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ON THE DETERMINATION OF A FUNCTION FROM ITS CONICAL RADON TRANSFORM WITH A FIXED CENTRAL AXIS*

SUNGHWAN MOON†

Abstract. Over the past decade, a Radon-type transform called a conical Radon transform, which assigns to a given function its integral over various sets of cones, has arisen in the context of Compton cameras used in single photon emission computed tomography. Here, we study the conical Radon transform for which the central axis of the cones of integration is fixed. We present many of its properties, such as two inversion formulas, a stability estimate, and uniqueness and reconstruction for a local data problem. An existing inversion formula is generalized and a stability estimate is presented for general dimensions. The other properties are completely new results.

Key words. Radon transform, Compton camera, tomography, conical Radon transform, SPECT

AMS subject classifications. 44A12, 65R10, 92C55

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1. Introduction.

Single photon emission computed tomography (SPECT), a useful medical diagnostic tool, inspects internal organs and produces pictures of internal processes using the distribution of an isotope. SPECT typically provides the information as a cross-sectional slice, but it is easy to reformat or manipulate this information into other types of images. To obtain the image in SPECT, a gamma-emitting radioisotope is injected into the patient, usually via the bloodstream. This radioisotope passes through the body and is detected by the scan. For use in SPECT, a *Compton camera* was introduced [22, 24]. This Compton camera has very high sensitivity and flexibility of geometrical design, so it has attracted a lot of interest in many areas, including nuclear power plant monitoring and astronomy.

A typical Compton camera consists of two planar detectors: a scatter detector and an absorption detector, positioned one behind the other. A photon emitted in the direction of the camera undergoes Compton scattering in the scatter detector positioned ahead and is absorbed in the absorption detector (see Figure 1). In each detector, the position of the hit and energy of the photon are measured. A difference vector between two device positions determines the central axis of a cone. The scattering angle ψ from the central axis can be computed from the measured energies and electron mass as follows:

$$\cos\psi = 1 - \frac{mc^2\Delta E}{(E-\Delta E)E},$$

where m is the mass of the electron, c is the speed of light, E is the initial gamma ray energy, and ΔE is the energy transferred to the electron in the scattering process [1, 16]. Therefore, we get the surface integral of the distribution of the radiation source over cones with a central axis, a vertex $\mathbf{u}$ at the position of the scatter detector, and a scattering angle ψ. We called this the *conical Radon transform*. Here we study

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†Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Ulsan 44919, Republic of Korea (shmoon@unist.ac.kr).

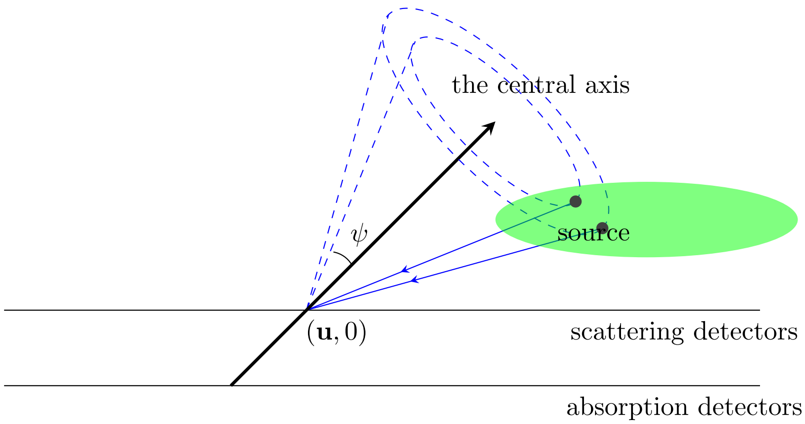

FIG. 1. *Schematic representation of a Compton camera.*

the conical Radon transform with the central axis fixed perpendicular to the detector plane.

Many inversion formulas for various types of conical Radon transforms have been derived [5, 11, 12, 15, 16, 21]. In particular, the conical Radon transform with a fixed central axis was studied in [6, 13, 14, 19, 20, 26]. Cree and Bones derived the inversion formula for the conical Radon transform in [6], and Nguyen, Truong, and Grangeat obtained another inversion formula in [20]. Haltmeier first defined an n-dimensional conical Radon transform and found the inversion formula in [13]. In [14] Jung and Moon discussed a relation between two existing formulas: one derived from Cree and Bones and the other from Nguyen, Truong, and Grangeat. In the same paper, they also obtained stability estimates for a more general form of the 3-dimensional conical Radon transform. A 2-dimensional conical Radon transform becomes a V-line Radon transform, which integrates a function along coupled rays with a common vertex. This V-line Radon transform has been studied in the context of single scattering optical tomography [7, 8, 9]. Many works [1, 2, 3, 17, 25] derived inversion formulas for various versions of the V-line Radon transform.

In this article we generalize the conical Radon transform with a fixed central axis to n-dimensions and study its two inversion formulas, a stability estimate, and uniqueness and reconstruction for a local data problem; an existing inversion formula derived in [6, 13] is generalized and an existing stability estimate derived in [14] is generalized for general dimensions.

The definition of the conical Radon transform is formulated precisely in section 2. Section 3 is devoted to elementary properties of the conical Radon transform including an analogue of the Fourier slice theorem. Two inversion formulas are presented in section 4. We describe the range of the conical Radon transform in one special case in section 5. In section 6, we show that taking a certain linear operator on the conical Radon transform is an isometry and discuss a stability estimate. In section 7, uniqueness and reconstruction for a partial data problem are studied.

2. Definition. Let $\mathbf{f}$ be a function on $\mathbb{R}^3$ with compact support in the upper half space $\mathbb{R}^2 \times [0, \infty)$. We define the conical Radon transform by

$$\mathcal{C}\mathbf{f}(\mathbf{u}, s) := \int_0^{2\pi} \int_0^{\infty} \mathbf{f}(\mathbf{u} + zs\boldsymbol{\theta}, z) z dz d\theta$$

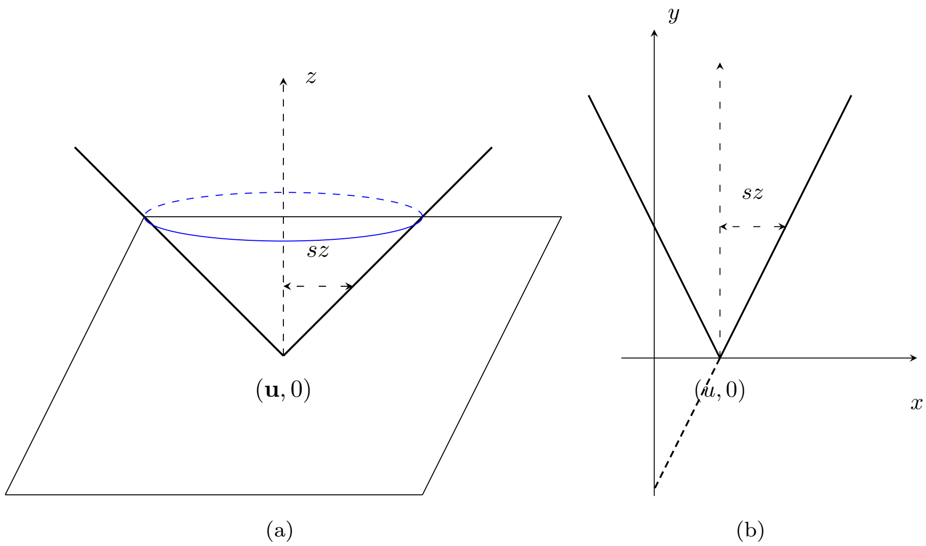

FIG. 2. (a) *A cone of integration;* (b) *a V-line and a line of integration.*

for $(\mathbf{u}, s) = (u_1, u_2, s) \in \mathbb{R}^2 \times [0, \infty)$ (see Figure 2(a)). Here $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in S^1$ and s means the opening extent of the cone of integration when $z = 1$, i.e., the tangent of the scattering angle ψ ($s = \tan\psi$). (In fact, $\mathcal{C}\mathbf{f}$ is not exactly the surface integral of f over a family of cones. It misses a weight factor $\sin\psi/\cos^2\psi$. However, from the surface integrals over cones, we can obtain it.) To more easily formulate and prove the results for $\mathcal{C}\mathbf{f}$, we introduce a related transform. Let the function f on $\mathbb{R}^4$ satisfy $f(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}, |\mathbf{y}|)$ for $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, and let the conical Radon transform of f be defined by

