization of ML estimation. In addition as c goes to 0 in the ZO loss function, the Bayes estimator approaches the MAP estimator, provided that the distribution of the parameter is unimodal. But generally a MAP estimator is not

a Bayes estimator unless parameter is discrete. Also note that, generally, the sign and magnitude of c in LINEX loss function reflect the direction and degree of asymmetry. This has been introduced by Varian (1975) and further properties of this loss function have been investigated by Zellner (1986). For small values of c (near to zero), the LINEX loss function is almost the same as the SE loss function, and for the choice of negative or positive values of c, the LINEX loss function gives more weight to overestimation or underestimation (for details, see Zellner 1986).

In Bayesian approach, we need to integrate over the posterior distribution and the problem is that the integrals are usually impossible to evaluate analytically. Markov chain Monte Carlo (MCMC) technique is a Monte Carlo integration method which draws samples from the target posterior distribution. MCMC methodology provided a convenient and efficient way to sample from complex, high-dimensional statistical distributions. The one of the main objective of this research is to estimate the two unknown parameters of the $\mathcal{GCEIE}$, that is, α_1 and α_2. We use the maximum likelihood and Bayes methods to derive such estimates. The estimators are obtained by using the data of type II censoring with random removals. Also the asymptotic confidence intervals for the parameters are also derived from the Fisher Information matrix. It is observed that the Bayes estimators can not be expressed in explicit forms and they can be obtained by two dimensional numerical integrations only. We use the idea of Lindley to compute the approximate Bayes estimators of the unknown parameters and it is observed that the approximation works quite well with the general class of exponentiated inverted exponential distributions. We compute the approximate Bayes estimators under the assumption of independent gamma priors of the unknown parameters and compare them with the MLEs by Monte Carlo simulations. We also propose Markov Chain Monte Carlo (MCMC) techniques to generate samples from the posterior distributions and in turn computing the Bayes estimators. The posterior density functions match quite well with the histograms of the asymptotic confidence intervals of the samples obtained by MCMC methods.

Although the classical and Bayesian frameworks of inferences are well-established and have been in the statistical arena for a long period of time, the generalized variable method and its affiliated generalized p-value were recently introduced by Tsui and Weerahandi (1989), and generalized confidence interval (CI) and generalized estimators by Weerahandi (1993, 2012) presenting them as extensions of–rather than alternatives to–classical methods of statistical evaluation. The concepts of generalized CI and generalized p-value have been widely applied to a wide variety of practical settings such as regression, Analysis of Variance (ANOVA), Analysis of Reciprocals (ANORE), Analysis of Covariance (ANCOVA), Analysis of Frequency (ANOFRE), Multivariate Analysis of Variance (MANOVA), Multivariate Analysis of Covariance (MANCOVA), mixed models, and growth curves where standard methods failed to produce satisfactory results obliging practitioners to settle for asymptotic results and approximate solutions. For example, see Weerahandi (1995, 2004), Krishnamoorthy et al. (2007), Gunasekera (2015, 2016 a,b), and Gunasekera and Ananda (2015). For a recipe of constructing generalized pivotal quantities, see Iyer and Patterson (2002). Moreover, in an effort to build a very robust discussion on the advantages and better performances of these procedures over the existing statistical procedures, reliability in the multi-component stress-strength model, in the presence of the randomly removed Type-II censored data, is derived and their inferences are also performed in the classical and Bayesian frameworks.

This theisis is organized as follows. In CHAPTER II, classical procedures for $R_{s,k}$ are reviewed. In CHAPTER III, the generalized variable method is reviewed, and a test based on the generalized test variable and a point and an interval estimate based on the generalized pivotal quantity for the $R_{s,k}$ is presented. In CHAPTER IV, Bayesian procedures, under SE and LINEX loss functions, are derived for the reliability parameter in the multicomponent stress-strength model. In CHAPTER V, simulation results on bias, coverage probability, mean confidence length, type I error control, unadjusted and adjusted power are presented. Concluding remarks are summarized in CHAPTER VI, and it is followed by the list of References.

CHAPTER II

THE CLASSICAL METHOD

Maximum likelihood estimator of $R_{s,k}$

Let $X_1, X_2, ..., X_k$ denote strength components that are statistically distributed with $\mathcal{GCEIE}$ (α_1, λ) whose *pdf* is given by $f_{X_j}(x_j) = \alpha_1 \lambda Q^{'}(1/x_j) \exp\{-\lambda Q(1/x_j)\}[1 - \exp\{\lambda Q(1/x_j)\}]^{\alpha_1 - 1}$; $\alpha_1, \lambda > 0$ with its *cdf* $F_{X_j}(x_j) = 1 - [1 - \exp\{\lambda Q(1/x_j)\}]^{\alpha_1}$;$x_j > 0, \alpha_1, \lambda > 0$; for $j = 1, 2, ..., k$. Consider that $X_{1j:n:N} \leq X_{2j:n:N} \leq \cdots \leq X_{nj:n:N}$, $j = 1, 2, ..., k$ is the corresponding progressively type II right censored sample, with censoring scheme $\mathbf{R}_j = \mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$; where n denote the number of failures observed before termination from N items that are on test, and $r_{1j}, r_{2j}, ..., r_{nj}$ denote the corresponding numbers of units randomly removed (withdrawn) from the jth test, where $j = 1, 2, ..., k$. Furthermore, let $x_{1j:n:N} \leq x_{2j:n:N} \leq \cdots \leq x_{nj:n:N}$, $j = 1, 2, ..., k$ be the observed ordered lifetimes. Let r_{ij} denote the number of strength components removed at the time of the ith failure of the jth strength component, $0 \leq r_{ij} \leq N - n - \sum_{l=1}^{i-1} r_{lj}, i = 2, 3, ..., n-1$; $j = 1, 2, ..., k$ with $0 \leq r_{1j} \leq N - n$ and $r_{nj} = N - n - \sum_{l=1}^{n-1} r_{lj}$, where r_{ij}'s are non-pre-specified integers and n are pre-specified integers and $j = 1, 2, ..., k$. Note that if $r_{1j}, r_{2j}, ..., r_{n-1,j} = 0$, so that $r_{nj} = N - n$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $r_{1j} = r_{2j} = ... = r_{nj} = 0$, so that $n = N$, the progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case). Similarly, let

$Y_1, Y_2, ..., Y_M$ denote stress of a component that is statistically distributed with $\mathcal{GCEIE}$ (α_2, λ) whose *pdf* is given by $f_Y(y) = \alpha_2 \lambda Q^{'}(1/y) \exp\{-\lambda Q(1/y)\}[1 - \exp\{\lambda Q(1/y)\}]^{\alpha_2 - 1}; \alpha_2, \lambda > 0$ with its *cdf* $F_Y(y) = 1 - [1 - \exp\{\lambda Q(1/y)\}]^{\alpha_2}\,; y > 0, \alpha_2, \lambda > 0$. Consider that $Y_{1:m:M} \leq Y_{2:m:M} \leq \cdots \leq Y_{m:m:M}$ is the corresponding progressively type II right censored sample, with censoring scheme $\mathbf{R}' = \mathbf{r}' = (r'_1, r'_2, ..., r'_m)$; where m denote the number of failures observed before termination from M items that are on test, and $r'_1, r'_2, ..., r'_m$ denote the corresponding numbers of units randomly removed (withdrawn) from the test. Furthermore, let $y_{1:m:M} \leq y_{2:m:M} \leq \cdots \leq y_{m:m:M}$ be the observed ordered lifetimes. Let r'_i denote the number of strength components removed at the time of the ith failure of the stress component, $0 \leq r'_i \leq M - m - \sum_{l=1}^{i-1} r'_l, i = 2, 3, ..., m-1$ with $0 \leq r'_1 \leq M - m$ and $r'_m = M - m - \sum_{l=1}^{m-1} r'_l$, where r'_i's are non-pre-specified integers and m are pre-specified integers. Note that if $r'_1, r'_2, ..., r'_{m-1} = 0$, so that $r'_m = M - m$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $r'_1 = r'_2 = ... = r'_m = 0$, so that $m = M$, the progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case).

The likelihood function of the unknown parameters based on the observed sample is then given as

$$
\begin{aligned}
L(\boldsymbol{\beta}, \lambda; \mathbf{x}, \mathbf{y}) &= \left[\prod_{i=1}^{n}\prod_{j=1}^{k} C_{ij} f(x_{ij})[1-F(x_{ij})]^{r_{ij}}\right] \times \left[\prod_{i=1}^{m} C_i g(y_i)[(1-G(y_i)]^{r_i'}\right] \\
&= \alpha_1^{nk}\alpha_2^{n}\lambda^{n(k+1)} \exp\left[\sum_{i=1}^{n}\sum_{j=1}^{k} \ln Q^{'}(1/x_{ij}) + \sum_{i=1}^{m} \ln Q^{'}(1/y_i)\right] \times \\
&\quad \exp\left[-\lambda\left\{\sum_{i=1}^{n}\sum_{j=1}^{k} Q(1/x_{ij}) + \sum_{i=1}^{m} Q(1/y_i)\right\}\right. \\
&\quad \left.-(\alpha_1-1)w_\lambda - (\alpha_2-1)v_\lambda\right] \qquad (2.1)
\end{aligned}
$$

and the log-likelihood is as

$$
\begin{aligned}
l(\boldsymbol{\beta}, \lambda; \mathbf{x}, \mathbf{y}) &= nk\ln\alpha_1 + m\ln\alpha_2 + m(k+1)\ln\lambda - \\
\lambda\left\{\sum_{i=1}^{n}\sum_{j=1}^{k} Q(1/x_{ij}) + \sum_{i=1}^{m} Q(1/y_i)\right\} & \quad -(\alpha_1-1)w_\lambda - (\alpha_2-1)v_\lambda, \qquad (2.2)
\end{aligned}
$$

where $\boldsymbol{\beta}$ =(α_1, α_1), $w_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1+r_{ij})\ln[1-e^{-\lambda Q(1/x_{ij})}]$, $v_\lambda = -\sum_{i=1}^{m}(1+r_i')\ln[1-e^{-\lambda Q(1/y_i)}]$, $C_{ij} = N - \sum_{l=1}^{i-1}(1+r_{lj})$, and $C_i = M - \sum_{l=1}^{i-1}(1+r_l')$.

The MLEs of α_1 and α_2; interchangeably denoted by $\widehat{\alpha}_1$ or A_1, and $\widehat{\alpha}_2$ or A_2, respectively; are given by

$$
A_1 = \frac{mk}{W_\lambda} \quad \text{and} \quad A_2 = \frac{m}{V_\lambda}, \qquad (2.3)
$$

where $W_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1+R_{ij})\ln(1-e^{-\lambda Q(1/X_{ij})})$ and $V_\lambda = -\sum_{i=1}^{m}(1+R_i')\ln(1-e^{-\lambda Q(1/Y_i)})$.

Note that observed values of the MLEs of α_1 and α_2 are also interchangeably denoted by $\widehat{\alpha}_{1_{obs}}$ or a_1, and $\widehat{\alpha}_{2_{obs}}$ or a_2, respectively. It can be seen from (2.3) that (W_λ, V_λ) is a complete sufficient statistics for (α_1, α_2). Moreover, W_λ and V_λ have gamma distributions with parameters (nk, α_1) and (m, α_2), respectively. Let $\Lambda = 2nk(A_1\alpha_1)^{-1}$ and $\Delta = 2m(A_2\alpha_2)^{-1}$, then

$$\Lambda \sim \chi^2_{2nk} \text{ and } \Delta \sim \chi^2_{2n}, \tag{2.4}$$

where $\sim$ denotes "distributed as" and χ^2_v denotes a central chi-square distribution with v degrees of freedom.

Hence, the MLE of $R_{s,k}$ that has been obtained from (1.1) by using the invariance property of MLEs is given by

$$\begin{aligned}
\widehat{R}^M_{s,k} &= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^j A_2}{A_1(i+j)+A_2} \\
&= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^j}{1+\frac{A_1}{A_2}(i+j)} \\
&= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\widehat{R}_{ij},
\end{aligned} \tag{2.5}$$

where $\widehat{R}_{ij} = (-1)^j/[1 + A_1(i + j)\ /A_2]$.

Since $2nk(A_1\alpha_1)^{-1} \sim \chi^2_{2nk}$ and $2m(A_2\alpha_2)^{-1} \sim \chi^2_{2n}$,

$$\widehat{R}_{ij} = \frac{(-1)^j}{1 + \frac{\alpha_1}{\alpha_2}(i\ +\ j)F_{ij}},$$

where

$$F_{ij} = \frac{R_{ij}}{1 - R_{ij}} \times \frac{1 - \widehat{R}_{ij}}{\widehat{R}_{ij}} \sim F_{2n,2m},$$

with $F_{\upsilon_1,\upsilon_2}$ denotes a central F-distribution with υ_1 numerator df and υ_2 denominator df, and $\widehat{R}'_{ij}$s *pdf* is given by

$$\begin{aligned} f_{\widehat{R}_{ij}}(\chi) &= \frac{1}{\chi^2 B(m,n)}\left(\frac{m\alpha_2}{n\alpha_1}\right)^m \times \frac{\left(\frac{1-\chi}{\chi}\right)^m}{\left(1 + \frac{m\alpha_2}{n\alpha_1}\left(\frac{1-\chi}{\chi}\right)\right)^{(m+n)}}; \\ 0 &\leq \chi \leq 1; \alpha_1, \alpha_2 > 0, \end{aligned} \tag{2.6}$$

where $B(\gamma, \delta)$ is the beta function given by $\int_0^1 w^{(\gamma-1)}(1-w)^{(\delta-1)}dw$.

Asymptotic distribution of $\widehat{R}_{s,k}$

Suppose that $\boldsymbol{\beta}$ =(α_1, α_2) is a vector of parameters of interest and $\widehat{\boldsymbol{\beta}} = (A_1, A_2)$ be its MLE. Therefore, it is known that $R_{s,k}$ is a function of $\boldsymbol{\beta} = (\alpha_1, \alpha_1)$, i.e., $R_{s,k} = g(\boldsymbol{\beta})$, then by the invariance property of MLEs, $\widehat{R}_{s,k} = g(\widehat{\boldsymbol{\beta}}) = g(A_1, A_2)$. The classical pivotal quantity; denoted by

$T^c_{R_{s,k}}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\beta})$ or simply by $T^c_{R_{s,k}}$, where $\mathbf{X} = \{X_{ij}\}_{i-1,2,\ldots,n;j=1,2,\ldots,k}$ and $\mathbf{Y} = (Y)_{i=1,2,\ldots,m}$, based on the large sample procedure; where for testing

$$H_0 : R_{s,k} \leq R_0 \text{ vs. } H_a : R_{s,k} > R_0\,, \text{ where } R_0 \text{ is a given quantity,} \tag{2.7}$$

is given by

$$T^c_{R_{s,k}}(\mathbf{X}, \mathbf{Y}, \boldsymbol{\beta}) = T^c_{R_{s,k}} = (\widehat{R}_{s,k} - R_{s,k})\sqrt{I^*_n(R_{s,k})^{-1}} \xrightarrow{D} N(0, 1), \tag{2.8}$$

here $\xrightarrow{D}$ denotes the "convergence in distribution" and $\sigma^2_{\widehat{R}_{s,k}} = I^*_n(R_{s,k})^{-1}$ is the asymptotic variance (or the mean squared error (MSE) for unbiased $\widehat{R}_{s,k}$) of $\widehat{R}_{s,k}$ with $I^*_n(R_{s,k})$ being the the Fisher information (or the expected Fisher information) matrix.

$I^*_n(R_{s,k})$ for the new parameterization $R_{s,k}$ is obtained using the chain rule as

$$I^*_n(R_{s,k}) = J(R_{s,k})^T I_n(\boldsymbol{\beta}))J(R_{s,k}),$$

where $J(R_{s,k})$ is the Jacobian matrix with elements $J(R_{s,k}) = (\partial R_{s,k}/\partial\alpha_1, \partial R_{s,k}/\partial\partial\alpha_2)$ and $I_n(\boldsymbol{\beta})$ is the observed information matrix of $\boldsymbol{\beta}$, whose ij^{th} element is given by $I_n(\boldsymbol{\beta})_{ij}$=$-E[\partial^2 l$ $(\boldsymbol{\beta})/\partial i\partial j]$, for $i, j = \alpha_1, \alpha_2$, with $l(\boldsymbol{\beta}) = l(\boldsymbol{\beta}, \lambda; \mathbf{x}, \mathbf{y})$ as in (2.2). Therefore, the asymptotic variance of $\widehat{R}_{s,k}$ is given by

$$\sigma^2_{\widehat{R}_{s,k}} = \left(\frac{\partial R_{s,k}}{\partial\alpha_1}\right)^2 \frac{\alpha_1^2}{nk} + \left(\frac{\partial R_{s,k}}{\partial\alpha_2}\right)^2 \frac{\alpha_2^2}{n},$$

where

$$\frac{\partial R_{s,k}}{\partial \alpha_1} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^{j+1}\alpha_2(i+j)}{(\alpha_1(i+j)+\alpha_2)^2} \quad \text{and}$$

$$\frac{\partial R_{s,k}}{\partial \alpha_2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^{j}\alpha_1(i+j)}{(\alpha_1(i+j)+\alpha_2)^2}.$$

The Asymptotic variance as well as the asymptotic one-and two-sided confidence intervals for $R_{s,k}$ can also be achieved through the following procedure. Let us consider $\mathbf{X} = \{X_{ij}\}_{i-1,2,\ldots,n;j=1,2,\ldots,k}$ and $\mathbf{Y} = (Y)_{i=1,2,\ldots,m}$. To compute the confidence interval of $R_{s,k}$, consider the log-likelihood function of the observed sample, which is given by

$$\begin{aligned} l(\boldsymbol{\beta}, \lambda; \mathbf{x}, \mathbf{y}) = \; & nk \ln \alpha_1 + m \ln \alpha_2 + m(k+1)\ln \lambda - \\ & \lambda \left\{ \sum_{i=1}^{n}\sum_{j=1}^{k} Q(1/x_{ij}) + \sum_{i=1}^{m} Q(1/y_i) \right\} \\ & -(\alpha_1 - 1)w_\lambda - (\alpha_2 - 1)v_\lambda. \end{aligned}$$

We denote the expected Fisher information matrix of $\boldsymbol{\delta}$ =(α_1, α_2) as $\mathbb{I}(\boldsymbol{\delta}) = E[\mathbb{I}^{\dagger}(\boldsymbol{\delta})]$, where

$\mathbb{I}^{\dagger}(\boldsymbol{\delta}) = \left[I^{\dagger}{}_{ij}\right]_{i,j\,=\,1,2} = \left[-\frac{\partial^2 l(\boldsymbol{\delta})}{\partial i \partial j}\right]_{i,j\,=\alpha_1,\alpha_2}$ is the observed information matrix. That is

$$\mathbb{I}^{\dagger}(\boldsymbol{\delta}) = -\begin{bmatrix} \frac{\partial^2 l(\boldsymbol{\delta})}{\partial^2 \alpha_1} & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \boldsymbol{\alpha_1} \partial \alpha_2} \\ \frac{\partial^2 l(\boldsymbol{\delta})}{\partial \boldsymbol{\alpha_1} \partial \alpha_2} & \frac{\partial^2 l(\boldsymbol{\delta})}{\partial^2 \alpha_2} \end{bmatrix}.$$

The following theorems will aid in our construction of the above mentioned confidence intervals.

Theorem 1:

As $n \longrightarrow \infty$ and $m \longrightarrow \infty$ and $n/m \longrightarrow p$

Then $[\sqrt{n}(\widehat{\alpha}_1 - \alpha_1), \sqrt{m}(\widehat{\alpha}_2 - \alpha_2)] \xrightarrow{D} N_2(\mathbf{0}, \mathbb{W}^{-1}(\boldsymbol{\delta}))$,

where $\xrightarrow{D}$ denotes the convergence in distribution, and

$$\mathbb{W}(\boldsymbol{\delta}) = -\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

and

$$w_{11} = \frac{1}{\sqrt{n}\sqrt{n}} I_{11} = \frac{1}{n} I_{11} = -\frac{nk}{\alpha_1^2}$$

$$w_{12} = w_{21} = \frac{1}{\sqrt{m}\sqrt{n}} I_{12} = 0$$

$$w_{22} = \frac{1}{\sqrt{m}\sqrt{m}} I_{22} = \frac{1}{n} I_{22} = -\frac{m}{\alpha_2^2}$$

Theorem 2:

As $n \longrightarrow \infty$ and $m \longrightarrow \infty$ so that $n/m \longrightarrow p$

$\sqrt{n}(\widehat{R}_{s,k}-R_{s,k})\xrightarrow{D} N(0,\sigma^2_{\widehat{R}_{s,k}})$

where $R_{s,k}$ is a function $\boldsymbol{\delta}$, i.e., $R_{s,k}=g(\boldsymbol{\delta})$,$\sigma^2_{\widehat{R}_{s,k}}=[\mathbb{W}(R_{s,k})]^{-1}$ and $\mathbb{W}(R_{s,k})=\mathbb{J}(R_{s,k})^T\mathbb{W}$ $(R_{s,k}))\mathbb{J}(R_{s,k})$, where $\mathbb{J}(R_{s,k})^T=(\partial R_{s,k}/\partial\alpha_1,\partial R_{s,k}/\partial\theta\alpha_2)$ is the Jacobian matrix with elements $J(R_{s,k})_{ij}(i,j=\alpha_1,\alpha_2\ )$, $\boldsymbol{\delta}(R_{s,k})=(g^{-1}(R_{s,k})_{ij})_{i,j=\alpha_1,\alpha_2}$.

In order to construct confidence interval for, and testing of, $R_{s,k}$, the variance $\sigma^2_{\widehat{R}_{s,k}}$ needs to be estimated. To estimate it, the empirical Fisher information matrix and $\widehat{\alpha}_1,\widehat{\alpha}_2$ are used. The estimator of $\sigma^2_{\widehat{R}_{s,k}}$ is denoted by $s^2_{\widehat{R}_{s,k}}$, and its observed value by $s^2_{\widehat{R}_{s,k}}$.

The p-value for testing hypotheses in (2.7), based on the asymptotic distribution of $R_{s,k}$, is given by

$$p_{R_{s,k}}=1-\Phi(q_{\widehat{R}_{s,k}}),\tag{2.9}$$

where $q_{\widehat{R}_{s,k}}=(\widehat{r}_{s,k}-R_0)\,s^{-1}_{\widehat{R}_{s,k}}$, and $q^c_{\widehat{R}_{s,k}},\widehat{r}_{s,k}$, respectively, are the observed values of $Q_{\widehat{R}_{s,k}}=\left(\widehat{R}_{s,k}-R_0\right)S^{-1}_{\widehat{R}_{s,k}}$ and $\widehat{R}_{s,k}$; $\Phi(.)$ is the distribution function of the standard normal distributi on.

