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PSEUDOSPECTRAL LEAST SQUARES METHOD FOR STOKES–DARCY EQUATIONS*

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Abstract. We investigate the first order system least squares Legendre and Chebyshev pseudospectral method for coupled Stokes–Darcy equations. A least squares functional is defined by summing up the weighted L^2-norm of residuals of the first order system for coupled Stokes–Darcy equations and that of Beavers–Joseph–Saffman interface conditions. Continuous and discrete homogeneous functionals are shown to be equivalent to a combination of weighted $H(\mathrm{div})$ and H^1-norms for Stokes and Darcy equations. The spectral convergence for the Legendre and Chebyshev methods is derived. Some numerical experiments are demonstrated to validate our analysis.

Key words. coupled Stokes–Darcy equation, pseudospectral method, first order system least squares method, interface problem

AMS subject classifications. 65F10, 65M30

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1. Introduction.

The coupled Stokes–Darcy equation which connects fluid and porous media flow has become a very active area of research recently, from both mathematical and numerical points of view. Filtration of blood through arterial vessel walls in physiology, air and oil filtration in industrial processes, and percolation of water of hydrological basin through rocks and sand are indications of practical applications.

Let Ω be an open bounded domain subdivided into two subdomains Ω_S and Ω_D with the curve Γ, such that $\overline{\Omega} = \overline{\Omega_S} \cup \overline{\Omega_D} \cup \Gamma$. Here, Γ is referred to as the *interface*. The boundary of Ω is denoted by $\partial\Omega$ and $\partial\Omega_S = \overline{\Omega_S} \cap \partial\Omega$, $\partial\Omega_D = \overline{\Omega_D} \cap \partial\Omega$ (see Figure 1).

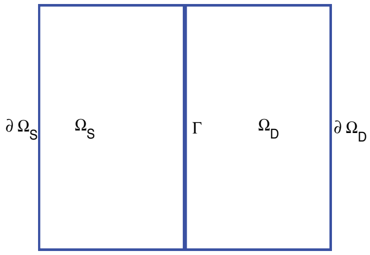

Fig. 1. *Schematic of domain for Stokes–Darcy equations with interface* Γ.

Assume that flow in Ω_S is governed by the Stokes equation

$$(1.1)\qquad \begin{cases} -\nabla\cdot\mathbf{T} = \mathbf{f} & \text{in} \quad \Omega_S, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega_S, \end{cases}$$

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where $\mathbf{T} := -p\,\mathbf{I} + 2\nu\,\mathcal{E}(\mathbf{u})$ is the stress tensor, $\mathbf{u}$ is velocity, and $\mathbf{f}$ is external force function. Here, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain tensor, p is pressure, and ν is kinetic viscosity of the fluid. Suppose also that flow in the porous medium domain Ω_D is governed by the Darcy equation

$$\begin{cases} \mathbf{w} + K\nabla q = 0 & \text{in} \quad \Omega_D, \\ \nabla\cdot\mathbf{w} = g & \text{in} \quad \Omega_D, \end{cases} \tag{1.2}$$

where $\mathbf{w}$ is velocity, q is pressure, g is a source (sink) function, and K is a symmetric and positive definite permeability tensor with component bounded from above which depends on the properties of fluid as well as on the characteristics of porous medium and can be diagonalized by introducing three mutually orthogonal axes called principal directions of anisotropy. For simplicity of exposition, we assume that K is constant, but a nonconstant and sufficiently smooth K can be treated without essential changes. We consider the following boundary conditions:

$$\begin{cases} \mathbf{u} = 0 & \text{on} \quad \partial\Omega_S, \\ \mathbf{w}\cdot\mathbf{n} = 0 & \text{on} \quad \partial\Omega_D, \end{cases} \tag{1.3}$$

where $\mathbf{n}$ is outward unit normal vector on $\partial\Omega_D$. Additional conditions on interface are the necessity of a well-posed formulation. On the interface Γ, the Beavers–Joseph–Saffman conditions are imposed,

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n} = 0, \\ \mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n}) + q = 0, \\ \beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n}) + \mathbf{u}\times\mathbf{n} = 0, \end{cases} \tag{1.4}$$

where β is a positive constant and $\mathbf{n}$ is unit normal vector pointing from Γ into Ω_D. For details of proper choice of β, see [6]. The first and second conditions of (1.4) impose continuity of flux and normal stress, respectively, and the last one is known as the Beavers–Joseph–Saffman law. The interface condition (1.4) was proposed by Beavers and Joseph [2] based on experiments, and a mathematical justification of these conditions was given by Saffman [24] and Jäger and Mikelić [14].

The coupled Stokes–Darcy equation has been investigated from mathematical and numerical analysis viewpoints ([16, 22, 7, 18] are some references in finite element framework). A survey of finite element method for coupling of Navier–Stokes and Stokes–Darcy flow can be found in [8]. The discontinuous Galerkin method [23] and edge stabilized method [4] are proposed for coupled Stokes–Darcy problems. Least squares finite element methods for Stokes–Darcy equations have been studied in [19]. In this work we wish to solve Stokes–Darcy equations based on the pseudospectral least squares method, which is known to be very accurate, by giving a rigorous analysis.

On one hand, the accuracy of spectral methods which employ the global polynomial for discretization makes it a popular method to approximate solutions of partial differential equations. Spectral collocation method also has been used to approximate the solution of interface problems. Shin and Jung [25] presented a spectral collocation method for one-dimensional interface problems. Hessari, Kim, and Shin [12] have developed an algorithm to approximate the solutions of second order elliptic interface problems. On the other hand, least squares methods have received much attention in past decades, due to their advantages. Among the advantages of least squares methods is that the algebraic system which must be solved to compute the discrete solution is always symmetric and positive definite. One of the motivations to investigate least squares methods is that the choice of approximation spaces for velocity and pressure is not subject to the LBB compatibility condition and one can use equal order interpolation polynomials to approximate all variables. Hessari [26] has investigated least squares pseudospectral methods for the interface problem of Stokes equations. The least squares pseudospectral method for the Navier–Stokes equation is also analyzed in [13].

The aim of this paper is to combine least squares and pseudospectral methods to approximate the solution of Stokes–Darcy equations, and the significant contribution is the combination of spectral space and the first order system least squares method. To achieve this combination, the least squares functionals are defined to be the sum of weighted L^2-norm of residuals of the first order system and that of Beavers–Joseph–Saffman interface conditions. The continuity and coercivity of the least squares functionals are proposed. For the continuous Legendre least squares functional, we follow the continuous least squares functional developed in [19]. The continuous and discrete Legendre least squares functionals are shown to be equivalent to product norm $\|\mathbf{T}\|^2_{\mathrm{div},\Omega_S}+\|\mathbf{T}\cdot\mathbf{n}\|^2_{\Gamma}+\|\mathbf{u}\|^2_{1,\Omega_S}+\|\mathbf{w}\|^2_{\mathrm{div},\Omega_D}+\|\mathbf{w}\cdot\mathbf{n}\|^2_{\Gamma}+\|q\|^2_{1,\Omega_D}$. The continuous and discrete Chebyshev least squares functionals are also demonstrated to be equivalent to product norm $\|\mathbf{T}\|^2_{w,\mathrm{div},\Omega_S}+\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|\mathbf{u}\|^2_{1,w,\Omega_S}+\|\mathbf{w}\|^2_{w,\mathrm{div},\Omega_D}+\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|q\|^2_{1,w,\Omega_D}$. Spectral convergence of the proposed method for both Legendre and Chebyshev cases is presented.

Numerical methods for coupled Stokes–Darcy equations lead to coupled discrete problems, implementation is complicated, and numerical difficulties arise when the mesh size is decreased. However, the algebraic system resulting from least squares pseudospectral discretization is symmetric and positive definite and can be efficiently solved by iterative and direct methods.

The structure of this paper is as follows. Some preliminaries, facts, and definitions are given in the following section. In section 3, the first order system of the Stokes–Darcy equation is provided. The Legendre and Chebyshev least squares functionals are defined and shown to be equivalent to some appropriate product norm. Spectral convergence of the proposed methods is also presented. Some numerical examples are given in section 4 to demonstrate spectral convergence of our method. The paper is finalized by some concluding remarks in section 5.

