The Astrophysical Journal, 589:338–346, 2003 May 20

THE EFFECT OF COSMIC-RAY DIFFUSION ON THE PARKER INSTABILITY

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Received 2002 November 18; accepted 2003 January 29

ABSTRACT

The Parker instability, which has been considered as a process governing the structure of the interstellar medium, is induced by the buoyancy of magnetic fields and cosmic rays. In previous studies, while the magnetic field has been fully incorporated in the context of isothermal magnetohydrodynamics, cosmic rays have normally been treated with the simplifying assumption of infinite diffusion along magnetic field lines but no diffusion across them. The cosmic-ray diffusion is, however, finite. In this work, we fully take into account the diffusion process of cosmic rays in a linear stability analysis of the Parker instability. Cosmic rays are described with the diffusion-convection equation. With realistic values of cosmic-ray diffusion coefficients expected in the interstellar medium, we show that the result of previous studies with the simplifying assumption about cosmic-ray diffusion applies well. The finiteness of the parallel diffusion decreases the growth rate of the Parker instability, while the relatively smaller perpendicular diffusion has no significant effect. We discuss the implication of our result on the role of the Parker instability in the interstellar medium.

Subject headings: cosmic rays — instabilities — ISM: magnetic fields — MHD

1. INTRODUCTION

In stability analyses of the interstellar medium (ISM), Parker (1966, 1967) put forward a simple model of the ISM that is composed of a single-phase gas, magnetic field, and cosmic rays (CRs) under external, uniform, vertical gravity. It was assumed that gas pressure originates from the ram motion of cloudlets rather than the thermal motion of atoms or molecules, so the velocity dispersion of cloudlets was taken as the sound speed. In equilibrium, the magnetic field has only a regular component, and the ratios of the pressures of the magnetic field and CRs to gas pressure are constant. In addition, the CR dynamics were simplified by setting $d\delta P_c/dt = 0$, based on the assumption that the CR pressure is uniform along magnetic field lines.[5] Then, he showed that the equilibrium state is subject to an instability, which is now known as the Parker instability.

Parker's work has since been elaborated on. For instance, Giz & Shu (1993), Kim, Hong, & Ryu (1997), and Kim & Hong (1998) investigated the modification of the Parker instability under nonuniform gravities. It was found that the linear growth rate increases under the gravities described by linear and hyperbolic-tangent functions. Kim et al. (2000) and Santillán et al. (2000) incorporated the multicomponent nature of the ISM in a realistic gravity model, in which the growth timescale turns out to be $\sim 3 \times 10^7$ yr and the length scale enlarges up to ~ 3 kpc. The effect of an irregular, random component of the magnetic field was studied in Parker & Jokipii (2000) and Kim & Ryu (2001). With the strength of the random component comparable to the regular one (see, e.g., Beck et al. 1996; Zweibel & Heiles 1997), it was shown that the Parker instability can be completely stabilized.

CRs form an important constituent of the ISM, with their energy density comparable to those of gas and magnetic fields (see, e.g., Blandford & Eichler 1987). The analyses of Parker (1966, 1967) and Shu (1974) showed that CRs play a significant role in the development of the Parker instability by widening the range of unstable wavelengths and increasing the growth rate under the limit of $\kappa_\parallel \rightarrow \infty$ (very large diffusion along the magnetic field lines) and $\kappa_\perp = 0$ (negligible diffusion across field lines). However, it is certainly true that $\kappa_\parallel$ and $\kappa_\perp$ are finite (see § 2.2 for details), but there has been no follow-up work on the effect of CRs with finite diffusion in the Parker instability. In an approach based on a different perspective, Nelson (1985) incorporated the dynamics of CRs by approximating their pressure as

$$P_{c,ij} = P_{c,\perp}\delta_{ij} - \left(P_{c,\perp} - P_{c,\parallel}\right)\frac{B_i B_j}{B^2} \quad (1)$$

(for comparison, see eq. [8] for the spatial diffusion tensor). Although the diffusion process was not included, the anisotropic nature of CR dynamics was taken into account. The surprising result was that the anisotropic CR pressure works toward stabilizing the instability, contrary to the common belief that CRs act as one of the agents to induce the instability itself.

In this paper, we describe a linear analysis in which CR dynamics are incorporated into the diffusion-convection equation (see, e.g., Skilling 1975), and their effect on the Parker instability is addressed. Previously, Kuznetsov & Ptuskin (1983) attempted a similar analysis. They argued that CRs enhance the instability by showing that the critical value of the gas adiabatic index for the instability increases because of CRs. Here, we first estimate the values of $\kappa_\parallel$ and $\kappa_\perp$ that are applicable

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[5] Subsequently, Shu (1974) refined this treatment in a more intuitive form in which the diffusion along the magnetic field lines is very large but the diffusion across the field lines is negligible. Both formulations of Parker (1966, 1967) and Shu (1974) turn out to be the same in the linear regime (see § 3.2).

in the analysis and then derive the dispersion relation. We show that the growth rate and the range of unstable wavelengths increase because of CRs. Our result confirms the validity of the analyses of Parker (1966) and Shu (1974) at a quantitative level but disagrees with that of Nelson (1985). Recently, Hanasz & Lesch (2000) studied the Parker instability triggered by the CRs injected in supernova remnants. They solved numerically the flux tube equation for the magnetic field along with the diffusion-convection equation for CRs. Their work is the first example that took into account CR diffusion in the Parker instability. However, it still needs to be quantified how much CR diffusion affects the range of unstable wavelengths and the growth rate. This paper addresses that specific issue.

In § 2 the stability analysis is described and the dispersion relation is derived. In addition, discussion of the CR diffusion tensor is presented. Interpretation of the dispersion relation is described in § 3. A summary and discussion of the implications of our result are given in § 4.