A $100(1-\gamma)\%$, asymptotic confidence interval (ACI) for $R_{s,k}$, based on the above asymptotic distribution, is given by

$$\mathrm{ACI}^{1-\gamma}_{R_{s,k}}=\left(\widehat{r}_{s,k}-Z_{\gamma/2}s_{\widehat{R}_{s,k}},\widehat{r}_{s,k}+Z_{\gamma/2}s_{\widehat{R}_{s,k}}\right),\tag{2.10}$$

where Z_η is ηth quantile (or 100ηth percentile) of the standard normal distribution. A one-sided

$100(1-\gamma)\%$ asymptotic lower confidence interval (ALCI) for $R_{s,k}$ is given by

$$\mathrm{ALCI}_{R_{s,k}}^{1-\gamma} = \left(\widehat{r}_{s,k}, \widehat{r}_{s,k} + Z_{\gamma/2} s_{\widehat{R}_{s,k}}\right).$$

Unformly minimum variance unbiased estimator of $R_{s,k}$

Furthermore, uniformly minimum variance unbiased estimator (UMVUE) of $R_{s,k}$, say $\widehat{R}^U_{s,k}$, will be derived. Since $\widehat{R}_{s,k}$ is a linear function of α_1 and α_2, it is sufficient to find the UMVUE of $\psi(\alpha_1,\alpha_2) = \alpha_2/(\alpha_1(i+j)+\alpha_2)$. It has already seen that (w_λ, v_λ) is a complete sufficient statistics for (α_1,α_2) from (2.3). Moreover, w_λ, v_λ have gamma distributions with parameters (nk,α_1) and (m,α_2), respectively. Let

$$\phi(V,W) = \begin{cases} 1, & W > (i+j)V \\ 0 & \text{otherwise} \end{cases},$$

where $W_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1+R_{ij})\ln(1-e^{-\lambda Q(1/X_{ij})})$ and $V_\lambda = -\sum_{i=1}^{m}(1+R'_i)\ln(1-e^{-\lambda Q(1/Y_i)})$. It is clear that W and V have exponential distributions with means $1/\alpha_1$ and $1/\alpha_2$, respectively. Then, $\phi(V,W)$ is an unbiased estimator for $\psi(\alpha_1,\alpha_2)$. The UMVUE of $\psi(\alpha_1,\alpha_2)$, say $\widehat{\psi}_U(\alpha_1,\alpha_2)$, can be obtained by using

Lehmann-Scheffé Theorem (1950) and is given by

$$\begin{aligned}
\widehat{\psi}_U(\alpha_1, \alpha_2) &= E(\phi(V,W)|W_\lambda = w_\lambda, V_\lambda = v_\lambda) \\
&= P(W > (i+j)V|W_\lambda = w_\lambda, V_\lambda = v_\lambda) \\
&= \int_C \int f_{W|W_\lambda = w_\lambda}(w|w_\lambda) f_{V|V_\lambda = v_\lambda}(v|v_\lambda) dv dw, \qquad (2.11)
\end{aligned}$$

where $C = \{(v, w) : 0 < v < v_\lambda, 0 < w < w_\lambda, v(i+j) < w\}$. Notice that $f_{W|W_\lambda = w_\lambda}(w|w_\lambda)$ and $f_{V|V_\lambda = v_\lambda}(v|v_\lambda)$ are easily obtained by using Lemma 1 in Basirat et al. (2015). This double integral is considered in two cases i.e. $(i+j)v_\lambda/w_\lambda < 1$ and $(i+j)v_\lambda/w_\lambda > 1$.

When $(i+j)v_\lambda/w_\lambda < 1$, the double integral in (2.11) can be expressed as

$$\begin{aligned}
\widehat{\psi}^{\dagger}_U(\alpha_1, \alpha_2) &= \int_0^{v_\lambda} \int_{v(i+j)}^{w_\lambda} \frac{(n-1)(nk-1)}{v_\lambda w_\lambda} \left(1 - \frac{v}{v_\lambda}\right)^{n-2} \left(1 - \frac{w}{w_\lambda}\right)^{nk-2} dw dv \\
&= (n-1) \int_0^1 (1-t)^{n-2}(1-ct)^{nk-1} dt, \\
\text{where } c &= (i+j)v_\lambda/w_\lambda < 1, t = v/v_\lambda \\
&= \sum_{z=s}^{nk-1} (-1)^z \left(\frac{(i+j)v_\lambda}{w_\lambda}\right)^z \frac{\begin{pmatrix} nk-1 \\ i \end{pmatrix}}{\begin{pmatrix} n+z-1 \\ z \end{pmatrix}}. \qquad (2.12)
\end{aligned}$$

When $(i+j)v_\lambda/w_\lambda > 1$, the double integral in (2.11) can be expressed as

$$\begin{aligned}
\widehat{\psi}^{\ddagger}_U(\alpha_1,\alpha_2) &= \int\limits_0^{w_\lambda}\int\limits_{v(i+j)}^{w/(i+j)} \frac{(n-1)(nk-1)}{v_\lambda w_\lambda}\left(1-\frac{v}{v_\lambda}\right)^{n-2}(1-\frac{w}{w_\lambda})^{n-1}dt,\\
&= 1-(nk-1)\int\limits_0^1 (1-t)^{nk-2}\left(1-\frac{t}{c}\right)^{n-1} dt,\\
\text{where } c &= (i+j)v_\lambda/w_\lambda > 1, t = w/w_\lambda\\
&= 1-\sum_{z\,=\,s}^{n-1}(-1)^z\left(\frac{w_\lambda}{(i+j)v_\lambda}\right)^z \frac{\begin{pmatrix} n-1 \\ i \end{pmatrix}}{\begin{pmatrix} nk+z-1 \\ z \end{pmatrix}}.
\end{aligned} \tag{2.13}$$

Therefore, the $\widehat{R}^U_{s,k}$ is obtained by using (2.12) and (2.13)

$$\widehat{R}^U_{s,k} = \sum_{i\,=\,s}^{k}\sum_{j\,=\,0}^{k\,-\,i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}(-1)^j \times \begin{cases} \widehat{\psi}^{\dagger}_U(\alpha_1,\alpha_2) & \text{if} \quad (i+j)v_\lambda/w_\lambda < 1 \\ \widehat{\psi}^{\ddagger}_U(\alpha_1,\alpha_2) & \text{if} \quad (i+j)v_\lambda/w_\lambda > 1 \end{cases} . \tag{2.14}$$

Bootstrap confidence intervals for $R_{s,k}$

It is clear that the confidence intervals for $R_{s,k}$ based on the asymptotic results do not perform very well for small sample sizes. So, two confidence intervals based on the parametric bootstrap methods for estimating $R_{s,k}$ are proposed: (i) percentile bootstrap method ((Efron 1979) (we call it from now on as boot-p), and (ii) studentized bootstrap method or bootstrap-t method (we call it for now on as boot-t) (Hall 1988).

(i) Percentile Bootstrap Method (Efron 1979)

Algorithm 1:

For given $(\alpha_1, \alpha_2, \lambda)$, $(m, n, k, s), \mathbf{R} = \mathbf{r} = \{r_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{R}' = \mathbf{r}' = (r'_1, r'_2, ...,$

$r'_m)$:

Step 1: Generate inverse exponenetiated x_{ij} from

$$\mathcal{GCEIE}\,(\alpha_1, \lambda) \sim \alpha_1\lambda[Q^{'}(1/x_{ij})/x_{ij}^2]e^{-\lambda Q(1/x_{ij})}) \times (1 - e^{-\lambda Q(1/x_{ij})})^{\alpha_1\,-\,1}$$

for $i = 1, 2, ..., n; j = 1, 2, ..., k$, and y_i from $\mathcal{GCEIE}\,(\alpha_2, \lambda) \sim$

$\alpha_2\lambda[Q^{'}(1/y_i)/y_i^2]e^{-\lambda Q(1/y_i)})(1 - e^{-\lambda Q(1/y_i)})^{\alpha_2\,-\,1}$ for $i = 1, 2, ..., m,$

Step 2: From the samples $\mathbf{x} = \{x_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{y} = (y_1, y_2..., y_n)$,

compute the estimates of (α_1, α_2), say (a_1, a_2)

$a_1 = nkw_\lambda^{-1}$, where $w_\lambda = -\sum_{i=1}^n \sum_{j=1}^k (1 + r_{ij}) \ln(1 - e^{-\lambda Q(1/x_{ij})})$

and $a_2 = nv_\lambda^{-1}$, where $v_\lambda = -\sum_{i=1}^m (1 + r'_i) \ln(1 - e^{-\lambda Q(1/y_i)})$

Step 3 : Generate bootstrap inverse exponenetiated x_{ij}^* from $\mathcal{GCEIE}\,(a_1, \lambda) \sim$

$a_1\lambda[Q^{'}(1/x_{ij})/x_{ij}^2]e^{-\lambda Q(1/x_{ij})}) \times (1 - e^{-\lambda Q(1/x_{ij})})^{a_1\,-\,1}$ for

$i = 1, 2, ..., n; j = 1, 2, ..., k$, and y_i^* from $\mathcal{GCEIE}\,(\alpha_2, \lambda) \sim$

$a_2\lambda[Q^{'}(1/y_i)/y_i^2]\ e^{-\lambda Q(1/y_i)})(1 - e^{-\lambda Q(1/y_i)})^{a_2\,-\,1}$ for

$i = 1, 2, ..., m,$

Then, compute bootstrap sample estimates of α_1 and α_2 :

$\widehat{\alpha}^*_{1_{obs}} = a^*_1 = nk(w^*_\lambda)^{-1}$, where $w^*_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1+r_{ij})\ln(1-e^{-\lambda Q(1/x^*_{ij})})$

and

$\widehat{\alpha}^*_{2_{obs}} = a^*_2 = n(v^*_\lambda)^{-1}$, where $v^*_\lambda = -\sum_{i=1}^{m}(1+r'_i)\ln(1-e^{-\lambda Q(1/y^*_i)})$

Based on $\mathbf{x}^*$ $\{x^*_{ij}\}_{i=1,2,\ldots,n;j=1,2,\ldots,k}$ and $\mathbf{y} = (y^*_1, y^*_2 \ldots, y^*_n)$ compute the bootstrap sample estimate of $R_{s,k}$, denoted by $\widehat{R}^*_{s,k}$, using

$$\widehat{R}^*_{s,k} = \sum_{i\,=\,s}^{k}\sum_{j\,=\,0}^{k\,-\,i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^j\widehat{\alpha}^*_{2_{obs}}}{\widehat{\alpha}^*_{1_{obs}}(i\ +\ j)\ \ +\ \widehat{\alpha}^*_{2_{obs}}},$$

Step 4: Repeat step 3, N boot times and get the bootstrap distribution given by

${}^1\widehat{R}^*_{s,k}, {}^2\,\widehat{R}^*_{s,k}, \ldots, {}^N\,\widehat{R}^*_{s,k}$.

The bootstrap distribution of the statistic $\widehat{R}^*_{s,k}$ that is based on many resamples represents the sampling distribution of the statistic $\widehat{R}^*_{s,k}$ that is based on many samples.

Step 5: After ranking from bottom to top, let us denote these bootstrap values as

${}^{(1)}R^*_{s,k}, {}^{(2)}\,R^*_{s,k}, \ldots, {}^{(N)}\,R^*_{s,k}$.

Let $G(R^*_{s,k}) = P(R^*_{s,k} \leq r^*_{s,k})$, where $r^*_{s,k}$ is the observed value of $R^*_{s,k}$, be the cumulative distribution of $R^*_{s,k}$. Define ${}^{boot-p}R^*_{s,k} = G^{-1}(\xi)$ for a given

ξ. The approximate $100(1-\gamma)\%$ percentile-bootstrap CI (PBCI) for $R_{s,k}$ is then given by

$$\text{PBCI} = \left({}^{boot-p}\widehat{R}^{*}_{s,k}\left(\frac{\gamma}{2}\right), {}^{boot-p}\,\widehat{R}^{*}_{s,k}\left(\frac{1-\gamma}{2}\right)\right) \tag{2.15}$$

When the distributions are skewed we need do some adjustment. One method which is proved to be reliable is BCa method (BCa stands for Bias-corrected and accelerated). For the details please refer to DiCiccio and Efron (1996). When the distribution of $R^{*}_{s,k}$ is skewed, we instead use the $q.low$ and $q.up$ percentiles of the bootstrap replicates of $R^{*}_{s,k}$ to calculate the lower bound and upper bound of the confidence intervals. Formally, for confidence level 95%, the bootstrap bias-corrected and accelerated CI(BBCACI) for $R_{s,k}$ is

$$\text{BBCACI} = (q.low, q.up)\,,$$

where

$$\begin{aligned} q.low &= \Phi\left(z_0 + \frac{z_0 + z_{0.025}}{1-b(z_0 + az_{0.025})}\right) \quad \text{and} \\ q.up &= \Phi\left(z_0 + \frac{z_0 + z_{0.975}}{1-b(z_0 + az_{0.975})}\right), \end{aligned}$$

where z_γ is the γth quantile of standard normal distribution, z_0 and b, namely bias-correction and acceleration, are two parameters to be estimated, by (2.8) and (6.6) in DiCiccio and Efron (1996).

(ii) Bootstrap-t Method (Hall 1988) : The method was suggested in Efron (1979), but some poor numerical results reduced its appeal. Hall's (1988) paper showing the bootstrap-t's good second-order properties has revived interest in its use. Babu and Singh (1983) gave the first proof of second-order accuracy for the bootstrap-t.

Algorithm *2:*

Step 1: Do steps 1–3 in **Algorithm 1.** Also, compute the following statistic

$$t^* = \frac{\sqrt{n}(\widehat{r}^*_{s,k} - \widehat{r}_{s,k})}{s_{R^*_{s,k}}},$$

where

$$T^* = \frac{\sqrt{n}(\widehat{R}^*_{s,k} - \widehat{R}_{s,k})}{S_{R^*_{s,k}}},$$

and $S_{R^*_{s,k}}$ is the standard deviation of the bootstrap distribution and $s_{R^*_{s,k}}$ is its observed value. $S_{R^*_{s,k}}$ is obtained using the Fisher (or expected Fisher) information matrix. Moreover, $r^*_{s,k}$ is the estimate (or the observed estimator) of $R_{s,k}$ based on the bootstrap resamples and $\widehat{r}_{s,k}$ is the estimate of $R_{s,k}$ based on the original observed sample, and $\widehat{R}^*_{s,k}$ is the estimator of $R_{s,k}$ based on the bootstrap random resamples and $\widehat{R}_{s,k}$ is the estimator of $R_{s,k}$ based on the original random sample.

Step 2: Compute N bootstrap replications of t^*. Denote t^* by $t^*_1, ..., t^*_N$.

Step 3: After ranking from bottom to top, let us denote these bootstrap values as $t^*_{(1)}, ..., t^*_{(N)}$.

Step 4: For t^* values obtained in step 1, determine the upper and lower bounds of the $100(1-\gamma)\%$ confidence interval of $R^*_{s,k}$ as follows:

Let $H(t^*) = P(T^* \leq t^*)$ be the cumulative distribution function of T^*.

For a given ξ,

define

$$^{boot-t}\widehat{R}^*_{s,k}(\xi) = \widehat{r}^*_{s,k} + H^{-1}(\xi)\frac{s_{R^*_{s,k}}}{\sqrt{n}}.$$

The $100(1-\gamma)\%$ bootstrap-t CI (BTCI) for $R_{s,k}$ is then given by

$$\text{BTCI} = \left(^{boot-t}\widehat{R}^*_{s,k}\left(\frac{\gamma}{2}\right), ^{boot-t}\widehat{R}^*_{s,k}\left(\frac{1-\gamma}{2}\right)\right) \tag{2.16}$$

CHAPTER III

THE GENERALIZED VARIABLE METHOD

A review

Suppose that $\mathbf{Z} = (Z_1, Z_2, ..., Z_n)$ form a random sample from a distribution, which depends on the parameters $\xi = (\theta, \boldsymbol{\delta})$ where θ is the parameter of interest and $\boldsymbol{\delta}^{\mathrm{T}}$ is a vector of nuisance parameters. Also, suppose that $\mathbf{z} = (z_1, z_2, ..., z_n)$ be its observed value. Let Ψ be the sample space of possible values of $\mathbf{Y}$, Ξ be the parameter space of ξ, and Θ be the parameter space of θ. Consider testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$, where θ_0 is a specified quantity. A *Generalized Test Variable* (Tusi and Weerahandi, 1989) of the form $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$, a map of $\Psi\times \Psi\times \Xi$ to a Euclidean space, is chosen to satisfy the following three conditions.

1. For fixed $\mathbf{z}$, the distribution of $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is free of the vector of nuisance parameter $\boldsymbol{\delta}$.
2. The value of $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ at $\mathbf{Z} = \mathbf{z}$ is free of any unknown parameters.
3. For fixed $\mathbf{z}$ and $\boldsymbol{\delta}$, and for all t, $\Pr[T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}) \geq t]$ is either an increasing or a decreasing function of θ.

If $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is stochastically increasing in θ, the *Generalized P-Value* for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$ is given by $\mathrm{Sup}_{H_0} \Pr[T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}) \geq t]$, and if $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is stochastically decreasing in θ, the *Generalized P-Value* for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$ is given by $\text{Sup}_{H_0} \Pr\left[T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}) \leq t\right]$.

Furthermore, a $100(1-\gamma)\%$ two-sided generalized CI for θ is given by $(R(\mathbf{z}; \frac{\gamma}{2}),R(\mathbf{z}; 1-\frac{\gamma}{2}))$, where $R(\mathbf{z}; p)$ is the pth quantile of the *Generalized Pivotal Quantity* $R(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$, a map of $\Psi\times \Psi\times \Xi$ to a Euclidean space, that has a relationship $T(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}) = R(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}) - \boldsymbol{\theta}$ and satisfy the following two conditions.

1. For fixed $\mathbf{z}$, the distribution of $R(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is free of unknown parameters.
2. The value of $R(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ at $\mathbf{Z} = \mathbf{z}$ is free of the nuisance parameter $\boldsymbol{\delta}$

   and equals to the parameter of interest θ, i.e., $R(\mathbf{z}; \mathbf{z}, \theta, \boldsymbol{\delta}) = \theta$.

A *Generalized Point Estimator* (Weerahandi 2012) of the form $Q(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$, a map of $\Psi\times \Psi\times \Xi$ to a Euclidean space, is chosen to satisfy the following three conditions.

1. The cumulative distribution of $Q(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is a monotonic function of θ
2. $Q(\mathbf{z}; \mathbf{z}, \theta, \boldsymbol{\delta})) = c$; where c is a constant free of nuisance parameters, but possibly

   could depend on $\mathbf{z}$ and θ.

An optional, but desirable additional property to have is:

3. The distribution of $Q(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta})$ is free of nuisance parameters $\boldsymbol{\delta}$.

Remarks:

Property 3 of the above *Generalized Point Estimator* is essential when we need to make additional inferences such as statistical tests and interval estimation based on $Q(\mathbf{Z}; \mathbf{z}, \theta, \boldsymbol{\delta}))$. In fact, it is easily verified that if $Q(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta})$ is a *Generalized Point Estimator* satisfying Property 3, then it is also a *Generalized Pivotal Quantity* $R(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta})$, as defined by Weerahandi (1993). Moreover, $T(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = Q(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) - Q(\mathbf{z};\mathbf{z},\,\theta,\boldsymbol{\delta})$ is a *Generalized Test Variable* as defined by Tsui and Weerahandi (1989), where $Q(\mathbf{z};\mathbf{z},\,\theta,\boldsymbol{\delta})$ is the observed value of $Q(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta})$. For example, in sampling from a normal population, say $N(\mu,\sigma^2)$, the random variable $Q_1(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = \overline{Z} - \mu$ is a *Generalized Point Estimator* having just the first two properties above, whereas the *Generalized Point Estimator* $Q_2(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = \overline{z} - s(\overline{Z} - \mu)/S$ has all three properties, where $\overline{Z}$ is the sample mean and S is the sample standard deviation. Therefore, $Q_2(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta})$ is a *Generalized Pivotal Quantity* $R_2(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = \overline{z} - s(\overline{Z} - \mu)/S$ and $T_2(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = Q_2(\mathbf{Z};\mathbf{z},\,\theta,\boldsymbol{\delta}) - Q_2(\mathbf{z};\mathbf{z},\,\theta,\boldsymbol{\delta}) = [\overline{z}-s(\overline{Z}-\mu)/S]-[\overline{z}-s(\overline{z}-\mu)/s] = \overline{z}-s(\overline{Z}-\mu)/S-\mu$ is a *Generalized Test Variable* for estimating μ. This $T_2(\mathbf{Y};\mathbf{y};\theta,\boldsymbol{\beta})$ can also be derived from $T_2(\mathbf{Y};\mathbf{y};\theta,\boldsymbol{\beta}) = R_2(\mathbf{Y};\mathbf{y};\theta,\boldsymbol{\beta})-\mu = \overline{z} - s(\overline{Z} - \mu)/S - \mu$ as explained under the *Generalized Pivotal Quantity.* Please also note that $Q_2(\mathbf{Y};\mathbf{y};\theta,\boldsymbol{\beta})$ was derived using the *Substitution Method* described in pp. 13-16 in Weerahandi (2004). For further details on the concepts of generalized p-values, we refer readers to the books by Weerahandi (1995, 2004), and references therin.

Generalized inferences for α_1 and α_2

Since the reliability parameter in (1.1) is a function of both α_1 and α_2, we first develop generalized variables for α_1 and α_2 for the one-sample case. Even though it is not our primary interest, knowing the results of the one-sample case will make it easier to understand the approach and results for the multicomponent stress-strength reliability .Let $X_{1j:n:N} \leq X_{2j:n:N} \leq \cdots \leq X_{nj:n:N}$, $j = 1, 2, ..., k$ denote n number of strength components observed from $\mathcal{GCEIE}$ (α_1, λ) before termination and $\mathbf{R}_j = (R_{1j}, R_{2j}, ..., R_{nj})$ denote the corresponding numbers of strength units removed (withdrawn) from the jth test, where $j = 1, 2, ..., k$. Furthermore, let $x_{1j:n:N} \leq x_{2j:n:N} \leq \cdots \leq x_{nj:n:N}$, $j = 1, 2, ..., k$ and $\mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$ be the observed ordered strengths and observed strength removals, respectively. Let r_{ij} denote the number of strength components removed at the time of the ith failure of the jth strength component, $0 \leq r_{ij} \leq N - n - \sum_{l=1}^{i-1} r_{lj}, i = 2, 3, ..., n-1; \; j = 1, 2, ..., k$ with $0 \leq r_{1j} \leq N - n$ and $r_{nj} = N - n - \sum_{l=1}^{n-1} r_{lj}$, where r_{ij}'s are non-pre-specified integers and n are pre-specified integers and $j = 1, 2, ..., k$. Note that if $r_{1j}, r_{2j}, ..., r_{n-1,j} = 0$, so that $r_{nj} = N - n$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $r_{1j} = r_{2j} = ... = r_{nj} = 0$, so that $n = N$, the progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case).

