Generalized Kapchinskij-Vladimirskij Distribution and Envelope Equation for High-Intensity Beams in a Coupled Transverse Focusing Lattice

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(Received 26 August 2009; published 25 November 2009)

In an uncoupled lattice, the Kapchinskij-Vladimirskij (KV) distribution function first analyzed in 1959 is the only known exact solution of the nonlinear Vlasov-Maxwell equations for high-intensity beams including self-fields in a self-consistent manner. The KV solution is generalized here to high-intensity beams in a coupled transverse lattice using the recently developed generalized Courant-Snyder invariant for coupled transverse dynamics. This solution projects to a rotating, pulsating elliptical beam in transverse configuration space, determined by the generalized matrix envelope equation.

DOI: 10.1103/PhysRevLett.103.224802 PACS numbers: 29.27.Bd

Modern high-intensity beams have many important applications ranging from high energy density physics and ion-beam-driven fusion to high-flux neutron sources and light sources. It is becoming increasingly important to understand the self-field effects of high-intensity beams including self-electric and self-magnetic fields in a fully self-consistent manner, from the nonlinear Vlasov-Maxwell equations [1]. In an uncoupled lattice, the Kapchinskij-Vladimirskij (KV) distribution function analyzed in 1959 [2] is the only known exact self-consistent solution of the nonlinear Vlasov-Maxwell equations for high-intensity beams. In practical accelerators and beam transport systems, the transverse coupling between the horizontal and vertical directions, induced by error fields and misalignments, is always a significant effect [3–8]. Strong coupling of the transverse dynamics is introduced intentionally in certain types of cooling channels [9] and in the final focusing system for high energy density physics experiments [10], as well as in the conceptual design of the Möbius accelerator [11]. In this Letter, we generalize the KV solution to describe high-intensity beam dynamics in a coupled transverse focusing lattice using the recently developed generalized Courant-Snyder invariant [12,13] for coupled transverse dynamics.

In a coupled transverse focusing lattice, the Vlasov-Maxwell equations that govern the evolution of the distribution function f of a high-intensity beam and the corresponding space-charge potential ψ are [1]

$$\frac{\partial f}{\partial s} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - (\nabla\psi + \kappa_q \mathbf{x})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \qquad (1)$$

$$\nabla^2\psi = \frac{-2\pi K_b}{N_b}\int f dv_x dv_y. \qquad (2)$$

Here, particle motion in the beam frame is assumed to be nonrelativistic, ψ is the space-charge potential normalized by $\gamma_b^3 m\beta_b^2 c^2/q_b$, $\beta_b c$ is the directed beam velocity in the longitudinal direction, $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, s is the time variable normalized by $1/\beta_b c$, $K_b = 2N_b q_b^2/\gamma_b^3 m\beta_b^2 c^2$ is the beam self-field perveance, $N_b = \int f dx dy dv_x dv_y$ is the line density, $\mathbf{x} = (x, y)^T$ represents the normalized transverse displacement of a beam particle, $\mathbf{v} = d\mathbf{x}/ds = (v_x, v_y)^T = (\dot{x}, \dot{y})^T$ is the normalized transverse velocity in the beam frame, and $\kappa_q \mathbf{x}$ is the coupled linear focusing force. In Eq. (1)

$$\kappa_q = \begin{pmatrix} \kappa_{qx} & \kappa_{qxy} \\ \kappa_{qyx} & \kappa_{qy} \end{pmatrix} \qquad (3)$$

is the matrix of coupling coefficients, κ_{qx} and κ_{qy} are the focusing coefficients for the quadrupole lattice, and $\kappa_{qxy} = \kappa_{qyx}$ are the coupling coefficients produced by the skew-quadrupole component of the lattice. In general, the coupled linear focusing force can also depend on transverse velocity, as in the case of a solenoidal lattice, which can be transformed into the form of Eqs. (1)–(3) if we choose the local Lamor frame [1,13]. For simplicity of presentation, we consider here only the coupling due to skew quadrupoles given by Eq. (3). The $-\nabla\psi$ term in Eq. (1) describes the self-field force, and is nonlinearly coupled to f through Eq. (2). Equations (1) and (2) form a set of nonlinear integro-differential equations, whose analytical solutions are difficult to find in general.