$$Cf(\mathbf{u}, \mathbf{v}) = \int_{\mathbb{R}^2} f(\mathbf{u} + |\mathbf{v}|\mathbf{y}, \mathbf{y}) d\mathbf{y}.$$

Then we have $Cf(\mathbf{u}, \mathbf{v}) = \mathcal{C}\mathbf{f}(\mathbf{u}, |\mathbf{v}|)$. In fact, making a change of the variables gives

$$\begin{aligned} Cf(\mathbf{u}, \mathbf{v}) &= \int_{\mathbb{R}^2} f(\mathbf{u} + |\mathbf{v}|\mathbf{y}, y_1, y_2) d\mathbf{y} = \int_0^{2\pi} \int_0^\infty \mathbf{f}(\mathbf{u} + |\mathbf{v}| r\boldsymbol{\theta}, r) r dr d\theta \\ &= \mathcal{C}\mathbf{f}(\mathbf{u}, |\mathbf{v}|). \end{aligned}$$

Let us consider a natural n-dimensional analogue of the conical Radon transform for the function $\mathbf{f}$ on $\mathbb{R}^n$ with compact support in the upper half space $\mathbb{R}^{n-1} \times [0, \infty)$. As in the 3-dimensional case, the conical Radon transform is defined by

$$\mathcal{C}\mathbf{f}(\mathbf{u}, s) := \begin{cases} \displaystyle\int_{S^{n-2}} \int_0^\infty \mathbf{f}(\mathbf{u} + zs\boldsymbol{\theta}, z) z^{n-2} dz dS(\boldsymbol{\theta}) & \text{if } n \geq 3, \\ \displaystyle\int_0^\infty \mathbf{f}(\mathbf{u} + zs, z) + \mathbf{f}(\mathbf{u} - zs, z) dz & \text{if } n = 2, \end{cases}$$

for $(\mathbf{u}, s) = (u_1, u_2, \ldots, u_{n-1}, s) \in \mathbb{R}^{n-1} \times [0, \infty)$. Here $dS(\boldsymbol{\theta})$ is the standard measure on the unit sphere S^{n-2}. When $n = 2$, $\mathcal{C}\mathbf{f}$ is the V-line Radon transform (without a weight factor $\sqrt{1 + s^2}$ or $\cos^{-1}\psi$) whose integral domain is the set of V-shape lines (see Figure 2(b)). Similar to the definition of Cf for the 3-dimensional case, we define Cf for a function f on $\mathbb{R}^{2(n-1)}$ with $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$ for $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ by

$$Cf(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^{n-1}} f(\mathbf{u}+|\mathbf{v}|\mathbf{y},\mathbf{y})d\mathbf{y} \qquad \text{for } (\mathbf{u},\mathbf{v}) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}. \tag{1}$$

Again, we have $Cf(\mathbf{u},\mathbf{v}) = \mathcal{C}\mathbf{f}(\mathbf{u},|\mathbf{v}|)$. Our goals are to reconstruct f (or $\mathbf{f}$) from Cf (or $\mathcal{C}\mathbf{f}$) and to study properties of this conical Radon transform.[1]

Remark 1. The above definition (1) includes the $n = 2$ case. When $n = 2$, Cf becomes an integral of f along the line perpendicular to

$$(1,-|v|)/\sqrt{1+v^2}$$

with signed distance

$$u/\sqrt{1+v^2} \quad \text{(see Figure 2(b)).}$$

In this case the measure for the line becomes

$$\sqrt{1+v^2}dy.$$

Hence we have a relation between Cf and Rf,

$$Cf(u,v) = (1+v^2)^{-\frac{1}{2}} Rf\left(\frac{(1,-|v|)}{\sqrt{1+v^2}}, \frac{u}{\sqrt{1+v^2}}\right), \tag{2}$$

where Rf is the regular Radon transform defined by

$$Rf(\boldsymbol{\omega},t) = \int_{\mathbb{R}} f(t\boldsymbol{\omega}+s\boldsymbol{\omega}^\perp)ds \qquad \text{for } (\boldsymbol{\omega},t)\in S^1\times\mathbb{R}.$$

Notice that

$$Rf\left(\frac{(1,-|v|)}{\sqrt{1+v^2}}, \frac{u}{\sqrt{1+v^2}}\right) = Rf\left(\frac{(1,|v|)}{\sqrt{1+v^2}}, \frac{u}{\sqrt{1+v^2}}\right).$$

❖

3. Elementary properties. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of infinitely differentiable functions $\mathbf{f}$ with $\sup\{|\mathbf{x}^\alpha\partial^\beta\mathbf{f}(\mathbf{x})| : \mathbf{x}\in\mathbb{R}^n\} < \infty$ for any multi-indices α and β. We introduce

$$\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) = \{f\in\mathcal{S}(\mathbb{R}^{2(n-1)}) : f(\mathbf{x},\mathbf{y}) = f(\mathbf{x},U\mathbf{y}) \text{ for any } (\mathbf{x},\mathbf{y})\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1} \text{ and for all orthonormal transformations } U\}.$$

THEOREM 1. *For $f\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, we have*

$$\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v}) = \mathcal{F}f(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi}) = \mathcal{F}f(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|),$$

where $\mathcal{F}_1(Cf)$ and $\mathcal{F}f$ are the $n-1$-dimensional and $2(n-1)$-dimensional Fourier transforms of Cf and f with respect to $\mathbf{u}\in\mathbb{R}^{n-1}$ and $(\mathbf{x},\mathbf{y})\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$, respectively.

[1] In fact, the expansion of a function defined on $\mathbb{R}^n$ to a function on $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ has also been used by Andersson in [4] to easily derive many properties for the spherical mean Radon transform.

Proof. Taking the $n-1$-dimensional Fourier transform of $Cf(\mathbf{u},\mathbf{v})$ with respect to $\mathbf{u}$ yields

$$\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v}) = \int_{\mathbb{R}^{n-1}} \mathcal{F}_1 f(\boldsymbol{\xi},\mathbf{y}) e^{i|\mathbf{v}|\mathbf{y}\cdot\boldsymbol{\xi}} d\mathbf{y} = \mathcal{F}f(\boldsymbol{\xi}, -|\mathbf{v}|\boldsymbol{\xi}),$$

where $\mathcal{F}_1 f$ is the $n-1$-dimensional Fourier transform of f with respect to $\mathbf{x}$. Since the Fourier transform of a radial function is also radial, we have the assertion. ∎

Note that if f is a radial function on $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ in the second variable $\mathbf{y}$ and $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$, then we have

$$\mathcal{F}_2 f(\mathbf{x},\boldsymbol{\xi}) = (2\pi)^{\frac{n-1}{2}} |\boldsymbol{\xi}|^{\frac{3-n}{2}} \mathbf{H}_{\frac{n-3}{2}} \mathbf{f}(\mathbf{x},|\boldsymbol{\xi}|), \tag{3}$$

where $\mathcal{F}_2 f$ is the $n-1$-dimensional Fourier transform of f with respect to $\mathbf{y}$ and

$$\mathbf{H}_{\frac{n-2}{2}} \mathbf{f}(\mathbf{x},\rho) = \int_0^\infty \mathbf{f}(\mathbf{x},s) s^{\frac{n}{2}} J_{\frac{n-2}{2}}(s\rho) ds,$$

where J_k is the Bessel function of the first kind of order k (see [23, section 7.7]).