We know that a_1 is the observed value of A_1, or simply the estimate of α_1, where $A_1 = nk/W_\lambda$ with $W_\lambda = -\sum_{i=1}^{n} \sum_{j=1}^{k} (1 + R_{ij}) \ln(1 - e^{-\lambda Q(1/X_{ij})})$. The random variable $Q(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda) = 2nk(a_1 \Lambda)^{-1} = (a_1)^{-1}(A_1 \alpha_1)$ is then a generalized estimator who satisfy the three conditions to be a bona fide generalized point estimator as presented in the subsection (3.1). Therefore, this would also serve as a generalized pivotal quantity $R(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda) = 2nk(a_1 \Lambda)^{-1} = (a_1)^{-1}(A_1 \alpha_1)$, thus $T(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda) = 2nk(a_1 \Lambda)^{-1} - \alpha_1 = a_1^{-1} A_1 \alpha_1 - \alpha_1$ is the generalized test variable.

First, for fixed $\mathbf{x}$, the distribution $F_T(t)$ of $T(\mathbf{X};\mathbf{x},\,\alpha_1,\lambda)$, where $F_T(t) = \Pr[T(\mathbf{X};\mathbf{x},\alpha_1,\lambda) \leq t] = \Pr[2nk(a_1\Lambda)^{-1} - \alpha_1 \leq t] = \Pr[\Lambda \geq 2nk[a_1(t+\alpha_1)]^{-1}] = 1 - F_\Lambda(2nk[a_1(t+\alpha_1)]^{-1})$ with $F_\Lambda(\cdot)$ being the distribution function of χ^2_{2nk}, is free of nuisance parameters. Second, at $\mathbf{X} = \mathbf{x}$, $T(\mathbf{x};\mathbf{x},\alpha_1,\lambda) = (a_1)^{-1}(a_1\alpha_1) - \alpha_1 = 0$, thus $T(\mathbf{x};\mathbf{x},\alpha_1,\lambda)$ is free of any unknown parameters. Third, $F_T(t) = \Pr[T(\mathbf{X};\mathbf{x},\alpha_1,\lambda) \leq t] = 1 - F_\Lambda(2nk[a_1(t+\alpha_1)]^{-1}$ is a decreasing function of α_1. Hence, $Q(\mathbf{X};\mathbf{x},\alpha_1,\lambda)$, $R(\mathbf{X};\mathbf{x},\alpha_1,\lambda)$, and $T(\mathbf{x};\mathbf{x},\alpha_1,\lambda)$ are bona fide generalized point estimator for α_1, generalized pivotal quantity for constructing interval estimation for α_1, and the generalized test variable for testing $H_0 : \alpha_1 \leq \alpha_{1_0}$ vs. $H_a : \alpha_1 > \alpha_{1_0}$, where α_{1_0} is a known quantity, respectively.

The generalized p-value for the test is given by $p^g_{\alpha_1} = \Pr(T(\mathbf{X};\mathbf{x},\alpha_1,\lambda) \leq 0 | \alpha_1 = \alpha_{1_0}) = \Pr[\Lambda \geq 2nk(a_1\alpha_{1_0})^{-1}]$. The p-value can be computed by numerical integration with respect to the independent chi-squared random variable Λ with $2nk$ degrees of freedom. The probability of this inequality can also be evaluated by the Monte Carlo method by generating a large number of random numbers from Λ, and then finding the fraction of random numbers for which the inequality is satisfied. In fixed level testing, one can use this p-value for rejecting the null hypothesis, if the generalized p-value is less than the desired nominal level γ.The equal tail $100(1-\gamma)\%$ generalized confidence interval for α_1, where $1-\gamma$ is the confidence coefficient, is given by $(R_{\gamma/2}(\alpha_1;a_1,\lambda)$,$R_{1-\gamma/2}(\alpha_1;a_1,\lambda))$, where $R_\xi(\alpha_1;a_1,\lambda)$ is the ξth quantile of the random variable $R(\mathbf{X};\mathbf{x},\alpha_1,\lambda)$. These quantiles can be evaluated by the Monte Carlo Simulations. This can be done by generating a large number of random numbers from Λ, evaluating $R(\mathbf{X};\mathbf{x},\alpha_1,\lambda)$, and then looking at the empirical distribution of $R(\mathbf{X};\mathbf{x},\alpha_1,\lambda)$.

Similarly, let $Y_{1:m:M} \leq Y_{2:m:M} \leq \cdots \leq Y_{m:m:M}$ denote m number of stress components observed from $\mathcal{GCEIE}$ (α_2,λ) before termination and $\mathbf{R}'_j = (R'_1, R'_2, ..., R'_m)$ denote the corresponding numbers of stress units removed (withdrawn) from the test. Furthermore, let $y_{1:m:M} \leq y_{2:m:M} \leq \cdots \leq y_{m:m:M}$ and $\mathbf{r}' = (r'_1, r'_2, ..., r'_m)$ be the observed ordered stress and observed stress removals, respectively. Let r'_i denote the number of stress components removed at the time of the ith failure of the stress component, $0 \leq r'_i \leq M - m - \sum_{l=1}^{i-1} r'_l, i = 2, 3, ..., m-1$ with $0 \leq r'_1 \leq M - m$ and $r'_m = M - m - \sum_{l=1}^{m-1} r'_l$, where r'_i's are non-pre-specified integers and m are pre-specified integers. Similarly, we can then show that $R(\mathbf{Y};\mathbf{y},\alpha_2,\lambda)$ $=2n(a_2\Delta)^{-1}$, where from (2.5) $\Delta = 2n(A_2\alpha_2)^{-1} \sim \chi^2_{2n}$, where $A_2$$= mk/V_\lambda$ with $V_\lambda = -\sum_{i=1}^{m}(1+R'_i)\ln[1-e^{-\lambda Q(1/Y_i)}]$, is a generalized pivotal quantity for constructing $100(1-\gamma)\%$ confidence interval for α_2, whereas $Q(\mathbf{Y};\mathbf{y},\alpha_2,\lambda)$ $=2n(a_2\Delta)^{-1}$ is the generalized point estimator for α_2. The generalized test variable for testing $H_0 : \alpha_2 \leq \alpha_{2_0}$ vs. $H_a : \alpha_2 > \alpha_{2_0}$ is $T(\mathbf{Y};\mathbf{y},\alpha_2,\lambda) = R(\mathbf{Y};\mathbf{y},\alpha_2,\lambda) - \alpha_2 = 2n(a_2\Delta)^{-1} - \alpha_2 = \alpha_2 A_2^{-1} a_2 - \alpha_{2}.$, and generalized p-value for this test is given by $\Pr\left(\sup_{H_0:\,\alpha_2\,\leq\,\alpha_{2_0}} T(\mathbf{Y};\mathbf{y},\alpha_2,\lambda) > 0\right) = \Pr(T(\mathbf{Y};\mathbf{y},\alpha_2,\lambda) \leq 0 | \alpha_2 = \alpha_{2_0}) = \Pr\left[\Delta \geq 2n(a_2\alpha_{1_0})^{-1}\right]$.

Generalized inference for $R_{s,k}$

Let $X_{DATA} = (\mathbf{X},\mathbf{Y})$, where $\mathbf{X} = \{X_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{Y} = (Y_1, ..., Y_m)$, and let $x_{DATA} = (\mathbf{x},\mathbf{y})$, where $\mathbf{x} = \{x_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{y} = (y_1, ..., y_m)$, be its observed value. The generalized point estimator for $R_{s,k}$, denoted by $Q(X_{DATA}; x_{DATA},$ $\boldsymbol{\beta},\lambda)$, where $\boldsymbol{\beta} = (\alpha_1, \alpha_2)$, can then be obtained by replacing α_1, α_2 in $R_{s,k}$ given in (2.13) with their generalized variables $Q(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda)$ and $Q(\mathbf{Y}; \mathbf{y}, \alpha_2, \lambda)$ as:

$$Q(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda) = \sum_{i=s}^{k} \sum_{j=0}^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} k-i \\ j \end{pmatrix} \times \frac{(-1)R(\mathbf{Y}; \mathbf{y}, \alpha_2, \lambda)}{R(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda)(i + j) + R(\mathbf{Y}; \mathbf{y}, \alpha_2, \lambda)}, \tag{3.1}$$

where $R(\mathbf{X}; \mathbf{x}, \alpha_1, \lambda) = 2nk(a_1\Lambda)^{-1}$ and $R(\mathbf{Y}; \mathbf{y}, \alpha_2, \lambda) = 2m(a_2\Delta)^{-1}$ with $\Lambda = 2nk(A_1\alpha_1)^{-1}$ $\sim \chi^2_{2nk}$ and $\Delta = 2m(A_2\alpha_2)^{-1} \sim \chi^2_{2m}$.

We are now interested making inferences such as point and interval estimation, and statistical tests for $R_{s,k}$ based on the generalized variable method. The random variable $Q(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda)$, also denoted by $\widehat{R}^G_{s,k}$, is a generalized point estimator which satisfy the three conditions to be a bona fide Generalized Point Estimator as presented in the subsection (3.1). Therefore, this would also serve as a generalized pivotal quantity $R(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda)$ and $T(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda) =$ $R(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda) - R_{s,k}$ is the generalized test variable. First, for fixed x_{DATA}, the distribution $H_T(t)$ of $T(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda)$, where $H_T(t) = \Pr[T(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda) \leq t = \Pr[R(X_{DATA};$ $x_{DATA}, \boldsymbol{\beta},\lambda) \leq t + R_{s,k}] = F_{R(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda)}(t + R_{s,k})$ with $F_{R(X_{DATA}; x_{DATA}, \boldsymbol{\beta},\lambda)}(\cdot)$ being the distribution function of $R(\mathbf{X};\mathbf{x},\alpha_1,\lambda$, is free of nuisance parameters. Second, at $X_{DATA} = x_{DATA}$,

$$
\begin{aligned}
T(x_{DATA};x_{DATA},\boldsymbol{\beta},\lambda) &= R(x_{DATA};x_{DATA},\boldsymbol{\beta},\lambda) - R_{s,k} \\
&= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix} \times \\
&\quad \frac{(-1)R(\mathbf{y};\mathbf{y},\alpha_2,\lambda)}{R(\mathbf{x};\mathbf{x},\alpha_1,\lambda)(i\ +\ j)\ \ +\ R(\mathbf{y};\mathbf{y},\alpha_2,\lambda)} - R_{s,k} \\
&= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)\alpha_2}{\alpha_1(i\ +\ j)\ \ +\ \alpha_2} - \\
&\quad \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^j\alpha_2}{\alpha_1(i\ +\ j)+\alpha_2} \\
&= 0
\end{aligned}
$$

thus $T(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda)$ is free of any unknown parameters.Third, $F_T(t) =$ $\Pr\left[T(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda) \le t\right] = \Pr[R(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda) \le t + R_{s,k}] =$ $F_{R(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda)}(t + R_{s,k})$ is a decreasing function of $R_{s,k}$. Hence, $Q(X_{DATA};x_{DATA},$ $\boldsymbol{\beta},\lambda)$, $R(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda)$, and $T(X_{DATA};x_{DATA},\boldsymbol{\beta},\lambda)$ are, respectively, bona fide generalized point estimator of $R_{s,k}$, generalized pivotal quantity for constructing interval estimation for $R_{s,k}$, and the generalized test variable for testing $H_0 : R_{s,k} \le R_0$ vs. $H_a : R_{s,k} > R_0$, where R_0 is a known quantity.

Generalized confidence interval for $R_{s,k}$

Given the specified significance level γ, the level $(1-\gamma)$ two–sided generalized confidence interval for $R_{s,k}$ can be derived as follows:

For mathematical tractability and simplicity, we write $R^{\alpha_1} = R(\mathbf{X};\mathbf{x},\alpha_1,\lambda) = 2nk(a_1\Lambda)^{-1}$ and $R^{\alpha_2} = R(\mathbf{Y};\mathbf{y},\alpha_2,\lambda) = 2m(a_2\Delta)^{-1}$ with $\Lambda = 2nk(A_1\alpha_1)^{-1} \sim \chi^2_{2nk}$ and $\Delta = 2m(A_2\alpha_2)^{-1} \sim \chi^2_{2m}$, and $A_1 = nk/W_\lambda$ with $W_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1+R_{ij})\ln(1-e^{-\lambda Q(1/X_{ij})})$ and $A_2 = m/V_\lambda$ with $V_\lambda = -\sum_{i=1}^{m}(1+R_i')\ln(1-e^{-\lambda Q(1/Y_i)})$. Hence, a generalized pivotal statistic for $R_{s,k}$ in (1.1) is given by

$$R^{R_{s,k},} = R(x_{DATA};x_{DATA},\boldsymbol{\beta},\lambda) = \sum_{i\,=\,s}^{k}\sum_{j\,=\,0}^{k\,-\,i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^j R^{\alpha_2}}{R^{\alpha_1}(i\ +\ j)\ \ +\ R^{\alpha_2}}. \tag{3.2}$$

Let $R^{R_{s,k},}_{\gamma/2} = R^{R_{s,k},}_{\gamma/2}(x_{DATA};,\widehat{\boldsymbol{\beta}}_{obs},\lambda)$ and $R^{R_{s,k},}_{1-\gamma/2} = R^{R_{s,k},}_{1-\gamma/2}(x_{DATA};,\widehat{\boldsymbol{\beta}}_{obs},\lambda)$,where $\widehat{\boldsymbol{\beta}}_{obs} = (\alpha_1,\alpha_2)$, satisfy

$$P[R^{R_{s,k},}_{\gamma/2} \le R^{R_{s,k},} \le R^{R_{s,k},}_{1-\gamma/2}] = 1-\gamma$$

The $\left(R^{R_{s,k},}_{\gamma/2}, R^{R_{s,k},}_{1-\gamma/2}\right)$is a $100(1-\gamma)\%$ lower confidence limit for $R_{s,k}$. That is, confidence bounds for $R_{s,k}$

$$\mathrm{CI}^{G,}_{R_{s,k},} = \left(R^{R_{s,k},}_{\gamma/2}, R^{R_{s,k},}_{1-\gamma/2}\right). \tag{3.3}$$

Generalized testing procedure for $R_{s,k}$

Construct a statistical testing procedure to assess whether the reliability function adheres to the required level. The one-sided hypothesis testing for $R_{s,k}$ is obtained using the generalized test variable $T^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta}) = Q^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta}) - R_{s,k}$, or simply $T^{R_{s,k}} = R^{R_{s,k}} - R_{s,k}$, where $T = T^S(\mathbf{X};\mathbf{x};\boldsymbol{\delta})$. Assuming that the required reliability is larger than R_0, where R_0 denotes the target value, the null hypothesis $H_0 : R_{s,k} \leq R_0^*$ and the

alternative hypothesis $H_a : R_{s,k} > R_0$ are constructed. Then, the generalized p-value, denoted by p_g, is derived as follows:

$$p_g = \Pr\left(\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)R(\mathbf{Y};\mathbf{y},\alpha_2,\lambda)}{R(\mathbf{X};\mathbf{x},\alpha_1,\lambda)(i+j) + R(\mathbf{Y};\mathbf{y},\alpha_2,\lambda)} > R_0\right). \tag{3.4}$$

This p-value can be either computed by numerical integration exact up to a desired level of accuracy or well approximated by a Monte Carlo method. When there are a large number of random numbers from various random variables, the latter method is more desirable and computationally more efficient. p is an exact probability of a well-defined extreme region of the sample space and measures the evidence in favor of the null hypothesis. This is an exact test in significance testing. In fixed level testing, one can use this p-value by rejecting the null hypothesis, if δ, where δ a desired nominal level .

The following algorithm is useful in constructing p_g.

Algorithm 3

Step 1: Given $\lambda, k, \gamma, m, n, R_0$, $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, ..., \mathbf{R}_k)$,and $\mathbf{R}' = (R'_1, R'_2, ..., R'_m)$,

where $\mathbf{R}_j = (R_{1j}, R_{2j}, ..., R_{nj})$ for $j = 1, 2, ..., k$

(a) The generation of data U_{ij} is by the uniform distribution $U(0, 1)$,for

$i = 1, 2, ..., n;\ j = 1, 2, ..., k$

(b) By the transformation of $Z_{ij} = Q^{-1}\left[\ln(1 - U_{ij}^{\frac{1}{\alpha_1}})^{-\lambda}\right]^{-1}$,

$i = 1, 2, ..., n; j = 1, 2, ..., k$

$\{Z_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ is a random sample from the $\mathcal{GCEIE}$ with density as (1.1).

(c) Set $X_{ij:n:N} = \frac{Z_{1j}}{n} + \frac{Z_{2j}}{(n-R_{1j}-1)} + ... + \frac{Zij}{\left[n-\sum_{l=1}^{i-1} R_{lj}-i+1\right]}$, for $i = 1, 2, ..., n$;

$j = 1, 2, ..., k$

$\{X_{ij:n:N}\}_{i=1,2,...,n;j=1,2,...,k}$ is the progressively type II right censored sample from a two-parameter $\mathcal{GCEIE}$ distribution with density as in (1.1).

Step 2: Compute the maximum likelihood estimate of α_1

$a_1 = nk/w_\lambda$, where $w_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1 + r_{ij})\ln(1 - e^{-\lambda Q(1/x_{ij})})$

Step 3: (a) Similarly, generate data U_i from the uniform distribution $U(0, 1)$,

for $i = 1, 2, ..., m$

(b) By the transformation of $Z'_{ij} = Q^{-1}\left[\ln(1-U_{ij}^{\frac{1}{\alpha_2}})^{-\lambda}\right]^{-1}, i = 1, 2, ..., m,$

$\{Z'_i\}_{i=1,2,...,m}$ is a random sample from the $\mathcal{GCEIE}$ with density as in (1.1).

(c) Set $Y_{i:m:M} = \frac{Z'_1}{m} + \frac{Z'_2}{(n-R'_1-1)} + ... + \frac{Z'i}{\left[m-\sum_{l=1}^{i-1} R'_l - i+1\right]}$, for $i = 1, 2, ..., m$

$\{Y_{i:m:M}\}_{i=1,2,...,m}$ is the progressively type II right censored sample from a two-parameter $\mathcal{GCEIE}$ distribution with density in (1.1).

Step 4: Compute the maximum likelihood estimate of α_2

$a_2 = m/v_\lambda$, where $v_\lambda = -\sum_{i=1}^{m}(1+r'_i)\ln(1-e^{-\lambda Q(1/y_i)})$

Step 5: For $l = 1 : L$

(a) Generate $\Lambda \sim \chi^2_{2nk}$ and $\Delta \sim \chi^2_{2n}$

(b) Compute the quantities $R^{\alpha_1} = 2nk(a_1\Lambda)^{-1}$ and $R^{\alpha_2} = 2m(a_2\Delta)^{-1}$

(c) Compute $R^{R_{s,k'}} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)R^{\alpha_2}}{R^{\alpha_1}(i+j)\ +R^{\alpha_2}}$

(end l loop)

Generalized p-value is estimated by the proportion of $R^{R_{s,k'}}$ which are greater than R_0. The $100(1-\gamma/2)$th and $100\gamma/2$th percentile of $R^{R_{s,k'}}$; $R_{\gamma/2}^{R_{s,k'}}$ and $R_{1-\gamma/2}^{R_{s,k'}}$, respectively; are the lower and upper bounds of the two-sided $1-\gamma$ confidence interval. That is, $\mathrm{CI}^G_{R_{s,k}} = \left(R_{\gamma/2}^{R_{s,k'}}, R_{1-\gamma/2}^{R_{s,k'}}\right)$.

Coverage probabilities of the generalized confidence intervals and powers of generalized tests are computed using the Monte Carlo method given in the following algorithm.

Algorithm 4

For given $\boldsymbol{\beta} = (\alpha_1, \alpha_2), \lambda, k, \gamma, m, n, R_0$, $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, ..., \mathbf{R}_k)$,

and $\mathbf{R}' = (R_1', R_2', ..., R_m')$,

where $\mathbf{R}_j = (R_{1j}, R_{2j}, ..., R_{nj})$ for $j = 1, 2, ..., k$

For $p = 1 : P$

1. Generate $\Lambda \sim \chi^2_{2nk}$ and $\Delta \sim \chi^2_{2n}$

2. Set $\alpha_1 = 2nk(a_1\Lambda)^{-1}$ and $\alpha_2 = 2m(a_2\Delta)^{-1}$,

3.Use **Algorithm 3** to construct a $(1-\gamma)$ confidence interval C_p,

$$\xi_{R_{s,k}} = \begin{cases} 1, \text{if } C_p \text{ contains } R_{s,k} \\ 0, \text{if } C_p \text{ does not contain } R_{s,k} \end{cases},$$

4.Use **Algorithm 3** again to compute the generalized p-value, p_g.

$$\eta_{R_{s,k}} = \begin{cases} 1, \text{if } p_g < \gamma \\ 0, \text{if } p_g > \gamma \end{cases}.$$

(end p loop)

The proportion $\frac{1}{P}\sum_{p=1}^{P} \xi_{R_{s,k}}$ is the estimated coverage probability of the generalized confidence interval. It is evident that sometimes the coverage of the generalized confidence interval may not equal to the nominal level. But, when generalized confidence interval reduces to traditional classical confidence intervals, theoretical results are available on coverage properties of generalized confidence intervals. The proportion $\frac{1}{P}\sum_{p=1}^{Q} \eta_{R_{s,k}}$ is the estimated power of the generalized test.

CHAPTER IV

THE BAYESIAN METHOD

Preview

A Bayesian approach for statistical inferences of the reliability parameter of the multicomponent system $R_{s,k}$, contrasting the conventional classical approach and the newly introduced generalized variable approach, is introduced and discussed, and then the Monte Carlo method and commonly used Markov Chain Monte Carlo (MCMC) methods are introduced in this section.

The Bayesian statistics and Markov Chain Monte Carlo (MCMC) methods have been twins in statistical arena for more than 20 years as the former covers the philosophical aspect of the Bayesian approach and the latter is well suited for the calculations of probabilities and does not rely on conjugacy or asymptotic moment-based approximations. When marginal posterior distributions are impossible to be summarized analytically, Bayesian statisticians (or simply Bayesians or practitioners from the Bayesian School) tend to numerical approaches for the summarization of these marginal posterior distributions. The Monte Carlo method is the commonly used numerical approach in the Bayesian statistics. In order to use this method, it is necessary to have well-suited algorithms; there are two well-known algorithms:

1. the Gibbs sampling — uses a sequence of draws from conditional posterior
   distribution to characterize the joint posterior distribution: special case of

Metropolis-Hastings algorithm

2. the Metropolis-Hastings algorithm — used for all sorts of numerical integration and optimization.

For more details on this algorithm, interested parties are referred to Metropolis et al. (1953), Hastings (1979), and Chib and Greenberg (1995). In the Gibbs sampling technique incorporated with the Meta-analysis — a statistical approach adopted to summarize and integrate a collection studies using many familiar techniques to draw general conclusions that was first performed by Karl Pearson in 1904 — the information from several $\mathcal{GCEIE}$ populations are combined to estimate the common $R_{s,k}$ when scale common parameter λ for the strength and stress is known, and the unknown shape parameters, where common for strength components a_1, but different for the stress component a_2.