2. LINEAR STABILITY ANALYSIS

2.1. *Basic Equations*

The equation set for our purpose is the combination of the MHD equations and the CR diffusion-convection equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 , \tag{2}$$

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla \left(P_g + P_c + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \rho \boldsymbol{g} , \tag{3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) , \tag{4}$$

$$\frac{\partial P_g}{\partial t} + \boldsymbol{v} \cdot \nabla P_g + \gamma_g P_g \nabla \cdot \boldsymbol{v} = 0 , \tag{5}$$

$$\frac{\partial P_c}{\partial t} + \boldsymbol{v} \cdot \nabla P_c + \gamma_c P_c \nabla \cdot \boldsymbol{v} = \nabla \cdot \left(\langle \kappa_{ij} \rangle \nabla P_c \right) + S_0 , \tag{6}$$

where the subscripts g and c stand for gas and CRs. CRs are described by the two-fluid model, which is derived from the second particle momentum moment of the well-known CR Fokker-Planck equation (Skilling 1975). Hence, CRs are described in equation (6) by a pressure plus an equation of state for the CRs represented by the adiabatic index $\gamma_c = 1 + P_c/E_c$, instead of a full momentum distribution function (see Drury & Völk 1981; Jones & Kang 1990 for details of the two-fluid model). The quantity $\langle \kappa_{ij} \rangle$ is the energy-weighted mean diffusion tensor of the CRs (see § 2.2 for discussion). The source term S_0 in the CR pressure equation is introduced to set up an initial equilibrium state (see § 2.3), not to describe the injection from thermal particles to CR particles. We ignore that process, along with shock acceleration of CRs (see, e.g., Blandford & Eichler 1987 for details of shock acceleration). There are no shocks in the regime of linear stability analyses.

As pointed out by Parker (1966, 1967), on the scale at which the Parker instability is relevant, the dominant contribution to gas pressure would not come from the thermal motions of atoms or molecules but would come from the turbulent motions of cloudlets. Then, the value of the "effective" adiabatic index for gas, γ_g, should be determined by considering the detailed mechanisms involved, such as supernova explosions, stellar winds, the Galactic differential rotation, cloud-cloud collisions, and turbulence dissipation. Although there has been much progress in the studies of each mechanism, the determination of γ_g for an ensemble of cloudlets is less well understood. Hence, here we simply set $\gamma_g = 1$, assuming that the cloudlet random motions are constant.

The adiabatic index of the CRs, $\gamma_c = 1 + P_c/E_c$, can be simply related to the form of the CR momentum distribution if the latter is a power law with an index between 4 and 5. In particular, a momentum distribution

$$f(q) \propto p^{-q}, \quad q \simeq \frac{14}{3} , \tag{7}$$

appropriate for Galactic CRs (see, e.g., Blandford & Eichler 1987), leads to $P_c/E_c = (q-3)/3 \simeq 5/9$. Hence, $\gamma_c = 14/9$ is used in our analysis.

2.2. *Cosmic-Ray Diffusion Tensor*

The frequently used form of the CR diffusion tensor is

$$\kappa_{ij} = \kappa_\perp \delta_{ij} - \left(\kappa_\perp - \kappa_\parallel \right) \frac{B_i B_j}{B^2} + \epsilon_{ijk} \kappa_A \frac{B_k}{B} , \tag{8}$$

where B_i is the magnetic field vector, $\kappa_\parallel$ and $\kappa_\perp$ are the diffusion coefficients along and across the mean field, respectively, and κ_A represents the curvature and gradient drifts (see, e.g., Bieber & Matthaeus 1997; Giacalone & Jokipii 1999; Casse, Lemoine, & Pelletier 2002 for discussions on κ_{ij}).

Giacalone & Jokipii (1999) and Casse et al. (2002) used Monte Carlo simulations in modeling turbulent magnetic fields and estimated the diffusion properties. With the energy ratio of the random to total magnetic fields,

$$\chi = \frac{\delta B^2}{B_0^2 + \delta B^2} , \tag{9}$$

Casse et al. (2002) showed that whenever $\chi < 1$, Bohm diffusion $[\kappa \propto p/(p^2 + m^2c^2)^{1/2}]$ does not apply, but the quasi-linear approximation does for parallel diffusion. They found that

$$\kappa_{\parallel} = \frac{v r_{\rm L}}{3h} , \quad h \simeq 0.4\chi(r_{\rm L} k_{\rm min})^{2/3} , \tag{10}$$

for the Kolmogorov turbulence. Here, $r_{\rm L}$ and $k_{\rm min}$ represent the Larmor radius and the minimum wavenumber of the Kolmogorov spectrum, respectively. The formal quasi-linear result would replace 0.4 by $\pi/6$. Note that equation (10) applies even when $\chi = 0.99$.

Based on the GALPROP model of CR propagation in the ISM, Strong & Moskalenko (1998) found that $\kappa_{\parallel} \simeq 6 \times 10^{28}$ cm^2 s^{-1} at the rigidity of $r_{\rm L}/B_0 = 3$ GV using isotropic diffusion. Matching this with equation (10), $\kappa_{\parallel}$ can be estimated. Taking the momentum distribution in equation (7) and letting the energy-weighted mean diffusion be

$$\langle \kappa \rangle = \frac{\int \kappa \left(\sqrt{p^2+1} - 1\right) f(q) p^2 \, dp}{\int \left(\sqrt{p^2+1} - 1\right) f(q) p^2 \, dp} , \tag{11}$$

we get

$$\langle \kappa_{\parallel} \rangle \simeq 2.5 \times 10^{28} \left(\frac{0.2}{\chi}\right) \left(\frac{A}{Z}\right)^{1/3} \left(\frac{3\ \mu{\rm G}}{B_0}\right)^{1/3} \left(\frac{L_{\rm max}}{200\ {\rm pc}}\right)^{2/3} , \tag{12}$$

where A and Z are the CR atomic number and charge, respectively, and $L_{\rm max}$ is the coherent length of the regular component of the magnetic field. Note that equation (12) is derived with $\chi \simeq 0.2$, but the result of Strong & Moskalenko (1998) was based on isotropic diffusion ($\chi = 1$). However, the normalization seems to be uncertain, at least by a factor of a few anyway, since the details of CR propagation are not really understood.