2. Preliminaries.

In this section, we provide some definitions and facts which are needed in subsequent sections. We use the standard notation and definitions for the weighted Sobolev spaces as follows. We suppose that $\mathcal{D}=[-1,\ 1]^2$. The weighted $L^2_w(\mathcal{D})$ is defined as

$$L^2_w(\mathcal{D})=\{v:\mathcal{D}\to\mathbf{R}\,\big|\,v\text{ is measurable and }\ \|v\|_{0,w,\mathcal{D}}<\infty\} \tag{2.1}$$

equipped with the norm and the associated scalar product

$$\|v\|_{0,w,\mathcal{D}}=\left(\int_{\mathcal{D}}|v(\mathbf{x})|^2w(\mathbf{x})d\mathbf{x}\right)^{1/2},\qquad (u,v)_w=\int_{\mathcal{D}}u(\mathbf{x})v(\mathbf{x})w(\mathbf{x})d\mathbf{x}.$$

Define the weighted Sobolev space $H^s_w(\mathcal{D})$ for a nonnegative integer s as

$$H^s_w(\mathcal{D})=\{v\in L^2_w(\mathcal{D})\ \big|\ v^{(\alpha)}\in L^2_w(\mathcal{D}),\ |\alpha|=1,2,\dots,s\}, \tag{2.2}$$

where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1,\, \alpha_2 \geq 0,\ |\alpha| = \alpha_1 + \alpha_2$, and $v^{(\alpha)} = \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, equipped with the norm and the associated scalar product

$$\|v\|_{s,w,\mathcal{D}} = \left(\sum_{|\alpha| \leq s} \|v^{(\alpha)}\|^2_{0,w,\mathcal{D}} \right)^{1/2}, \qquad (u, v)_{s,w} = \sum_{|\alpha| \leq s} (u^{(\alpha)}, v^{(\alpha)})_w.$$

We note that $w(\mathbf{x}) = \hat{w}(x)\hat{w}(y)$ is either the Legendre weight function with $\hat{w}(t) = 1$, or the Chebyshev weight function with $\hat{w}(t) = \frac{1}{\sqrt{1-t^2}}$. The space $H^0_w(\mathcal{D})$ indicates $L^2_w(\mathcal{D})$, in which the norm and inner product will be denoted by $\|\cdot\|_{w,\mathcal{D}}$ and $(\cdot,\, \cdot)_{w,\mathcal{D}}$, respectively. Let $H^1_{0,w}(\mathcal{D})$ be the subspace of $H^1_w(\mathcal{D})$, consisting of the functions which vanish on the boundary. For the Legendre case, we will simply write the notation without the subscripts w, i.e., $\|\cdot\|_{\mathcal{D}}$, $(\cdot,\cdot)_{\mathcal{D}}$. Denote by $H^{-1}_w(\mathcal{D})$ (see page 18 in [3]) the dual space of the space $H^1_{0,w}(\mathcal{D})$ equipped with its norm

$$\|u\|_{-1,w,\mathcal{D}} := \sup_{\phi \in H^1_{0,w}(\mathcal{D})} \frac{(u, \phi)_{w,\mathcal{D}}}{\|\phi\|_{1,w,\mathcal{D}}}. \tag{2.3}$$

Let

$$H_w(\mathrm{div}, \mathcal{D}) = \{\mathbf{v} \in L^2_w(\mathcal{D})^2 \ :\ \nabla \cdot \mathbf{v} \in L^2_w(\mathcal{D})\},$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{w,\mathrm{div},\mathcal{D}} = \left(\|\mathbf{v}\|^2_{w,\mathcal{D}} + \|\nabla \cdot \mathbf{v}\|^2_{w,\mathcal{D}}\right)^{1/2}.$$

Define its subspaces

$$H_{0,w}(\mathrm{div}, \mathcal{D}) = \{\mathbf{v} \in H_w(\mathrm{div}, \mathcal{D}) \ :\ \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \partial\mathcal{D}\}.$$

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N. Let $\{\xi_i\}_{i=0}^N$ be the Legendre–Gauss–Lobatto (LGL) or Chebyshev–Gauss–Lobatto (CGL) points on $[-1, 1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$. For the Legendre case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)L'_N(t)$, where L_N is the Nth Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)} \frac{1}{[L_N(\xi_j)]^2}, \quad 1 \leq j \leq N-1. \tag{2.4}$$

For the Chebyshev case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)T'_N(t)$, where T_N is the Nth Chebyshev polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{\pi}{2N}, \quad w_j = \frac{\pi}{N}, \quad 1 \leq j \leq N-1. \tag{2.5}$$

We have the following exactness of numerical integration of the Gaussian quadrature rule:

$$\int_{-1}^{1} p(t)\hat{w}(t)dt = \sum_{i=0}^{N} w_i\, p(\xi_i) \quad \forall\, p \in \mathcal{P}_{2N-1}. \tag{2.6}$$

Let $\{\phi_i\}_{i=0}^N$ be the set of Lagrange polynomials of degree N with respect to LGL or CGL points $\{\xi_i\}_{i=0}^N$ which satisfy

$$\phi_i(\xi_j) = \delta_{ij} \qquad \forall i, j = 0, 1, \dots, N,$$

where δ_{ij} denotes the Kronecker delta function. For any continuous function v on $I = (-1, 1)$, denote by $I_N v \in \mathcal{P}_N$ its Lagrangian interpolant at the nodes $\{\xi_j\}_{j=0}^N$, i.e., $I_N v(\xi_j) = v(\xi_j)$. The interpolation error estimate [21] is given by

$$\|v - I_N v\|_{k,w,I} \le C\, N^{k-s} \|v\|_{s,w,I}, \quad k = 0, 1, \tag{2.7}$$

provided $v \in H_w^s(I)$ for some $s \ge 1$. Define the discrete scalar product and norm as

$$\langle u, v\rangle_{w,I,N} = \sum_{j=0}^{N} u(\xi_j) v(\xi_j) w_j, \qquad \|v\|_{w,I,N} = \langle v,\, v\rangle_{w,I,N}^2.$$

By (2.6), we have

$$\langle u, v\rangle_{w,I,N} = (u, v)_{w,I} \quad \text{for } uv \in \mathcal{P}_{2N-1}. \tag{2.8}$$

It is well known that

$$\|v\|_{w,I} \le \|v\|_{w,I,N} \le \gamma^* \|v\|_{w,I} \quad \forall\, v \in \mathcal{P}_N, \tag{2.9}$$

where $\gamma^* = \sqrt{2 + \frac{1}{N}}$ for the Legendre case and $\gamma^* = \sqrt{2}$ for the Chebyshev case [21]. For $u \in H_w^s(I)$, $s \ge 1$, and $v_N \in \mathcal{P}_N$,

$$|(u, v_N)_{w,I} - \langle u, v_N\rangle_{w,I,N}| \le C\, N^{-s}\, \|u\|_{s,w,I}\, \|v_N\|_{w,I}. \tag{2.10}$$

If the interval $[-1\,,\,1]$ is replaced by $[a\,,\,b]$, we can use the following linear transformation:

$$t = \frac{b-a}{2}(x+1) + a\ : [-1, 1] \to [a, b]$$

to find Gauss points $\{\hat{\xi}_j\}_{j=0}^N$ and the quadrature weights $\{\hat{w}_j\}_{j=1}^N$,

$$\hat{\xi}_j = \frac{b-a}{2}(\xi_j + 1) + a, \quad \text{and} \quad \hat{w}_j = \frac{b-a}{2} w_j.$$

The two-dimensional LGL or CGL nodes $\{\mathbf{x}_{ij}\}$ and the corresponding weights $\{\mathbf{w}_{ij}\}$ are denoted as

$$\mathbf{x}_{ij} = (\xi_i, \xi_j), \quad \mathbf{w}_{ij} = w_i w_j, \quad i, j = 0, 1, \dots, N.$$

Let $\mathcal{Q}_N$ be the space of all polynomials of degree less than or equal to N with respect to each single variable x and y. Define the basis for $\mathcal{Q}_N$ as

$$\psi_{ij}(x, y) = \phi_i(x)\phi_j(y), \qquad i, j = 0, 1, \dots, N.$$

For any continuous functions u and v in $\bar{\mathcal{D}}$, the associated discrete scalar product and norm are given by

$$\langle u, v\rangle_{w,\mathcal{D},N} = \sum_{i,j=0}^{N} \mathbf{w}_{ij}\, u(\mathbf{x}_{ij})\, v(\mathbf{x}_{ij}) \quad \text{and} \quad \|v\|_{w,\mathcal{D},N} = \langle v, v\rangle_{w,\mathcal{D},N}^{1/2}. \tag{2.11}$$