The marginal posterior distribution of a parameter of interest is the target distribution in the Bayesian analysis for the estimation of the parameter of interest. But, there are few possible difficulties incorporated with handling those distributions:

1. when the marginal posterior distribution is a non-standard distribution,
2. when the marginal posterior distribution is a poly standard distribution,
3. when the marginal posterior distribution is a poly non-standard distribution,
4. when the dimensionality problem causes the numerical integration is difficult

Bayes estimation

Now, we deal with the problem of estimating the parameters a_1 and a_2, and the reliability function $R_{s,k}$ of $\mathcal{GCEIE}$ distribution under mainly a SE loss function and LINEX loss functions found in CHAPTER I. Similar procedure can be adopted for estimating the reliability function $R_{s,k}$ under various other loss functions described in the Introduction Section. The Gibbs sampler provides considerable and fair robust solutions for such drastic and difficult situations. In this section, we assume that the parameters (α_1, α_2) are random variables and have statistically independent gamma prior distributions with parameters $(a_i, b_i), i = x, y$, respectively., that is, prior distributions for a_1 and a_2 are taken to be $\mathcal{G}(a_i, b_i), i = x, y$. The Gibbs sampler provides considerable and fair robust solutions for such drastic and difficult situations. In this section, we assume that the parameters (α_1, α_2) are random variables and have statistically independent gamma prior distributions with parameters $(a_i, b_i), i = x, y$, respectively. The pdf of a gamma random variable X with parameters (a_i, b_i) is

$$f(x) = \frac{b_i^{a_i}}{\Gamma(a_i)} x^{a_i - 1} e^{-x b_i}, \; x > 0, a_i, b_i > 0. \tag{4.1}$$

Then, the joint posterior density function of (α_1, α_2) is

$$\pi(\alpha_1, \alpha_2 | \lambda, \mathbf{x}, \mathbf{y}) = \frac{(b_1 + w_\lambda)^{nk + a_1} (b_2 + v_\lambda)^{m + a_2}}{\Gamma(nk + a_1)\Gamma(m + a_2)} \alpha_1^{nk + a_1 - 1} \alpha_2^{m + a_2 - 1} e^{-\alpha_1(b_1 + w_\lambda) - \alpha_2(b_2 + v_\lambda)} \tag{4.2}$$

where $\mathbf{x} = \{x_{ij}\}_{i=1,2,\ldots,n; j=1,2,\ldots,k}; \mathbf{y} = \{y_i\}_{i=1,2,\ldots,m}$; $w_\lambda = -\sum_{i=1}^{n} \sum_{j=1}^{k} (1 + r_{ij}) \ln[1 - e^{-\lambda Q(1/x_{ij})}]$,

$v_\lambda = -\sum_{i=1}^m (1+r_i') \ln[1-e^{-\lambda Q(1/y_i)}]$ with $\mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$ and $\mathbf{r}' = (r_1', r_2', ..., r_m'), j = 1, 2, ..., k$. Furthermore, the marginal posterior densities of α_1 and α_2 have gamma distributions with parameters $(nk + a_1, b_1 + w_\lambda)$ and $(m + a_2, b_2 + v_\lambda)$. The Bayes estimate of $R_{s,k}$ under the SE loss function, say $\widehat{R}_{s,k}^{B,SE}$, is

$$\begin{aligned} \widehat{R}_{s,k}^{B,SE} &= E_{\pi(\alpha_1,\alpha_2|\lambda,x_{DATA})}[R_{s,k}|x_{DATA}] \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \int_0^\infty \int_0^\infty \frac{\alpha_2}{\alpha_1(i+j)+\alpha_2} \times \\ &\quad \pi(\alpha_1, \alpha_2|\lambda, x_{DATA}) d\alpha_1 d\alpha_2, \end{aligned} \tag{4.3}$$

where $x_{DATA} = (\mathbf{x}, \mathbf{y})$ with $\mathbf{x} = \{x_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{y} = (y_1, ..., y_m)$ is the observed (or realized) value of $X_{DATA} = (\mathbf{X}, \mathbf{Y})$ with $\mathbf{X} = \{X_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{Y} = (Y_1, ..., Y_m)$.

We consider a one-to-one transformation $u_1 = \alpha_2/(\alpha_1(i+j)+\alpha_2)$ and $u_2 = \alpha_1(i+j)+\alpha_2$. Then, $0 < u_1 < 1$, $0 < u_2 < \infty, \alpha_1 = u_2(1-u_1)/(i+j), \alpha_2 = u_1 u_2$ and the Jacobian of (u_1, u_2) is $J(u_1, u_2) = -u_2/(i+j)$. Therefore, the double integral in (4.3) can be rewritten as

$$\begin{aligned} & \frac{(b_1+w_\lambda)^{nk+a_1}(b_2+v_\lambda)^{m+a_2}}{\Gamma(nk+a_1)\Gamma(m+a_2)(i+j)^{nk+a_1}} \left\{ \int_0^1 \int_0^\infty u_1^{m+a_2}(1-u_1)^{nk+a_1-1} u_2^{p-1} \times \right. \\ & \left. \exp\left(-u_2 \left\{ \frac{(1-u_1)(b_1+w_\lambda)}{(i+j)} + u_1(b_2+v_\lambda) \right\} \right) du_1 du_2 \right\} \\ = & \frac{(1-z)^{m+a_2}}{B(nk+a_1, m+a_2)} \int_0^1 u_1^{m+a_2}(1-u_1)^{nk+a_1-1}(1-u_1 z)^{-p} du_1, \end{aligned} \tag{4.4}$$

where $z = 1 - ((b_2 + v_\lambda)(i+j)/(b_1 + w_\lambda))$ and $p = nk + a_1 + m + a_2$. The integral representation of the hypergeometric series is (this was given by Euler in 1748 and implies Euler's and Pfaff's hypergeometric transformations. See Section 9.1 in Gradshteyn and Ryzhik (1994)

$$ {}_2F_1(\alpha, \beta; \gamma, z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt, $$

$$ |z| < 1 or\, |z| = 1, Re(\gamma) > Re(\beta) > 0. $$

Notice that the hypergeometric series converges in the unit circle $|z| < 1$. Then,

$$ \widehat{R}^B_{s,k} = \begin{cases} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} k-i \\ j \end{pmatrix} \frac{(-1)^j (1-z)^{(n+a_2)}(n+a_2)}{p} \times_2 F_1(p, m+a_2+1; p+1, z) \\ \text{if} \quad |z| < 1 \\ \sum_{i=s}^{k} \sum_{j=0}^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} k-i \\ j \end{pmatrix} \frac{(-1)^j (m+a_2)}{(1-z)^{nk+a_1} p} \times_2 F_1(p, nk+a_1; p+1, \frac{z}{z-1}) \\ \text{if} \quad z < -1. \end{cases} \tag{4.5} $$

The Bayes estimate of $R_{s,k}$ under the LINEX loss function, say $\widehat{R}^{B,LINEX}_{s,k}$, is

$$
\begin{aligned}
\widehat{R}_{s,k}^{B,LINEX} &= E_{\pi(\alpha_1,\alpha_2|\lambda,x_{DATA})}[\exp\{cR_{s,k}|x_{DATA}\}] \\
&= \int_0^\infty\int_0^\infty \exp\left\{-\alpha_1(b_1+w_\lambda)-\alpha_2(b_2+v_\lambda)+\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\times\right. \\
&\quad (-1)^j\frac{\alpha_2}{\alpha_1(i+j)+\alpha_2}\times\frac{(b_1+w_\lambda)^{nk+a_1}(b_2+v_\lambda)^{m+a_2}}{\Gamma(nk+a_1)\Gamma(m+a_2)}\times \\
&\quad \alpha_1^{nk+a_1-1}\alpha_2^{m+a_2-1}d\alpha_1 d\alpha_2,
\end{aligned}
\tag{4.6}
$$

where $x_{DATA}=(\mathbf{x},\mathbf{y})$ with $\mathbf{x}=\{x_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{y}=(y_1,...,y_m)$ is the observed (or realized) value of $X_{DATA}=(\mathbf{X},\mathbf{Y})$ with $\mathbf{X}=\{X_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$ and $\mathbf{Y}=(Y_1,...,Y_m)$.

Based on a type II censored sample, we obtained several Bayesian estimates, based on the type-II progressively censored data with random removals, of the reliability function $\widehat{R}_{s,k}$. These Bayesian estimates are derived against SE and LINEX loss functions. It is easily observed that all these estimates are in the form of ratio of two integrals for which simplified closed forms are not available. Thus to evaluate these estimates in practice intensive numerical techniques are required. Instead, one can apply approximation methods to evaluate these estimates such as Lindley's approximation and Markov Chain Monte Carlo (MCMC). However, the Bayes estimate under the SE loss function is obtained in the closed form, and alternative methods are also used to see how good the approximate methods compared with the exact one. We completely use the Lindley's method for the Bayes estimate under the LINEX loss function as has no closed forms. If these result are close, then it will be encouraging to use the approximate methods when the exact form can not be obtained in the all parameters are unknown case. These estimators will be compared in the simulation study section. Next, we give the Bayes estimates of $R_{s,k}$ using the Lindley's approximation and MCMC method.

Lindley's approximation

Lindley (1980) introduced an approximate procedure for the computation of the ratio of two integrals. This procedure, applied to the posterior expectation of the function $U(\boldsymbol{\theta})$ for a given $\mathbf{x}$, is

$$E(U(\boldsymbol{\theta})|\mathbf{x}) = \frac{\int_{\Theta} u(\boldsymbol{\theta}) e^{Q(\theta)} d\theta}{\int_{\Theta} e^{Q(\theta)} d\theta}, \tag{4.7}$$

where $Q(\theta) = l(\theta) + \rho(\theta)$, $l(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ, $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_L), i, j, k, l = 1, 2, ..., L$, and Θ is the parameter space. Using Lindley's approximation, $E(U(\boldsymbol{\theta})|\mathbf{x})$ is approximately estimated by

$$\begin{aligned} E(U(\boldsymbol{\theta})|\mathbf{x}) \;=\; & \left| u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right|_{\widehat{\boldsymbol{\theta}}} \\ & + \text{terms of order } n^{-2} \text{ or smaller}, \end{aligned} \tag{4.8}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_L), i, j, k, l = 1, 2, ..., L$, $\widehat{\boldsymbol{\theta}}$ is the MLE of θ, $u = u(\boldsymbol{\theta})$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $L_{ijk} = \partial^3 l/\partial\theta_i\partial\theta_j\ \partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$ and $\sigma_{ij} = (i, j)$th element in the inverse of the matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

For the two parameter case $\boldsymbol{\theta} = (\theta_1, \theta_2)$, Lindley's approximation leads to

$$\widehat{u}_{Lin} = u(\theta) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right],$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} u_{ij}\tau_{ij}, Q_{ij} = \partial^{i+j}u/\partial^i\theta_1\partial^j\theta_2$ for $i,j = 0,1,2,3, i+j = 3, u_i = \partial u/\partial\theta_i, u_{ij}$ $= \partial^2 u/\partial\theta_i\partial\theta_j$ for $i,j = 1,2$, and $B_{ij} = (u_i\tau_{ii} + u_j\tau_{ij})\tau_{ii}, C_{ij} = 3u_i\tau_{ii}\tau_{ij} + u_j(\tau_{ii}\tau_{ij} + 2\tau_{ij}^2)\tau_{ij}$ for $i \neq j$. τ_{ij} is the (i,j)th element in the inverse of matrix $Q^* = (Q^*_{ij}), i,j = 1,2$ such that $Q^*_{ij} =$ $\partial^2 Q/\partial\theta_i\partial\theta_j$. The approximate Bayes estimate $\widehat{u}_{Lin}$ is evaluated at $\widetilde{\boldsymbol{\theta}} = (\widetilde{\theta}_1, \widetilde{\theta}_2)$ which is the mode of the posterior density.

In our case, $\boldsymbol{\theta} = (\theta_1, \theta_2) = \boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and

$$Q = \ln \pi(\alpha_1, \alpha_2|\lambda, \mathbf{x}, \mathbf{y}) \propto (nk + a_1 - 1)\ln a_1 + (m + a_2 - 1)\ln a_2 - a_1(b_1 + w_\lambda) - \alpha_2(b_2 + v_\lambda),$$

where $\mathbf{x} = \{x_{ij}\}_{i=1,2,\ldots,n; j=1,2,\ldots,k}; \mathbf{y} = \{y_i\}_{i=1,2,\ldots,m}$; $w_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1 + r_{ij})\ln[1 - e^{-\lambda Q(1/x_{ij})}]$, $v_\lambda = -\sum_{i=1}^{m}(1 + r'_i)\ln[1 - e^{-\lambda Q(1/y_i)}]$ with $\mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$ and $\mathbf{r}' = (r'_1, r'_2, ..., r'_m), j = 1, 2, ..., k$.

The posterior mode of (α_1, α_2) is obtained from Q and is given by

$$\widetilde{\alpha}_1 = \frac{nk + a_1 - 1}{b_1 + w_\lambda} \quad \text{and} \quad \widetilde{\alpha}_2 = \frac{m + a_2 - 1}{b_2 + v_\lambda}.$$

We obtain that $\tau_{11} = \alpha_1^2/(nk + a_1 - 1), \tau_{22} = \alpha_2^2/(m + a_2 - 1), \tau_{12} = \tau_{21} = 0, Q_{12} = Q_{21} = 0, Q_{03} = 2/(m+a_2-1)/\alpha_2^3, Q_{30} = 2/(nk+a_1-1)/\alpha_1^3, B_{12} = u_1\tau_{11}^2, B_{21} = u_2\tau_{22}^2, B = u_{11}\tau_{11} + u_{22}\tau_{22}$, and

$$
\begin{aligned}
u_1 &= \frac{\partial R_{s,k}}{\partial a_1} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{(-1)^{j+1}(i+j)\alpha_2}{(\alpha_1(i+j)+\alpha_2)^2}, \\
u_2 &= \frac{\partial R_{s,k}}{\partial a_2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{\alpha_1(i+j)(-1)^{j}}{(\alpha_1(i+j)+\alpha_2)^2}, \\
u_{11} &= \frac{\partial^2 R_{s,k}}{\partial^2 a_1^2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{2(-1)^{j}(i+j)^2\alpha_2}{(\alpha_1(i+j)+\alpha_2)^3}, \\
u_{12} &= u_{21} = \frac{\partial^2 R_{s,k}}{\partial a_1 \partial a_2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix} \times \\
&\quad \frac{2(-1)^{j+1}(i+j)\alpha_2(\alpha_1(i+j)-\alpha_2)}{(\alpha_1(i+j)+\alpha_2)^3}, \\
u_{22} &= \frac{\partial^2 R_{s,k}}{\partial^2 a_2^2} = \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\begin{pmatrix} k-i \\ j \end{pmatrix}\frac{2(-1)^{j+1}(i+j)\alpha_1}{(\alpha_1(i+j)+\alpha_2)^3}.
\end{aligned}
$$

Therefore, the approximate Bayes estimate of the reliability function $R_{s,k}$ under SE loss function is given by

$$\widehat{R}_{s,k}^{B,Lin(SE)} = R_{s,k}\Big|_{(\alpha_1,\alpha_2)=(\widetilde{\alpha}_1,\widetilde{\alpha}_2)} + \frac{1}{2}\left[\frac{\alpha_1^2 u_{11} + 2\alpha_1 u_1}{nk + a_1 - 1} + \frac{\alpha_2^2 u_{22} + 2\alpha_2 u_2}{m + a_2 - 1}\right]_{(\alpha_1,\alpha_2)=(\widetilde{\alpha}_1,\widetilde{\alpha}_2)}, \tag{4.9}$$

where u_1, u_2, u_{11}, and u_{22} are given above.

With the same argument, we can obtain Bayes estimators under the LINEX loss function of the reliability function from Eq. ( ). They are obtained by the following forms:

if $u(\alpha_1, \alpha_2) = \exp[-cR_{s,k}]$, then

$$\begin{aligned}
u_1^* &= \frac{\partial \exp[-cR_{s,k}]}{\partial a_1} = -c\exp[-cR_{s,k}] \times \frac{\partial R_{s,k}}{\partial a_1} = -c\exp[-cR_{s,k}] \times u_1, \\
u_2^* &= \frac{\partial \exp[-cR_{s,k}]}{\partial a_2} = -c\exp[-cR_{s,k}] \times \frac{\partial R_{s,k}}{\partial a_2} = -c\exp[-cR_{s,k}] \times u_2, , \\
u_{11}^* &= \frac{\partial^2 \exp[-cR_{s,k}]}{\partial^2 a_1^2} = \frac{\partial}{\partial a_1}\{-c\exp[-cR_{s,k}] \times u_1\} = -c\left\{\exp[-cR_{s,k}]u_{11} + u_1 u_1^*\right\} \\
u_{12}^* &= u_{21}^* = \frac{\partial^2 \exp[-cR_{s,k}]}{\partial a_1 \partial a_2} = \frac{\partial}{\partial a_2}\{-c\exp[-cR_{s,k}] \times u_1\} \\
&= -c\left\{\exp[-cR_{s,k}]u_{12} + u_1 u_2^*\right\} \\
u_{22}^* &= \frac{\partial^2 \exp[-cR_{s,k}]}{\partial^2 a_2^2} = \frac{\partial}{\partial a_2}\{-c\exp[-cR_{s,k}] \times u_2\} = -c\left\{\exp[-cR_{s,k}]u_{22} + u_2 u_2^*\right\}
\end{aligned}$$

The approximate Bayes estimate of the reliability function $R_{s,k}$ under a LINEX loss function is given by

$$\widehat{R}_{s,k}^{B,Lin(LINEX)} = \frac{-1}{c}\ln\left\{E_{\pi(\alpha_1,\alpha_2|\lambda,x_{DATA})}[\exp(cR_{s,k}|x_{DATA})]\right\} \tag{4.10}$$

$$
\begin{aligned}
E_{\pi(\alpha_1,\alpha_2|\lambda,x_{DATA})}[\exp(cR_{s,k}|x_{DATA})] &= \exp(cR_{s,k}|x_{DATA}) + \frac{1}{2}\left[B^* + Q^*_{30}B^*_{12} + \right. \\
&\quad \left. Q^*_{21}C^*_{12} + Q^*_{12}C^*_{21} + Q^*_{03}B^*_{21}\right] \qquad (4.11)
\end{aligned}
$$

where $B^* = \sum_{i=1}^{2}\sum_{j=1}^{2} u^*_{ij}\tau_{ij}$, $Q_{ij} = \partial^{i+j}u^*/\partial^i\theta_1\partial^j\theta_2$ for $i,j = 0,1,2,3, i+j = 3, u^*_i = \partial u^*/\partial\theta_i, u^*_{ij}$ $= \partial^2 u^*/\partial\theta_i\partial\theta_j$ for $i,j = 1,2$, and $B^*_{ij} = (u^*_i\tau_{ii} + u^*_j\tau_{ij})\tau_{ii}, C^*_{ij} = 3u^*_i\tau_{ii}\tau_{ij} + u^*_j(\tau_{ii}\tau_{ij} + 2\tau^2_{ij})\tau_{ij}$ for $i \neq j$. τ_{ij} is the (i,j)th element in the inverse of matrix $Q^* = (Q^*_{ij}), i,j = 1,2$ such that $Q^*_{ij} = \partial^2 Q/\partial\theta_i\partial\theta_j$, and $\boldsymbol{\theta} = (\theta_1,\theta_2) = \boldsymbol{\alpha} = (\alpha_1,\alpha_2)$.

Markov chain Monte Carlo (MCMC) method

The MCMC algorithm is used for computing the Bayes estimates of the parameters α_1 and α_2 as well as the reliability function $R_{s,k}$. The joint posterior density function of α_1 and α_2 is given in (4.2). It is easily seen that the marginal posterior density functions of α_1 and α_2 are, respectively,

$$
\alpha_1|\lambda,\mathbf{x},\mathbf{y} \sim \mathcal{G}(nk + a_1, b_1 + w_\lambda) \quad \text{and} \quad \alpha_2|\lambda,\mathbf{x},\mathbf{y} \sim \mathcal{G}(m + a_2, b_2 + v_\lambda), \qquad (4.12)
$$

where $\mathbf{x} = \{x_{ij}\}_{i=1,2,...,n;j=1,2,...,k}$; $\mathbf{y} = \{y_i\}_{i=1,2,...,m}$; $w_\lambda = -\sum_{i=1}^{n}\sum_{j=1}^{k}(1 + r_{ij})\ln[1 - e^{-\lambda Q(1/x_{ij})}]$, $v_\lambda = -\sum_{i=1}^{m}(1+r'_i)\ln[1-e^{-\lambda Q(1/y_i)}]$ with $\mathbf{r}_j = (r_{1j}, r_{2j}, ..., r_{nj})$ and $\mathbf{r}' = (r'_1, r'_2, ..., r'_m), j = 1,2,...,k$.

In the event that the conditional posterior distribution of any parameter to be estimated is not in the closed form or well-known distribution, we then consider the Metropolis-Hastings algorithm to generate samples from the conditional posterior distributions and then compute the Bayes estimates. The Metropolis-Hastings (Metropolis et al. 1953) algorithm generate samples from an arbitrary proposal distribution (i.e., a Markov transition kernel), where most of the time the samples are drawn from normal distribution. So, as suggested by Tierney (1994), a common way to solve this problem is to use the

hybrid algorithm by combining a Metropolis sampling with the Gibbs sampling scheme using normal proposal distribution.

We assume that α_1 and α_2 can be generated from $\mathcal{G}(nk + a_1, b_1 + w_\lambda)$ and $\mathcal{G}(m + a_2, b_2 + v_\lambda)$, respectively, using a direct random generation scheme (see, for example, Devroye 1986) or a Markov Chain Monte Carlo (MCMC) sampling algorithm (see Gelfand and Smith 1990 for the Gibbs sampler, and Tierney 1994 for the Metropolis-Hastings algorithm).

Step 1: Set $l = 1$.

Step 2: Generate $\alpha_1^{(l)}$ from $\mathcal{G}(nk + a_1, b_1 + w_\lambda)$.

Step 3:Generate $\alpha_2^{(l)}$ from $\mathcal{G}(m + a_2, b_2 + v_\lambda)$.

Step 4: Compute the $R_{s,k}^{(l)}$ at $(\alpha_1^{(l)}, \alpha_2^{(l)})$

Step 5: Set $l = l + 1$.