For the case of an uncoupled lattice, i.e., $\kappa_{qxy} = \kappa_{qyx} = 0$, Eqs. (1) and (2) admit a remarkable solution known as the Kapchinskij-Vladimirskij (KV) distribution [2], which has played an important role in high-intensity beam physics [14–17]. The KV distribution function is constructed as a function of the Courant-Snyder (CS) invariants of the transverse dynamics [18]. Since the CS invariants are valid for linear, uncoupled transverse forces, the KV distribution must self-consistently generate a linear, uncoupled space-charge force. The KV distribution indeed satisfies this requirement. It is given by [1,2]

$$f_{\mathrm{KV}} = \frac{N_b}{\pi^2\varepsilon_x\varepsilon_y}\delta\left(\frac{I_x}{\varepsilon_x} + \frac{I_y}{\varepsilon_y} - 1\right), \qquad (4)$$

$$I_x = \frac{x^2}{w_x^2} + (w_x\dot{x} - x\dot{w}_x)^2, \qquad I_y = \frac{y^2}{w_y^2} + (w_y\dot{y} - y\dot{w}_y)^2. \tag{5}$$

Here, I_x and I_y are the CS invariants for the x- and y-motions, respectively, ε_x and ε_y are the constant transverse emittances, and w_x and w_y are the envelope functions satisfying the envelope equations,

$$\ddot{w}_x + \kappa_x w_x = w_x^{-3}, \qquad \ddot{w}_y + \kappa_y w_y = w_y^{-3}, \tag{6}$$

$$\kappa_x = \kappa_{qx} - \frac{2K_b}{a(a+b)}, \qquad \kappa_y = \kappa_{qy} - \frac{2K_b}{b(a+b)}, \tag{7}$$

$$a \equiv \sqrt{\varepsilon_x} w_x, \qquad b \equiv \sqrt{\varepsilon_y} w_y. \tag{8}$$

The density profile in the transverse configuration space projected by the distribution function f_{KV} in Eq. (4) is given by

$$n(x, y, s) = \int d\dot{x}d\dot{y} f_{\mathrm{KV}} = \begin{cases} N_b/\pi ab = \mathrm{const}, & 0 \le x^2/a^2 + y^2/b^2 < 1, \\ 0, & 1 < x^2/a^2 + y^2/b^2. \end{cases} \tag{9}$$

which corresponds to a constant-density beam with elliptical cross section and pulsating transverse dimensions a and b [see Fig. 1(a)]. The associated space-charge potential inside the beam, determined from Eq. (2), is given by

$$\psi = \frac{-K_b}{a+b}\left(\frac{x^2}{a} + \frac{y^2}{b}\right), \quad 0 \le x^2/a^2 + y^2/b^2 < 1. \tag{10}$$

The KV distribution (4) reduces the original nonlinear Vlasov-Maxwell equations (1) and (2) to the two envelope equations in Eq. (6) for w_x and w_y, or equivalently, for $a = \sqrt{\varepsilon_x} w_x$ and $b = \sqrt{\varepsilon_y} w_y$ [Eq. (8)]. As the only known solution of the nonlinear Vlasov-Maxwell equations (1) and (2), the KV distribution and the associated envelope equations provide very important elementary theoretical tools for our understanding of high-intensity beam dynamics [14–17]. The KV distribution in Eq. (4) is constructed from the exact dynamical invariants I_x and I_y in Eq. (5), and constitutes an exact solution of the Vlasov equation (1), which also generates the uncoupled linear space-charge force assumed *a priori*.