From (2.6), we have

$$\langle u, v\rangle_{w,\mathcal{D},N} = (u, v)_{w,\mathcal{D}} \quad \text{for } uv \in \mathcal{Q}_{2N-1}, \tag{2.12}$$

and it is well known that

$$\|v\|_{w,\mathcal{D}} \le \|v\|_{w,\mathcal{D},N} \le \gamma^* \|v\|_{w,\mathcal{D}} \quad \forall\, v \in \mathcal{Q}_N, \tag{2.13}$$

where $\gamma^* = \left(2 + \frac{1}{N}\right)$ for the Legendre case and $\gamma^* = 2$ for the Chebyshev case [21]. We recall some approximation properties given in [3, 5, 21]. The interpolation error estimate is given by

$$\|v - I_N v\|_{k,w,\mathcal{D}} \le C\, N^{k-s} \|v\|_{s,w,\mathcal{D}}, \quad k = 0, 1, \tag{2.14}$$

provided $v \in H^s_w(\mathcal{D})$ for some $s \ge 2$. For $u \in H^s_w(\mathcal{D})$, $s \ge 2$, and $v_N \in \mathcal{Q}_N$,

$$|(u, v_N)_{w,\mathcal{D}} - \langle u, v_N\rangle_{w,\mathcal{D},N}| \le C\, N^{-s}\, \|u\|_{s,w,\mathcal{D}}\, \|v_N\|_{w,\mathcal{D}}. \tag{2.15}$$

LEMMA 2.1. *For any* $\mathbf{v} \in [L^2_w(\mathcal{D})]^2$, *we have*

$$\|\nabla \cdot \mathbf{v}\|_{-1,w,\mathcal{D}} \le C \|\mathbf{v}\|_{w,\mathcal{D}}.$$

Proof. The proof is similar to Lemma 4.2 of [17]. □

We use the following bounds for traces from $H^1_w(\Omega_D)$ and $H^1_w(\Omega_S)$ [20]:

$$\|q\|^2_{1/2,w,\Gamma} \le C_T (\|q\|^2_{0,w,\Omega_D} + \|\nabla q\|^2_{0,w,\Omega_D}), \tag{2.16}$$

$$\|\mathbf{v}\|^2_{1/2,w,\Gamma} \le C_T (\|\mathbf{v}\|^2_{0,w,\Omega_S} + \|\nabla \mathbf{v}\|^2_{0,w,\Omega_S}). \tag{2.17}$$

Remark 1. If the domain $\mathcal{D}$ is replaced by a simply connected domain, then we can use the Gordon and Hall transformation [10, 11] to map the simply connected domain into $\mathcal{D}$ (see Appendix A).

The following a priori estimate holds for Stokes equations with homogeneous Dirichlet boundary condition on $\partial\mathcal{D}$:

$$\|\nu \mathbf{u}\|_{1,\omega,\mathcal{D}} + \|p\|_{\omega,\mathcal{D}} \le C\big(\| - \nu\Delta\mathbf{u} + \nabla p\|_{-1,\omega,\mathcal{D}} + \|\nu \nabla \cdot \mathbf{u}\|_{\omega,\mathcal{D}}\big). \tag{2.18}$$

Its proof can be found for the cases $\nu = 1$ and $\omega = 1$ in [9]. For the Chebyshev weight ω in [3], the case for general ν is then immediate. The a priori estimate for Poisson equation $-\Delta q = g$ with the Neumann boundary condition $\frac{\partial q}{\partial \mathbf{n}} = 0$ on $\partial\mathcal{D}$ is [1]

$$\|q\|_{1,w,\mathcal{D}} \le C\| - \Delta q\|_{-1,w,\mathcal{D}}, \tag{2.19}$$

subject to solvability condition

$$\int_{\mathcal{D}} g = 0 \qquad \text{or} \qquad \int_{\mathcal{D}} q = 0.$$

3. First order system least squares method. In this section, we investigate the first order system least squares pseudospectral method to approximate the solution of Stokes–Darcy equations (1.1)–(1.4). The pressure p is eliminated from the Stokes equation by taking the trace operator on $\mathbf{T} = 2\nu\,\mathcal{E}(\mathbf{u}) - p\mathbf{I}$ to get

$$p = -\frac{1}{2}\mathrm{tr}\,\mathbf{T}.$$

We impose the following condition:

$$\int_{\Omega_S} \mathrm{tr}\,\mathbf{T} = 0 \tag{3.1}$$

in order to get the unique solution. We also have

$$\mathbf{T} + p\,\mathbf{I} - 2\nu\,\mathcal{E}(\mathbf{u}) = \mathbf{T} - \frac{1}{2}\mathrm{tr}\,(\mathbf{T})\,\mathbf{I} - 2\nu\,\mathcal{E}(\mathbf{u}) = \hat{\mathbf{T}} - 2\nu\,\mathcal{E}(\mathbf{u}).$$

We consider the following first order system of equations for the Stokes–Darcy problem (1.1)–(1.2):

$$\begin{cases} -\nabla\cdot\mathbf{T} = \mathbf{f} & \text{in} \quad \Omega_S, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega_S, \\ \hat{\mathbf{T}} - 2\nu\,\mathcal{E}(\mathbf{u}) = 0 & \text{in} \quad \Omega_S, \\ \mathbf{w} + K\nabla q = 0 & \text{in} \quad \Omega_D, \\ \nabla\cdot\mathbf{w} = g & \text{in} \quad \Omega_D, \end{cases} \tag{3.2}$$

along with boundary condition (1.3) and interface condition (1.4).

3.1. Legendre pseudospectral least squares method. We consider the Legendre pseudospectral least squares method for the first order system of equations (3.2). Let

$$\begin{aligned} \mathbf{V}_S &= \{\mathbf{v} \in H(\mathrm{div}, \Omega_S) \,:\, \mathbf{v}\cdot\mathbf{n} \in L^2(\Gamma)\}, \\ \mathbf{V}_D &= \{\mathbf{v} \in H_{\partial\Omega_D}(\mathrm{div}, \Omega_D) \,:\, \mathbf{v}\cdot\mathbf{n} \in L^2(\Gamma)\}, \\ \mathbf{W} &= [\mathbf{V}_S]^2 \times [H^1_{\partial\Omega_S}(\Omega_S)]^2 \times \mathbf{V}_D \times H^1(\Omega_D), \end{aligned}$$

where

$$H^1_{\partial\Omega_S}(\Omega_S) = \{\mathbf{v} \in H^1(\Omega_S) \,:\, \mathbf{v} = 0 \,\mathrm{on}\, \partial\Omega_S\},$$

$$H_{\partial\Omega_D}(\mathrm{div}, \Omega_D) = \{\mathbf{v} \in H(\mathrm{div}, \Omega_D) \,:\, \mathbf{v}\cdot\mathbf{n} = 0 \,\mathrm{on}\, \partial\Omega_D\}.$$

Define the least squares functional as

$$\begin{aligned} \mathcal{G}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{f}) = {} & \|\nabla\cdot\mathbf{T} + \mathbf{f}\|^2_{\Omega_S} + \|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{\Omega_S} + \|\nabla\cdot\mathbf{u}\|^2_{\Omega_S} \\ & + \|\tfrac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q\|^2_{\Omega_D} + \|\nabla\cdot\mathbf{w} - g\|^2_{\Omega_D} + \|\mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n}\|^2_{\Gamma} \\ & + \|\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n}) + q\|^2_{\Gamma} + \|\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n}) + \mathbf{u}\times\mathbf{n}\|^2_{\Gamma} \end{aligned} \tag{3.3}$$

for $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}$. The first order system least squares variational problem for (3.3) is to minimize the quadratic function $\mathcal{G}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{f})$ over $\mathbf{W}$. That is, find $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}$ such that

$$\mathcal{G}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{f}) = \inf_{(\mathbf{S},\mathbf{v},\mathbf{z},r)\in\mathbf{W}} \mathcal{G}(\mathbf{S}, \mathbf{v}, \mathbf{z}, r; \mathbf{f}). \tag{3.4}$$