Corollary 1. Let $f(\mathbf{x},\mathbf{y}) \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, and let $\mathbf{f}$ be a function on $\mathbb{R}^{n-1}\times[0,\infty)$ with $\mathbf{f}(\mathbf{x},|\mathbf{y}|) = f(\mathbf{x},\mathbf{y})$. Then we have

$$\mathcal{F}_1(\mathcal{C}\mathbf{f})(\boldsymbol{\xi},s) = (2\pi)^{\frac{n-1}{2}} |\boldsymbol{\xi}|^{\frac{3-n}{2}} s^{\frac{3-n}{2}} \mathbf{H}_{\frac{n-3}{2}} \mathcal{F}_1 \mathbf{f}(\boldsymbol{\xi}, s|\boldsymbol{\xi}|). \tag{4}$$

Remark 2. Equation (4) was first derived in [6] for $n=3$ and in [13] for general n. ❖

Proposition 1. The conical Radon transforms C and $\mathcal{C}$ are self-adjoint, in the sense that for $f,g \in S_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ with $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$ and $g(\mathbf{u},\mathbf{v}) = \mathbf{g}(\mathbf{u},|\mathbf{v}|)$,

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} Cf(\mathbf{u},\mathbf{v}) g(\mathbf{u},\mathbf{v}) d\mathbf{u} d\mathbf{v} = \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y}) Cg(\mathbf{x},\mathbf{y}) d\mathbf{x} d\mathbf{y} \tag{5}$$

and

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty \mathcal{C}\mathbf{f}(\mathbf{u},s)\mathbf{g}(\mathbf{u},s) s^{n-2} ds d\mathbf{u} = \int_{\mathbb{R}^{n-1}}\int_0^\infty \mathbf{f}(\mathbf{x},z)\mathcal{C}\mathbf{g}(\mathbf{x},z) z^{n-2} dz d\mathbf{x}. \tag{6}$$

Proof. We start out from

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} Cf(\mathbf{u},\mathbf{v}) g(\mathbf{u},\mathbf{v}) d\mathbf{u} d\mathbf{v} &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{u}+|\mathbf{v}|\mathbf{y},\mathbf{y}) d\mathbf{y}\; g(\mathbf{u},\mathbf{v}) d\mathbf{u} d\mathbf{v} \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y}) \int_{\mathbb{R}^{n-1}} g(\mathbf{x}-\mathbf{y}|\mathbf{v}|,\mathbf{v}) d\mathbf{v} d\mathbf{y} d\mathbf{x} \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},-\mathbf{y}) \int_{\mathbb{R}^{n-1}} g(\mathbf{x}+\mathbf{y}|\mathbf{v}|,\mathbf{v}) d\mathbf{v} d\mathbf{y} d\mathbf{x},\end{aligned}$$

where in the second and last lines, we changed the variables $\mathbf{u}+|\mathbf{v}|\mathbf{y}\to\mathbf{x}$ and $\mathbf{y}\to-\mathbf{y}$, respectively. Since f is a radial function in $\mathbf{y}$, we have

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} Cf(\mathbf{u},\mathbf{v}) g(\mathbf{u},\mathbf{v}) d\mathbf{u} d\mathbf{v} = \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y}) \int_{\mathbb{R}^{n-1}} g(\mathbf{x}+\mathbf{y}|\mathbf{v}|,\mathbf{v}) d\mathbf{v} d\mathbf{y} d\mathbf{x},$$

which is our assertion. ∎

Proposition 2. For $f, g \in \mathcal{S}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, we have

$$C(f * g) = Cf * Cg.$$

This proposition follows from Theorem 1 and $\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$.

By the definition of Cf, we notice that Cf is the integral of f over an $n-1$-dimensional plane. It is a natural idea that Cf can be converted to the $2(n-1)$-dimensional regular Radon transform by an $n-2$-dimensional integration. (Indeed, we showed that the 2-dimensional conical Radon transform is converted to the 2-dimensional regular Radon transform in Remark 1.) Let the regular Radon transform be defined by

$$Rf(\boldsymbol{\omega}, t) = \int_{\boldsymbol{\omega}^\perp} f(t\boldsymbol{\omega} + \boldsymbol{\tau}) d\boldsymbol{\tau} \qquad \text{for } (\boldsymbol{\omega}, t) \in S^{2n-3} \times \mathbb{R}.$$

Then this can be represented by

$$Rf(\boldsymbol{\omega}, t) = \sqrt{1 + |\boldsymbol{\omega}'/\omega_1|^2} \int_{\mathbb{R}^{2n-3}} f\left(-\frac{\boldsymbol{\omega}' \cdot \boldsymbol{\tau}}{\omega_1} + \frac{t}{\omega_1}, \boldsymbol{\tau}\right) d\boldsymbol{\tau} \tag{7}$$

for $\omega_1 \neq 0$. Here $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{2n-2}) = (\omega_1, \boldsymbol{\omega}') \in S^{2n-3}$ and $\boldsymbol{\tau} \in \mathbb{R}^{2n-3}$.

Now we convert the $2(n-1)$-dimensional conical Radon transform Cf to the $2(n-1)$-dimensional regular Radon transform. For $(\mathbf{a}, b) \in \mathbb{R}^{n-2} \times \mathbb{R}$, we integrate $Cf(\mathbf{a} \cdot \mathbf{u}' + b, \mathbf{u}', \mathbf{v})$ with respect to $\mathbf{u}' = (u_2, u_3, \ldots, u_{n-1}) \in \mathbb{R}^{n-2}$:

(8)
$$\begin{aligned}\int_{\mathbb{R}^{n-2}} Cf(\mathbf{a} \cdot \mathbf{u}' + b, \mathbf{u}', \mathbf{v}) d\mathbf{u}' &= \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^{n-1}} f(\mathbf{a} \cdot \mathbf{u}' + b + |\mathbf{v}|y_1, \mathbf{u}' + |\mathbf{v}|\mathbf{y}', \mathbf{y}) d\mathbf{y} d\mathbf{u}' \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f(\mathbf{a} \cdot \mathbf{u}' + b + |\mathbf{v}|y_1, \mathbf{u}' + |\mathbf{v}|\mathbf{y}', \mathbf{y}) d\mathbf{u}' d\mathbf{y} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f(\mathbf{a} \cdot \mathbf{u}' + |\mathbf{v}|y_1 - \mathbf{a}|\mathbf{v}|\mathbf{y}' + b, \mathbf{u}' + |\mathbf{v}|\mathbf{y}', \mathbf{y}) d\mathbf{u}' d\mathbf{y},\end{aligned}$$

where we changed the variables $\mathbf{u}' + |\mathbf{v}|\mathbf{y}' \to \mathbf{u}'$. Equation (8) is equivalent to

$$\begin{aligned}&\int_{\mathbb{R}^{n-2}} Cf(\mathbf{a} \cdot \mathbf{u}' + b, \mathbf{u}', \mathbf{v}) d\mathbf{u}' \\ &= ((1 + |\mathbf{a}|^2)(1 + |\mathbf{v}|^2))^{-1/2} Rf\left(\frac{(-1, \mathbf{a}, |\mathbf{v}|, -\mathbf{a}|\mathbf{v}|)}{\sqrt{(1 + |\mathbf{a}|^2)(1 + |\mathbf{v}|^2)}}, \frac{b}{\sqrt{(1 + |\mathbf{a}|^2)(1 + |\mathbf{v}|^2)}}\right)\end{aligned} \tag{9}$$