Step 6: Repeat Steps 2 through 5, L times, and obtain the posterior sample

$R_{s,k}^{(l)}, l = 1, ..., L.$

Now the approximate posterior mean, and posterior variance of $R_{s,k}$ become

$$\widehat{E}(R_{s,k}|x_{DATA}) = \frac{1}{L-S}\sum_{l=S+1}^{L} R_{s,k}^{(l)},$$

where $\widehat{R}_{s,k}^{B,MC^2} = \widehat{E}(R_{s,k}|x_{DATA})$ is the Bayes estimate of $R_{s,k}$, and

$$\widehat{V}(R_{s,k}|x_{DATA}) = \frac{1}{L-S}\sum_{l=S+1}^{L} (R_{s,k}^{(l)} - \widehat{E}(R_{s,k}|x_{DATA}))^2,$$

respectively. Then a $100(1-\gamma)\%$ HPD interval (HPDI) of $R_{s,k}$ can be approximated (Chen and Shao 1999) by

$$C_{p^*}(L)_{R_{s,k}} = \left(R_{s,k}^{(p^*)}, R_{s,k}^{(p^*+[(1-\gamma)L])}\right), \tag{4.13}$$

where p^* is chosen so that

$$R_{s,k}^{(p^*+[(1-\gamma)L])} - R_{s,k}^{(p^*)} = \min_{1\, \leq\, p \leq\, [(1-\gamma)L]} \left(R_{s,k}^{(p^*+[(1-\gamma)L])} - R_{s,k}^{(p^*)}\right).$$

Furthermore, approximate $100(1-\gamma)\%$ Bayesian credible interval (BCI) of Ψ can be obtained by

$$\mathrm{BCI}_{R_{s,k}} = \widehat{E}(R_{s,k}|x_{DATA}) \pm Z_{\gamma/2}\sqrt{\frac{\widehat{V}(R_{s,k}|x_{DATA}}{L}}, \tag{4.14}$$

where Z_ζ is the ζ^{th} quantile of the standard normal distribution and S is the burn-in period. It well known that rapid convergence is facilitated by choosing appropriate starting values. In order to guarantee the convergence and to remove the affection of the selection of initial value, the first S simulated variates are discarded. Then the selected sample are $\alpha_1^{(l)}$ and $\alpha_2^{(l)}, l = 1, ..., L.$, for sufficiently large L, forms an approximate posterior sample which can be used to develop the Bayesian inference. Furthermore,

Similarly, the Bayes estimate of $R_{s,k}$ under a LINEX loss function is given by

$$\widehat{R}_{s,k}^{B,MC^2} = -\frac{1}{c}\ln\left\{\frac{1}{L-S}\sum_{l=S+1}^{L}\exp[-cR_{s,k}^{(l)}]\right\}, \tag{4.15}$$

and in a similar fashion, we can easily find the BCI as well as HPDI $R_{s,k}$ under LINEX function.

CHAPTER V

EXAMPLES

Practical application study

The monthly water capacity of the Shasta reservoir of the Shasta Dam (USBR SHA operated by the U.S. Bureau of Reclamation, United States Department of the Interior) in Sacramento, California, USA, especially the month of April for the maximum water level, and the mean annual capacity from 1974 to 2016 are considered (see, http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA; **Source:** *California Data Exchange Center, Department of Water Resources (DWR), Government of California)*. The maximum and the minimum water levels of the reservoir are generally observed on April and October (or November), respectively. To take the precautions for the excessive drought, the following scenario can be constructed. In the five-year period, if the water capacity of the reservoir on each April is more than the average water capacity of the previous year (which is the preceding year of the five-year period) at least three (3) times, it is claimed that there will be no excessive drought in the months of October and November afterwards. Using these data, an s-out-of-$k : G$ system as given above has the following description.

We assume that $s = 3$ and $k = 5$, and X denotes the water capacity in April, and this has been taken fro seven ($N = 7$) five-year periods such as $1975-1979$, $1981-1985$, $1987-1991$, $1993-1997$, $1999-2003$, $2005-2009$, and $2011-2016$ thus X_{ij} represents the water capacity in April for the

jth year of the ith five-year period; where $i = 1, 2, ..., N = 7, j = 1, 2, ...k = 5$. Nevertheless, due to the time limitation and/or other restrictions (such as financial, material resources, mechanical or experimental difficulties) on data collection, we observe type-II progressively censored data with random removals, thus we have the $X'_{ij}s$ for $i = 1, 2, ..., n = 4, j = 1, 2, ...k = 5$ with random removals $\mathbf{R} = (R_1 = 2, R_2 = 0, R_3 = 0, R_4 = 1)$ creating four $(n = 4)$ five-year periods $1975 - 1979$, $1993 - 1997$, $1999 - 2003$, and $2005 - 2009$ Similarly, Y_i is the mean annual water capacity of the ith year in-between two consecutive five-year periods, where $i = 1, 2, ..., M = 7$, but again due to the restrictions on data collection and to keep the consistency with the water capacity in April of each five-year period, we consider the mean annual capacity of only four $(m = 4)$ years such as 1974, 1992, 1998, and 2004. To remove (or to reduce) the dependency between X_{ij} and Y_i; the years of Y_i are not used for obtaining the data X_{ij}. Thus, we obtain the 3-out-of-5 : G system and observed data $(\mathbf{X}, \mathbf{Y})$. For computational ease, all of the values divided by the total capacity of Shasta reservoir 4:552:000 acre-foot and these transformed data are obtained as

$$\mathbf{X} = \begin{pmatrix} 0.9366 & 0.7763 & 0.9150 & 0.9463 & 0.8649 \\ 0.9350 & 0.9124 & 0.8831 & 0.9439 & 0.9966 \\ 0.9243 & 0.8913 & 0.8570 & 0.6490 & 0.6587 \\ 0.9372 & 0.9754 & 0.8322 & 0.5292 & 0.5849 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 0.4529 \\ 0.8222 \\ 0.6730 \\ 0.7985 \end{pmatrix}$$

First we want to check whether the $\mathcal{GCEIE}$ distribution fits the data $(\mathbf{X}, \mathbf{Y})$ or not. For this reason, the MLE of the unknown parameters are obtained separately for $\mathbf{X}$ and $\mathbf{Y}$.

In the case of real-world data, use the least squares estimation method which is based on the minimum Error Sum of Squares (SSE) for various values of λ and the "shape-first" approach (that is to fit the shape parameter λ before fitting the parameter α) to fit the optimal value of λ and estimate of α such that SSE is minimized for progressively type-II right censored data. Then, λ is defined as known. The procedure is as follows:

Step 1. Let $X_j \sim \mathcal{GCEIE}\,(\alpha_1, \lambda), j = 1, 2, ..., k$ whose common *pdf* is given by

$$\begin{aligned} f_X(x; \alpha_1, \lambda) &= \alpha_1 \lambda Q^{'}(1/x) \exp\{-\lambda Q(1/x)\}[1 - \exp\{\lambda Q(1/x)\}]^{\alpha_1 - 1}; \\ x &> 0,\ \alpha_1 > 0,\ \lambda > 0, \end{aligned}$$

and the common *cdf* is

$$F_X(x; \alpha_1, \lambda) = 1 - [1 - \exp\{\lambda Q(1/x)\}]^{\alpha_1}\,; x > 0,\ \alpha_1 > 0,\ \lambda > 0,$$

and $F_X(x; \alpha_1, \lambda)$ satisfies

$$\ln\left[1 - F_X(x; \alpha_1, \lambda)\right] = \alpha_1 \ln[1 - \exp\{\lambda Q(1/x)\}],\ x > 0,\ \alpha_1 > 0,\ \lambda > 0,$$

Consider that $X_{1:n:N} \leq X_{2:n:N} \leq \cdots \leq X_{n:n:N}$ is the corresponding progressively type-II right censored sample, with observed censoring scheme $\mathbf{r} = (r_1, r_2, ..., r_n)$. The expectation of $F_X(x_{i:n:N}; \alpha_1, \lambda)$ is $1 - \prod_{j\,=\,n-i+1}^{n} (a_j/(a_j+1)), i = 1, ..., n,$ where $a_j = j + \sum_{i\,=\,n-j+1}^{n} R_i$ (see, Gail and Gastwirth 1978). By using the approximate equation $\ln\left(1 - \left(1 - \prod_{j\,=\,n-i+1}^{n} (a_j/(a_j+1))\right)\right) \approx \alpha_{1_i} \times$ $\ln[1 - \exp\{\lambda Q(1/x_{i:n:N})\}],\ i = 1, ..., n,$ we get

$$\alpha_{1_i} \approx -\frac{\ln\left(1 - \left(1 - \prod_{j\,=\,n-i+1}^{n} (a_j/(a_j+1))\right)\right)}{\ln[1 - \exp\{\lambda Q(1/x_{i:n:N})\}]} \text{ for } i = 1, ..., n.$$

Then using the least squares estimation method for various values of λ and the "shape-first" approach to fit the optimal value of λ,calculate the SSE for given each value of λ, that is,

$$\begin{aligned} SSE_\lambda &= \sum_{i=1}^{n} (\alpha_{1_i} - \widehat{\alpha}_1)^2, \text{ where } \widehat{\alpha}_1 = \frac{nk}{w_\lambda} \\ \text{with } w_\lambda &= -\sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{k} (1 + R_{ij}) \ln(1 - e^{-\lambda Q(1/x_{ij})}) \end{aligned}$$

Now, find the optimal value of λ (say $\lambda_{fit)}$) and estimate α_1 such that

SSE is minimized. The density of the fitted $\mathcal{GCEIE}$ distribution is now

$$\begin{aligned} f_X(x;\lambda_{fit},\alpha_1) &= \alpha_1\lambda_{fit)}Q^{'}(1/x)\exp\{-\lambda_{fit)}Q(1/x)\}[1-\exp\{\lambda_{fit)}Q(1/x)\}]^{\alpha_1\,-\,1}; \\ x &> 0,\ \alpha_1 > 0. \end{aligned}$$

Step 2. Use the scale-free goodness-of-fit test for $\mathcal{GCEIE}$ distribution based on the Gini statistic due to Gail and Gastwirth (1978) for the progressively type-II right censored data $X_{1:n:N} \leq X_{2:n:N} \leq \cdots \leq X_{n:n:N}$. The procedure is as follows:

The null hypothesis is H_0: $X \sim \mathcal{GCEIE}$ distribution with the *pdf*

$f_X(x;\lambda_{fit},\alpha_1) = \alpha_1\lambda_{fit}Q^{'}(1/x)\exp\{-\lambda_{fit}Q(1/x)\}\times$

$[1-\exp\{\lambda_{fit}Q(1/x)\}]^{\alpha_1\,-\,1}$

The Gini statistic given as follows:

$$G_n = \frac{\sum_{i\,=\,1}^{n-1} iW_{i+1}}{(n-1)\sum_{i\,=\,1}^{n} W_i},$$

where $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)})$, $Z_{(0)} = 0, i = 1, ..., n, Z_1 = NY_1, Z_i =$ $[N - \sum_{j\,=\,1}^{i-1}(R_j+1)](Y_i - Y_{i-1}), i = 1, ..., n,$ and the data transformation $Y_i = 1 - [1-\exp\{\lambda_{fit}Q(1/x)\}]^{\alpha_1}$.

For $n = 3, ..., 20$, the rejection region is given by $\left\{G_n > \xi_{1-\gamma/2} \text{ or } G_n < \xi_{\gamma/2}\right\}$, where the critical value $\xi_{\gamma/2}$ is the $100(\gamma/2)$th percentile of the G_n statistic and is available on p. 352 in Gail and Gastwirth (1978).

$Y \sim \mathcal{GCEIE}\,(\alpha_2, \lambda)$ is also treated in a similar fashion to see whether Y values are fitted to a $\mathcal{GCEIE}$.

Once the procedure for handling real-world data described above, the value of λ (out of various λ values) that minimizes SSE_{λ}^{X} is found to be $\lambda = 1.4$ which is very close to the optimum (minimum) value of the graph of SSE versus λ. (These graphs have been omitted for saving space and can be produced upon request). Further, $\widehat{\alpha}_1$ value corresponds to $\lambda = 1.4$ is 0.22. Then, λ is defined as known. That is,

$$\begin{aligned} f_X(x; \lambda_{fit}, \alpha_1) &= 1.4\alpha_1 Q^{'}(1/x)\exp\{-1.4Q(1/x)\}[1 - \exp\{1.4Q(1/x)\}]^{\alpha_1 - 1}; \\ x &> 0,\ \alpha_1 > 0. \end{aligned}$$

The goodness of fit test for testing H_0: $X \sim \mathcal{GCEIE}$ distribution with the *pdf* $f_X(x; 1.4, \alpha_1)$ $= 1.4\alpha_1 Q^{'}(1/x)\exp\{-1.4Q(1/x)\}[1 - \exp\{1.4Q(1/x)\}]^{\alpha_1 - 1}$ at level $\gamma = 0.05$ based on the Gini statistic for the progressively type-II right censored observed sample. Since the theory has been built up for the general class of exponentiated exponential distribution, we have to make sure that one of the members in this family would be the best distributional candidate for this particular data. Therefore, we hypothesized that H_0: $X \sim$ *exponentiated inverted Rayleigh distribution* ($\mathcal{EIR}$

distribution) with the *pdf* $f_X(x;1.4,\alpha_1) = (2.8\alpha_1/x^3)\exp\{-1.4/x^2)\}[1-\exp\{1.4/x^2)\}]^{\alpha_1 - 1}$, where $Q(1/x) = 1/x^2$ and $Q'(1/x) = -2/x^3$.

This procedure has been explained in the previous section, and the Gini statistics is found to be

$$G_4 = \frac{\sum_{i=1}^{(4-1)} iW_{i+1}}{(4-1)\sum_{i=1}^{4} W_i} = \frac{\sum_{i=1}^{3} iW_{i+1}}{3\sum_{i=1}^{4} W_i} = 0.41920.$$

where $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)})$, $Z_{(0)} = 0, i = 1, ..., n$, $Z_1 = nY_1$, $Z_i = [N - \sum_{j=1}^{i-1}(R_j+1)](Y_i - Y_{i-1}), i = 1, ..., n$, and the data transformation $Y_i = 1-[1-\exp\{\lambda_{fit}Q(1/x)\}]^{\alpha_1}$

Since $\xi_{0.025} = 0.28748 < G_4 = 0.41920 < \xi_{0.975} = 0.71252$,we cannot reject H_0 at the 0.05 level of significance, and we can conclude the observed strength components are from the $\mathcal{EIR}$ distribution with the *pdf* is $f_X(x;1.4,\alpha_1) = (2.8\alpha_1/x^3)$
$\exp\{-1.4/x^2)\}[1-\exp\{1.4/x^2)\}]^{\alpha_1 - 1}$, $x > 0$, $\alpha_1 > 0$, at level $\gamma = 0.05$. $Y \sim \mathcal{GCEIE}\,(\alpha_2, \lambda)$ (or simply $Y \sim \mathcal{EIR}\,(\alpha_2, \lambda)$) is also treated in a similar fashion to see whether Y values are fitted to an $\mathcal{EIR}$. Then,

$$\begin{aligned}
\widehat{\alpha}_1 &= 0.2433, \text{where } w_\lambda = -\sum_{i=1}^{3}\sum_{j=1}^{5}(1+R_{ij})\ln(1-e^{-1.4Q(1/x_{ij})}) = 32.8879 \\
\widehat{\alpha}_2 &= 0.8314, \text{where } v_\lambda = -\sum_{i=1}^{3}(1+R'_i)\ln(1-e^{-1.4Q(1/y_i)}) = 25.7612
\end{aligned}$$

To fully explore the advantage of the newly introduced generalized variable method, classical and generalized point and 95% interval estimates are compared for the reliability function $R_{s,k}$. In addition, p-values for testing reliability function are also compared. The numerical results for these data are presented in Table 5.1 and 5.2. Posterior distributions are obtained from 10,000 Gibbs samplings after a burn-in period of 1,000 iterations.

Table 5.1 Comparison of Point Estimates of $R_{s,k}$

Bayesian		Classical		Generalized	
$\widehat{R}_{s,k}^{SE}$	0.6781	$\widehat{R}_{s,k}^{M}$	0.6987	$\widehat{R}_{s,k}^{G}$	0.6781
$\widehat{R}_{s,k}^{LINEX}$	0.6875	$\widehat{R}_{s,k}^{U}$	0.6988		
${}^{SE}\widehat{R}_{s,k}^{Lin}$	0.6985	${}^{BP}\widehat{R}_{s,k}^{*}$	0.6701		
${}^{LINEX}\widehat{R}_{s,k}^{Lin}$	0.6855	${}^{BT}\widehat{R}_{s,k}^{*}$	0.6898		
${}^{SE}\widehat{R}_{s,k}^{MCMC}$	0.7101				
${}^{LINEX}\widehat{R}_{s,k}^{MCMC}$	0.6998				

Table 5.2 Comparison of Interval Estimates of $R_{s,k}$

Bayesian		Classical		Generalized	
${}^{SE}\mathrm{BCI}^{MCMC}$	$(0.57 - 0.95)$	ACI	$(0.51 - 1.5)$	GCI	$(0.65 - 0.75)$
${}^{SE}\mathrm{HDPI}^{MCMC}$	$(0.51 - 0.88)$	PBCI	$(0.58 - 0.95)$		
${}^{LINEX}\mathrm{BCI}^{MCMC}$	$(0.55 - 1.5)$	BBCACI	$(0.61 - 1.00)$		
${}^{LINEX}\mathrm{HPDI}^{MCMC}$	$(0.55 - 1.5)$	BTCI	$(0.51 - 1.7)$		

Both these arguments clearly show that the generalized variable method (GV-Method) provides accurate, reliable, and non-misleading results, while the classical method (C-Method) and Bayesian method (B-Method) approaches fail to do so for this particular case. Hence, the GV-Method outperforms the C–and B-Method for this particular practical application.

Simulation study

In this section, to illustrate the benefit of the generalized variable method for this problem, we present some numerical results for the inverted exponentiated Rayleigh distribution ($Q(1/x) = 1/x^2$). Those random variables are simulated in the following manner.

For given $\boldsymbol{\beta} = (\alpha_1, \alpha_2)$ and λ, and (n, k) :

1. Generate uniform random numbers, i.e.„ $U \sim U(n, 0, 1)$, where $U(n, 0, 1)$ is the standard continuous uniform distribution with boundary parameters 0 and 1,

and n is the sample size,

2. Generate pseudo general inverse exponentiated random variates for X :

   $\{x_{ij}\}_{i=1,2,\ldots,n;j=1,2,\ldots,k} = \sqrt{\lambda\left[\ln(-u^{1/\alpha_1})^{-1}\right]^{-1}}$,

3. Generate pseudo general inverse exponentiated random variates for Y : $y_{i=1,2,\ldots,n}$

   $= \sqrt{\lambda\left[\ln(-u^{1/\alpha_2})^{-1}\right]^{-1}}$.

The performances of the point estimators are compared by using estimated risks (ERs) or estimate of the mean squared errors (MSE's), and biases. The ER and bias of $\widehat{\theta}$ relative to an known parameter θ, when it is estimated by $\widehat{\theta}$, is given by

$$ER(\widehat{\theta}) = \widehat{MSE(\widehat{\theta})} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta)^2 \quad \text{and} \quad \widehat{Bias(\widehat{\theta})} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}_i - \theta),$$

under ER has been calculated under the squared error function.

The performances of the confidence intervals are compared by using average confidence lengths and coverage probabilities. The coverage probability (CP) of a confidence interval is the proportion of the time that the interval contains the true value of interest. That is,

$$CP = \frac{[\text{Number of intervals that contain the true value of interest } \theta]}{\text{The total number of simulations}}$$

(the total number of intervals that contain the true value of interest)/the total number of simulations. The performances of the hypothesis testing are compared by using average empirical Type-I error rate (or the actual size) of the test, and the unadjusted and adjusted powers of the test.

Actual size (AS) for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$ is the proportion of p-values that are less than the nominal value γ. That is,

$$AS = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0 \ vs.\ H_a : \theta > \theta_0 \text{ that are less than } \gamma}{\text{The total number of simulations}}.$$

When $\theta = \theta_0$, unadjusted power (UP) for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$, where $\theta_0^* < \theta_0$, is the proportion of p-values that are less than the nominal value γ. That is,

$$UP = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0^* \ \text{ vs. } H_a : \theta > \theta_0^* \text{ that are less than } \gamma}{\text{The total number of simulations}},$$

where $\theta_0^* < \theta_0$.

When $\theta = \theta_0$, adjusted power (AP) for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$, where $\theta_0^* < \theta_0$, is the proportion of p-values for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$ that are less than the p-value (p_γ) for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$. That is,

$$AP = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0^* \ \text{ vs. } H_a : \theta > \theta_0^* \text{ that are less than } p_\gamma}{\text{The total number of simulations}},$$

where $\theta_0^* < \theta_0$.

The performance of the estimates of $R_{s,k}$ are obtained by using the classical and generalized methods for different sample sizes. All of the computations are performed by using $R \times 643.1.3$. All the results are based on $N = 10,000$ replications

In Table 5.3(a),(b),(c),and (d) when the common scale parameter is known ($\lambda = 3$), strength and stress populations are generated for $\boldsymbol{\beta} = (\alpha_1, \alpha_2) = (4,2), (4,4), (4,6)$, and $(4,8)$ and different sample sizes $n = 10, 15, 25$ and 35. The corresponding true values of reliability in multicomponent stress-strength with the given combinations for $(s;k) = (1,3)$ are $0.5429, 0.7500, 0.8476$ and 0.9000; and for $(s;k) = (2,4)$ are $0.3905, 0.6000$, 0.7229 and 0.8000.

In Table 5.4(a),(b), (c), and (b) when $\lambda = 10$, strength and stress populations are generated for $\boldsymbol{\beta} = (\alpha_1, \alpha_2) = (18,5), (12,5), (6,5), (1,5)$ and different sample sizes $n = 10, 15, 25$ and 35. The corresponding true values of reliability in multicomponent stress-strength with the given combinations for $(s;k) = (1,3)$ are $0.3711, 0.4871, 0.6987$ and 0.9821; and for $(s,k) = (2,4)$ are $0.2485, 0.3419, 0.5428$ and 0.9524.

From Table 5.3(a),(b0-4.5(b), we observe that the average ERs for the estimates of $R_{s,k}$ decrease as the sample size increases in all cases and all tables, as expected. The ERs of the ML, UMVU and generalized estimates have generally following order of $ER(\widehat{R}^{G}_{s,k}) < ER(\widehat{R}^{MLE}_{s,k}) < ER(\widehat{R}^{U}_{s,k})$ except for the cases when the true value of $R_{s,k}$ is not close to extreme values. On the other hand, when the true value of $R_{s,k}$ approaches the extreme values, we have following order of $ER(\widehat{R}^{G}_{s,k}) < ER(\widehat{R}^{U}_{s,k})$ $< ER(\widehat{R}^{MLE}_{s,k})$ and all ERs are close the each other as the sample size increases. The average lengths of the intervals decrease as the sample size increases. The average lengths of the generalized intervals are smaller than those of the classical confidence intervals. Furthermore, the coverage probabilities of the generalized intervals are more close to the nominal level 95% than the classical confidence intervals.

Table 5.3(a), (b), 5.4 (a), (b), 5.5 (a), (b), and 5.6 (a),and (b).show the point and interval estimates when $(s;k) = \{(1,3),(2,4)\}$ and $\lambda = 3$. The first rows under the point estimates represent the average estimates and the second row represents corresponding ERs. The first row under the interval estimates represent a 95%confidence interval and the second rows represent their expected lengths and coverage probabilities.