The corresponding variational problem is to find $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}$ such that

$$\mathcal{A}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{S}, \mathbf{v}, \mathbf{z}, r) = \mathcal{F}(\mathbf{S}, \mathbf{v}, \mathbf{z}, r) \qquad \forall (\mathbf{S}, \mathbf{v}, \mathbf{z}, r) \in \mathbf{W}, \tag{3.5}$$

where

$$\begin{aligned} &\mathcal{A}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{S}, \mathbf{v}, \mathbf{z}, r) \\ &\quad = (\nabla \cdot \mathbf{T}, \nabla \cdot \mathbf{S})_{\Omega_S} + \big(\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u}),\ \tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{S}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{v})\big)_{\Omega_S} \\ &\quad + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{\Omega_S} + \big(\tfrac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q, \tfrac{1}{\sqrt{K}}\mathbf{z} + \sqrt{K}\nabla r\big)_{\Omega_D} + (\nabla \cdot \mathbf{w}, \nabla \cdot \mathbf{z})_{\Omega_D} \\ &\quad + (\mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} - \mathbf{z} \cdot \mathbf{n})_\Gamma + \big(\mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{n}) + q, \mathbf{n} \cdot (\mathbf{S} \cdot \mathbf{n}) + r\big)_\Gamma \\ &\quad + \big(\beta\mathbf{n} \times (\mathbf{T} \cdot \mathbf{n}) + \mathbf{u} \times \mathbf{n}, \beta\mathbf{n} \times (\mathbf{S} \cdot \mathbf{n}) + \mathbf{v} \times \mathbf{n}\big)_\Gamma \end{aligned} \tag{3.6}$$

and

$$\mathcal{F}(\mathbf{S}, \mathbf{v}, \mathbf{z}, r) = (g,\, \nabla \cdot \mathbf{z})_{\Omega_D} - (\mathbf{f}, \nabla \cdot \mathbf{S})_{\Omega_S}. \tag{3.7}$$

For $U = (\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}$, define

$$|\!|\!| U |\!|\!| = \Big(\|\mathbf{T}\|^2_{\mathrm{div},\Omega_S} + \|\mathbf{T} \cdot \mathbf{n}\|^2_\Gamma + \|\mathbf{u}\|^2_{1,\Omega_S} + \|\mathbf{w}\|^2_{\mathrm{div},\Omega_D} + \|\mathbf{w} \cdot \mathbf{n}\|^2_\Gamma + \|q\|^2_{1,\Omega_D} \Big)^{1/2}.$$

The following theorem, which shows coercivity and continuity of variational problem (3.5), can be found in [19].

THEOREM 3.1. *There are positive constants c and C such that*

$$c|\!|\!| U |\!|\!|^2 \le \mathcal{G}(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; 0) \le C|\!|\!| U |\!|\!|^2 \tag{3.8}$$

hold for all $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}$ *which satisfy* (3.1).

To consider the discrete least squares Legendre method, let

$$\mathbf{V}_{S,N} = \mathbf{V}_S \cap \mathcal{Q}^2_N(\Omega_S), \qquad \mathbf{V}_{D,N} = \mathbf{V}_D \cap \mathcal{Q}^2_N(\Omega_D),$$

and

$$\mathbf{W}_N = [\mathbf{V}_{S,N}]^2 \times [H^1_{\partial\Omega_S}(\Omega_S) \cap \mathcal{Q}_N(\Omega_S)]^2 \times \mathbf{V}_{D,N} \times [H^1(\Omega_D) \cap \mathcal{Q}_N(\Omega_D)].$$

We now define the discrete least squares functional using the discrete Legendre spectral norm as

$$\begin{aligned} \mathcal{G}_N(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{f}) &= \|\nabla \cdot \mathbf{T} + \mathbf{f}\|^2_{\Omega_S,N} + \|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{\Omega_S,N} + \|\nabla \cdot \mathbf{u}\|^2_{\Omega_S,N} \\ &+ \|\tfrac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q\|^2_{\Omega_D,N} + \|\nabla \cdot \mathbf{w}\|^2_{\Omega_D,N} + \|\mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n}\|^2_{\Gamma,N} \\ &+ \|\mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{n}) + q\|^2_{\Gamma,N} + \|\beta\mathbf{n} \times (\mathbf{T} \cdot \mathbf{n}) + \mathbf{u} \times \mathbf{n}\|^2_{\Gamma,N} \end{aligned} \tag{3.9}$$

for $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}_N$. The discrete least squares problem associated with (3.9) is to minimize the quadratic functional $\mathcal{G}_N(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; \mathbf{f})$ over $\mathbf{W}_N$, and the corresponding variational problem is to find $(\mathbf{T}_N, \mathbf{u}_N, \mathbf{w}_N, q_N) \in \mathbf{W}_N$ such that

$$\mathcal{A}_N(\mathbf{T}_N, \mathbf{u}_N, \mathbf{w}_N, q_N; \mathbf{S}, \mathbf{v}, \mathbf{z}, r) = \mathcal{F}_N(\mathbf{S}, \mathbf{v}, \mathbf{z}, r) \qquad \forall (\mathbf{S}, \mathbf{v}, \mathbf{z}, r) \in \mathbf{W}_N, \tag{3.10}$$

where

$$\begin{aligned} &\mathcal{A}_N(\mathbf{T}_N, \mathbf{u}_N, \mathbf{w}_N, q_N; \mathbf{S}, \mathbf{v}, \mathbf{z}, r) = \langle \nabla \cdot \mathbf{T}_N, \nabla \cdot \mathbf{S} \rangle_{\Omega_S,N} \\ &\quad + \big\langle \tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}}_N - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u}_N),\ \tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{S}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{v}) \big\rangle_{\Omega_S,N} + \langle \nabla \cdot \mathbf{u}_N, \nabla \cdot \mathbf{v} \rangle_{\Omega_S,N} \\ &\quad + \big\langle \tfrac{1}{\sqrt{K}}\mathbf{w}_N + \sqrt{K}\nabla q_N, \tfrac{1}{\sqrt{K}}\mathbf{z} + \sqrt{K}\nabla r \big\rangle_{\Omega_D,N} + \langle \nabla \cdot \mathbf{w}_N, \nabla \cdot \mathbf{z} \rangle_{\Omega_D,N} \\ &\quad + \langle \mathbf{u}_N \cdot \mathbf{n} - \mathbf{w}_N \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} - \mathbf{z} \cdot \mathbf{n} \rangle_{\Gamma,N} + \langle \mathbf{n} \cdot (\mathbf{T}_N \cdot \mathbf{n}) + q_N, \mathbf{n} \cdot (\mathbf{S} \cdot \mathbf{n}) + r \rangle_{\Gamma,N} \\ &\quad + \langle \beta\mathbf{n} \times (\mathbf{T}_N \cdot \mathbf{n}) + \mathbf{u}_N \times \mathbf{n}, \beta\mathbf{n} \times (\mathbf{S} \cdot \mathbf{n}) + \mathbf{v} \times \mathbf{n} \rangle_{\Gamma,N} \end{aligned} \tag{3.11}$$

and

$$\mathcal{F}_N(\mathbf{S},\mathbf{v},\mathbf{z},r) = \langle g, \nabla\cdot\mathbf{z}\rangle_{\Omega_D,N} - \langle \mathbf{f}, \nabla\cdot\mathbf{S}\rangle_{\Omega_S,N}. \tag{3.12}$$

Now, we show continuity and coercivity of the $\mathcal{G}_N(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0})$.

THEOREM 3.2. *There are positive constants c and C such that*

$$c|\!|\!|U|\!|\!|^2 \le \mathcal{G}_N(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le C|\!|\!|U|\!|\!|^2 \tag{3.13}$$

hold for all $(\mathbf{T},\mathbf{u},\mathbf{w},q)\in\mathbf{W}_N$ which satisfy (3.1).

Proof. We have $\nabla\cdot\mathbf{T}\in[\mathcal{Q}_{N-1}]^2$, $\nabla\cdot\mathbf{u}$, $\nabla\cdot\mathbf{w}\in\mathcal{Q}_{N-1}$, $\frac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\in[\mathcal{Q}_N]^4$, $\frac{1}{\sqrt{K}}\mathbf{w}+\sqrt{K}\nabla q\in[\mathcal{Q}_N]^2$, and $\mathbf{u}\cdot\mathbf{n}-\mathbf{w}\cdot\mathbf{n}$, $\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n})+q$, $\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n})+\mathbf{u}\times\mathbf{n}\in\mathcal{P}_N$; by using (2.13) and (2.9), there are constants c and C such that

$$c\,\mathcal{G}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le \mathcal{G}_N(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le C\,\mathcal{G}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}).$$

Hence the bounds (3.13) are consequences of Theorem 3.1. ∎

Now, we show the spectral convergence of the Legendre pseudospectral least squares method.