(see (7)). Since $f(\mathbf{x}, \mathbf{y})$ is radial in $\mathbf{y}$, we have for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in \mathbb{R}^{n-1}$

$$f(\mathbf{a} \cdot \mathbf{u}' + \boldsymbol{\alpha} \cdot \mathbf{y} + b, \mathbf{u}', \mathbf{y}) = f(\mathbf{a} \cdot \mathbf{u}' + \boldsymbol{\alpha} \cdot \mathbf{y} + b, \mathbf{u}', U\mathbf{y}),$$

where $U = (U_{\{i,j\}})$ is any $n-1 \times n-1$ orthogonal matrix. Then we have

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f(\mathbf{a} \cdot \mathbf{u}' + \boldsymbol{\alpha} \cdot \mathbf{y} + b, \mathbf{u}', \mathbf{y}) d\mathbf{u}' d\mathbf{y} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f(\mathbf{a} \cdot \mathbf{u}' + \boldsymbol{\alpha} \cdot \mathbf{y} + b, \mathbf{u}', U\mathbf{y}) d\mathbf{u}' d\mathbf{y} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-2}} f(\mathbf{a} \cdot \mathbf{u}' + U\boldsymbol{\alpha} \cdot \mathbf{y} + b, \mathbf{u}', \mathbf{y}) d\mathbf{u}' d\mathbf{y},\end{aligned} \tag{10}$$

where we changed the variables $U\mathbf{y} \to \mathbf{y}$. Combining the three equations (8), (9), and (10), we have the following proposition.

Proposition 3. Let $f \in S_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $(\boldsymbol{\alpha}, \mathbf{a}, b) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-2} \times \mathbb{R}$. Then we have for $|\boldsymbol{\alpha}| = |\mathbf{v}|^2(1 + |\mathbf{a}|^2)$ and $\mathbf{u}' = (u_2, u_3, \dots, u_{n-1}) \in \mathbb{R}^{n-2}$

$$\begin{aligned}(11)\quad &\int_{\mathbb{R}^{n-2}} Cf(\mathbf{a}\cdot\mathbf{u}' + b, \mathbf{u}', \mathbf{v})d\mathbf{u}' \\ &= ((1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2))^{-1/2} Rf\left(\frac{(-1,\mathbf{a},\boldsymbol{\alpha})}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}}, \frac{b}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}}\right).\end{aligned}$$

Remark 3. Taking an inversion formula for the regular Radon transform on (11), f can be recovered from Cf. ❖

4. Inversion formulas. Although we already show how to recover f from Cf in Remark 3, we present two explicit inversion formulas for the conical Radon transform in this section.

For $k < n$ we define the linear operator I^k by

$$\mathcal{F}(I^k\mathbf{f})(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-k}\mathcal{F}\mathbf{f}(\boldsymbol{\xi}).$$

The linear operator I^k is called the Riesz potential. For $\mathbf{f} \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}(I^k\mathbf{f}) \in L^1(\mathbb{R}^n)$, and hence $I^k\mathbf{f}$ makes sense and $I^{-k}I^k\mathbf{f} = \mathbf{f}$. When I_1^k is applied to functions on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, it acts on the first $n-1$-dimensional variable $\mathbf{u}$ or $\mathbf{x}$. Also, for $f \in \mathcal{S}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, we have $\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}) = \mathcal{F}f(\boldsymbol{\xi}, \boldsymbol{\xi}|\mathbf{v}|)$ by Theorem 1, so for a fixed $\mathbf{v}$, $\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}) \in \mathcal{S}(\mathbb{R}^{n-1})$, and therefore $I_1^k(Cf)$ makes sense.

THEOREM 2. *Let $f \in S_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ with $f(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}, |\mathbf{y}|)$. For $k < n-1$, we have*

$$(12)\qquad f = (2\pi)^{1-n} I_1^{-k} C I_1^{k+1-n} F, \qquad F = Cf,$$

and

$$\mathbf{f} = (2\pi)^{1-n} I_1^{-k} \mathcal{C} I_1^{k+1-n} \mathbf{F}, \qquad \mathbf{F} = \mathcal{C}\mathbf{f}.$$

Proof. We start out from the Fourier inversion formula

$$\begin{aligned}I_1^k f(\mathbf{x},\mathbf{y}) &= (2\pi)^{2(1-n)} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}|^{-k}\mathcal{F}f(\boldsymbol{\xi},\boldsymbol{\eta}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} e^{i\mathbf{y}\cdot\boldsymbol{\eta}} d\boldsymbol{\eta} d\boldsymbol{\xi} \\ &= (2\pi)^{2(1-n)} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}|^{n-1-k}\mathcal{F}f(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} e^{i\mathbf{y}\cdot\mathbf{v}|\boldsymbol{\xi}|} d\mathbf{v} d\boldsymbol{\xi},\end{aligned}$$

where in the last line we changed the variables $\boldsymbol{\eta} \to \mathbf{v}|\boldsymbol{\xi}|$. Since $\mathcal{F}f(\boldsymbol{\xi},\boldsymbol{\eta})$ and $I_1^k f(\mathbf{x},\mathbf{y})$ are radial on $\boldsymbol{\eta}$ and $\mathbf{y}$, respectively, we obtain

$$I_1^k f(\mathbf{x},\mathbf{y}) = (2\pi)^{2(1-n)} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}|^{n-1-k}\mathcal{F}f(\boldsymbol{\xi},\boldsymbol{\xi}|\mathbf{v}|) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} e^{i\mathbf{v}\cdot\boldsymbol{\xi}|\mathbf{y}|} d\mathbf{v} d\boldsymbol{\xi},$$

which is equivalent to

$$I_1^k f(\mathbf{x},\mathbf{y}) = (2\pi)^{2(1-n)} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}|^{n-1-k}\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v}) e^{i(\mathbf{x}+|\mathbf{y}|\mathbf{v})\cdot\boldsymbol{\xi}} d\mathbf{v} d\boldsymbol{\xi}.$$

Here we used Theorem 1. Since the inner integral can be represented by the Riesz potential, we have

$$\begin{aligned} I_1^k f(\mathbf{x},\mathbf{y}) &= (2\pi)^{2(1-n)} \int_{\mathbb{R}^{n-1}} (I_1^{k+1-n} Cf)(\mathbf{x}+|\mathbf{y}|\mathbf{v},\mathbf{v})d\mathbf{v} \\ &= (2\pi)^{2(1-n)} C(I_1^{k+1-n}Cf)(\mathbf{x},\mathbf{y}), \end{aligned}$$

and the inversion formula for C follows by applying I_1^{-k}. ∎

Remark 4. Putting $k=0$ in (12) yields

$$f = (2\pi)^{1-n} C I_1^{1-n} F, \qquad F = Cf. \tag{13}$$

When n is odd, (13) is actually equivalent to

$$f(\mathbf{x},\mathbf{y}) = (2\pi)^{1-n}(-1)^{\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \triangle_{\mathbf{u}}^{\frac{n-1}{2}} F(\mathbf{x}+|\mathbf{y}|\mathbf{v},\mathbf{v})d\mathbf{v},$$

where $\triangle_{\mathbf{u}}$ is the Laplacian operator with respect to $\mathbf{u}$. Thus the problem of reconstructing a function from its integrals over cones is local in odd dimensions, in the sense that computing the function at a point $(\mathbf{x},\mathbf{y})$ needs the integrals over cones passing through neighborhood of that point $(\mathbf{x},\mathbf{y})$. On the other hand, when n is even, the inversion (13) is nonlocal because the fractional Laplacian is nonlocal. Also, the reconstruction problem for $\mathcal{C}\mathbf{f}$ is local in odd dimensions and nonlocal in even dimensions, as Haltmeier also mentioned in [13]. ❖

Combining Proposition 1 and Theorem 2, we have an analogue of the Plancherel formula.