Casse et al. (2002) found the perpendicular diffusion to behave according to

$$\kappa_{\perp} \simeq 0.2\chi^{2/3} \kappa_{\parallel} , \tag{13}$$

rather than $\kappa_{\perp} \sim 10^{-6}\kappa_{\parallel}$, which is predicted in the quasi-linear result. In addition, from the argument based on the escape of the Galactic CRs and their lifetime, Giacalone & Jokipii (1999) drew a consistent value for the perpendicular diffusion, $\kappa_{\perp} \simeq (0.02\text{–}0.04)\kappa_{\parallel}$. On the other hand, Bieber & Matthaeus (1997) argued that

$$\kappa_A \simeq \left(\frac{c}{r_{\rm L}} \tau_{\rm decorr}\right) \kappa_{\perp} , \quad \frac{c}{r_{\rm L}} \tau_{\rm decorr} \sim 2 \frac{r_{\rm L}}{L_{\rm max}} \frac{1}{\chi} , \tag{14}$$

where $\tau_{\rm decorr}$ is the CR decorrelation time. Therefore, with $r_{\rm L}/L_{\rm max} \ll 1$, it is expected that $\kappa_A \ll \kappa_{\perp}$ and κ_A can be neglected to a first approximation.

In the rest of the paper, brackets are dropped in the mean diffusion coefficients, for simplicity.

2.3. *Initial Equilibrium State*

A stability analysis is started by setting up the initial equilibrium configuration. We employ the one originally suggested by Parker (1966, 1967). In Cartesian coordinates (x, y, z), the azimuthal magnetic field is set to lie along the y-direction [0, $B_0(z)$, 0] and the externally given uniform gravity to accelerate in the negative z-direction (0, 0, $-g$). Then, the initial states of mass density ρ_0, gas pressure P_{g0}, CR pressure P_{c0}, and magnetic field B_0 are described by an exponential function:

$$\frac{\rho_0(z)}{\rho_0(0)} = \frac{P_{g0}(z)}{P_{g0}(0)} = \frac{P_{c0}(z)}{P_{c0}(0)} = \frac{B_0^2(z)}{B_0^2(0)} = \exp\left(-\frac{z}{H}\right) , \tag{15}$$

where $H = (1 + \alpha + \beta)a^2/g$ and a is the isothermal sound speed (with $\gamma_g = 1$). The scale height (160 pc) and the velocity dispersion (6.4 km s^{-1}) of interstellar clouds are used for H and a (see, e.g., Falgarone & Lequeux 1973). The quantity α is the ratio of initial magnetic to gas pressures and β is the ratio of initial CR to gas pressures, respectively, and they are assumed to be constant.

Special attention needs to be given to the initial equilibrium of equation (6). Nonzero $\kappa_{\perp}$ would cause CRs to diffuse upward. The source term, $S_0 = -\kappa_{\perp} P_{c0}(z)/H^2$, was included to balance it. This ad hoc treatment, however, would introduce spurious features in the stability properties, and the interpretation of it should be done with caution (see §§ 3.3 and 3.4). Of course, with $\kappa_{\perp} = 0$, this problem does not appear.

2.4. *Linearized Perturbation Equations*

Here, we focus as in Parker (1966) on the stability in the (y, z)-plane defined by the initial magnetic field and gravity. The analysis becomes simplified if the linearization of equations (2)–(6) is performed with dimensionless quantities (Shu 1974). With H and H/a as the normalization units of length and time, we define dimensionless coordinates and time,

$$y' = \frac{y}{H}, \quad z' = \frac{z}{H}, \quad t' = \frac{at}{H} , \tag{16}$$

and introduce dimensionless perturbations of density, s, velocity, $\boldsymbol{u}$, magnetic field, $\boldsymbol{b}$, gas pressure, p_g, and CR pressure, p_c. Then, the perturbed state can be written as

$$\rho = \rho_0(z)(1+s), \quad \boldsymbol{v} = a\boldsymbol{u}, \quad \boldsymbol{B} = B_0(z)(\hat{\boldsymbol{e}}_y + \boldsymbol{b}), \quad P_g = P_{g0}(z)(1+p_g), \quad P_c = P_{c0}(z)(1+p_c) , \tag{17}$$

where $\hat{\boldsymbol{e}}_y$ is the unit vector along the y-direction. For simplicity, primes have been dropped in equation (17) and in the rest of the paper.

Substituting equation (17) into equations (2)–(6) and keeping terms only up to the linear order of the perturbations, the linearized perturbation equations are written as

$$\frac{\partial s}{\partial t} - u_z + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 , \tag{18}$$

$$\frac{\partial u_y}{\partial t} + \frac{\partial p_g}{\partial y} + \beta\frac{\partial p_c}{\partial y} + \alpha b_z = 0 , \tag{19}$$

$$\frac{\partial u_z}{\partial t} + (1+\alpha+\beta)s - (p_g + \beta p_c + 2\alpha b_y) + \frac{\partial}{\partial z}(p_g + \beta p_c + 2\alpha b_y) - 2\alpha\frac{\partial b_z}{\partial y} = 0 , \tag{20}$$

$$\frac{\partial b_y}{\partial t} - \frac{1}{2}u_z + \frac{\partial u_z}{\partial z} = 0 , \tag{21}$$

$$\frac{\partial b_z}{\partial t} - \frac{\partial u_z}{\partial y} = 0 , \tag{22}$$

$$\frac{\partial p_g}{\partial t} - u_z + \gamma_g\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 , \tag{23}$$

$$\frac{\partial p_c}{\partial t} - u_z + \gamma_c\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \kappa_\parallel\frac{\partial^2 p_c}{\partial y^2} - \kappa_\perp\left(p_c - 2\frac{\partial p_c}{\partial z} + \frac{\partial^2 p_c}{\partial z^2}\right) - (\kappa_\perp - \kappa_\parallel)\frac{\partial b_z}{\partial y} = 0 . \tag{24}$$

Here again, primes have been dropped in the normalized $\kappa_\parallel$ and $\kappa_\perp$ for simplicity.