THEOREM 3.3. *Suppose that the solution $(\mathbf{T},\mathbf{u},\mathbf{w},q)$ of* (3.5) *is in $\mathbf{W}\cap\big[H^s(\Omega_S)^4\cap H^s(\Omega_S)^2\cap H^s(\Omega_D)^2\cap H^s(\Omega_D)\big]$ for some $s\ge 1$, and $g\in H^k(\Omega_D)$, $\mathbf{f}\in[H^\ell(\Omega_S)]^2$, for ℓ, $k\ge 2$. Let $(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N)\in\mathbf{W}_N$ be the approximate solution of* (3.10). *Then there exists a constant C such that*

$$\begin{aligned}
&\|\mathbf{T}-\mathbf{T}_N\|^2_{\mathrm{div},\Omega_S} + \|\mathbf{u}-\mathbf{u}_N\|^2_{1,\Omega_S} + \|\mathbf{w}-\mathbf{w}_N\|^2_{\mathrm{div},\Omega_D} + \|q-q_N\|^2_{1,\Omega_D}\\
&\quad\le C\Big(N^{1-s}\big[\|\mathbf{T}\|_{s,\Omega_S} + \|\mathbf{u}\|_{s,\Omega_S} + \|\mathbf{w}\|_{s,\Omega_D} + \|q\|_{s,\Omega_D}\big]\\
&\qquad\qquad + N^{-\ell}\|\mathbf{f}\|_{\ell,\Omega_S} + N^{-k}\|g\|_{k,\Omega_D}\Big).
\end{aligned} \tag{3.14}$$

Proof. Using the first Strang lemma [3], we have

$$\begin{aligned}|\!|\!|U-U_N|\!|\!| \le \inf_{V\in\mathbf{W}_N}\Big[|\!|\!|U-V|\!|\!| + \sup_{W\in\mathbf{W}_N}\tfrac{|\mathcal{A}(V,W)-\mathcal{A}_N(V,W)|}{|\!|\!|W|\!|\!|}\Big]\\ + \sup_{W\in\mathbf{W}_N}\tfrac{|\mathcal{F}(W)-\mathcal{F}_N(W)|}{|\!|\!|W|\!|\!|},\end{aligned}$$

where $U=(\mathbf{T},\mathbf{u},\mathbf{w},q)$, $U_N=(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N)$, and so on. Using inequality (2.15) yields

$$\begin{aligned}|\mathbf{F}(W)-\mathcal{F}_N(W)| &= \big|(g,\nabla\cdot\mathbf{z})_{\Omega_D} - (\mathbf{f},\nabla\cdot\mathbf{S})_{\Omega_S} - \langle g,\nabla\cdot\mathbf{z}\rangle_{\Omega_D,N} + \langle\mathbf{f},\nabla\cdot\mathbf{S}\rangle_{\Omega_S,N}\big|\\
&\le CN^{-\ell}\|\mathbf{f}\|_{\ell,\Omega_S}\|\nabla\cdot S\|_{\Omega_S} + CN^{-k}\|g\|_{k,\Omega_D}\|\nabla\cdot\mathbf{z}\|_{\Omega_D}\\
&\le C(N^{-\ell}\|\mathbf{f}\|_{\ell,\Omega_S} + N^{-k}\|g\|_{k,\Omega_D})|\!|\!|W|\!|\!|.\end{aligned}$$

If we take $V\in\mathbf{W}_{N-1}$, then by using (2.8) for the interior inner product and (2.12) for the interface inner product, we have

$$\big|\mathcal{A}(V,W)-\mathcal{A}_N(V,W)\big| = 0,\qquad W\in\mathbf{W}_N. \tag{3.15}$$

Hence we arrive at

$$|\!|\!|U-U_N|\!|\!| \le \inf_{V\in\mathbf{W}_N}|\!|\!|U-V|\!|\!| + C(N^{-\ell}\|\mathbf{f}\|_{\ell,\Omega_S} + N^{-k}\|g\|_{k,\Omega_D}).$$

Applying (2.7) and (2.14) to the above inequality results in (3.14) and the proof is complete. ∎

3.2. Chebyshev pseudospectral least squares method. We now consider the Chebyshev pseudospectral least squares method for first order system of equations (3.2) of the Stokes–Darcy equations. To do so, let

$$\mathbf{V}_{w,S} = \{\mathbf{v} \in H_w(\mathrm{div}, \Omega_S) \,:\, \mathbf{v}\cdot\mathbf{n} \in L^2_w(\Gamma)\},$$
$$\mathbf{V}_{w,D} = \{\mathbf{v} \in H_{w,\partial\Omega_D}(\mathrm{div}, \Omega_D) \,:\, \mathbf{v}\cdot\mathbf{n} \in L^2_w(\Gamma)\},$$
$$\mathbf{W}_w = [\mathbf{V}_{w,S}]^2 \times [H^1_{w,\partial\Omega_S}(\Omega_S)]^2 \times \mathbf{V}_{w,D} \times H^1_w(\Omega_D),$$

where

$$H^1_{w,\partial\Omega_S}(\Omega_S) = \{\mathbf{v} \in H^1_w(\Omega_S) \,:\, \mathbf{v} = 0 \,\mathrm{on}\, \partial\Omega_S\},$$

$$H_{w,\partial\Omega_D}(\mathrm{div}, \Omega_D) = \{\mathbf{v} \in H_w(\mathrm{div}, \Omega_D) \,:\, \mathbf{v}\cdot\mathbf{n} = 0 \,\mathrm{on}\, \partial\Omega_D\}.$$

Define the least squares functional as

$$\begin{aligned}\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{f},\,g) = {}& \|\nabla\cdot\mathbf{T}+\mathbf{f}\|^2_{w,\Omega_S} + \|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} \\ & + \|\nabla\cdot\mathbf{u}\|^2_{w,\Omega_S} + \|\tfrac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q\|^2_{w,\Omega_D} \\ & + \|\nabla\cdot\mathbf{w} - g\|^2_{w,\Omega_D} + \|\mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma} \\ & + \|\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n}) + q\|^2_{w,\Gamma} + \|\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n}) + \mathbf{u}\times\mathbf{n}\|^2_{w,\Gamma}\end{aligned} \tag{3.16}$$

for $(\mathbf{T},\mathbf{u},\mathbf{w},q) \in \mathbf{W}_w$. The first order system least squares variational problem for (3.16) is to minimize the quadratic function $\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{f})$ over $\mathbf{W}_w$; that is, find $(\mathbf{T},\mathbf{u},\mathbf{w},q) \in \mathbf{W}_w$ such that

$$\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{f},\,g) = \inf_{(\mathbf{S},\mathbf{v},\mathbf{z},r)\in\mathbf{W}_w} \mathcal{G}_w(\mathbf{S},\mathbf{v},\mathbf{z},r;\mathbf{f},\,g). \tag{3.17}$$

The corresponding variational problem is to find $(\mathbf{T},\mathbf{u},\mathbf{w},q) \in \mathbf{W}_w$ such that

$$\mathcal{A}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{S},\mathbf{v},\mathbf{z},r) = \mathcal{F}_w(\mathbf{S},\mathbf{v},\mathbf{z},r) \qquad \forall(\mathbf{S},\mathbf{v},\mathbf{z},r) \in \mathbf{W}_w, \tag{3.18}$$

where

(3.19)
$$\begin{aligned}&\mathcal{A}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{S},\mathbf{v},\mathbf{z},r)\\ &\quad= (\nabla\cdot\mathbf{T},\nabla\cdot\mathbf{S})_{w,\Omega_S} + \big(\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u}),\, \tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{S}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{v})\big)_{w,\Omega_S}\\ &\quad+(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{w,\Omega_S} + \big(\tfrac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q,\, \tfrac{1}{\sqrt{K}}\mathbf{z} + \sqrt{K}\nabla r\big)_{w,\Omega_D} + (\nabla\cdot\mathbf{w},\nabla\cdot\mathbf{z})_{w,\Omega_D}\\ &\quad+(\mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n},\mathbf{v}\cdot\mathbf{n} - \mathbf{z}\cdot\mathbf{n})_{w,\Gamma} + \big(\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n}) + q, \mathbf{n}\cdot(\mathbf{S}\cdot\mathbf{n}) + r\big)_{w,\Gamma}\\ &\quad+\big(\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n}) + \mathbf{u}\times\mathbf{n}, \beta\mathbf{n}\times(\mathbf{S}\cdot\mathbf{n}) + \mathbf{v}\times\mathbf{n}\big)_{w,\Gamma}\end{aligned}$$

and

$$\mathcal{F}_w(\mathbf{S},\mathbf{v},\mathbf{z},r) = (g,\,\nabla\cdot\mathbf{z})_{w,\Omega_D} - (\mathbf{f},\nabla\cdot\mathbf{S})_{w,\Omega_S}. \tag{3.20}$$