Table 5.3(a) Classical and Generalized Point Estimates of $R_{s,k}$ when the Common Scale Parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{1,3}$	$\widehat{R}^M_{s,k}$	$\widehat{R}^U_{s,k}$	$\widehat{R}^*_{s,k}$	$\widehat{R}^G_{s,k}$
10	$(4, 2)$	0.5429	0.5595	0.5488	0.5480	0.5450
			0.0135	0.0138	0.0088	0.0056
15			0.5521	0.5444	0.5450	0.5446
			0.0086	0.0090	0.0061	0.0062
25			0.5485	0.5439	0.5447	0.5446
			0.0050	0.0051	0.0040	0.0040
35			0.5461	0.5429	0.5435	0.5434
			0.0036	0.0036	0.0031	0.0031
10	$(4, 4)$	0.7500	0.7522	0.7521	0.7389	0.7392
			0.0091	0.0102	0.0049	0.0049
15			0.7518	0.7517	0.7421	0.7423
			0.0060	0.0065	0.0039	0.0039
25			0.7502	0.7501	0.7441	0.7442
			0.0038	0.0040	0.0029	0.0029
35			0.7505	0.7504	0.7459	0.7459
			0.0026	0.0027	0.0021	0.0021
10	$(4, 6)$	0.8476	0.8453	0.8499	0.8345	0.8348
			0.0053	0.0057	0.0026	0.0026
15			0.8449	0.8480	0.8367	0.8370
			0.0036	0.0038	0.0022	0.0022
25			0.8462	0.8481	0.8405	0.8406
			0.0023	0.0024	0.0017	0.0017
35			0.8454	0.8467	0.8412	0.8412
			0.0017	0.0017	0.0013	0.0013
10	$(4, 8)$	0.9000	0.8954	0.9015	0.8880	0.8885
			0.0033	0.0034	0.0015	0.0015
15			0.8941	0.8982	0.8886	0.8888
			0.0023	0.0023	0.0013	0.0013
25			0.8974	0.8999	0.8929	0.8929
			0.0013	0.0013	0.0009	0.0009
35			0.8992	0.9010	0.8954	0.8955
			0.0009	0.0009	0.0007	0.0007

Table 5.3(b) Bayesian Point Estimates of $R_{1,3}$ when the Common Scale Parameter λ is Known ($\lambda = 3$)

Sample size	Parameters	Reliability	Bayesian					
n	β	$R_{1,3}$	$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE\,\widehat{R}_{s,k}^{Lin}$	$LINEX\,\widehat{R}_{s,k}^{Lin}$	$SE\,\widehat{R}_{s,k}^{MCMC}$	$LINEX\,\widehat{R}_{s,k}^{MCMC}$
10	(4, 2)	0.5429	0.5593	0.5489	0.5486	0.5454	0.5454	0.5454
			0.0145	0.0138	0.0088	0.0056	0.0138	0.0017
15			0.5521	0.5444	0.5450	0.5446	0.5446	0.5446
			0.0086	0.0090	0.0061	0.0062	0.0138	0.0138
25			0.5485	0.5439	0.5447	0.5446	0.5446	0.5446
			0.0050	0.0051	0.0040	0.0040	0.0138	0.0138
35			0.5461	0.5429	0.5435	0.5434	0.5454	0.5429
			0.0036	0.0036	0.0031	0.0031	0.0031	0.0031
10	(4, 4)	0.7500	0.7522	0.7521	0.7389	0.7392	0.5461	0.0039
			0.0091	0.0102	0.0049	0.0049	0.0031	0.0031
15			0.7518	0.7517	0.7421	0.7423	0.5454	0.5461
			0.0060	0.0065	0.0039	0.0039	0.0039	0.0039
25			0.7502	0.7501	0.7441	0.7442	0.5444	0.7504
			0.0038	0.0040	0.0029	0.0029	0.0029	0.0017
35			0.7505	0.7504	0.7459	0.7459	0.7504	0.7504
			0.0026	0.0027	0.0021	0.0021	0.0029	0.0029
10	(4, 6)	0.8476	0.8453	0.8499	0.8345	0.8348	0.7442	0.7442
			0.0053	0.0057	0.0026	0.0026	0.0029	0.0029
15			0.8449	0.8480	0.8367	0.8370	0.8370	0.8370
			0.0036	0.0038	0.0022	0.0022	0.0029	0.0017
25			0.8462	0.8481	0.8405	0.8406	0.7442	0.7442
			0.0023	0.0024	0.0017	0.0017	0.0017	0.0029
35			0.8454	0.8467	0.8412	0.8412	0.8370	0.8370
			0.0017	0.0017	0.0013	0.0013	0.0057	0.0057
10	(4, 8)	0.9000	0.8954	0.9015	0.8880	0.8885	0.8449	0.8449
			0.0033	0.0034	0.0015	0.0015	0.0057	0.0057
15			0.8941	0.8982	0.8886	0.8888	0.8941	0.8941
			0.0023	0.0023	0.0013	0.0013	0.0023	0.0007
25			0.8974	0.8999	0.8929	0.8929	0.8992	0.8992
			0.0013	0.0013	0.0009	0.0009	0.0007	0.0007
35			0.8992	0.9010	0.8954	0.8955	0.8955	0.8955
			0.0009	0.0009	0.0007	0.0007	0.0023	0.0007

Table 5.3(c) Classical and generalized interval estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{1,3}$	ACI	PBCI	BTCI	GCI
10	(4, 2)	0.5429	(0.3475, 0.7704)	(0.3459, 0.7745)	(0.3474, 0.7733)	(0.3591, 0.7319)
			0.4229/0.9092	0.4230/0.9070	0.4230/0.9091	0.3728/0.9552
15			(0.3764, 0.7269)	(0.3721, 0.7261)	(0.3765, 0.7270)	(0.3844, 0.7030)
			0.3506/0.9228	0.3498/0.9201	0.3501/0.9212	0.3186/0.9476
25			(0.4109, 0.6860)	(0.4095, 0.6840)	(0.4100, 0.6862)	(0.4152, 0.6726)
			0.2752/0.9412	0.2725/0.9422	0.2735/0.9410	0.2573/0.9520
35			(0.4293, 0.6630)	(0.4278, 0.6622)	(0.4292, 0.6638)	(0.4321, 0.6538)
			0.2336/0.9400	0.2331/0.9401	0.2333/0.9395	0.2217/0.9456
10	(4, 4)	0.7500	(0.5698, 0.9346)	(0.5648, 0.9326)	(0.5658, 0.9336)	(0.5763, 0.8872)
			0.3648/0.9076	0.3635/0.9065	0.3644/0.9070	0.3109/0.9692
15			(0.5998, 0.9038)	(0.5988, 0.9030)	(0.5998, 0.9033)	(0.6022, 0.8712)
			0.3040/0.9212	0.3035/0.9200	0.3025/0.9211	0.2690/0.9588
25			(0.6304, 0.8700)	(0.6300, 0.8701)	(0.6302, 0.8705)	(0.6307, 0.8505)
			0.2395/0.9312	0.2394/0.9300	0.2388/0.9310	0.2197/0.9568
35			(0.6486, 0.8524)	(0.6477, 0.8511)	(0.6478, 0.8520)	(0.6480, 0.8385)
			0.2038/0.9416	0.2033/0.9410	0.2028/0.9405	0.1905/0.9552
10	(4, 6)	0.8476	(0.7036, 0.9869)	(0.7030, 0.9861)	(0.7031, 0.9861)	(0.7097, 0.9431)
			0.2833/0.8892	0.2828/0.8878	0.2830/0.8890	0.2334/0.9832
15			(0.7269, 0.9628)	(0.7255, 0.9618)	(0.7263, 0.9629)	(0.7290, 0.9323
			0.2359/0.9068	0.2355/0.9063	0.2360/0.9065	0.2033/0.9712
25			(0.7541, 0.9384)	(0.7538, 0.9378)	(0.7540, 0.9380)	(0.7532, 0.9196)
			0.1843/0.9224	0.1837/0.9219	0.1838/0.9225	0.1664/0.9628
35			(0.7667, 0.9241)	(0.7665, 0.9237)	(0.7670, 0.9240)	(0.7656, 0.9109
			0.1574/0.9336	0.1565/0.9330	0.1570/0.9340	0.1453/0.9564
10	(4, 8)	0.9000	(0.7862, 1.0046)	(0.7858, 1.0044)	(0.7860, 1.0040)	(0.7941, 0.9667)
			0.2183/0.8844	0.2178/0.8840	0.2180/0.8840	0.1726/0.9824
15			(0.8024, 0.9857)	(0.8017, 0.9844)	(0.8019, 0.9851)	(0.8064, 0.9593)
			0.1834/0.9116	0.1831/0.9109	0.1833/0.9115	0.1528/0.9768
25			(0.8269, 0.9679)	(0.8258, 0.9677)	(0.8255, 0.9677)	(0.8264, 0.9514)
			0.1410/0.9188	0.1405/0.9177	0.1405/0.9177	0.1250/0.9708
35			(0.8399, 0.9585)	(0.8387, 0.9578)	(0.8389, 0.9584)	(0.8384, 0.9465)
			0.1186/0.9352	0.1179/0.9348	0.1179/0.9350	0.1081/0.9660

Table 5.3(d) Bayesian interval estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Bayesian			
n	β	$R_{1,3}$	$SE_{\mathrm{BCI}}{}^{MCMC}$	$SE_{\mathrm{HDPI}}{}^{MCMC}$	$LINEX_{\mathrm{BCI}}{}^{MCMC}$	$LINEX_{\mathrm{HPDI}}{}^{MCMC}$
10	(4, 2)	0.5429	(0.3475, 0.7704)	(0.3459, 0.7745)	(0.3474, 0.7733)	(0.3591, 0.7319)
			0.4229/0.9092	0.4230/0.9070	0.4230/0.9091	0.3728/0.9552
15			(0.3764, 0.7269)	(0.3721, 0.7261)	(0.3765, 0.7270)	(0.3844, 0.7030)
			0.3506/0.9228	0.3498/0.9201	0.3501/0.9212	0.3186/0.9476
25			(0.4109, 0.6860)	(0.4095, 0.6840)	(0.4100, 0.6862)	(0.4152, 0.6726)
			0.2752/0.9412	0.2725/0.9422	0.2735/0.9410	0.2573/0.9520
35			(0.4293, 0.6630)	(0.4278, 0.6622)	(0.4292, 0.6638)	(0.4321, 0.6538)
			0.2336/0.9400	0.2331/0.9401	0.2333/0.9395	0.2217/0.9456
10	(4, 4)	0.7500	(0.5698, 0.9346)	(0.5648, 0.9326)	(0.5658, 0.9336)	(0.5763, 0.8872)
			0.3648/0.9076	0.3635/0.9065	0.3644/0.9070	0.3109/0.9692
15			(0.5998, 0.9038)	(0.5988, 0.9030)	(0.5998, 0.9033)	(0.6022, 0.8712)
			0.3040/0.9212	0.3035/0.9200	0.3025/0.9211	0.2690/0.9588
25			(0.6304, 0.8700)	(0.6300, 0.8701)	(0.6302, 0.8705)	(0.6307, 0.8505)
			0.2395/0.9312	0.2394/0.9300	0.2388/0.9310	0.2197/0.9568
35			(0.6486, 0.8524)	(0.6477, 0.8511)	(0.6478, 0.8520)	(0.6480, 0.8385)
			0.2038/0.9416	0.2033/0.9410	0.2028/0.9405	0.1905/0.9552
10	(4, 6)	0.8476	(0.7036, 0.9869)	(0.7030, 0.9861)	(0.7031, 0.9861)	(0.7097, 0.9431)
			0.2833/0.8892	0.2828/0.8878	0.2830/0.8890	0.2334/0.9832
15			(0.7269, 0.9628)	(0.7255, 0.9618)	(0.7263, 0.9629)	(0.7290, 0.9323
			0.2359/0.9068	0.2355/0.9063	0.2360/0.9065	0.2033/0.9712
25			(0.7541, 0.9384)	(0.7538, 0.9378)	(0.7540, 0.9380)	(0.7532, 0.9196)
			0.1843/0.9224	0.1837/0.9219	0.1838/0.9225	0.1664/0.9628
35			(0.7667, 0.9241)	(0.7665, 0.9237)	(0.7670, 0.9240)	(0.7656, 0.9109
			0.1574/0.9336	0.1565/0.9330	0.1570/0.9340	0.1453/0.9564
10	(4, 8)	0.9000	(0.7862, 1.0046)	(0.7858, 1.0044)	(0.7860, 1.0040)	(0.7941, 0.9667)
			0.2183/0.8844	0.2178/0.8840	0.2180/0.8840	0.1726/0.9824
15			(0.8024, 0.9857)	(0.8017, 0.9844)	(0.8019, 0.9851)	(0.8064, 0.9593)
			0.1834/0.9116	0.1831/0.9109	0.1833/0.9115	0.1528/0.9768
25			(0.8269, 0.9679)	(0.8258, 0.9677)	(0.8255, 0.9677)	(0.8264, 0.9514)
			0.1410/0.9188	0.1405/0.9177	0.1405/0.9177	0.1250/0.9708
35			(0.8399, 0.9585)	(0.8387, 0.9578)	(0.8389, 0.9584)	(0.8384, 0.9465)
			0.1186/0.9352	0.1179/0.9348	0.1179/0.9350	0.1081/0.9660

Table 5.4(a) Classical and generalized point estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{2,4}$	$\widehat{R}^{M}_{s,k}$	$\widehat{R}^{U}_{s,k}$	$\widehat{R}^{*}_{s,k}$	$\widehat{R}^{G}_{s,k}$
10		0.3905	0.4071	0.3905	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066
15			0.3998	0.3886	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046
25			0.3980	0.3913	0.3955	0.3957
			0.0038	0.0038	0.0031	0.0031
35			0.3957	0.3909	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025
10		0.6000	0.6085	0.5986	0.5950	0.5953
			0.0115	0.0126	0.0059	0.0059
15			0.6074	0.6006	0.5981	0.5983
			0.0077	0.0081	0.0047	0.0047
25			0.6045	0.6004	0.5989	0.5990
			0.0048	0.0050	0.0036	0.0036
35			0.6031	0.6001	0.5992	0.5992
			0.0034	0.0035	0.0027	0.0027
10		0.7229	0.7267	0.7235	0.7130	0.7129
			0.0093	0.0105	0.0043	0.0042
15			0.7257	0.7235	0.7158	0.7158
			0.0064	0.0069	0.0036	0.0036
25			0.7246	0.7232	0.7181	0.7181
			0.0040	0.0042	0.0027	0.0027
35			0.7239	0.7229	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022
10		0.8000	0.7995	0.8005	0.7880	0.7878
			0.0070	0.0077	0.0029	0.0029
15			0.8016	0.8024	0.7922	0.7921
			0.0048	0.0052	0.0025	0.0025
25			0.7989	0.7992	0.7929	0.7928
			0.0030	0.0031	0.0019	0.0019
35			0.7998	0.8000	0.7952	0.7952
			0.0020	0.0021	0.0015	0.0015

Table 5.4(b) Bayesian point estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Bayesian					
n	β	$R_{2,4}$	$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE\,\widehat{R}_{s,k}^{Lin}$	$LINEX\,\widehat{R}_{s,k}^{Lin}$	$SE\,\widehat{R}_{s,k}^{MCMC}$	$LINEX\,\widehat{R}_{s,k}^{MCMC}$
10		0.3905	0.4071	0.3905	0.3982	0.3989	0.3982	0.3982
			0.0108	0.0108	0.0066	0.0066	0.0046	0.0077
15			0.3998	0.3886	0.3952	0.3956	0.3982	0.3982
			0.0064	0.0064	0.0046	0.0046	0.0046	0.0077
25			0.3980	0.3913	0.3955	0.3957	0.3982	0.5950
			0.0038	0.0038	0.0031	0.0031	0.0025	0.0031
35			0.3957	0.3909	0.3942	0.3942	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025	0.0025	0.5986
10		0.6000	0.6085	0.5986	0.5950	0.5953	0.3942	0.5950
			0.0115	0.0126	0.0059	0.0059	0.0025	0.0025
15			0.6074	0.6006	0.5981	0.5983	0.5983	0.0025
			0.0077	0.0081	0.0047	0.0047	0.0047	0.0077
25			0.6045	0.6004	0.5989	0.5990	0.5992	0.5992
			0.0048	0.0050	0.0036	0.0036	0.0025	0.0025
35			0.6031	0.6001	0.5992	0.5992	0.5992	0.5992
			0.0034	0.0035	0.0027	0.0027	0.0025	0.0042
10		0.7229	0.7267	0.7235	0.7130	0.7129	0.5992	0.7192
			0.0093	0.0105	0.0043	0.0042	0.0042	0.0042
15			0.7257	0.7235	0.7158	0.7158	0.5992	0.7192
			0.0064	0.0069	0.0036	0.0036	0.0042	0.0042
25			0.7246	0.7232	0.7181	0.7181	0.5992	0.7192
			0.0040	0.0042	0.0027	0.0027	0.0027	0.0042
35			0.7239	0.7229	0.7192	0.7192	0.7181	0.7181
			0.0029	0.0030	0.0022	0.0022	0.0019	0.0019
10		0.8000	0.7995	0.8005	0.7880	0.7878	0.7181	0.7181
			0.0070	0.0077	0.0029	0.0029	0.0027	0.0018
15			0.8016	0.8024	0.7922	0.7921	0.7181	0.7880
			0.0048	0.0052	0.0025	0.0025	0.0019	0.0042
25			0.7989	0.7992	0.7929	0.7928	0.7952	0.7880
			0.0030	0.0031	0.0019	0.0019	0.0019	0.0017
35			0.7998	0.8000	0.7952	0.7952	0.7953	(0.7952
			0.0020	0.0021	0.0015	0.0015	0.0019	0.0019

Table 5.4(c) Classical and generalized interval estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{2,4}$	ACI	PBCI	BTCI	GCI
10		0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.218, 0.5949)	(0.2340, 0.5663)
			0.3779/0.9272	0.3770/0.9270	0.3775/0.9265	0.3322/0.9548
15			(0.2447, 0.5550)	(0.2444, 0.5555)	(0.2441, 0.5548)	(0.2553, 0.5375)
			0.3103/0.9432	0.3100/0.9428	0.3103/0.9432	0.2822/0.9604
25			(0.2771, 0.5190)	(0.2758, 0.5188)	(0.2765, 0.5175)	(0.2830, 0.5100)
			0.2419/0.9448	0.2415/0.9440	0.2410/0.9440	0.2269/0.9556
35			(0.2935, 0.4980)	(0.2930, 0.4975)	(0.2933, 0.4977)	(0.2975, 0.4920)
			0.2045/0.9412	0.2044/0.9412	0.2040/0.9410	0.1945/0.9432
10		0.6000	(0.4018, 0.8153)	(0.4015, 0.8150)	(0.4010, 0.8152)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9189	0.4128/0.9190	0.3495/0.9672
15			(0.4357, 0.7790)	(0.4351, 0.7777)	(0.4355, 0.7788)	(0.4447, 0.7467)
			0.3432/0.9372	0.3429/0.9365	0.3425/0.9365	0.3020/0.9716
25			(0.4699, 0.7391)	(0.4688, 0.7389)	(0.4688, 0.7389)	(0.4740, 0.7206)
			0.2692/0.9364	0.2689/0.9365	0.2687/0.9360	0.2466/0.9572
35			(0.4886, 0.7175)	(0.4883, 0.7177)	(0.4885, 0.7170)	(0.4911, 0.7047)
			0.2289/0.9480	0.2282/0.9479	0.2284/0.9477	0.2136/0.9576
10		0.7229	(0.5394, 0.9141)	(0.5388, 0.9138)	(0.5389, 0.9138)	(0.5537, 0.8601)
			0.3747/0.9032	0.3739/0.9031	0.3744/0.9030	0.3065/0.9752
15			(0.5699, 0.8816)	(0.5688, 0.8810)	(0.5688, 0.8811)	(0.5776, 0.8450)
			0.3117/0.9172	0.3111/0.9165	0.3115/0.9165	0.2674/0.9700
25			(0.6019, 0.8472)	(0.6018, 0.8468)	(0.6010, 0.8468)	(0.6051, 0.8254)
			0.2452/0.9308	0.2444/0.9300	0.2449/0.9300	0.2203/0.9596
35			(0.6196, 0.8283)	(0.6190, 0.8280)	(0.6195, 0.8282)	(0.6209, 0.8127)
			0.2087/0.9332	0.2085/0.9331	0.2080/0.9328	0.1918/0.9540
10		0.8000	(0.6376, 0.9614)	(0.6372, 0.9601)	(0.6365, 0.9610)	(0.6531, 0.9092)
			0.3238/0.8992	0.3228/0.8985	0.3230/0.8960	0.2561/0.9844
15			(0.6680, 0.9353)	(0.6677, 0.9344)	(0.6677, 0.9350)	(0.6750, 0.8988)
			0.2674/0.9040	0.2670/0.9039	0.2666/0.9033	0.2239/0.9664
25			(0.6929, 0.9048)	(0.6920, 0.9040)	(0.6918, 0.9040)	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9248	0.2108/0.9238	0.1866/0.9648
35			(0.7098, 0.8897)	(0.7098, 0.8897)	(0.7088, 0.8885)	(0.7110, 0.8739)
			0.1799/0.9372	0.1799/0.9372	0.1789/0.9365	0.1630/0.9628

Table 5.4(d) Bayesian interval estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 3$)

Sample size	Parameters	Reliability	Bayesian			
n	β	$R_{2,4}$	$SE_{\text{BCI}}{}^{MCMC}$	$SE_{\text{HDPI}}{}^{MCMC}$	$LINEX_{\text{BCI}}{}^{MCMC}$	$LINEX_{\text{HPDI}}{}^{MCMC}$
10		0.3905	$(0.2182, 0.5960)$	$(0.2180, 0.5959)$	$(0.218, 0.5949)$	$(0.2340, 0.5663)$
			0.3779/0.9272	0.3770/0.9270	0.3775/0.9265	0.3322/0.9548
15			$(0.2447, 0.5550)$	$(0.2444, 0.5555)$	$(0.2441, 0.5548)$	$(0.2553, 0.5375)$
			0.3103/0.9432	0.3100/0.9428	0.3103/0.9432	0.2822/0.9604
25			$(0.2771, 0.5190)$	$(0.2758, 0.5188)$	$(0.2765, 0.5175)$	$(0.2830, 0.5100)$
			0.2419/0.9448	0.2415/0.9440	0.2410/0.9440	0.2269/0.9556
35			$(0.2935, 0.4980)$	$(0.2930, 0.4975)$	$(0.2933, 0.4977)$	$(0.2975, 0.4920)$
			0.2045/0.9412	0.2044/0.9412	0.2040/0.9410	0.1945/0.9432
10		0.6000	$(0.4018, 0.8153)$	$(0.4015, 0.8150)$	$(0.4010, 0.8152)$	$(0.4172, 0.7667)$
			0.4134/0.9200	0.4130/0.9189	0.4128/0.9190	0.3495/0.9672
15			$(0.4357, 0.7790)$	$(0.4351, 0.7777)$	$(0.4355, 0.7788)$	$(0.4447, 0.7467)$
			0.3432/0.9372	0.3429/0.9365	0.3425/0.9365	0.3020/0.9716
25			$(0.4699, 0.7391)$	$(0.4688, 0.7389)$	$(0.4688, 0.7389)$	$(0.4740, 0.7206)$
			0.2692/0.9364	0.2689/0.9365	0.2687/0.9360	0.2466/0.9572
35			$(0.4886, 0.7175)$	$(0.4883, 0.7177)$	$(0.4885, 0.7170)$	$(0.4911, 0.7047)$
			0.2289/0.9480	0.2282/0.9479	0.2284/0.9477	0.2136/0.9576
10		0.7229	$(0.5394, 0.9141)$	$(0.5388, 0.9138)$	$(0.5389, 0.9138)$	$(0.5537, 0.8601)$
			0.3747/0.9032	0.3739/0.9031	0.3744/0.9030	0.3065/0.9752
15			$(0.5699, 0.8816)$	$(0.5688, 0.8810)$	$(0.5688, 0.8811)$	$(0.5776, 0.8450)$
			0.3117/0.9172	0.3111/0.9165	0.3115/0.9165	0.2674/0.9700
25			$(0.6019, 0.8472)$	$(0.6018, 0.8468)$	$(0.6010, 0.8468)$	$(0.6051, 0.8254)$
			0.2452/0.9308	0.2444/0.9300	0.2449/0.9300	0.2203/0.9596
35			$(0.6196, 0.8283)$	$(0.6190, 0.8280)$	$(0.6195, 0.8282)$	$(0.6209, 0.8127)$
			0.2087/0.9332	0.2085/0.9331	0.2080/0.9328	0.1918/0.9540
10		0.8000	$(0.6376, 0.9614)$	$(0.6372, 0.9601)$	$(0.6365, 0.9610)$	$(0.6531, 0.9092)$
			0.3238/0.8992	0.3228/0.8985	0.3230/0.8960	0.2561/0.9844
15			$(0.6680, 0.9353)$	$(0.6677, 0.9344)$	$(0.6677, 0.9350)$	$(0.6750, 0.8988)$
			0.2674/0.9040	0.2670/0.9039	0.2666/0.9033	0.2239/0.9664
25			$(0.6929, 0.9048)$	$(0.6920, 0.9040)$	$(0.6918, 0.9040)$	$(0.6961, 0.8826)$
			0.2119/0.9252	0.2111/0.9248	0.2108/0.9238	0.1866/0.9648
35			$(0.7098, 0.8897)$	$(0.7098, 0.8897)$	$(0.7088, 0.8885)$	$(0.7110, 0.8739)$
			0.1799/0.9372	0.1799/0.9372	0.1789/0.9365	0.1630/0.9628

Table 5.5(a), (b), (c), (d), and 5.6 (a), (b), (c), (d) show the point and interval estimates when

$(s;k) = \{(1,3),(2,4)\}$ and $\lambda = 10$. The first rows under the point estimates represent the average estimates and the second row represents corresponding ERs. The first row under the interval estimates represent a 95%confidence interval and the second rows represent their expected lengths and coverage probabilities.