Proposition 4. Let $f,g \in S_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ satisfy $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$ and $g(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{x},|\mathbf{y}|)$. For any $k < n-1$, we have

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y})g(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} = (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} I_1^{-k}Cf(\mathbf{u},\mathbf{v})I_1^{k+1-n}Cg(\mathbf{u},\mathbf{v})d\mathbf{u}d\mathbf{v}$$

and

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^{n-1}} \mathbf{f}(\mathbf{x},z)\mathbf{g}(\mathbf{x},z)z^{n-2}d\mathbf{x}dz \\ &= (2\pi)^{1-n}\int_0^\infty\int_{\mathbb{R}^{n-1}} I_1^{-k}\mathcal{C}\mathbf{f}(\mathbf{u},s)I_1^{k+1-n}\mathcal{C}\mathbf{g}(\mathbf{u},s)s^{n-2}d\mathbf{u}ds. \end{aligned}$$

Proof. By Proposition 1 and Theorem 2, we have

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y})g(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &= (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} f(\mathbf{x},\mathbf{y})CI_1^{1-n}Cg(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &= (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} Cf(\mathbf{x},\mathbf{y})I_1^{1-n}Cg(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y} \\ &= (2\pi)^{2-2n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\boldsymbol{\xi}|^k\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{y})|\boldsymbol{\xi}|^{-k}\mathcal{F}_1(I_1^{1-n}Cg)(\boldsymbol{\xi},\mathbf{y})d\boldsymbol{\xi}d\mathbf{y}. \end{aligned}$$

Here in the last line, we used the Plancherel formula. ∎

A completely different inversion formula for the conical Radon transform is derived by expanding $\mathcal{F}_1 f$ and $\mathcal{F} f$ in spherical harmonics,

$$\mathcal{F}_1 f(\varrho\boldsymbol{\varphi},\mathbf{y}) = \sum_{l=0}^{\infty}\sum_{k=0}^{N(n-1,l)} (\mathcal{F}_1 f)_{kl}(\varrho,\mathbf{y}) Y_{lk}(\boldsymbol{\varphi})$$

and

$$\mathcal{F} f(\varrho\boldsymbol{\varphi},\boldsymbol{\eta}) = \sum_{l=0}^{\infty}\sum_{k=0}^{N(n-1,l)} (\mathcal{F} f)_{kl}(\varrho,\boldsymbol{\eta}) Y_{lk}(\boldsymbol{\varphi}),$$

where $Y_{lk}(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in S^{n-2}$ are spherical harmonics and

$$N(n-1,l) = \frac{(2l+n-3)(n+l-4)}{l!(n-3)!}, \qquad N(n-1,0) = 1.$$

Notice that

$$\int_{\mathbb{R}^{n-1}} (\mathcal{F}_1 f)_{kl}(\varrho,\mathbf{y}) e^{-i\mathbf{y}\cdot\boldsymbol{\eta}} d\mathbf{y} = (\mathcal{F} f)_{kl}(\varrho,\boldsymbol{\eta}).$$

From Theorem 1, we have the following relation between $(\mathcal{F} f)_{kl}$ and $(\mathcal{F}_1 F)_{kl}$, where $F = Cf$:

$$(\mathcal{F} f)_{kl}(\varrho,\mathbf{v}\varrho) = (\mathcal{F}_1 F)_{kl}(\varrho,\mathbf{v}). \tag{14}$$

Taking the inverse Fourier transform with respect to $\mathbf{v}$, we have the following theorem.

THEOREM 3. *Let $f \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. If $F = Cf$, then we have*

$$(\mathcal{F}_1 f)_{kl}(\varrho,\mathbf{y}) = \frac{\varrho^{n-1}}{(2\pi)^{n-1}} (\mathcal{F} F)_{kl}(\varrho,\mathbf{y}\varrho). \tag{15}$$

Proof. Taking the inverse Fourier transform of $(\mathcal{F} f)_{kl}(\varrho,\mathbf{v}\varrho)$ with respect to $\mathbf{y}$ gives

$$\begin{aligned}(\mathcal{F}_1 f)_{kl}(\varrho,\mathbf{y}) &= \frac{\varrho^{n-1}}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} (\mathcal{F} f)_{kl}(\varrho,\mathbf{v}\varrho) e^{i\mathbf{v}\varrho\cdot\mathbf{y}} d\mathbf{v}\\ &= \frac{\varrho^{n-1}}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} (\mathcal{F}_1 F)_{kl}(\varrho,\mathbf{v}) e^{i\mathbf{v}\varrho\cdot\mathbf{y}} d\mathbf{v}\\ &= \frac{\varrho^{n-1}}{(2\pi)^{n-1}} (\mathcal{F} F)_{kl}(\varrho,\mathbf{y}\varrho),\end{aligned}$$

where we used (14) in the second equality. ∎

Together with (3), (15) yields that that

$$(\mathcal{F}_1 \mathbf{f})_{kl}(\varrho,s) = \frac{\varrho^{\frac{n+1}{2}} s^{\frac{3-n}{2}}}{(2\pi)^{\frac{n-1}{2}}} \mathbf{H}_{\frac{n-3}{2}} (\mathcal{F}_1 \mathbf{F})_{kl}(\varrho,s\varrho), \quad \mathbf{F} = \mathcal{C}\mathbf{f}. \tag{16}$$

Remark 5. Equation (16) was already derived in [14, 20] for $n = 3$. ❖

5. The range. In this section, we shall determine the range of the conical Radon transform in one special case.

THEOREM 4. *The conical Radon transform C is a injection of $\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ into $\mathcal{S}_{r,c}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, where*

$$\begin{aligned}&\mathcal{S}_{r,c}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\\ &=\left\{F\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}):\int_{\mathbb{R}^{n-1}}F(\mathbf{u},\mathbf{v})d\mathbf{u}\ \textit{is a constant for any}\ \mathbf{v}\in\mathbb{R}^{n-1}\right\}.\end{aligned}$$