For $U = (\mathbf{T},\mathbf{u},\mathbf{w},q) \in \mathbf{W}_w$, we define

$$|\!|\!|U|\!|\!|_w = \left(\|\mathbf{T}\|^2_{w,\mathrm{div},\Omega_S} + \|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma} + \|\mathbf{u}\|^2_{1,w,\Omega_S} + \|\mathbf{w}\|^2_{w,\mathrm{div},\Omega_D} + \|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma} + \|q\|^2_{1,w,\Omega_D}\right)^{1/2}.$$

We assume that the a priori estimate (2.18) holds for the Stokes equation in Ω_S; that is,

$$\|\nu\mathbf{u}\|_{1,\omega,\Omega_S} + \|p\|_{\omega,\Omega_S} \le C\big(\|-\nu\Delta\mathbf{u} + \nabla p\|_{-1,\omega,\Omega_S} + \|\nu\nabla\cdot\mathbf{u}\|_{\omega,\Omega_S}\big). \tag{3.21}$$

We note that the elimination of $\mathbf{w}$ in (1.2) gives the Poisson equation $-K\Delta q = g$ with the Neumann boundary condition $\frac{\partial q}{\partial \mathbf{n}} = 0$ on $\partial\Omega_D$, and we assume that a priori estimate (2.19) holds; that is,

$$\|q\|_{1,w,\Omega_D} \le C\| - K\Delta q\|_{-1,w,\Omega_D}. \tag{3.22}$$

The coercivity and continuity of variational problem (3.18) is given in the following theorem.

THEOREM 3.4. *Assume that inequalities* (3.21) *and* (3.22) *hold. Then there exist positive constants c and C such that*

$$c|\!|\!|U|\!|\!|_w^2 \le \mathcal{G}_w(\mathbf{T}, \mathbf{u}, \mathbf{w}, q; 0) \le C|\!|\!|U|\!|\!|_w^2 \tag{3.23}$$

hold for all $(\mathbf{T}, \mathbf{u}, \mathbf{w}, q) \in \mathbf{W}_w$ *which satisfy* (3.1).

Proof. By definitions, we have

$$\|\mathcal{E}(\mathbf{u})\|_w^2 \le \|\nabla \mathbf{u}\|_w^2. \tag{3.24}$$

The upper bound of (3.23) is a consequence of using (3.24) and triangle inequality. To proof the lower bound of (3.23), we have

$$\begin{aligned}
\|\mathbf{T}\|^2_{w,\Omega_S} &= (\mathbf{T},\mathbf{T})_{w,\Omega_S} = (\hat{\mathbf{T}},\mathbf{T})_{w,\Omega_S} + (\tfrac{1}{2}\mathrm{tr}\mathbf{T},\mathbf{T})_{w,\Omega_S} \\
&= \sqrt{2\nu}(\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u}),\mathbf{T})_{w,\Omega_S} + 2\nu(\mathcal{E}(\mathbf{u}),\mathbf{T})_{w,\Omega_S} + (\tfrac{1}{2}\mathrm{tr}\mathbf{T},\mathbf{T})_{w,\Omega_S} \\
&\le \sqrt{2\nu}\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|_{w,\Omega_S}\|\mathbf{T}\|_{w,\Omega_S} + 2\nu\|\mathcal{E}(\mathbf{u})\|_{w,\Omega_S}\|\mathbf{T}\|_{w,\Omega_S} \\
&\quad + \|\tfrac{1}{2}\mathrm{tr}T\|_{w,\Omega_S}\|\mathbf{T}\|_{w,\Omega_S} \\
&= \big(\sqrt{2\nu}\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|_{w,\Omega_S} + 2\nu\|\mathcal{E}(\mathbf{u})\|_{w,\Omega_S} + \|\tfrac{1}{2}\mathrm{tr}T\|_{w,\Omega_S}\big)\|\mathbf{T}\|_{w,\Omega_S}.
\end{aligned} \tag{3.25}$$

Using ϵ inequality with $\epsilon = 1$ gives

$$\begin{aligned}
\|\mathbf{T}\|^2_{w,\Omega_S} &\le \big(\sqrt{2\nu}\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|_{w,\Omega_S} + 2\nu\|\mathcal{E}(\mathbf{u})\|_{w,\Omega_S} + \tfrac{1}{2}\|\mathrm{tr}T\|_{w,\Omega_S}\big)^2 \\
&= 6\nu\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 12\nu^2\|\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 3\|\tfrac{1}{2}\mathrm{tr}T\|^2_{w,\Omega_S}.
\end{aligned} \tag{3.26}$$

By (3.24), we have

$$\begin{aligned}
\|\mathbf{T}\|^2_{w,\Omega_S} &\le 6\nu\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 12\nu^2\|\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 3\|\tfrac{1}{2}\mathrm{tr}T\|^2_{w,\Omega_S} \\
&\le 6\nu\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 12(\nu^2\|\nabla\mathbf{u}\|^2_{w,\Omega_S} + \|\tfrac{1}{2}\mathrm{tr}T\|^2_{w,\Omega_S}) \\
&\le 6\nu\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}} - \sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S} + 12(\nu^2\|\mathbf{u}\|^2_{1,w,\Omega_S} + \|\tfrac{1}{2}\mathrm{tr}T\|^2_{w,\Omega_S}) \\
&\le C\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0).
\end{aligned} \tag{3.27}$$

Obviously,

$$\|\nabla\cdot\mathbf{T}\|_w^2 \le \mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0). \tag{3.28}$$

By using (3.22), and Lemma 2.1, we have

$$\begin{aligned}
\|q\|^2_{1,w,\Omega_D} &\le C\| - K\Delta q\|^2_{-1,w,\Omega_D} \\
&\le C\| - \nabla\cdot(K\nabla q + \mathbf{w})\|^2_{-1,w,\Omega_D} + C\|\nabla\cdot\mathbf{w}\|^2_{-1,w,\Omega_D} \\
&\le C\|K\nabla q + \mathbf{w}\|^2_{w,\Omega_D} + C\|\nabla\cdot\mathbf{w}\|^2_{w,\Omega_D} \\
&\le C\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0).
\end{aligned}$$

The triangle inequality gives

$$\|\mathbf{w}\|^2_{w,\Omega_D} \le C\|\mathbf{w} + K\nabla q\|^2_{w,\Omega_D} + C\|q\|^2_{1,w,\Omega_D} \le C\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0); \tag{3.29}$$

then

$$\|\mathbf{w}\|^2_{\mathrm{div},w,\Omega_D} \le C\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0). \tag{3.30}$$

With the similar idea of [19], by using trace inequalities (2.16) and (2.17), for $\alpha \in (0,\frac{1}{2}]$, we have

$$\begin{aligned}
\|\mathbf{u}\cdot\mathbf{n}-\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma} &+\|\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n})+q\|^2_{w,\Gamma}+\|\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n})+\mathbf{u}\times\mathbf{n}\|^2_{w,\Gamma}\\
&\ge \alpha(\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n})\|^2_{w,\Gamma}+\|\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n})\|^2_{w,\Gamma})\\
&\quad -2\alpha(\|\mathbf{u}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|q\|^2_{w,\Gamma}+\|\mathbf{u}\times\mathbf{n}\|^2_{w,\Gamma})\\
&\ge \alpha(\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\min\{1,\beta^2\}\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma})\\
&\quad -2\alpha(\|\mathbf{u}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|q\|^2_{w,\Gamma}+\|\mathbf{u}\times\mathbf{n}\|^2_{w,\Gamma})\\
&\ge \alpha(\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\min\{1,\beta^2\}\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma})\\
&\quad -2\alpha(\|\mathbf{u}\|^2_{w,\Gamma}+\|q\|^2_{w,\Gamma})\\
&\ge \alpha(\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\min\{1,\beta^2\}\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma})\\
&\quad -2\alpha C_T(\|\mathbf{u}\|^2_{1,w,\Omega_S}+\|q\|^2_{w,\Omega_D}+\|\nabla q\|^2_{w,\Omega_D}).
\end{aligned} \tag{3.31}$$