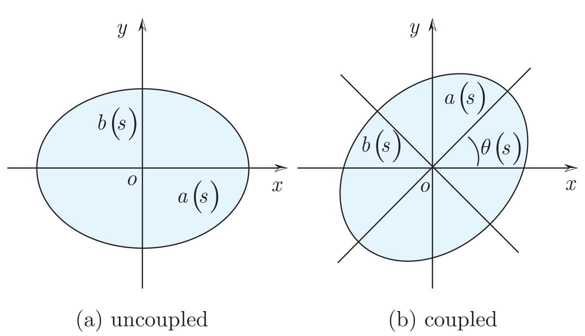

FIG. 1 (color online). Beam cross sections for the KV distribution. (a) Uncoupled lattice: the cross section is determined by $0 \le x^2/a^2 + y^2/b^2 < 1$; and (b) coupled lattice: the cross section is determined by $0 \le \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x} < \varepsilon$.

We now show how to generalize this KV solution to the case of coupled transverse dynamics when $\kappa_{qxy} = \kappa_{qyx} \neq 0$, using the recently developed generalized CS invariant for coupled transverse lattice [12,13]. In the coupled case, the generalized KV distribution that solves the nonlinear Vlasov-Maxwell system (1) and (2) projects to a rotating, pulsating beam with elliptical cross section in transverse configuration space with constant density inside the beam. Both the dimensions a and b, and the tilt angle θ are functions of $s = \beta_b ct$ [see Fig. 1(b)], in contrast with the pulsating upright elliptical beam cross section for the uncoupled case [see Fig. 1(a)]. The rotating, pulsating beam with elliptical cross section in transverse configuration space, and constant density inside the beam, generates a coupled linear space-charge force of the form

$$-\nabla\psi = -\kappa_s \mathbf{x}, \qquad \kappa_s = \begin{pmatrix} \kappa_{sx} & \kappa_{sxy} \\ \kappa_{syx} & \kappa_{sy} \end{pmatrix}, \tag{11}$$

where $\kappa_{sxy} = \kappa_{syx}$, which allows us to apply the generalized CS invariant for the coupled transverse dynamics. The exact form of κ_s will be determined self-consistently [see Eq. (24)]. Our strategy is to use the generalized CS invariant to construct a generalized KV solution of the Vlasov equation (1), which also projects to a rotating, pulsating elliptical beam with constant density inside the beam. In this manner, a self-consistent solution of the nonlinear Vlasov-Maxwell equations (1) and (2) is found for high-intensity beams in a coupled transverse focusing lattice.

For a charged particle subject to the coupled linear focusing force and the coupled linear space-charge force

$$-\nabla\psi - \kappa_q \mathbf{x} = -\kappa \mathbf{x}, \qquad \kappa = \kappa_q + \kappa_s, \tag{12}$$

the generalized CS invariant is given by [12,13]

$$I_{\mathrm{CS}} = \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x} + (\dot{\mathbf{x}}^T w^T - \mathbf{x}^T \dot{w}^T)(w\dot{\mathbf{x}} - \dot{w}\mathbf{x}), \tag{13}$$

where

$$w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$$

is the 2×2 envelope matrix determined from the matrix envelope equation

$$\ddot{w} + w\kappa = (w^{-1})^T w^{-1} (w^{-1})^T. \tag{14}$$

Since I_{CS} is an invariant of the particle dynamics, any function of I_{CS} is a solution of the Vlasov equation (1). However, in order to solve the nonlinear Vlasov-Maxwell equations (1) and (2), the distribution function must generate the coupled linear space-charge force of the form in Eq. (11) as well. For this purpose, we select the distribution function to be the following generalized KV distribution

$$f_{\mathrm{KV}} = \frac{N_b|w|}{A\varepsilon\pi}\delta\left(\frac{I_{\mathrm{CS}}}{\varepsilon} - 1\right). \tag{15}$$