2.5. *Dispersion Relation*

The normal mode of perturbations has the form

$$\begin{bmatrix} s(y,\ z,\ t) \\ u_y(y,\ z,\ t) \\ u_z(y,\ z,\ t) \\ b_y(y,\ z,\ t) \\ b_z(y,\ z,\ t) \\ p_g(y,\ z,\ t) \\ p_c(y,\ z,\ t) \end{bmatrix} = \begin{bmatrix} s \\ u_y \\ u_z \\ b_y \\ b_z \\ p_g \\ p_c \end{bmatrix} \exp(nt)\exp(-i\eta y)\exp\left(\tfrac{1}{2}z - i\zeta z\right) , \tag{25}$$

where n is the dimensionless growth rate and η and ζ are the dimensionless wavenumbers along the azimuthal (y) and vertical (z) directions, respectively. The $\exp(z/2)$ factor was included because of the stratified background. The same notations were used for the perturbations themselves in the left-hand side and their amplitudes in the right-hand side, because no confusion arises in later algebra. Substituting equation (25) into equations (18)–(24) results in the following set of equations:

$$ns - i\eta u_y - \left(\tfrac{1}{2} + i\zeta\right)u_z = 0 , \tag{26}$$

$$nu_y - i\eta p_g - i\eta\beta p_c + \alpha b_z = 0 , \tag{27}$$

$$nu_z + (1+\alpha+\beta)s - \left(\tfrac{1}{2} + i\zeta\right)(p_g + \beta p_c + 2\alpha b_y) + i\eta 2\alpha b_z = 0 , \tag{28}$$

$$nb_y - i\zeta u_z = 0 , \tag{29}$$

$$nb_z + i\eta u_z = 0 \,, \tag{30}$$

$$np_g - i\eta\gamma_g u_y - \left[1 - \gamma_g\left(\tfrac{1}{2} - i\zeta\right)\right]u_z = 0 \,, \tag{31}$$

$$\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]p_c - i\eta\gamma_c u_y - \left[1 - \gamma_c\left(\tfrac{1}{2} - i\zeta\right)\right]u_z + \left(\kappa_\perp - \kappa_\parallel\right)i\eta b_z = 0 \,. \tag{32}$$

The dispersion relation is derived by combining the above seven equations. Although straightforward, it entails tedious algebra. Here we present a few intermediate steps. First, by eliminating b_z in equation (32) with equation (30), we have

$$\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]np_c - i\eta\gamma_c n u_y - \left\{\left[1 - \gamma_c\left(\tfrac{1}{2} - i\zeta\right)\right]n - \left(\kappa_\perp - \kappa_\parallel\right)\eta^2\right\}u_z = 0 \,, \tag{33}$$

which expresses p_c in terms of u_y and u_z. Then, on substituting s in equation (26), b_y in equation (29), b_z in equation (30), and p_c in equation (33) into equations (27) and (28), we obtain two equations for u_y and u_z:

$$\begin{aligned}\left\{\left(n^2 + \gamma_g\eta^2\right)\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right] + \beta\gamma_c\eta^2 n\right\}u_y - i\eta\Big\{\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]\left[1 + \alpha - \gamma_g\left(\tfrac{1}{2} - i\zeta\right)\right]& \\ + \beta n\left[1 - \gamma_c\left(\tfrac{1}{2} - i\zeta\right)\right] - \left(\kappa_\perp - \kappa_\parallel\right)\beta\eta^2\Big\}u_z = 0 \,,&\end{aligned} \tag{34}$$

$$\begin{aligned}i\eta\Big\{\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]\left[1 + \alpha + \beta - \gamma_g\left(\tfrac{1}{2} + i\zeta\right)\right] - \beta\gamma_c n\left(\tfrac{1}{2} + i\zeta\right)\Big\}u_y + \Big(\left[n + \kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]& \\ \times\left[n^2 + 2\alpha\eta^2 + \beta\left(\tfrac{1}{2} + i\zeta\right) + (2\alpha + \gamma_g)\left(\tfrac{1}{4} + \zeta^2\right)\right] + \beta\left(\tfrac{1}{2} + i\zeta\right)\Big\{\left(\kappa_\perp - \kappa_\parallel\right)\eta^2 - \left[1 - \gamma_c\left(\tfrac{1}{2} - i\zeta\right)\right]n\Big\}\Big)u_z = 0 \,.&\end{aligned} \tag{35}$$

Finally, by combining equations (34) and (35), we get the dispersion relation, which is a sixth-order polynomial of n:

$$n^6 + C_5 n^5 + C_4 n^4 + C_3 n^3 + C_2 n^2 + C_1 n + C_0 = 0 \,, \tag{36}$$

where

$$C_5 = 2\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right] , \tag{37}$$

$$C_4 = \left(2\alpha + \gamma_g + \beta\gamma_c\right)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) + \left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]^2 , \tag{38}$$

$$C_3 = \left(4\alpha + 2\gamma_g + \beta\gamma_c\right)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right)\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right] + \beta\kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)\left[\eta^2 - \left(\tfrac{1}{2} + i\zeta\right)^2\right] , \tag{39}$$

$$\begin{aligned}C_2 = &\left[2\alpha(\gamma_g + \beta\gamma_c)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1 + \alpha + \beta)\left(1 + \alpha + \beta - \gamma_g - \beta\gamma_c\right)\right]\eta^2 \\ &+ (2\alpha + \gamma_g)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right)\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]^2 + \beta\kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)\left[\eta^2 - \left(\tfrac{1}{2} + i\zeta\right)^2\right]\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right] ,\end{aligned} \tag{40}$$

$$\begin{aligned}C_1 = &\left[2\alpha\left(2\gamma_g + \beta\gamma_c\right)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1 + \alpha + \beta)\left(2 + 2\alpha + \beta - 2\gamma_g - \beta\gamma_c\right)\right] \\ &\times\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]\eta^2 + \beta(1 + \alpha + \beta)\eta^4\left(\kappa_\perp - \kappa_\parallel\right) ,\end{aligned} \tag{41}$$

$$\begin{aligned}C_0 = &\left[2\alpha\gamma_g\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1 + \alpha + \beta)\left(1 + \alpha - \gamma_g\right)\right]\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]^2\eta^2 + \beta(1 + \alpha + \beta)\left(\kappa_\perp - \kappa_\parallel\right) \\ &\times\left[\kappa_\parallel\eta^2 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]\eta^4 \,.\end{aligned} \tag{42}$$