Therefore, there exists a constant C such that

$$\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma} \le C\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;0). \tag{3.32}$$

The lower bound of (3.23) follows from inequalities (3.24)–(3.32). ◻

To investigate the discrete Chebyshev least squares method, let

$$\mathbf{V}_{w,S,N}=\mathbf{V}_{w,S}\cap\mathcal{Q}^2_N(\Omega_S), \qquad \mathbf{V}_{w,D,N}=\mathbf{V}_{w,D}\cap\mathcal{Q}^2_N(\Omega_D),$$

and

$$\mathbf{W}_{w,N}=[\mathbf{V}_{w,S,N}]^2\times[H^1_{w,\partial\Omega_S}(\Omega_S)\cap\mathcal{Q}_N(\Omega_S)]^2\times\mathbf{V}_{w,D,N}\times[H^1_w(\Omega_D)\cap\mathcal{Q}_N(\Omega_D)].$$

We now define the discrete least squares functional using discrete Chebyshev spectral norm as

$$\begin{aligned}
&\mathcal{G}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{f})\\
&\quad =\|\nabla\cdot\mathbf{T}+\mathbf{f}\|^2_{w,\Omega_S,N}+\|\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}}-\sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\|^2_{w,\Omega_S,N}+\|\nabla\cdot\mathbf{u}\|^2_{w,\Omega_S,N}\\
&\quad +\|\tfrac{1}{\sqrt{K}}\mathbf{w}+\sqrt{K}\nabla q\|^2_{w,\Omega_D,N}+\|\nabla\cdot\mathbf{w}\|^2_{w,\Omega_D,N}+\|\mathbf{u}\cdot\mathbf{n}-\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma,N}\\
&\quad +\|\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n})+q\|^2_{w,\Gamma,N}+\|\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n})+\mathbf{u}\times\mathbf{n}\|^2_{w,\Gamma,N}
\end{aligned} \tag{3.33}$$

for $(\mathbf{T},\mathbf{u},\mathbf{w},q)\in\mathbf{W}_{w,N}$. The discrete least squares problem of (3.33) is to minimize the quadratic functional $\mathcal{G}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{f})$ over $\mathbf{W}_{w,N}$. Hence the corresponding variational problem is to find $(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N)\in\mathbf{W}_{w,N}$ such that

$$\mathcal{A}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{S},\mathbf{v},\mathbf{z},r)=\mathcal{F}_{w,N}(\mathbf{S},\mathbf{v},\mathbf{z},r) \qquad \forall(\mathbf{S},\mathbf{v},\mathbf{z},r)\in\mathbf{W}_{w,N}, \tag{3.34}$$

where

(3.35)

$$\begin{aligned}
&\mathcal{A}_{w,N}(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N;\mathbf{S},\mathbf{v},\mathbf{z},r)=\langle\nabla\cdot\mathbf{T}_N,\nabla\cdot\mathbf{S}\rangle_{w,\Omega_S,N}\\
&\quad +\big\langle\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{T}}_N-\sqrt{2\nu}\,\mathcal{E}(\mathbf{u}_N),\,\tfrac{1}{\sqrt{2\nu}}\hat{\mathbf{S}}-\sqrt{2\nu}\,\mathcal{E}(\mathbf{v})\big\rangle_{w,\Omega_S,N}+\langle\nabla\cdot\mathbf{u}_N,\nabla\cdot\mathbf{v}\rangle_{w,\Omega_S,N}\\
&\quad +\big\langle\tfrac{1}{\sqrt{K}}\mathbf{w}_N+\sqrt{K}\nabla q_N,\,\tfrac{1}{\sqrt{K}}\mathbf{z}+\sqrt{K}\nabla r\big\rangle_{w,\Omega_D,N}+\langle\nabla\cdot\mathbf{w}_N,\nabla\cdot\mathbf{z}\rangle_{w,\Omega_D,N}\\
&\quad +\langle\mathbf{u}_N\cdot\mathbf{n}-\mathbf{w}_N\cdot\mathbf{n},\mathbf{v}\cdot\mathbf{n}-\mathbf{z}\cdot\mathbf{n}\rangle_{w,\Gamma,N}+\langle\mathbf{n}\cdot(\mathbf{T}_N\cdot\mathbf{n})+q_N,\mathbf{n}\cdot(\mathbf{S}\cdot\mathbf{n})+r\rangle_{w,\Gamma,N}\\
&\quad +\langle\beta\mathbf{n}\times(\mathbf{T}_N\cdot\mathbf{n})+\mathbf{u}_N\times\mathbf{n},\beta\mathbf{n}\times(\mathbf{S}\cdot\mathbf{n})+\mathbf{v}\times\mathbf{n}\rangle_{w,\Gamma,N}
\end{aligned}$$

and

$$\mathcal{F}_{w,N}(\mathbf{S},\mathbf{v},\mathbf{z},r) = \langle g, \nabla\cdot\mathbf{z}\rangle_{w,\Omega_D,N} - \langle \mathbf{f}, \nabla\cdot\mathbf{S}\rangle_{w,\Omega_S,N}. \tag{3.36}$$

Now, we show continuity and coercivity of $\mathcal{G}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0})$.

THEOREM 3.5. *Suppose that* (3.21) *and* (3.22) *hold. There are positive constants c and C such that*

$$c|\!|\!|U|\!|\!|_w^2 \le \mathcal{G}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le C|\!|\!|U|\!|\!|_w^2 \tag{3.37}$$

hold for all $(\mathbf{T},\mathbf{u},\mathbf{w},q)\in\mathbf{W}_{w,N}$ *which satisfy* (3.1).

Proof. We have $\nabla\cdot\mathbf{T}\in[\mathcal{Q}_{N-1}]^2$, $\nabla\cdot\mathbf{u}$, $\nabla\cdot\mathbf{w}\in\mathcal{Q}_{N-1}$, $\frac{1}{\sqrt{2\nu}}\hat{\mathbf{T}}-\sqrt{2\nu}\,\mathcal{E}(\mathbf{u})\in[\mathcal{Q}_N]^4$, $\frac{1}{\sqrt{K}}\mathbf{w}+\sqrt{K}\nabla q\in[\mathcal{Q}_N]^2$, and $\mathbf{u}\cdot\mathbf{n}-\mathbf{w}\cdot\mathbf{n}$, $\mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n})+q$, $\beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n})+\mathbf{u}\times\mathbf{n}\in\mathcal{P}_N$; by using (2.13) and (2.9), there are constants c and C such that

$$c\,\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le \mathcal{G}_{w,N}(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}) \le C\,\mathcal{G}_w(\mathbf{T},\mathbf{u},\mathbf{w},q;\mathbf{0}).$$

Hence the bounds (3.37) are a consequence of Theorem 3.4. ∎

The spectral convergence of the Chebyshev pseudospectral least squares method is investigated in the following theorem.