Here, N_b and ε are constants, where N_b is the line-density, and ε is the transverse emittance. Moreover, $|w|$ is the determinant of the envelope matrix w, and A is the area of the beam cross section determined by $|w|$ and ε. Both $|w|$ and A are functions of $s = \beta_b ct$. The beam density profile in transverse configuration space is

$$\begin{aligned} n(x, y, s) &= \int d\dot{x}d\dot{y}f_{\mathrm{KV}} = \int d\left(\frac{r^2}{\varepsilon}\right)\frac{N_b}{A}\delta\left(\frac{I_{\mathrm{CS}}}{\varepsilon} - 1\right) \\ &= \begin{cases} N_b/A, & 0 \le \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x} < \varepsilon, \\ 0, & \varepsilon < \mathbf{x}^T w^{-1} w^{-1T}\mathbf{x}. \end{cases} \end{aligned} \tag{16}$$

In the above calculation, the velocity integration with respect to $d\dot{x}d\dot{y}$ is carried out in the new velocity coordinates (p, q) through the transformation

$$d\dot{x}d\dot{y} = \frac{1}{|w|}dpdq = \frac{2\pi}{|w|}rdr, \tag{17}$$

$$p \equiv w_1\dot{x} + w_2\dot{y} - \dot{w}_1 x - \dot{w}_2 y, \tag{18}$$

$$q \equiv w_3\dot{x} + w_4\dot{y} - \dot{w}_3 x - \dot{w}_4 y, \tag{19}$$

$$r^2 \equiv p^2 + q^2. \tag{20}$$

The density profile $n\ (x, y, s)$ obtained in Eq. (16) is indeed of the desired form. That is, $n\ (x, y, s)$ is constant inside the ellipse defined by

$$\mathbf{x}^T\beta^*\mathbf{x} = \varepsilon, \qquad \beta^* \equiv w^{-1}w^{-1T}, \tag{21}$$

and $n(x, y, s) = 0$ outside the ellipse. The ellipse defined by Eq. (21) is pulsating and rotating. Its transverse dimensions $a(s)$ and $b(s)$, and tilt angle $\theta(s)$ depend on $s = \beta_b ct$ and are determined from the matrix β^*. Because β^* is obviously real, symmetric, and positive definite, the two eigenvectors v_1 and v_2 of β^* are orthogonal with two positive eigenvalues λ_1 and λ_2. It is an elementary result [19] that the transverse dimensions of the ellipse are given by $a = \sqrt{\varepsilon/\lambda_1}$ and $b = \sqrt{\varepsilon/\lambda_2}$, and the tilt angle θ is that of v_1. The principal axis theorem [19] states that the diagonalizing matrix Q of β^* can be constructed as $Q = (v_1, v_2)$ with $Q^{-1} = Q^T$ and

$$Q^{-1}\beta^*Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We now introduce the rotating frame

$$\begin{pmatrix} X \\ Y \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The ellipse in (X, Y) coordinates is given

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \tag{22}$$

and the self-field force is

$$-\begin{pmatrix} \partial\psi/\partial X \\ \partial\psi/\partial Y \end{pmatrix} = \frac{2K_b}{a+b}\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{23}$$

Transforming back to (x, y) coordinate, the self-field force can be expressed as

$$\begin{aligned} -\begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \end{pmatrix} &= -\kappa_s\begin{pmatrix} x \\ y \end{pmatrix}, \\ \kappa_s &= \frac{-2K_b}{a+b}Q\begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}Q^{-1}. \end{aligned} \tag{24}$$

The coupled linear space-charge coefficient κ_s is a function of the envelope matrix w and the constant emittance ε. When Eq. (24) is substituted back into Eq. (12), the envelope equation (14) becomes a closed nonlinear matrix equation for the envelope matrix w. Therefore, we have succeeded in finding a class of self-consistent solutions of the nonlinear Vlasov-Maxwell equations for high-intensity beams in a coupled transverse focusing lattice. The solution reduces to a nonlinear matrix ordinary differential equation for the envelope matrix w, which determines the geometry of the pulsating and rotating beam ellipse. The matrix envelope equation (14) can be numerically solved in a straightforward manner. We note that the self-consistent solution constructed for the coupled lattice has one emittance ε in the transverse directions [11], whereas in an uncoupled lattice, the standard KV distribution contains two emittances, i.e., ε_x and ε_y. This should not come as a surprise because a coupled lattice is more complex than an uncoupled lattice, and it is natural for the self-consistent solution to have less freedom in a coupled lattice than in an uncoupled lattice. In accelerators and storage rings with coupling, a single emittance in the transverse directions implies an equilibrium between the x direction and the y direction, which can be reached in certain situations, but not always. Therefore, the self-consistent distribution constructed only applies to those cases where such an equilibrium is reached, such as in a strongly coupled system or in the lattice of transport lines.