3. RESULTS

3.1. *Parameters*

The dispersion relation, equation (36) augmented by equations (37)–(42), gives the linear growth rate n as a function of the azimuthal wavenumber η and the vertical wavenumber ζ. It involves the parameters α, β, γ_g, γ_c, $\kappa_\parallel$, and $\kappa_\perp$. In addition to $\gamma_g = 1$ and $\gamma_c = 14/9$, which were specified in § 2.1, $\alpha = 1$ and $\beta = 1$ are used in the calculation of the growth rate. Setting $\alpha = 1$ and $\beta = 1$ means that initially the magnetic and CR pressures are the same as the gas pressure. For the energy-weighted CR diffusion coefficients, $\kappa_\parallel = 3 \times 10^{28}$ cm^2 s^{-1} and $\kappa_\perp = 0.02\kappa_\parallel$ are taken as the fiducial values (§ 2.2). Then, in units of aH, $(\kappa_\parallel,\ \kappa_\perp) \sim (100,\ 2)$. Other values of $(\kappa_\parallel, \kappa_\perp)$ are also considered in demonstrating the effect of CR diffusion.

3.2. *Parallel Diffusion* (*Nonzero* $\kappa_\parallel$ *and* $\kappa_\perp = 0$)

We first check whether our dispersion relation recovers the ones of previous works with simplified treatments of CR dynamics. The quantity $d\delta P_c/dt = 0$ of Parker (1966) is translated to $\gamma_c = 0$ and $\kappa_\parallel = \kappa_\perp = 0$ in our formulation. The assumption introduced by Shu (1974) corresponds to $\kappa_\parallel \to \infty$ with $\kappa_\perp = 0$. Although they look different, both limits end with the same dispersion relation:

$$n^4 + (2\alpha + \gamma_g)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right)n^2 + \left[2\alpha\gamma_g\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1+\alpha+\beta)(1+\alpha+\beta-\gamma_g)\right]\eta^2 = 0\ , \qquad (43)$$

which matches with those of Parker (1966) and Shu (1974).

We now study the case of nonzero parallel diffusion, but no perpendicular diffusion. Note that with $\kappa_\perp = 0$, the initial equilibrium is exact, and the ad hoc inclusion of a source is not necessary (see § 2.3). Hence, the dispersion relation is exact with its limit. Taking $\kappa_\perp = 0$, equations (36)–(42) reduce to

$$\begin{aligned} & n^5 + \kappa_\parallel\eta^2 n^4 + (2\alpha + \gamma_g + \beta\gamma_c)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right)n^3 + (2\alpha + \gamma_g)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right)\kappa_\parallel\eta^2 n^2 \\ & + \left[2\alpha(\gamma_g + \beta\gamma_c)\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1+\alpha+\beta)(1+\alpha+\beta-\gamma_g-\beta\gamma_c)\right]\eta^2 n \\ & + \left[2\alpha\gamma_g\left(\eta^2 + \zeta^2 + \tfrac{1}{4}\right) - (1+\alpha+\beta)(1+\alpha+\beta-\gamma_g)\right]\kappa_\parallel\eta^4 = 0\ , \end{aligned} \qquad (44)$$

after factoring out $n + \kappa_\parallel\eta^2 = 0$, which represents diffusive decay.

Figure 1 plots the growth rate (the largest n) as a function of the azimuthal wavenumber η for zero vertical wavenumber $\zeta = 0$ and several different values of $\kappa_\parallel$ including $\kappa_\parallel \to \infty$ and $\kappa_\parallel = 0$. Nonzero ζ reduces n. We note that the dispersion relation (44) is of fifth order and has five roots. As $\kappa_\parallel \to \infty$, four reduce to the roots of equation (43), while the last one becomes $n + \kappa_\parallel\eta^2 = 0$, representing another mode of diffusive decay. Figure 1 also plots the growth rate of the case without CRs ($\beta = 0$). The figure shows that the growth rate increases with increasing $\kappa_\parallel$. The finiteness of $\kappa_\parallel$ reduces the growth of the instability over that of $\kappa_\parallel \to \infty$, because there is a gradient of CR pressure along the magnetic field lines. The gradient hinders the falling motion of gas from arc regions to valleys and slows down the development of the instability. However, with the value of $\kappa_\parallel$ expected in the ISM, $\sim$100, the growth rate is close to that of $\kappa_\parallel \to \infty$. Therefore, we conclude that the treatments $d\delta P_c/dt = 0$ or $\kappa_\parallel \to \infty$ with $\kappa_\perp = 0$ used in previous analyses were good approximations and produced quantitatively correct results. With $\kappa_\parallel = 100$, the maximum growth rate and the critical wavenumber where $n \to 0$ are about twice as large as those for $\beta = 0$, indicating that CRs can enhance the instability significantly.