Table 5.5(a) Classical and generalized point estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{1,3}$	$\widehat{R}^M_{s,k}$	$\widehat{R}^U_{s,k}$	$\widehat{R}^*_{s,k}$	$\widehat{R}^G_{s,k}$
10	(4, 2)	0.5429	0.5590	0.5483	0.5478	0.5473
			0.0128	0.0138	0.0078	0.0079
15			0.5517	0.5442	0.5449	0.5446
			0.0086	0.0090	0.0061	0.0062
25			0.5485	0.5439	0.5447	0.5446
			0.0050	0.0051	0.0040	0.0040
35			0.5461	0.5429	0.5435	0.5434
			0.0036	0.0036	0.0031	0.0031
10	(4, 4)	0.7500	0.7522	0.7521	0.7389	0.7392
			0.0091	0.0102	0.0049	0.0049
15			0.7518	0.7517	0.7421	0.7423
			0.0060	0.0065	0.0039	0.0039
25			0.7502	0.7501	0.7441	0.7442
			0.0038	0.0040	0.0029	0.0029
35			0.7505	0.7504	0.7459	0.7459
			0.0026	0.0027	0.0021	0.0021
10	(4, 6)	0.8476	0.8453	0.8499	0.8345	0.8348
			0.0053	0.0057	0.0026	0.0026
15			0.8449	0.8480	0.8367	0.8370
			0.0036	0.0038	0.0022	0.0022
25			0.8462	0.8481	0.8405	0.8406
			0.0023	0.0024	0.0017	0.0017
35			0.8454	0.8467	0.8412	0.8412
			0.0017	0.0017	0.0013	0.0013
10	(4, 8)	0.9000	0.8954	0.9015	0.8880	0.8885
			0.0033	0.0034	0.0015	0.0015
15			0.8941	0.8982	0.8886	0.8888
			0.0023	0.0023	0.0013	0.0013
25			0.8974	0.8999	0.8929	0.8929
			0.0013	0.0013	0.0009	0.0009
35			0.8992	0.9010	0.8954	0.8955
			0.0009	0.0009	0.0007	0.0007

Table 5.5(b) Bayesian point estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Bayesian					
n	β	$R_{1,3}$	$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE\,\widehat{R}_{s,k}^{Lin}$	$LINEX\,\widehat{R}_{s,k}^{Lin}$	$SE\,\widehat{R}_{s,k}^{MCMC}$	$LINEX\,\widehat{R}_{s,k}^{MCMC}$
10	$(4,2)$	0.5429	0.5590	0.5483	0.5478	0.5473	0.5473	0.3475
			0.0128	0.0138	0.0078	0.0079	0.0061	0.0061
15			0.5517	0.5442	0.5449	0.5446	0.5473	0.3764
			0.0086	0.0090	0.0061	0.0062	0.0061	0.0061
25			0.5485	0.5439	0.5447	0.5446	(0.5590	0.4109
			0.0050	0.0051	0.0040	0.0040	0.0061	0.0061
35			0.5461	0.5429	0.5435	0.5434	0.5590	0.4293
			0.0036	0.0036	0.0031	0.0031	0.0031	0.0031
10	$(4,4)$	0.7500	0.7522	0.7521	0.7389	0.7392	0.5473	0.5429
			0.0091	0.0102	0.0049	0.0049	0.0031	0.0031
15			0.7518	0.7517	0.7421	0.7423	0.5590	0.5429
			0.0060	0.0065	0.0039	0.0039	0.0031	0.0031
25			0.7502	0.7501	0.7441	0.7442	0.7501	0.5429
			0.0038	0.0040	0.0029	0.0029	0.0029	0.0022
35			0.7505	0.7504	0.7459	0.7459	0.7501	0.7502
			0.0026	0.0027	0.0021	0.0021	0.0029	0.0029
10	$(4,6)$	0.8476	0.8453	0.8499	0.8345	0.8348	0.7442	0.7459
			0.0053	0.0057	0.0026	0.0026	0.0029	0.0029
15			0.8449	0.8480	0.8367	0.8370	0.8370	0.7502
			0.0036	0.0038	0.0022	0.0022	0.0029	0.0029
25			0.8462	0.8481	0.8405	0.8406	0.7442	0.7459
			0.0023	0.0024	0.0017	0.0017	0.0029	0.0022
35			0.8454	0.8467	0.8412	0.8412	0.7502	0.7459
			0.0017	0.0017	0.0013	0.0013	0.8954	0.0022
10	$(4,8)$	0.9000	0.8954	0.9015	0.8880	0.8885	0.7502	0.7502
			0.0033	0.0034	0.0015	0.0015	0.8954	0.0022
15			0.8941	0.8982	0.8886	0.8888	0.8954	0.8954
			0.0023	0.0023	0.0013	0.0013	0.0009	0.0009
25			0.8974	0.8999	0.8929	0.8929	(0.8269,	0.8269
			0.0013	0.0013	0.0009	0.0009	0.0009	0.0013
35			0.8992	0.9010	0.8954	0.8955	(0.8269,	0.8399
			0.0009	0.0009	0.0007	0.0007	0.0009	0.0013

Table 5.5(c) Classical and generalized interval estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{1,3}$	ACI	PBCI	BTCI	GCI
10	(4, 2)	0.5429	(0.3475, 0.7704)	(0.3470, 0.7700)	(0.3471, 0.7700)	(0.3591, 0.7319)
			0.4229/0.9092	0.4230/0.9100	0.4228/0.9090	0.3728/0.9552
15			(0.3764, 0.7269)	(0.3760, 0.7260)	(0.3760, 0.7266)	(0.3844, 0.7030)
			0.3506/0.9228	0.3501/0.9222	0.3501/0.9223	0.3186/0.9476
25			(0.4109, 0.6860)	(0.4100, 0.6858)	(0.4101, 0.6857)	(0.4152, 0.6726)
			0.2752/0.9412	0.2751/0.9401	0.2747/0.9401	0.2573/0.9520
35			(0.4293, 0.6630)	(0.4289, 0.6628)	(0.4289, 0.6625)	(0.4321, 0.6538)
			0.2336/0.9400	0.2335/0.9389	0.2333/0.9387	0.2217/0.9456
10	(4, 4)	0.7500	(0.5698, 0.9346)	(0.5695, 0.9340)	(0.5693, 0.9344)	(0.5763, 0.8872)
			0.3648/0.9076	0.3641/0.9074	0.3644/0.9070	0.3109/0.9692
15			(0.5998, 0.9038)	(0.5993, 0.9035)	(0.5995, 0.9031)	(0.6022, 0.8712)
			0.3040/0.9212	0.3035/0.9210	0.3035/0.9209	0.2690/0.9588
25			(0.6304, 0.8700)	(0.6301, 0.8698)	(0.6298, 0.8697)	(0.6307, 0.8505)
			0.2395/0.9312	0.2393/0.9308	0.2394/0.9301	0.2197/0.9568
35			(0.6486, 0.8524)	0.6481, 0.8521)	0.6481, 0.8520)	(0.6480, 0.8385)
			0.2038/0.9416	0.2031/0.9413	0.2031/0.9410	0.1905/0.9552
10	(4, 6)	0.8476	(0.7036, 0.9869)	(0.7028, 0.9861)	(0.7033, 0.9850)	(0.7097, 0.9431)
			0.2833/0.8892	0.2830/0.8889	0.2830/0.8889	0.2334/0.9832
15			(0.7269, 0.9628)	(0.7261, 0.9625)	(0.7263, 0.9621)	(0.7290, 0.9323
			0.2359/0.9068	0.2350/0.9061	0.2355/0.9065	0.2033/0.9712
25			(0.7541, 0.9384)	(0.7539, 0.9380)	(0.7539, 0.9384)	(0.7532, 0.9196)
			0.1843/0.9224	0.1838/0.9218	0.1843/0.9220	0.1664/0.9628
35			(0.7667, 0.9241)	(0.7661, 0.9228)	(0.7666, 0.9240)	(0.7656, 0.9109
			0.1574/0.9336	0.1571/0.9332	0.1571/0.9331	0.1453/0.9564
10	(4, 8)	0.9000	(0.7862, 1.0046)	(0.7859, 1.0044)	(0.7860, 1.0040)	(0.7941, 0.9667)
			0.2183/0.8844	0.2181/0.8840	0.2180/0.8844	0.1726/0.9824
15			(0.8024, 0.9857)	(0.8022, 0.9851)	(0.8020, 0.9857)	(0.8064, 0.9593)
			0.1834/0.9116	0.1830/0.9111	0.1834/0.9101	0.1528/0.9768
25			(0.8269, 0.9679)	(0.8264, 0.9674)	(0.8266, 0.9676)	(0.8264, 0.9514)
			0.1410/0.9188	0.1408/0.9185	0.1409/0.9183	0.1250/0.9708
35			(0.8399, 0.9585)	(0.8395, 0.9581)	(0.8391, 0.9581)	(0.8384, 0.9465)
			0.1186/0.9352	0.1180/0.9350	0.1182/0.9348	0.1081/0.9660

Table 5.5(d) Bayesian interval estimates of $R_{1,3}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Bayesian			
n	β	$R_{1,3}$	$SE_{\mathrm{BCI}}{}^{MCMC}$	$SE_{\mathrm{HDPI}}{}^{MCMC}$	$LINEX_{\mathrm{BCI}}{}^{MCMC}$	$LINEX_{\mathrm{HPDI}}{}^{MCMC}$
10	(4, 2)	0.5429	(0.3475, 0.7704)	(0.3470, 0.7700)	(0.3471, 0.7700)	(0.3591, 0.7319)
			0.4229/0.9092	0.4230/0.9100	0.4228/0.9090	0.3728/0.9552
15			(0.3764, 0.7269)	(0.3760, 0.7260)	(0.3760, 0.7266)	(0.3844, 0.7030)
			0.3506/0.9228	0.3501/0.9222	0.3501/0.9223	0.3186/0.9476
25			(0.4109, 0.6860)	(0.4100, 0.6858)	(0.4101, 0.6857)	(0.4152, 0.6726)
			0.2752/0.9412	0.2751/0.9401	0.2747/0.9401	0.2573/0.9520
35			(0.4293, 0.6630)	(0.4289, 0.6628)	(0.4289, 0.6625)	(0.4321, 0.6538)
			0.2336/0.9400	0.2335/0.9389	0.2333/0.9387	0.2217/0.9456
10	(4, 4)	0.7500	(0.5698, 0.9346)	(0.5695, 0.9340)	(0.5693, 0.9344)	(0.5763, 0.8872)
			0.3648/0.9076	0.3641/0.9074	0.3644/0.9070	0.3109/0.9692
15			(0.5998, 0.9038)	(0.5993, 0.9035)	(0.5995, 0.9031)	(0.6022, 0.8712)
			0.3040/0.9212	0.3035/0.9210	0.3035/0.9209	0.2690/0.9588
25			(0.6304, 0.8700)	(0.6301, 0.8698)	(0.6298, 0.8697)	(0.6307, 0.8505)
			0.2395/0.9312	0.2393/0.9308	0.2394/0.9301	0.2197/0.9568
35			(0.6486, 0.8524)	0.6481, 0.8521)	0.6481, 0.8520)	(0.6480, 0.8385)
			0.2038/0.9416	0.2031/0.9413	0.2031/0.9410	0.1905/0.9552
10	(4, 6)	0.8476	(0.7036, 0.9869)	(0.7028, 0.9861)	(0.7033, 0.9850)	(0.7097, 0.9431)
			0.2833/0.8892	0.2830/0.8889	0.2830/0.8889	0.2334/0.9832
15			(0.7269, 0.9628)	(0.7261, 0.9625)	(0.7263, 0.9621)	(0.7290, 0.9323
			0.2359/0.9068	0.2350/0.9061	0.2355/0.9065	0.2033/0.9712
25			(0.7541, 0.9384)	(0.7539, 0.9380)	(0.7539, 0.9384)	(0.7532, 0.9196)
			0.1843/0.9224	0.1838/0.9218	0.1843/0.9220	0.1664/0.9628
35			(0.7667, 0.9241)	(0.7661, 0.9228)	(0.7666, 0.9240)	(0.7656, 0.9109
			0.1574/0.9336	0.1571/0.9332	0.1571/0.9331	0.1453/0.9564
10	(4, 8)	0.9000	(0.7862, 1.0046)	(0.7859, 1.0044)	(0.7860, 1.0040)	(0.7941, 0.9667)
			0.2183/0.8844	0.2181/0.8840	0.2180/0.8844	0.1726/0.9824
15			(0.8024, 0.9857)	(0.8022, 0.9851)	(0.8020, 0.9857)	(0.8064, 0.9593)
			0.1834/0.9116	0.1830/0.9111	0.1834/0.9101	0.1528/0.9768
25			(0.8269, 0.9679)	(0.8264, 0.9674)	(0.8266, 0.9676)	(0.8264, 0.9514)
			0.1410/0.9188	0.1408/0.9185	0.1409/0.9183	0.1250/0.9708
35			(0.8399, 0.9585)	(0.8395, 0.9581)	(0.8391, 0.9581)	(0.8384, 0.9465)
			0.1186/0.9352	0.1180/0.9350	0.1182/0.9348	0.1081/0.9660

Table 5.6(a) Classical and generalized point estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{2,4}$	$\widehat{R}^{M}_{s,k}$	$\widehat{R}^{U}_{s,k}$	$\widehat{R}^{*}_{s,k}$	$\widehat{R}^{G}_{s,k}$
10	(4, 2)	0.3905	0.4071	0.3905	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066
15			0.3998	0.3886	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046
25			0.3980	0.3913	0.3955	0.3957
			0.0038	0.0038	0.0031	0.0031
35			0.3957	0.3909	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025
10	(4, 4)	0.6000	0.6085	0.5986	0.5950	0.5953
			0.0115	0.0126	0.0059	0.0059
15			0.6074	0.6006	0.5981	0.5983
			0.0077	0.0081	0.0047	0.0047
25			0.6045	0.6004	0.5989	0.5990
			0.0048	0.0050	0.0036	0.0036
35			0.6031	0.6001	0.5992	0.5992
			0.0034	0.0035	0.0027	0.0027
10	(4, 6)	0.7229	0.7267	0.7235	0.7130	0.7129
			0.0093	0.0105	0.0043	0.0042
15			0.7257	0.7235	0.7158	0.7158
			0.0064	0.0069	0.0036	0.0036
25			0.7246	0.7232	0.7181	0.7181
			0.0040	0.0042	0.0027	0.0027
35			0.7239	0.7229	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022
10	(4, 8)	0.8000	0.7995	0.8005	0.7880	0.7878
			0.0070	0.0077	0.0029	0.0029
15			0.8016	0.8024	0.7922	0.7921
			0.0048	0.0052	0.0025	0.0025
25			0.7989	0.7992	0.7929	0.7928
			0.0030	0.0031	0.0019	0.0019
35			0.7998	0.8000	0.7952	0.7952
			0.0020	0.0021	0.0015	0.0015

Table 5.6(b) Bayesian point estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Bayesian					
n	β	$R_{2,4}$	$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE\,\widehat{R}_{s,k}^{Lin}$	$LINEX\,\widehat{R}_{s,k}^{Lin}$	$SE\,\widehat{R}_{s,k}^{MCMC}$	$LINEX\,\widehat{R}_{s,k}^{MCMC}$
10	$(4, 2)$	0.3905	0.4071	0.3905	0.3982	0.3989	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066	0.0066	0.0066
15			0.3998	0.3886	0.3952	0.3956	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046	0.0046	0.0046
25			0.3980	0.3913	0.3955	0.3957	0.3980	0.3958
			0.0038	0.0038	0.0031	0.0031	0.0038	0.0040
35			0.3957	0.3909	0.3942	0.3942	0.3957	0.3958
			0.0029	0.0029	0.0025	0.0025	0.0029	0.0022
10	$(4, 4)$	0.6000	0.6085	0.5986	0.5950	0.5953	0.5953	0.5953
			0.0115	0.0126	0.0059	0.0059	0.0059	0.0060
15			0.6074	0.6006	0.5981	0.5983	0.5983	0.5987
			0.0077	0.0081	0.0047	0.0047	0.0048	0.0047
25			0.6045	0.6004	0.5989	0.5990	0.5990	0.5990
			0.0048	0.0050	0.0036	0.0036	0.0037	0.0038
35			0.6031	0.6001	0.5992	0.5992	0.5989	0.5992
			0.0034	0.0035	0.0027	0.0027	0.0027	0.0097
10	$(4, 6)$	0.7229	0.7267	0.7235	0.7130	0.7129	0.7129	0.7129
			0.0093	0.0105	0.0043	0.0042	0.0042	0.0042
15			0.7257	0.7235	0.7158	0.7158	0.7159	0.7159
			0.0064	0.0069	0.0036	0.0036	0.0036	0.0036
25			0.7246	0.7232	0.7181	0.7181	0.7232	0.7232
			0.0040	0.0042	0.0027	0.0027	0.0042	0.0042
35			0.7239	0.7229	0.7192	0.7192	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022	0.0029	0.0029
10	$(4, 8)$	0.8000	0.7995	0.8005	0.7880	0.7878	0.7878	0.7879
			0.0070	0.0077	0.0029	0.0029	0.0077	0.0077
15			0.8016	0.8024	0.7922	0.7921	0.7921	0.7921
			0.0048	0.0052	0.0025	0.0025	0.0025	0.0026
25			0.7989	0.7992	0.7929	0.7928	0.7992	0.7992
			0.0030	0.0031	0.0019	0.0019	0.0019	0.0019
35			0.7998	0.8000	0.7952	0.7952	0.7953	0.7954
			0.0020	0.0021	0.0015	0.0015	0.0015	0.0015

Table 5.6(c) Classical and generalized interval estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Classical			Generalized
n	β	$R_{2,4}$	ACI	PBCI	BTCI	GCI
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.2178, 0.5959)	(0.2340, 0.5663)
			0.3778/0.9271	0.3779/0.9272	0.3773/0.9271	0.3322/0.9548
15			(0.2443, 0.5550)	(0.2447, 0.5550)	(0.2444, 0.5549)	(0.2553, 0.5375)
			0.3103/0.9432	0.3098/0.9430	0.3100/0.9430	0.2822/0.9604
25			(0.2771, 0.5190)	(0.2767, 0.5189)	(0.2770, 0.5189)	(0.2830, 0.5100)
			0.2419/0.9448	0.2411/0.9444	0.2411/0.9445	0.2269/0.9556
35			(0.2935, 0.4980)	(0.2931, 0.4979)	(0.2932, 0.4982)	(0.2975, 0.4920)
			0.2045/0.9412	0.2043/0.9401	0.2041/0.9411	0.1945/0.9432
10	(4, 4)	0.6000	(0.4018, 0.8153)	(0.4012, 0.8151)	(0.4017, 0.8153)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9197	0.4134/0.9203	0.3495/0.9672
15			(0.4357, 0.7790)	(0.4355, 0.7789)	(0.4356, 0.7791)	(0.4447, 0.7467)
			0.3432/0.9372	0.3430/0.9370	0.3431/0.9372	0.3020/0.9716
25			(0.4699, 0.7391)	(0.4691, 0.7388)	(0.4693, 0.7390)	(0.4740, 0.7206)
			0.2692/0.9364	0.2688/0.9361	0.2691/0.9361	0.2466/0.9572
35			(0.4886, 0.7175)	(0.4885, 0.7175)	(0.4885, 0.7175)	(0.4911, 0.7047)
			0.2289/0.9480	0.2283/0.9478	0.2284/0.9475	0.2136/0.9576
10	(4, 6)	0.7229	(0.5394, 0.9141)	(0.5391, 0.9140)	(0.5391, 0.9140)	(0.5537, 0.8601)
			0.3747/0.9032	0.3742/0.9031	0.3744/0.9030	0.3065/0.9752
15			(0.5699, 0.8816)	(0.5695, 0.8811)	(0.5697, 0.8815)	(0.5776, 0.8450)
			0.3117/0.9172	0.3112/0.9170	0.3116/0.9170	0.2674/0.9700
25			(0.6019, 0.8472)	(0.6009, 0.8467)	(0.6011, 0.8470)	(0.6051, 0.8254)
			0.2452/0.9308	0.2451/0.9303	0.2450/0.9300	0.2203/0.9596
35			(0.6196, 0.8283)	(0.6191, 0.8282)	(0.6195, 0.8280)	(0.6209, 0.8127)
			0.2087/0.9332	0.2088/0.9333	0.2080/0.9330	0.1918/0.9540
10	(4, 8)	0.8000	(0.6376, 0.9614)	(0.6371, 0.9611)	(0.6371, 0.9614)	(0.6531, 0.9092)
			0.3238/0.8992	0.3233/0.8989	0.3233/0.8992	0.2561/0.9844
15			(0.6680, 0.9353)	(0.6678, 0.9350)	(0.6680, 0.9352)	(0.6750, 0.8988)
			0.2674/0.9040	0.2671/0.9037	0.2673/0.9040	0.2239/0.9664
25			(0.6929, 0.9048)	(0.6928, 0.9041)	(0.6929, 0.9045	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9251	0.2111/0.9250	0.1866/0.9648
35			(0.7098, 0.8897)	(0.7092, 0.8892)	(0.7099, 0.8897)	(0.7110, 0.8739)
			0.1799/0.9372	0.1793/0.9371	0.1799/0.9371	0.1630/0.9628