As a specific example, we consider a periodic quadrupole FODO lattice with the middle magnet being misaligned by a small angle ξ. The misaligned magnet induces a skew-quadrupole component of the form [4] $\kappa_{qxy} = \kappa_{qyx} = \kappa_q \sin 2\xi$. The strength of the quadrupole component of the misaligned magnet is reduced to $\kappa_{qx} = -\kappa_{qy} = \kappa_q \cos 2\xi$. The normalized quadrupole focusing field is $\kappa_q \equiv q_b B_q'/\gamma_b m\beta_b c^2 = 15$ with a filling factor $\eta = 0.15$. The misalignment is $\xi = 11.4°$, and the normalized self-field perveance is $K_b/\varepsilon = 0.1$. The matrix envelope equation (14) has been solved numerically to find a matched solution. The numerical result, plotted in

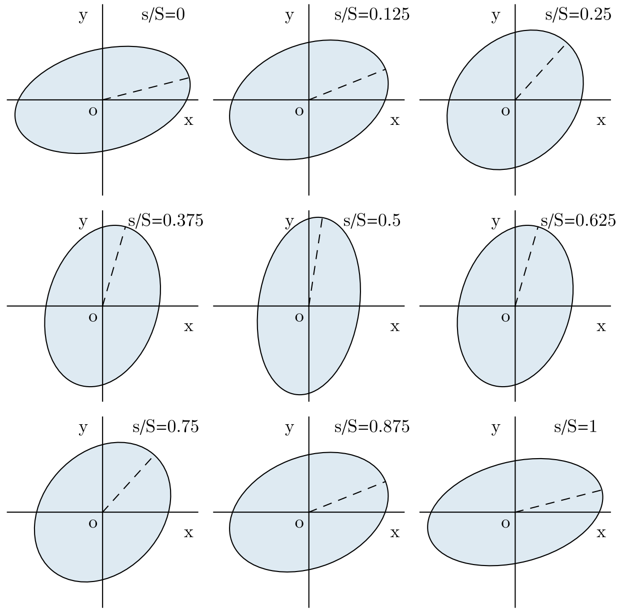

FIG. 2 (color online). Beam cross sections as a function of $s/S = \beta_b ct/S$ over the interval $0 \leq s/S \leq 1$. The dynamics of the beam pulsation and rotation is evident from the figure.

Fig. 2, shows the beam cross section as a function of $s/S = \beta_b ct/S$, where S is the lattice period. The dynamics of the beam pulsation and rotation is clearly demonstrated in the plots. The rotation dynamics result in a wobbling motion of the tilt angle between $\theta = 14.28°$ at $s/S = 0$ and $\theta = 81.35°$ at $s/S = 0.5$. As expected, in the rotating frame the transverse dimensions a and b of the beam ellipse oscillate with time. Note that the dynamics of beam rotation and pulsation is matched with the lattice period.

In conclusion, the KV distribution function, the exact self-consistent solution of the nonlinear Vlasov-Maxwell equations for high-intensity charged particle beams in an uncoupled focusing lattice including self-electric and self-magnetic fields, has been generalized to describe high-intensity beam dynamics in a coupled transverse focusing lattice using the recently developed generalized Courant-Snyder invariant [12,13] for coupled transverse dynamics. The fully self-consistent solution reduces the nonlinear Vlasov-Maxwell equations to a nonlinear matrix ordinary differential equation for the envelope matrix w, which determines the geometry of the pulsating and rotating beam ellipse. This result provides us with a new theoretical tool to investigate the dynamics of high-intensity beams in a coupled transverse lattice.

This research was supported by the U. S. Department of Energy.

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