One thing to note is that the critical wavenumber is independent of $\kappa_\parallel$, once $\kappa_\parallel > 0$, and is given as

$$\eta_c^2 = \frac{(1+\alpha+\beta)(1+\alpha+\beta-\gamma_g)}{2\alpha\gamma_g} - \zeta^2 - \frac{1}{4}\ . \qquad (45)$$

The reason is the following: It takes infinitely long for such a marginally stable state to be developed, because of the zero growth rate. This means that, however small the diffusion is, the system has enough time to diffuse across any gradient of CR pressure along the magnetic field lines. However, with no diffusion, $\kappa_\parallel = 0$, the critical wavenumber is different and is given as

$$\eta_c^2 = \frac{(1+\alpha+\beta)(1+\alpha+\beta-\gamma_g-\beta\gamma_c)}{2\alpha(\gamma_g + \beta\gamma_c)} - \zeta^2 - \frac{1}{4}\ , \qquad (46)$$

which is much smaller for the parameters we employed, and as a matter of fact, the growth rate is much smaller too, even smaller than that for $\beta = 0$. This is because without diffusion, CRs accumulate at valleys along with gas, and their pressure pushes the gas out of the valleys, exerting a stabilizing effect.

3.3. *Perpendicular Diffusion* ($\kappa_\parallel = 0$ *and Nonzero* $\kappa_\perp$)

We start to investigate the effect of nonzero perpendicular diffusion on the Parker instability by looking at the case of $\kappa_\parallel = \eta = 0$. This case describes the acoustic instability of CR-mediated gas, which was studied by Drury & Falle (1986) and Kang, Jones, & Ryu (1992), but with a magnetic field. From the full dispersion relation in equations (36)–(42), we get in this case

$$\left[n - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2\right]\left\{n^3 - \kappa_\perp\left(\tfrac{1}{2} + i\zeta\right)^2 n^2 + (2\alpha + \gamma_g + \beta\gamma_c)\left(\tfrac{1}{4} + \zeta^2\right)n - \kappa_\perp\left[\beta + (2\alpha + \gamma_g)\left(\tfrac{1}{2} - i\zeta\right)\right]\left(\tfrac{1}{2} + i\zeta\right)^3\right\} = 0\ . \qquad (47)$$

In the limit of zero diffusion, the dispersion relation $n = \pm i\left[(2\alpha + \gamma_g + \beta\gamma_c)(1/4 + \zeta^2)\right]^{1/2}$ describes a pair of magnetosonic waves propagating upward and downward. The term of $\beta\gamma_c$ represents the effect of the additional support of CR pressure on top of the magnetosonic waves. The $\frac{1}{4}$ term comes from the $\exp(z/2)$ factor in the normal mode of perturbations in equation (25), added to account for the stratified background. As a matter of fact, all the $\frac{1}{2}$ and $\frac{1}{4}$ terms in the dispersion relation (47) come from the same origin.

Figure 2 plots the growth rate as a function of the vertical wavenumber ζ for two nonzero $\kappa_\perp$'s. For other parameters, the values listed in § 3.1 were used. As pointed out in Drury & Falle (1986) and Kang et al. (1992), because of the ad hoc source term, which is necessary to sustain the initial equilibrium state, the analysis is justified rigorously only in the limit of $k_z \gg 1/H$, or $\zeta \gg 1$ in our normalized units. In that limit, the growth rate of nontrivial modes is

$$n = \pm i\sqrt{2\alpha + \gamma_g}\,\zeta - \frac{\beta\gamma_c}{2\kappa_\perp} \mp \frac{\beta}{2\sqrt{2\alpha + \gamma_g}}\ . \qquad (48)$$

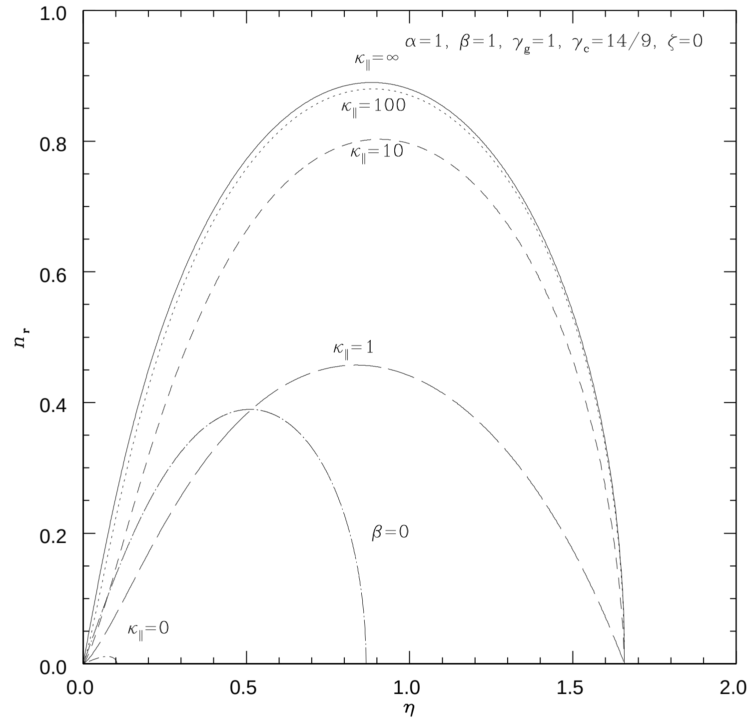

FIG. 1.—Dispersion relations of the Parker instability with nonzero $\kappa_\parallel$ and $\kappa_\perp = 0$. The growth rate (the largest n) is presented as a function of the wavenumber along the initial magnetic field direction. The vertical wavenumber along the direction of gravity was set to be 0. The normalization units of time and length are 2.4×10^7 yr (H/a) and 160 pc (H), respectively. Each curve is labeled by the value of $\kappa_\parallel$. The values of other parameters are specified within the frame. The growth rate of the case without CRs ($\beta = 0$) is also presented for comparison, in which the same normalization was applied.