Proof. Putting $\boldsymbol{\xi}=0$ in Theorem 1, we have that $\mathcal{F}f(0,0)=\int_{\mathbb{R}^{n-1}}Cf(\mathbf{u},\mathbf{v})d\mathbf{u}$ is a constant. Hence it is enough to show $Cf\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. To demonstrate this, we show that $\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Also, $\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})$ is infinitely differentiable since $\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})=\mathcal{F}f(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi})=\mathcal{F}f(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|)$, and we have for any multi-indices $\alpha_1,\alpha_2,\beta_1$, and β_2,

$$\begin{aligned}&\sup\{|\boldsymbol{\xi}^{\alpha_1}\mathbf{v}^{\alpha_2}\partial_{\boldsymbol{\xi}}^{\beta_1}\partial_{\mathbf{v}}^{\beta_2}\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})|:(\boldsymbol{\xi},\mathbf{v})\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\}\\ &=\sup\{|\boldsymbol{\xi}^{\alpha_1}\mathbf{v}^{\alpha_2}|\boldsymbol{\xi}|^{|\beta_2|}|\mathbf{v}|^{|\beta_1|}\partial_{\boldsymbol{\xi}}^{\beta_1}\partial_{\mathbf{v}}^{\beta_2}\mathcal{F}f(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|)|(\boldsymbol{\xi},\mathbf{v})\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\}<\infty.\end{aligned}$$

Hence, Cf belongs to $\mathcal{S}_{r,c}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. ∎

When n is odd, I_1^{1-n} is equal to $(-1)^{\frac{n-1}{2}}\triangle_{\mathbf{u}}^{\frac{n-1}{2}}$ and in this case, we can say more.

THEOREM 5. *When n is odd, the conical Radon transform C is a bijection of $\mathcal{S}_{r,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, where*

$$\mathcal{S}_{r,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})=\left\{f\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}):\int_{\mathbb{R}^{n-1}}f(\mathbf{x},\mathbf{y})d\mathbf{x}=0\ \textit{for any}\ \mathbf{y}\in\mathbb{R}^{n-1}\right\}.$$

Proof. By Theorem 1, we have

$$\mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})=\mathcal{F}f(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|)=\int_{\mathbb{R}^{n-1}}\mathcal{F}_1f(\boldsymbol{\xi},\mathbf{y})e^{-i\mathbf{y}\cdot\mathbf{v}|\boldsymbol{\xi}|}d\mathbf{y},$$

so $\mathcal{F}_1(Cf)(0,\mathbf{v})$ is equal to zero. Therefore, the range of $S_{r,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ under the conical Radon transform C is a subset of $S_{r,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.

To show C is onto, let $F\in\mathcal{S}_{r,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. In view of Theorem 2, it appears natural to define

$$f=(2\pi)^{1-n}(-1)^{\frac{n-1}{2}}C\triangle_{\mathbf{u}}^{\frac{n-1}{2}}F.$$

We know that $\triangle_{\mathbf{u}}^{\frac{n-1}{2}}F\in\mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. By Theorem 4, f belongs to $\mathcal{S}_{r,c}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. In particular, $\mathcal{F}_1f(\boldsymbol{\xi},\mathbf{y})$ is equal to

$$(2\pi)^{1-n}|\boldsymbol{\xi}|^{n-1}\mathcal{F}F(\boldsymbol{\xi},\mathbf{y}|\boldsymbol{\xi}|),$$

so for any $\mathbf{y}\in\mathbb{R}^{n-1}$,

$$\mathcal{F}_1f(0,\mathbf{y})=\int_{\mathbb{R}^{n-1}}f(\mathbf{x},\mathbf{y})d\mathbf{x}=0.$$ ∎

6. An isometry property and Sobolev space estimates. In this section, we show that $I_1^{-\frac{n-1}{2}}C$ extends to an isometry and that the problem of reconstructing from the conical Radon transform is well-posed in the following sense: if f satisfying $Cf = F$ is uniquely determined for any F belonging to a certain space, the function f depends continuously on F.

Let $L^2(\mathbb{R}^{2(n-1)})$ be the regular L^2 space. For any $\gamma \geq \mathbb{R}$, let the spaces $L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $H^\gamma L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ be defined by

$$\begin{aligned} L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) = \{ & f \in L^2(\mathbb{R}^{2(n-1)}) : f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, U\mathbf{y}) \\ & \text{for any } (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \text{ and for all orthonormal transformations } U \} \end{aligned}$$

and

$$H^\gamma L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) = \{ f \in L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) : ||f||_\gamma < \infty \},$$

where

$$||f||_\gamma^2 = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\mathcal{F}f(\boldsymbol{\xi}, \boldsymbol{\eta})|^2 (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\boldsymbol{\eta}.$$

Notice that $H^\gamma L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ is a Hilbert space with the norm $||\cdot||_\gamma$ and $H^0 L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) = L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

THEOREM 6. *The mapping* $f \to I_1^{-\frac{n-1}{2}} Cf$ *extends to an isometry of* $H^\gamma L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ *onto itself.*

Proof. We start with $||f||_\gamma^2$:

$$\begin{aligned} ||f||_\gamma^2 &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\mathcal{F}f(\boldsymbol{\xi}, \mathbf{v})|^2 (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}/|\boldsymbol{\xi}|)|^2 (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v})|^2 (1 + |\boldsymbol{\xi}|^2)^\gamma |\boldsymbol{\xi}|^{n-1} d\boldsymbol{\xi} d\mathbf{v} \\ &= (2\pi)^{1-n} ||I_1^{-\frac{n-1}{2}} Cf||_\gamma^2. \end{aligned} \tag{17}$$

Here in the second and third lines, we used Theorem 1 and changed variables $\mathbf{v}/|\boldsymbol{\xi}| \to \mathbf{v}$, respectively. It remains to prove that the mapping is surjective. It is enough to show that if $g \in H^\gamma L_r^2(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ satisfies

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \mathcal{F}_1 g(\boldsymbol{\xi}, \mathbf{v}) \mathcal{F}_1(I_1^{-\frac{n-1}{2}} Cf)(\boldsymbol{\xi}, \mathbf{v}) (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v} = 0$$

for all $f \in \mathcal{S}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, then $g = 0$. Theorem 1 gives us

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \mathcal{F}_1 g(\boldsymbol{\xi}, \mathbf{v}) |\boldsymbol{\xi}|^{\frac{n-1}{2}} \mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}) (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \mathcal{F}_1 g(\boldsymbol{\xi}, \mathbf{v}) |\boldsymbol{\xi}|^{\frac{n-1}{2}} \mathcal{F}f(\boldsymbol{\xi}, \mathbf{v}|\boldsymbol{\xi}|) (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \mathcal{F}_1 g(\boldsymbol{\xi}, \mathbf{v}/|\boldsymbol{\xi}|) |\boldsymbol{\xi}|^{\frac{1-n}{2}} \mathcal{F}f(\boldsymbol{\xi}, \mathbf{v}) (1 + |\boldsymbol{\xi}|^2)^\gamma d\boldsymbol{\xi} d\mathbf{v}. \end{aligned}$$

Since $\mathcal{F}f \in \mathcal{S}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $\mathcal{F}_1 g$ is equal to zero almost everywhere, and so is g. ☐

Notice that if $f \in L^2_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ with $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$, then

$$||f||_0^2 = \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |f(\mathbf{x},\mathbf{y})|^2 d\mathbf{x}d\mathbf{y} = |S^{n-2}|\int_0^\infty\int_{\mathbb{R}^{n-1}} |\mathbf{f}(\mathbf{x},z)|^2 z^{n-2} d\mathbf{x}dz.$$

Let us define $H^\gamma L^2_{n-2}(\mathbb{R}^{n-1}\times[0,\infty))$ by

$$H^\gamma L^2_{n-2}(\mathbb{R}^{n-1}\times[0,\infty)) = \{\mathbf{f} : \mathbf{f}(\mathbf{x},|\mathbf{y}|) = f(\mathbf{x},\mathbf{y}),\ f \in L^2_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}), \text{ and } ||\mathbf{f}||_{\gamma,n-2} < \infty\},$$

where

$$||\mathbf{f}||^2_{\gamma,n-2} = \int_{\mathbb{R}^{n-1}}\int_0^\infty |\mathcal{F}_1\mathbf{f}(\boldsymbol{\xi},z)|^2(|\boldsymbol{\xi}|^2+1)^\gamma z^{n-2} dz d\boldsymbol{\xi}.$$

Then $H^\gamma L^2_{n-2}(\mathbb{R}^{n-1}\times[0,\infty))$ is a Hilbert space with the norm $||\cdot||_{\gamma,n-2}$.