Table 5.6(d) Bayesian interval estimates of $R_{2,4}$ when the common scale parameter λ is known ($\lambda = 10$)

Sample size	Parameters	Reliability	Bayesian			
n	β	$R_{2,4}$	$SE_{\mathrm{BCI}}{}^{MCMC}$	$SE_{\mathrm{HDPI}}{}^{MCMC}$	$LINEX_{\mathrm{BCI}}{}^{MCMC}$	$LINEX_{\mathrm{HPDI}}{}^{MCMC}$
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.2178, 0.5959)	(0.2340, 0.5663)
			0.3778/0.9271	0.3779/0.9272	0.3773/0.9271	0.3322/0.9548
15			(0.2443, 0.5550)	(0.2447, 0.5550)	(0.2444, 0.5549)	(0.2553, 0.5375)
			0.3103/0.9432	0.3098/0.9430	0.3100/0.9430	0.2822/0.9604
25			(0.2771, 0.5190)	(0.2767, 0.5189)	(0.2770, 0.5189)	(0.2830, 0.5100)
			0.2419/0.9448	0.2411/0.9444	0.2411/0.9445	0.2269/0.9556
35			(0.2935, 0.4980)	(0.2931, 0.4979)	(0.2932, 0.4982)	(0.2975, 0.4920)
			0.2045/0.9412	0.2043/0.9401	0.2041/0.9411	0.1945/0.9432
10	(4, 4)	0.6000	(0.4018, 0.8153)	(0.4012, 0.8151)	(0.4017, 0.8153)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9197	0.4134/0.9203	0.3495/0.9672
15			(0.4357, 0.7790)	(0.4355, 0.7789)	(0.4356, 0.7791)	(0.4447, 0.7467)
			0.3432/0.9372	0.3430/0.9370	0.3431/0.9372	0.3020/0.9716
25			(0.4699, 0.7391)	(0.4691, 0.7388)	(0.4693, 0.7390)	(0.4740, 0.7206)
			0.2692/0.9364	0.2688/0.9361	0.2691/0.9361	0.2466/0.9572
35			(0.4886, 0.7175)	(0.4885, 0.7175)	(0.4885, 0.7175)	(0.4911, 0.7047)
			0.2289/0.9480	0.2283/0.9478	0.2284/0.9475	0.2136/0.9576
10	(4, 6)	0.7229	(0.5394, 0.9141)	(0.5391, 0.9140)	(0.5391, 0.9140)	(0.5537, 0.8601)
			0.3747/0.9032	0.3742/0.9031	0.3744/0.9030	0.3065/0.9752
15			(0.5699, 0.8816)	(0.5695, 0.8811)	(0.5697, 0.8815)	(0.5776, 0.8450)
			0.3117/0.9172	0.3112/0.9170	0.3116/0.9170	0.2674/0.9700
25			(0.6019, 0.8472)	(0.6009, 0.8467)	(0.6011, 0.8470)	(0.6051, 0.8254)
			0.2452/0.9308	0.2451/0.9303	0.2450/0.9300	0.2203/0.9596
35			(0.6196, 0.8283)	(0.6191, 0.8282)	(0.6195, 0.8280)	(0.6209, 0.8127)
			0.2087/0.9332	0.2088/0.9333	0.2080/0.9330	0.1918/0.9540
10	(4, 8)	0.8000	(0.6376, 0.9614)	(0.6371, 0.9611)	(0.6371, 0.9614)	(0.6531, 0.9092)
			0.3238/0.8992	0.3233/0.8989	0.3233/0.8992	0.2561/0.9844
15			(0.6680, 0.9353)	(0.6678, 0.9350)	(0.6680, 0.9352)	(0.6750, 0.8988)
			0.2674/0.9040	0.2671/0.9037	0.2673/0.9040	0.2239/0.9664
25			(0.6929, 0.9048)	(0.6928, 0.9041)	(0.6929, 0.9045	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9251	0.2111/0.9250	0.1866/0.9648
35			(0.7098, 0.8897)	(0.7092, 0.8892)	(0.7099, 0.8897)	(0.7110, 0.8739)
			0.1799/0.9372	0.1793/0.9371	0.1799/0.9371	0.1630/0.9628

Tables 5.7 (a) and (b) show the classical and generalized empirical (actual) type-I error rates or

the sizes of the test (the rejection rate of the null hypothesis: the fraction of times the p-value is less than the nominal level) for testing t $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) type-I error rate is at $\gamma = 0.05$.

Table 5.7(a) Empirical (true) Type-I error rates for testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) level is $\gamma = 0.05$ with the known common scale parameter $\lambda = 3$

n	β	$R_{1,3}$	R_0	Generalized	Bayesian	Classical	$R_{2,4}$	R_0	Generalized	Bayesian	Classical
10	$(4,2)$	0.5429	0.50	0.0490	0.0059	0.0070	0.3905	0.35	0.0510	0.0510	0.0145
15				0.0450	0.0050	0.0058			0.0489	0.0541	0.0125
25				0.0510	0.0480	0.0060			0.0485	0.0478	0.0128
35				0.0491	0.0480	0.0063			0.0510	0.0478	0.088
10	$(4,4)$	0.7500	0.70	0.0481	0.0030	0.0031	0.6000	0.55	0.0510	0.0510	0.0412
15				0.0510	0.0500	0.0281			0.0478	0.0510	0.0415
25				0.0503	0.0050	0.0017			0.0512	0.0478	0.0325
35				0.0540	0.0570	0.0125			0.0499	0.0510	0.0324
10	$(4,6)$	0.8476	0.80	0.0479	0.0590	0.0254	0.7229	0.65	0.0510	0.0541	0.0254
15				0.0486	0.0480	0.0123			0.0502	0.0499	0.0213
25				0.0512	0.0480	0.0325			0.0513	0.0499	0.0215
35				0.01487	0.0059	0.0327			0.0499	0.0510	0.0113
10	$(4,8)$	0.9000	0.85	0.0489	0.0590	0.0400	0.8000	0.75	0.0501	0.0541	0.0413
15				0.0466	0.0570	0.0328			0.4888	0.0510	0.0077
25				0.0485	0.0059	0.0214			0.4789	0.0541	0.0012
35				0.0512	0.0059	0.0415			0.0541	0.0510	0.0045

Table 5.7(b) Empirical (true) Type-I error rates for testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) level is $\gamma = 0.05$ with the known common scale parameter $\lambda = 10$

n	β	$R_{1,3}$	R_0	Generalized	Bayesian	Classical	$R_{2,4}$	R_0	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.50	0.0511	0.0478	0.0012	0.3905	0.35	0.0512	0.0498	0.0124
15				0.0513	0.0231	0.0045			0.0548	0.0088	0.0128
25				0.0489	0.0478	0.0078			0.0510	0.0145	0.0088
35				0.0485	0.0511	0.0099			0.0555	0.0498	0.0099
10	(4, 4)	0.7500	0.70	0.0478	0.0231	0.0012	0.6000	0.55	0.0478	0.0128	0.0100
15				0.0498	0.0222	0.0100			0.0498	0.0145	0.0099
25				0.0478	0.0125	0.0125			0.0457	0.0145	0.0145
35				0.0456	0.0231	0.0123			0.0498	0.0145	0.0179
10	(4, 6)	0.8476	0.80	0.0511	0.0511	0.0236	0.7229	0.65	0.0478	0.0400	0.0258
15				0.0509	0.0222	0.0223			0.0513	0.0088	0.0248
25				0.0478	0.0222	0.0145			0.0511	0.0145	0.0325
35				0.0499	0.0222	0.0128			0.0547	0.0547	0.0125
10	(4, 8)	0.9000	0.85	0.0456	0.0511	0.0222	0.8000	0.75	0.0555	0.0498	0.0410
15				0.0477	0.0478	0.0114			0.0547	0.0400	0.0124
25				0.0518	0.0231	0.0231			0.0512	0.0400	0.0400
35				0.0498	0.0231	0.0224			0.0478	0.0478	0.0128

When hypothesis $R_{s,k} > 0.50$ is tested when nominal (intended) level is $\gamma = 0.05$ with the common parameter $\lambda = 3$ for $\boldsymbol{\beta} = (4,2)$,the generalized Type-I error rate is 0.0511, which is very close to the nominal value. However, the classical Type-I error rate is 0.007, a value way off from the nominal value. This suggests that the generalized variable method is size-guaranteed. When $R_{s,k} > R_0$ is tested in a similar fashion for various parameter combinations such as $\lambda = (3, 10)$, $(s,k) = \{(1,3),(2,4)\}$, $\boldsymbol{\beta} = (\alpha_1, \alpha_2) = \{(4,2),(4,4),(4,6),(4,8)\}$, $n = \{10, 15, 25,\ 35\}$, and $R_0 = \{0.35, 0.50, 0.55, 0.65, 0.70, 0.75, 0.80, 0.85\}$, all these arguments clearly show that the generalized variable method (GV-Method) is size-guaranteed, while the classical method (C-Method) approach fails to do so. Hence, the GV-Method outperforms the C-Method for this particular case.

Tables 5.8 (a), (b), (c), and (d) show the power comparison for testing $R_{s,k} \leq 0.50$ vs. $R_{s,k} > 0.50$ before and after adjusting the actual type-I error rate at $\gamma = 0.05$ based on $10,000$ replications.

Table 5.8(a) Comparison of powers for testing $H_0 : R_{1,3} \leq 0.5429$ vs $H_a : R_{1,3} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common parameter is known ($\lambda = 3$)

Parameters			Without adjusting the size			After adjusting the size		
n	β	$R_{1,3}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.1151	0.0630	0.0630	0.0500	0.0500	0.0500
15			0.1142	0.1217	0.0592	0.1138	0.0500	0.0612
25			0.1178	0.1217	0.0698	0.1161	0.0500	0.0789
35			0.1154	0.0630	0.0657	0.1175	0.0754	0.0754
10	(4, 4)	0.7500	0.2481	0.1125	0.1125	0.2400	0.0754	0.1145
15			0.2441	0.1125	0.1127	0.2389	0.0500	0.1189
25			0.2145	0.0630	0.1217	0.2082	0.0754	0.1245
35			0.2345	0.2569	0.1354	0.2333	0.0500	0.1256
10	(4, 6)	0.8476	0.5879	0.2569	0.2569	0.5414	0.0754	0.2456
15			0.5789	0.2569	0.3512	0.5412	0.3542	0.2889
25			0.5887	0.3489	0.3489	0.5879	0.3542	0.3542
35			0.5456	0.4415	0.4415	0.5312	0.5312	0.4412
10	(4, 8)	0.9000	0.8011	0.4415	0.6123	0.7889	0.5312	0.5555
15			0.8951	0.4415	0.6879	0.8045	0.5312	0.6415
25			0.8561	0.6879	0.7412	0.8412	0.6889	0.6889
35			0.8893	0.6879	0.7425	0.8745	0.6889	0.7850

Table 5.8(b) Comparison of powers for testing $H_0 : R_{2,4} \le 0.0.5429$ vs $H_a : R_{2,4} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common scale parameter is known $(\lambda = 3)$

Parameters			Without adjusting the size			After adjusting the size		
n	β	$R_{2,4}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.1180	0.1180	0.0660	0.0500	0.0553	0.0553
15			0.1010	0.1180	0.0712	0.0998	0.0553	0.0621
25			0.1021	0.1180	0.0722	0.1000	0.0553	0.0702
35			0.1225	0.0712	0.0741	0.1198	0.0553	0.0715
10	(4, 4)	0.7500	0.2222	0.0712	0.1215	0.1998	0.2000	0.1125
15			0.2112	0.0712	0.1015	0.2000	0.0715	0.1001
25			0.3125	0.0712	0.2451	0.2145	0.2000	0.1198
35			0.3546	0.2451	0.2415	0.3212	0.0715	0.2356
10	(4, 6)	0.8476	0.4115	0.3999	0.3874	0.3998	0.2000	0.3789
15			0.4899	0.2451	0.3899	0.3454	0.0715	0.3877
25			0.5551	0.3999	0.3999	0.4597	0.2000	0.3845
35			0.5789	0.3999	0.4521	0.5412	0.4852	0.4511
10	(4, 8)	0.9000	0.6889	0.3999	0.4887	0.5778	0.4852	0.4852
15			0.7888	0.8888	0.6552	0.6589	0.4852	0.5879
25			0.8888	0.8888	0.7858	0.7777	0.6666	0.6666
35			0.8994	0.8888	0.7889	0.8412	0.6666	0.6894

Table 5.8(c) omparison of powers for testing $H_0 : R_{1,3} \leq 0.5429$ vs $H_a : R_{1,3} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common scale parameter is known ($\lambda = 10$)

Parameters			Without adjusting the size			After adjusting the size		
n	β	$R_{1,3}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.1001	0.1001	0.0125	0.0500	0.0500	0.0500
15			0.1254	0.1001	0.0245	0.1356	0.0500	0.0235
25			0.1540	0.1001	0.325	0.1389	0.0500	0.0245
35			0.1656	0.0245	0.0458	0.1478	0.0500	0.0889
10	(4, 4)	0.7500	0.2789	0.0245	0.1225	0.1899	0.0500	0.0999
15			0.2889	0.0245	0.1458	0.2458	0.0500	0.1225
25			0.3211	0.0245	0.2451	0.2589	0.1899	0.1889
35			0.3489	0.3211	0.2898	0.3458	0.1899	0.1997
10	(4, 6)	0.8476	0.4569	0.3211	0.3254	0.3888	0.1899	0.2458
15			0.4689	0.3211	0.3589	0.4125	0.1899	0.2789
25			0.4889	0.3211	0.3789	0.5478	0.2589	0.3458
35			0.5879	0.3211	0.4558	0.6521	0.2589	0.3333
10	(4, 8)	0.9000	0.6655	0.3254	0.4789	0.7415	0.2589	0.3889
15			0.7889	0.3254	0.5511	0.8888	0.4215	0.4215
25			0.8994	0.6789	0.6654	0.8995	0.4215	0.4887
35			0.9994	0.6789	0.6789	0.9885	0.4215	0.6987

Table 5.8(d) omparison of powers for testing $H_0 : R_{2,4} \leq 0.5429$ vs $H_a : R_{2,4} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common scale parameter is known ($\lambda = 10$)

Parameters			Without adjusting the size			After adjusting the size		
n	β	$R_{1,3}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.1001	0.1001	0.0125	0.0500	0.0500	0.0500
15			0.1254	0.1001	0.0245	0.1356	0.0500	0.0235
25			0.1540	0.1001	0.325	0.1389	0.0500	0.0245
35			0.1656	0.0245	0.0458	0.1478	0.0500	0.0889
10	(4, 4)	0.7500	0.2789	0.0245	0.1225	0.1899	0.0500	0.0999
15			0.2889	0.0245	0.1458	0.2458	0.0500	0.1225
25			0.3211	0.0245	0.2451	0.2589	0.1899	0.1889
35			0.3489	0.3211	0.2898	0.3458	0.1899	0.1997
10	(4, 6)	0.8476	0.4569	0.3211	0.3254	0.3888	0.1899	0.2458
15			0.4689	0.3211	0.3589	0.4125	0.1899	0.2789
25			0.4889	0.3211	0.3789	0.5478	0.2589	0.3458
35			0.5879	0.3211	0.4558	0.6521	0.2589	0.3333
10	(4, 8)	0.9000	0.6655	0.3254	0.4789	0.7415	0.2589	0.3889
15			0.7889	0.3254	0.5511	0.8888	0.4215	0.4215
25			0.8994	0.6789	0.6654	0.8995	0.4215	0.4887
35			0.9994	0.6789	0.6789	0.9885	0.4215	0.6987

Without adjusting the size, the generalized powers for testing $H_0 : R_{s,k} \leq 0.5429$ vs. $H_a : R_{s,k} > 0.5429$ clearly suggest that the generalized variable method outperforms the classical method. Even after adjusting the size, the generalized variable method still maintains a light advantage over the classical method. The size of the test has to be adjusted to get a meaningful comparison of power of tests. But, in reality practitioners, being less-concern about the size, are not interested in adjusting the nominal size in order to get the desired level γ. In terms of computational time, it takes less than few minutes to run the proposed procedure for either of the examples on Dell Optiplex 3020 with processor 3.20 GHz and 8.00 GB RAM.

CHAPTER VI

OVERVIEW, SUMMARY, AND FUTURE WORKS

Overview

A number of authors have proposed and developed various inferential techniques for the reliability in multicomponent stress-strength system using various underlying distributions; see Hanagal (1999), Eryilmaz (2010), Rao, et al. (2015). For a comprehensive discussion on different stress-strength models, along with more theories and examples, readers are referred to the monograph of Kotz et al. (2003). In these studies, maximum likelihood estimator (MLE), moment estimator, and asymptotic confidence interval were obtained, but the generalized variable method (GVM)\ due to Tusi and Weerahandi (1989) was not taken into consideration. The purpose of this research is to develop, firstly, under the classical framework of inference, a pivotal quantity based on MLEs and the uniformly minimum variance unbiased estimators (UMVUEs) for the hypothesis testing of, and a pivotal quantity for constructing confidence intervals for $R_{s,k}$. Secondly, under the Bayesian framework of inference, exact and approximate point estimators of $R_{s,k}$ with the aid of the Markov Chain Monte Carlo (MCMC) procedure using the Gibbs sampler and Metropolis-Hasting sampler, and Lindley's approximation (1980) procedure will be discussed. Bayesian confidence intervals (BCIs) as well as highest posterior density intervals (HPDIs) are also computed. Finally, under the generalized variable framework of inferences, generalized point estimators and generalized CIs for, and hypothesis testing of $R_{s,k}$ are discussed. Toward this, we develop methods based on the concept of generalized variable.

The diagnostic testing procedures found in reliability analyses have a wide variety of applications in economics, engineering, biostatistics, biomedical, and various other related-fields of research. It is the opinion of the author of this research that the intensive and extensive research like these are to be carried out to broaden the scope of, and to open new avenues for, the critical and rational thinking needed to produce new statistical methodologies and procedures to tackle the complex and complicated statistical problems found in aforementioned fields. Collaborative and independent research based on these new procedure with other interested parties will contribute in a great deal to the success and advancement of the statistical research. A statistics major with a background in this material will be at a competitive advantage whether the students, majoring or minoring in statistics, plan to enter the work force directly, or plan to pursue doctoral degrees. Over the years we have seen an increase in the number of students pursuing advanced degrees in statistics after graduation. This research will broaden the statistical knowledge of those students who are pursuing Ph.D. and are interested in doing research to contribute to the statistical arena, and also those who seek employment or internships in various institutions.

Summary

In Chapter II, we review and suggest remedies for the problem of making classical inferences for $R_{s,k}$.

In Chapter II, we review and suggest remedies for the problem of making inferences in the face of nuisance parameters from different populations by using generalized p-value approach introduced by Tsui and Weerahandi (1987). This new development, which has a promising approach for data modeling in reliability and survivability has revolutionized modern society by its advanced techniques - may be very useful for practitioners who have been performing inferences for small samples with the large sample approach for their research work. Reliability experts who encounter several various systems are exposed to a model which has a longer right tails. Inferences of functions of parameters of such heavy-tailed distributions, especially several distributions are performed using this new model. In addition to reliability found in engineering, this methodology is heavily used in agricultural, mechanical engineering, econometrics fields, etc. This generalized p-value approach can easily be used to overcome the drawbacks of F-test's failure to detect significant experimental results. Practitioners in biomedical research where each sample point is vital and expensive can comfortably use this generalized variable method to provide a significant test with power of testing procedures.

In Chapter IV, we review and suggest remedies for the problem of making Bayesian inferences for $R_{s,k}$

In Chapter V simulation results on bias, coverage probability, mean confidence length, type I error control, unadjusted and adjusted power are presented. In addsiton , practical application analysis based on the monthly water capacity of the Shasta reservoir of the Shasta Dam (USBR

SHA operated by the U.S. Bureau of Reclamation, United States Department of the Interior) in Sacramento, California, USA, especially the month of April for the maximum water level, and the mean annual capacity from 1974 to 2016 are considered (see, http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA; **Source:** *California Data Exchange Center, Department of Water Resources (DWR), Government of California).*

Complicated functions of parameters are not easily inferred exactly using classical approach; in that sense here we emphasize the importance of using generalized variable method which outperforms other available inferential methodologies in the face of nuisance parameters

Future research

One of the major weaknesses and the drawbacks of generalized variable method is that its non-applicability when the pivotal quantities are not distributed with standard distributions. But such situations are also tackled by using intensive and tedious numerical approaches which is to be explored as future works. Moreover, the power guarantee has not been mathematically proved and is a topic to be discussed too. Advantages and drawbacks are, furthermore, summarized as follows;

Advantages of the proposed method:

1. Can handle complicated functions of parameters.
2. Various distribution-driven tests.
3. Valid for smaller samples as well as for the larger samples.
4. Can easily avoid the unnecessary large sample assumption.

5. Can avoid the unnecessary large sample assumption.

6. Can find exact solutions in the face of nuisance parameters.

Drawbacks of the proposed procedure:

1. p-values are not uniformly distributed,

2. If the estimators are not distributed with distributions with closed forms, intensive numerical analysis has to be carried out.

3. Can not be remedied all situations unless the test variable satisfy the properties of Generalized Test Variable.

A compact and comprehensive final version of the thesis will be submitted to the Graduate Coordinating Committee of the Department of Mathematics and to the university's Graduate School. Collaborating with my advisor Dr. Gunasekera, several high quality advanced papers stemming from this research will be submitted to top peer-reviewed statistical/mathematical journals. In addition, a paper will be submitted to the 2017 Joint Statistical Meetings (JSM) for the oral presentation. JSM\ is the largest gathering of statisticians in North America, attended by more than 6000 across the globe, held jointly with the American Statistical Association (ASA), Institute of Mathematical Statistics (IMS), International Biometric Society (IBS) (Eastern North American Region - ENAR and Western North American Region –WNAR), Statistical Society of Canada (SSC), International Chinese Statistical Association (ICSA), International Indian Statistical Association (IISA), International Society for Bayesian Analysis (ISBA), and Korean International Statistical

Association (KISA). It will be held at the Baltimore Convention Center, Baltimore, Maryland from July 29 to August 03, 2017.

Furthermore, rather than just analysing the two-component system, we can analyze three-component or many-compoent systems. Another development in analysis of reliability is taking different type of censored, truncated, grouped, or merged data under Type-or -II left-and right-censored data rather than taking type-II progressively right censored data uniformly removals thus paving the way for different aspects to be discussed.

Applicability, accessibility, and usability of exact nonparametric procedures in reliability are in consideration and hope to explore nonparametric new approaches coupled with the old ones to come up with methodology to tackle drastic, vague situations without taking the underlying distributions into account. Furthermore, seek the applications of this generalized p-value methodology not only reliability but also in other areas and fields such as data networking, econometrics, agriculture, actuarial field, insurance, etc.

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VITA

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