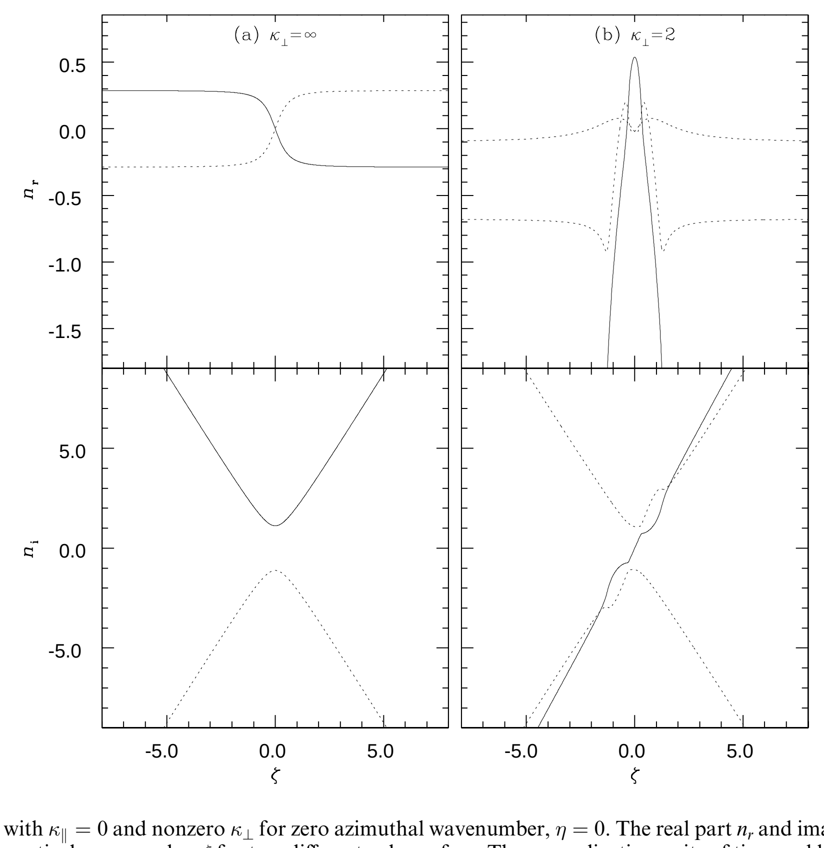

FIG. 2.—Dispersion relations with $\kappa_\parallel = 0$ and nonzero $\kappa_\perp$ for zero azimuthal wavenumber, $\eta = 0$. The real part n_r and imaginary part n_i of the growth rate are presented as functions of the vertical wavenumber ζ for two different values of $\kappa_\perp$. The normalization units of time and length are 2.4×10^7 yr (H/a) and 160 pc (H), respectively. The values of other parameters are $\alpha = 1$, $\beta = 1$, $\gamma_g = 1$, and $\gamma_c = 14/9$.

The first term on the right-hand side represents magnetosonic waves, which are shown in the bottom panels of Figure 2. With nonzero diffusion, the CR perturbation associated with small wavelengths is wiped out because of the diffusive nature. Therefore, CR pressure (β) does not contribute to the speed of the waves. The second and third terms are attributed to the acoustic instability, if their sum is positive or if $\kappa_\perp$ is sufficiently large, which is shown in the top left panel of Figure 2. The above limiting growth rate matches exactly with that of Kang et al. (1992), if the magnetosonic waves are replaced by sound waves.

The physical mechanism of the acoustic instability is the following: Perturbations in a gas mediated by CRs with a gradient can become unstable, because the CR pressure perturbation is reduced by diffusion. In the limit of large diffusion, CRs are completely decoupled from the gas for small-scale perturbations, but the gradient of CR pressure remains the same. In this limit, let us suppose a constant volume force F is exerted on the gas so that the acceleration is F/ρ. Then, with $-F\delta\rho/\rho^2$, compressed regions in a wave train will be accelerated in the opposite direction of the applied force, while decompressed regions will be accelerated in the direction of the force. As a result, oscillating density disturbances (or magnetosonic/sound waves) moving opposite to the direction of the force will suffer an extra restoring force, and their amplitude will grow. On the other hand, waves propagating in the other direction will be damped. This explains why disturbances traveling in one direction will be amplified, while those traveling in the opposite direction will decay. The instability works only if the following two conditions are satisfied: (1) the perturbation wavelength is shorter than the scale height of the CR pressure, or $\eta > 1$, and (2) the scale height of the CR pressure is smaller than the diffusion length associated with the sound speed, or $\kappa_\perp > 1$, in our normalized units.

In addition to the acoustic instability (*top left panel*), Figure 2 shows the unstable nature in the regime of $\kappa_\perp \sim 1$ and $\zeta \lesssim 1$, especially for $\zeta = 0$, which corresponds to the state of no perturbation at all (*top right panel*). Without any viable mechanism, we attribute this to the artifact of the ad hoc source term along with the $\exp(z/2)$ factor in the normal mode. A detailed analysis shows that it is dominated by the perturbation of CRs, confirming that it is not related to the acoustic instability.

3.4. *Nonzero Parallel and Perpendicular Diffusions*

Finally, we consider the effect of nonzero perpendicular diffusion on top of nonzero parallel diffusion. Figure 3 plots, from the dispersion relation in equations (36)–(42), the growth rate (the largest n) as a function of the azimuthal wavenumber η for two different sets of values for κ and zero vertical wavenumber $\zeta = 0$. Comparing Figures 1 and 3, it can be seen that although $\kappa_\perp$ is expected to reduce n, the growth rates with the same $\kappa_\parallel$ are almost identical except around $\eta = 0$. This is because $\kappa_\perp \ll \kappa_\parallel$ in the ISM. The peak at $\zeta = 0$ is again attributed to the artifact of the ad hoc source term along with the $\exp(z/2)$ factor in the normal mode, as explained in § 3.3. Therefore, we conclude that with realistic values of CR diffusion in the ISM, the effect of nonzero perpendicular diffusion is mostly negligible.