Corollary 2. The mapping $\mathbf{f} \to I_1^{-\frac{n-1}{2}}\mathcal{C}\mathbf{f}$ extends to an isometry of $H^\gamma L^2_{n-2}(\mathbb{R}^{n-1}\times[0,\infty))$ onto itself.

The next corollary shows the Sobolev estimates.

Corollary 3. For each $\gamma \in \mathbb{R}$, we have that for $f \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ with $f(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x},|\mathbf{y}|)$,

$$||f||_\gamma \le (2\pi)^{\frac{1-n}{2}}||Cf||_{\gamma+\frac{n-1}{2}} \quad\text{and}\quad ||\mathbf{f}||_{\gamma,n-2} \le (2\pi)^{\frac{1-n}{2}}||\mathcal{C}\mathbf{f}||_{\gamma+\frac{n-1}{2},n-2}.$$

This corollary follows from (17).

Remark 6. When $n = 3$, $||\mathbf{f}||_{\gamma,1} \le (2\pi)^{-1}||\mathcal{C}\mathbf{f}||_{\gamma+1,1}$ was already discussed in [14]. ❖

7. The partial data problem. From the inversion formula in Theorem 2, $f \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ is uniquely determined by Cf. However, in many practical situations, we know only partial data, i.e., the values of Cf only on a subset of its domain. The question arises: Does this partial data still determine f uniquely?

THEOREM 7. *Let $f \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. The conical Radon transform $Cf(\mathbf{u},\mathbf{v})$ is equal to zero for $|\mathbf{u}| > \sqrt{1+|\mathbf{v}|^2}$ if and only if $f(\mathbf{x},\mathbf{y})$ is equal to zero for $|\mathbf{x}|^2+|\mathbf{y}|^2 > 1$.*

Also, for $\mathbf{f}(\mathbf{x},|\mathbf{y}|) = f(\mathbf{x},\mathbf{y})$, $\mathcal{C}\mathbf{f}(\mathbf{u},s)$ is equal to zero for $|\mathbf{u}| > \sqrt{1+s^2}$ if and only if $\mathbf{f}(\mathbf{x},z)$ is equal to zero for $|\mathbf{x}|^2+z^2 > 1$ (see Figure 3).

Proof. It is clear that if $f(\mathbf{x},\mathbf{y})$ is equal to zero for any $|\mathbf{x}|^2+|\mathbf{y}|^2 > 1$, then $Cf(\mathbf{u},\mathbf{v})$ is equal to zero for any $\mathbf{v} \in \mathbb{R}^{n-1}$ and $|\mathbf{u}| > \sqrt{1+|\mathbf{v}|^2}$. From Proposition 3, we know that for $|\boldsymbol{\alpha}| = |\mathbf{a}|^2(1+|\mathbf{v}|^2)$ and $(\boldsymbol{\alpha},\mathbf{a},b) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-2}\times\mathbb{R}$,

$$\int_{\mathbb{R}^{n-2}} Cf(\mathbf{a}\cdot\mathbf{u}'+b,\mathbf{u}',\mathbf{v})d\mathbf{u}' = ((1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2))^{-1/2}Rf\left(\frac{(-1,\mathbf{a},\boldsymbol{\alpha})}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}},\frac{b}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}}\right).$$

By assumption, we have that

$$Rf\left(\frac{(-1,\mathbf{a},\boldsymbol{\alpha})}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}},\frac{b}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}}\right) = \sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}\int_{\mathbb{R}^{n-2}} Cf(\mathbf{a}\cdot\mathbf{u}'+b,\mathbf{u}',\mathbf{v})d\mathbf{u}'$$

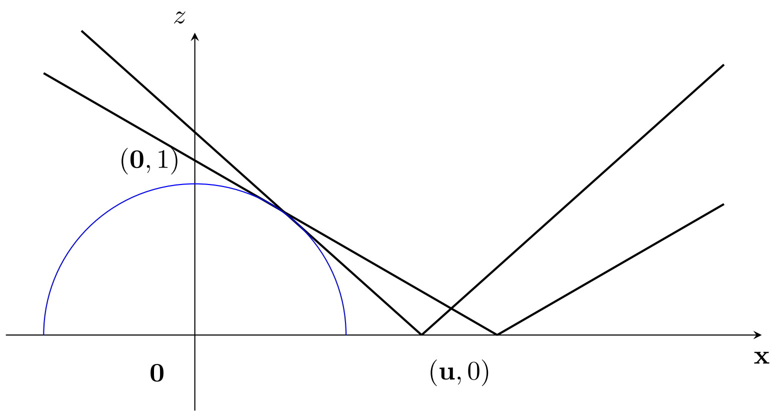

FIG. 3. *The upper half unit circle and V-shape lines.*

is equal to zero for any $b/\sqrt{1+|\mathbf{a}|^2} > \sqrt{1+|\mathbf{v}|^2}$, i.e.,

$$\frac{b}{\sqrt{(1+|\mathbf{a}|^2)(1+|\mathbf{v}|^2)}} > 1$$

because the hyperplane $\{(u_1,\mathbf{u}') \in \mathbb{R}\times\mathbb{R}^{n-2} : u_1 = \mathbf{a}\cdot\mathbf{u}' + b\}$ on $\mathbb{R}^{n-1}$ has the normal vector $(-1,\mathbf{a})$ and the distance $b/\sqrt{1+|\mathbf{a}|^2}$ from the origin. The support theorem [18, Theorem 3.2 in Chapter II] for the regular Radon transform completes our proof. ∎

In the next theorem, we exploit the analyticity of the Fourier transform of a smooth function with compact support.

THEOREM 8. *Let $f \in \mathcal{S}_r(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ have a compact support and $\mathbf{f}(\mathbf{x},|\mathbf{y}|) = f(\mathbf{x},\mathbf{y})$. Assume that $S \subset (0,\infty)$ is any uncountable set. If $Cf(\mathbf{u},\mathbf{v}) = 0$ for any $\mathbf{u}\in\mathbb{R}^{n-1}$ and $\mathbf{v}\in\{\mathbf{v}\in\mathbb{R}^{n-1} : |\mathbf{v}|\in S\}$, then $f = 0$. Also, if $\mathcal{C}\mathbf{f}(\mathbf{u},s) = 0$ for any $\mathbf{u}\in\mathbb{R}^{n-1}$ and $s\in S$, then $\mathbf{f} = 0$.*