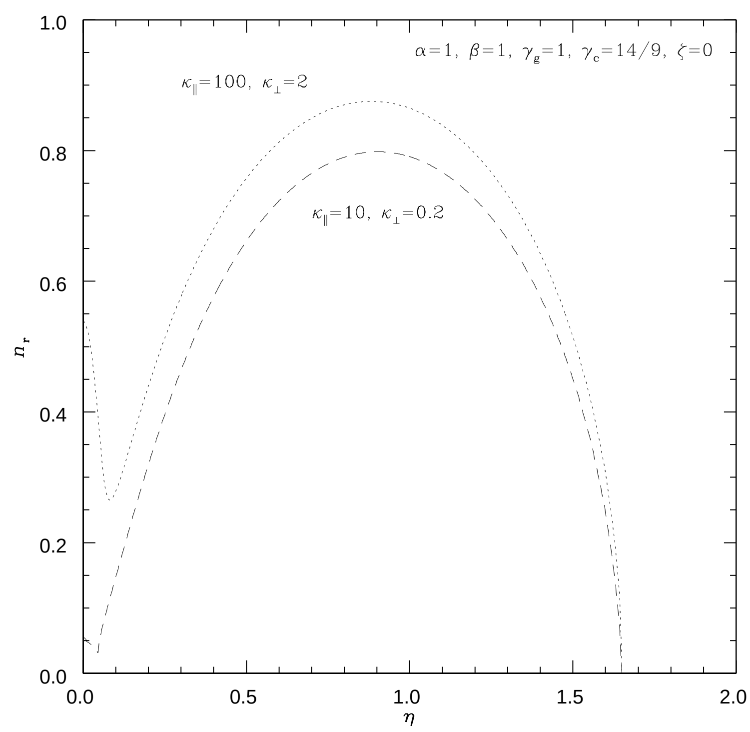

FIG. 3.—Dispersion relations of the Parker instability with nonzero $\kappa_\parallel$ and nonzero $\kappa_\perp$. The growth rate (the largest n) is presented as a function of the wavenumber along the initial magnetic field direction. The vertical wavenumber along the direction of gravity was set to be 0. The normalization units of time and length are 2.4×10^7 yr (H/a) and 160 pc (H), respectively. Each curve is labeled by the values of κ. The values of other parameters are specified within the frame.

4. SUMMARY AND DISCUSSIONS

The Parker instability in the ISM is induced by the buoyancy of magnetic fields as well as CRs. Hence, its analysis can be completed with a full treatment of CR dynamics. However, previous analyses incorporated CRs with simplified assumptions or ignored CRs completely, partially because of the lack of available CR physics. For instance, although the diffusion of CRs is finite in the ISM, Parker (1966) and Shu (1974) assumed that the diffusion along magnetic field lines is large enough that there is no CR pressure gradient along them, while the diffusion across field lines is neglected. In this contribution, we have relaxed this assumption and studied the role of full CR dynamics with finite CR diffusion in the Parker instability. First, the values of the energy-weighted mean diffusion coefficients, which are applicable to the scales relevant to the Parker instability, have been estimated as $\kappa_{\parallel} \simeq 3 \times 10^{28}$ cm^2 s^{-1} and $\kappa_{\perp} = 0.02\kappa_{\parallel}$. Then, a standard normal mode analysis has been performed in the two-dimensional plane defined by gravity and the initial magnetic field. Linearized perturbation equations have been combined into the dispersion relation (36) of a polynomial of sixth order in the growth rate n, with complex coefficients given by equations (37)–(42).

It has been shown that the finiteness of parallel diffusion slows down the development of the Parker instability. However, with $\kappa_{\parallel} = 3 \times 10^{28}$ cm^2 s^{-1} in the ISM, the maximum growth rate is smaller by only a couple percent than that for $\kappa_{\parallel} \rightarrow \infty$, and the range of unstable wavenumbers remains the same. The inclusion of perpendicular diffusion with $\kappa_{\perp} = 0.02\kappa_{\parallel}$ does not change the growth rate noticeably. That is, the original Parker approximation of infinite $\kappa_{\parallel}$ and $\kappa_{\perp} = 0$ was a good one and produced a quantitatively correct result. Hence, we conclude that CRs can enhance the Parker instability significantly, by increasing the maximum growth rate by a factor of up to 2 or so. We would like to note that this result disagrees with that of Nelson (1985), who found that CRs could stabilize, rather than destabilize, the Parker instability.

As noted in § 1, recent studies of the Parker instability, in which the random component as well as the regular component of the magnetic field were considered (Parker & Jokipii 2000; Kim & Ryu 2001), showed that a random component of strength comparable to that of the regular component ($\delta B^2/B_0^2 \gtrsim 0.5$) can stabilize the instability completely. For smaller $\delta B^2/B_0^2$, the instability is still operating, but with reduced growth rate and vanishing wavenumber along the radial direction of the Galaxy. Hence, it was concluded that the Parker instability alone has a difficult time forming Galactic structures such as giant molecular clouds. However, our new result suggests that the Parker instability might be preserved once CRs are incorporated, since inclusion of CRs increases not only the growth rate but also the range of unstable wavenumbers. Settling this issue would require a full three-dimensional analysis with the random component of the magnetic field, which we leave for future work.

Finally, we comment on the applicability of an analysis that assumes a smooth distribution of CRs, even though they are thought to be the products of discrete sources, namely, of supernova remnants. For the purposes of the present calculation, the approximation should be quite adequate, in fact. Observations of secondary pion-produced γ-rays show that the Galactic hadronic CR distribution is smooth on large scales (see, e.g., Bloeman et al. 1986). This is very reasonable in light of the long containment time of such CRs in the Galaxy ($\sim 10^7$ yr; see, e.g., Connell 1998) and their associated diffusion and advection. Recent sophisticated models of the CR distribution including stochastic sources along with numerous experimental constraints lead CRs at above a few hundred MeV to very smooth distributions outside the CR acceleration sites (Strong & Moskalenko 2001). Furthermore, there are good arguments that small, isolated supernova remnants expanding into dense media could be far less common and less important sources of CRs than supernova remnants inside large, low-density bubbles where the freshly accelerated CRs would be more broadly distributed (see, e.g., Higdon, Lingenfelter, & Ramady 1998).

The work was supported in part by the Korea Research Foundation through grant KRF-2000-015-DS0046. We thank the anonymous referee for constructive comments.

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