Proof. From Theorem 1, we have $\mathcal{F}f(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi}) = \mathcal{F}_1(Cf)(\boldsymbol{\xi},\mathbf{v})$. Thus we have $\mathcal{F}f(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi}) = 0$ for any $\boldsymbol{\xi}\in\mathbb{R}^{n-1}$ and $\mathbf{v}\in\{\mathbf{v}\in\mathbb{R}^{n-1} : |\mathbf{v}|\in S\}$. Since f is compactly supported, $\mathcal{F}f(\boldsymbol{\xi},\boldsymbol{\eta})$ is an analytic function in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ whose power series expansion can be written as

$$\mathcal{F}f(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{\alpha,\beta} a_{\alpha,\beta}\boldsymbol{\xi}^{\alpha}\boldsymbol{\eta}^{\beta},$$

where α and β are multi-indices and $a_{\alpha,\beta}$ are constants. Then we obtain for any $\boldsymbol{\xi}\in\mathbb{R}^{n-1}$ and $\mathbf{v}\in\{\mathbf{v}\in\mathbb{R}^{n-1} : |\mathbf{v}|\in S\}$

$$0 = \mathcal{F}f(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi}) = \sum_{\alpha,\beta} a_{\alpha,\beta}\boldsymbol{\xi}^{\alpha+\beta}|\mathbf{v}|^{|\beta|}.$$

For any $k = 0,1,2,\ldots$, let the polynomial P_k of degree k be defined by

$$P_k(t) := \sum_{|\alpha|+|\beta|=k} a_{\alpha,\beta}t^{|\beta|}.$$

Then we have for any $\xi\in\mathbb{R}$

$$0 = \mathcal{F}f(\xi(1,1,\ldots,1),|\mathbf{v}|\xi(1,1,\ldots,1)) = \sum_{k=0}^{\infty} P_k(|\mathbf{v}|)\xi^k,$$

and $P_k(|\mathbf{v}|)$ is zero for any $\mathbf{v} \in \{\mathbf{v} \in \mathbb{R}^{n-1} : |\mathbf{v}| \in S\}$. Since for an uncountable set S $P_k(|\mathbf{v}|)$ is equal to zero for any k, $a_{\alpha,\beta}$ is zero for any α, β and thus $\mathcal{F}f(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0$ for any $|(\boldsymbol{\xi}, \boldsymbol{\eta})| < \delta$ and for some $\delta > 0$. Since $\mathcal{F}f$ is analytic, $\mathcal{F}f$ is equal to zero. ◻

We study the reconstruction problem for the limited data. The data $Cf(\mathbf{u}, \mathbf{v})$ is known only for $\mathbf{v} \in \mathbb{R}^{n-1}$ with $0 \le a < |\mathbf{v}| < b \le \infty$. In order to compute a limited reconstruction, we have to deal with the limited conical Radon transform $C_{(a,b)}f(\mathbf{u}, \mathbf{v}) = \chi_{a<|\mathbf{v}|<b}(\mathbf{v})Cf(\mathbf{u}, \mathbf{v})$, where χ_A is the characteristic function of a set A. We define the projection operators by

$$P_{(a,b)}f(\mathbf{x}, \mathbf{y}) = \mathcal{F}^{-1}(\chi_{a|\boldsymbol{\xi}|<|\boldsymbol{\eta}|<b|\boldsymbol{\xi}|}(\boldsymbol{\eta})\mathcal{F}f(\boldsymbol{\xi}, \boldsymbol{\eta}))(\mathbf{x}, \mathbf{y})$$

and

$$\mathbf{P}_{(a,b)}\mathbf{f}(\mathbf{x}, z) = (2\pi)^{\frac{n-1}{2}}|z|^{\frac{2-n}{2}}\mathbf{H}_{\frac{n-2}{2}}\mathcal{F}_1^{-1}(\chi_{a|\boldsymbol{\xi}|<\rho<b|\boldsymbol{\xi}|}(\rho)|\rho|^{\frac{2-n}{2}}\mathbf{H}_{\frac{n-2}{2}}\mathcal{F}_1\mathbf{f}(\boldsymbol{\xi}, \rho))(\mathbf{x}, z).$$

THEOREM 9. *Let $f \in \mathcal{S}_r(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Then we have for $k < n-1$*

$$P_{(a,b)}I_1^k f = (2\pi)^{1-n}CI_1^{k+1-n}C_{(a,b)}f$$

and

$$\mathbf{P}_{(a,b)}I_1^k\mathbf{f} = (2\pi)^{1-n}\mathcal{C}I_1^{k+1-n}\mathcal{C}_{(a,b)}\mathbf{f}.$$

Proof. By Theorem 1, we have $\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}) = \mathcal{F}f(\boldsymbol{\xi}, \mathbf{v}|\boldsymbol{\xi}|)$, so

$$\begin{aligned}\mathcal{F}(P_{(a,b)}I_1^k f)(\boldsymbol{\xi}, \mathbf{v}|\boldsymbol{\xi}|) &= |\boldsymbol{\xi}|^{-k}\chi_{a<|\mathbf{v}|<b}(\mathbf{v})\mathcal{F}f(\boldsymbol{\xi}, \mathbf{v}|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^{-k}\chi_{a<|\mathbf{v}|<b}(\mathbf{v})\mathcal{F}_1(Cf)(\boldsymbol{\xi}, \mathbf{v}) \\ &= |\boldsymbol{\xi}|^{-k}\mathcal{F}_1(C_{(a,b)}f)(\boldsymbol{\xi}, \mathbf{v}). \qquad (18)\end{aligned}$$

Similar to the proof of Theorem 2, we have

$$\begin{aligned}P_{(a,b)}I_1^k f(\mathbf{x}, \mathbf{y}) &= (2\pi)^{2(1-n)}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\mathcal{F}(P_{(a,b)}I_1^k f)(\boldsymbol{\xi}, \boldsymbol{\eta})e^{i(\boldsymbol{\xi},\boldsymbol{\eta})\cdot(\mathbf{x},\mathbf{y})}d\boldsymbol{\eta}d\boldsymbol{\xi} \\ &= (2\pi)^{2(1-n)}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\mathcal{F}(P_{(a,b)}I_1^k f)(\boldsymbol{\xi}, \mathbf{v}|\boldsymbol{\xi}|)e^{i(\boldsymbol{\xi},\mathbf{v}|\boldsymbol{\xi}|)\cdot(\mathbf{x},\mathbf{y})}|\boldsymbol{\xi}|^{n-1}d\mathbf{v}d\boldsymbol{\xi} \\ &= (2\pi)^{2(1-n)}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\mathcal{F}_1(C_{(a,b)}f)(\boldsymbol{\xi}, \mathbf{v})e^{i(\boldsymbol{\xi},|\mathbf{v}|\boldsymbol{\xi})\cdot(\mathbf{x},\mathbf{y})}|\boldsymbol{\xi}|^{n-1-k}d\mathbf{v}d\boldsymbol{\xi},\end{aligned}$$

where in the second line we changed the variables $\boldsymbol{\eta} \to \mathbf{v}|\boldsymbol{\xi}|$, and in the third line we used (18) and the fact that $\mathcal{F}_1(C_{(a,b)}f)(\boldsymbol{\xi}, \mathbf{v})$ is radial in $\mathbf{v}$. Therefore, we have

$$P_{(a,b)}I_1^k f(\mathbf{x}, \mathbf{y}) = (2\pi)^{(1-n)}\int_{\mathbb{R}^{n-1}}I_1^{k-n+1}(C_{(a,b)}f)(\mathbf{x} + |\mathbf{v}|\mathbf{y}, \mathbf{v})d\mathbf{v}. \qquad ◻$$

8. Conclusion. Several types of conical Radon transforms have been studied since the Compton camera was introduced. Here we study the n-dimensional conical Radon transform with a fixed central axis. Two inversion formulas, range conditions, Sobolev space estimates, and uniqueness and reconstruction for a limited data problem are presented.

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