THEOREM 3.6. *Assume* (3.21) *and* (3.22) *hold. Suppose that the solution* $(\mathbf{T},\mathbf{u},\mathbf{w},q)$ *of* (3.18) *is in* $\mathbf{W}_w\cap\left[H_w^s(\Omega_S)^4\cap H_w^s(\Omega_S)^2\cap H_w^s(\Omega_D)^2\cap H_w^s(\Omega_D)\right]$ *for* $s\ge 1$, *and* $g\in H_w^k(\Omega_D)$, $\mathbf{f}\in[H_w^\ell(\Omega_S)]^2$, *for* $k,\ell\ge 2$. *Let* $(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N)\in\mathbf{W}_{w,N}$ *be the approximate solution of* (3.34). *Then there is a constant C such that*

$$\begin{aligned}
&\|\mathbf{T}-\mathbf{T}_N\|_{w,\mathrm{div},\Omega_S}^2 + \|\mathbf{u}-\mathbf{u}_N\|_{1,w,\Omega_S}^2 + \|\mathbf{w}-\mathbf{w}_N\|_{w,\mathrm{div},\Omega_D}^2 + \|q-q_N\|_{1,w,\Omega_D}^2\\
&\quad\le C\Big(N^{1-s}\big[\|\mathbf{T}\|_{s,w,\Omega_S}+\|\mathbf{u}\|_{s,w,\Omega_S}+\|\mathbf{w}\|_{s,w,\Omega_D}+\|q\|_{s,w,\Omega_D}\big]\\
&\qquad+N^{-\ell}\|\mathbf{f}\|_{\ell,w,\Omega_S}+N^{-k}\|g\|_{k,w,\Omega_D}\Big).
\end{aligned} \tag{3.38}$$

Proof. Using the first Strang lemma [3], we have

$$\begin{aligned}|\!|\!|U-U_N|\!|\!|_w \le\ &\inf_{V\in\mathbf{W}_{w,N}}\left[|\!|\!|U-V|\!|\!|_w + \sup_{W\in\mathbf{W}_N}\frac{|\mathcal{A}_w(V,W)-\mathcal{A}_{w,N}(V,W)|}{|\!|\!|W|\!|\!|_w}\right]\\
&+\sup_{W\in\mathbf{W}_N}\frac{|\mathcal{F}_w(W)-\mathcal{F}_{w,N}(W)|}{|\!|\!|W|\!|\!|_w},\end{aligned} \tag{3.39}$$

where $U=(\mathbf{T},\mathbf{u},\mathbf{w},q)$, $U_N=(\mathbf{T}_N,\mathbf{u}_N,\mathbf{w}_N,q_N)$, and so on. Using inequality (2.15) yields

$$\begin{aligned}
&\big|\mathcal{F}_w(W)-\mathcal{F}_{w,N}(W)\big|\\
&\quad=\big|(g,\nabla\cdot\mathbf{z})_{w,\Omega_D}-(\mathbf{f},\nabla\cdot\mathbf{S})_{w,\Omega_S}-\langle g,\nabla\cdot\mathbf{z}\rangle_{w,\Omega_D,N}+\langle\mathbf{f},\nabla\cdot\mathbf{S}\rangle_{w,\Omega_S,N}\big|\\
&\quad\le CN^{-\ell}\|\mathbf{f}\|_{\ell,w,\Omega_S}\|\nabla\cdot S\|_{w,\Omega_S}+CN^{-k}\|g\|_{k,w,\Omega_D}\|\nabla\cdot\mathbf{z}\|_{w,\Omega_D}\\
&\quad\le C(N^{-\ell}\|\mathbf{f}\|_{\ell,w,\Omega_S}+N^{-k}\|g\|_{k,w,\Omega_D})|\!|\!|W|\!|\!|_w.
\end{aligned} \tag{3.40}$$

If we take $V\in\mathbf{W}_{w,N-1}$, then by using (2.8) for the interior inner product and (2.12) for the interface inner product, we have

$$\big|\mathcal{A}_w(V,W)-\mathcal{A}_{w,N}(V,W)\big|=0,\qquad W\in\mathbf{W}_{w,N}. \tag{3.41}$$

So

$$|\!|\!|U-U_N|\!|\!|_w\le\inf_{V\in\mathbf{W}_{w,N}}|\!|\!|U-V|\!|\!|_w+C(N^{-\ell}\|\mathbf{f}\|_{\ell,w,\Omega_S}+N^{-k}\|g\|_{k,w,\Omega_D}).$$

Application of (2.7) and (2.14) in the above inequality gives the result and the proof is complete. ∎

4. Numerical experiments and implementation. In this section, we present implementation of the pseudospectral least squares method briefly (see [13, 15, 17] for more details) and give some numerical examples to support our theory. Let D_N be the one-dimensional pseudospectral derivative matrix associated to the $N+1$ values of $\{\partial_N v(\xi_j)\}_{j=0}^N$ at LGL or CGL points [5]. The entries of D_N can be computed by differentiating the Lagrange polynomials ϕ_j. First, we reorder the LGL and CGL points from bottom to top and then from left to right such that $\mathbf{x}_{k(N+1)+l} := \mathbf{x}_{kl} = (\xi_k,\, \xi_l)$ for $k, l = 1, \ldots, N$. The pseudospectral derivative matrix in two-dimensional space is defined via the Kronecker tensor product; that is,

$$S_x = D_N \otimes I_N \quad \text{and} \quad S_y = I_N \otimes D_N,$$

where I_N is identity matrix of the same order as D_N. Let $\hat{\mathbf{r}}$ be the vector containing the nodal values of a continuous function r; that is,

$$\hat{\mathbf{r}} = (r(x_0), \ldots, r(x_{(N+1)^2-1}))^T.$$

By using

$$\partial_t v(x_i) = \sum_{j=0}^{(N+1)^2-1} (S_t)_{ij} v(x_j) = (S_t \hat{\mathbf{v}})_i \quad \text{for} \quad t = x, \text{ or } y,$$

we have

$$\langle v, z\rangle_{w,N} = \hat{\mathbf{z}}^T W \hat{\mathbf{v}} \quad \text{and} \quad \langle \partial_{t_1} v, \partial_{t_2} z\rangle_{w,N} = (S_{t_2}\hat{\mathbf{z}})^T W (S_{t_1}\hat{\mathbf{v}}),$$

where t_1 and t_2 are x or y, and $W = \mathrm{diag}\{w_i\}$ is the diagonal weight matrix. Then the problems (3.10) and (3.34) can be easily assembled.

Remark 2. The proposed method can be applied to the Stokes–Darcy problem defined on curved domain with curve interface by using the Gordon–Hall transformation [10, 11]. A brief introduction of the Gordon–Hall transformation is given in Appendix A. The good feature of the Gordon–Hall map and pseudospectral method is that the collocation points always lie on the interface and two neighboring domains Ω_S and Ω_D share the same nodes on the interface, regardless of interface shape. Due to this property, the error discretization in our method does not include mismatch parameter introduced in [28].

In the following examples, we take $\Omega = (0,1) \times (0,2)$ with $\Omega_D = (0,1) \times (0,1)$, $\Omega_S = (0,1) \times (1,2)$, and $\Gamma = (0,1) \times \{1\}$. The results are given for $K = 1$, $\beta = 1$, and $\nu = 1$. The functions $\mathbf{T}$ and $\mathbf{w}$ can be computed by definition of $\mathbf{T}$ and (1.2), respectively. Let $(\mathbf{T}_N, \mathbf{u}_N, \mathbf{w}_N, q_N)$ be the approximate solution of Stokes–Darcy equations by Legendre or Chebyshev least squares method, and let $E_v = v - v_N$ for $v = \mathbf{T}, \mathbf{u}, \mathbf{w}$, or q.

Example 1. Let

$$\begin{cases} \mathbf{u}_1 = -\cos(\frac{\pi}{2}y)\sin(\frac{\pi}{2}x), \\ \mathbf{u}_2 = \sin(\frac{\pi}{2}y)\cos(\frac{\pi}{2}x) - 1 + x, \\ p = \frac{1}{2} - x, \\ q = (\frac{2}{\pi})\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - y(x-1) \end{cases} \tag{4.1}$$

be the exact solution of the Stokes–Darcy equations. Error discretization of Example 1 for Legendre and Chebyshev approximation are given in Figures 2 and 3, which show that the spectral errors decay exponentially with respect to N.

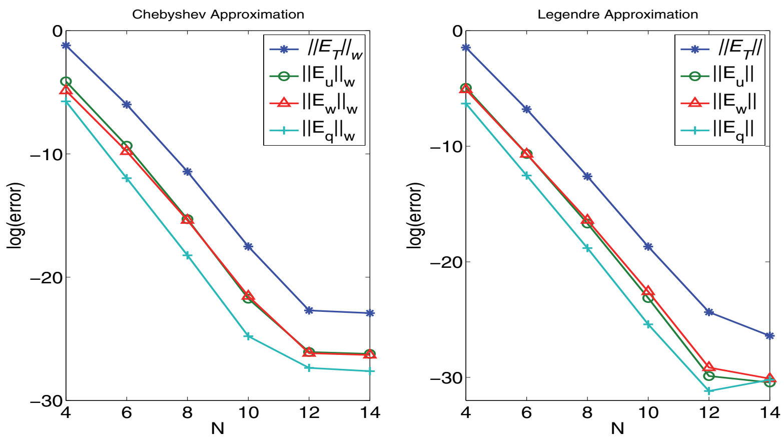

FIG. 2. L^2_w-*error discretization for Example* 1.

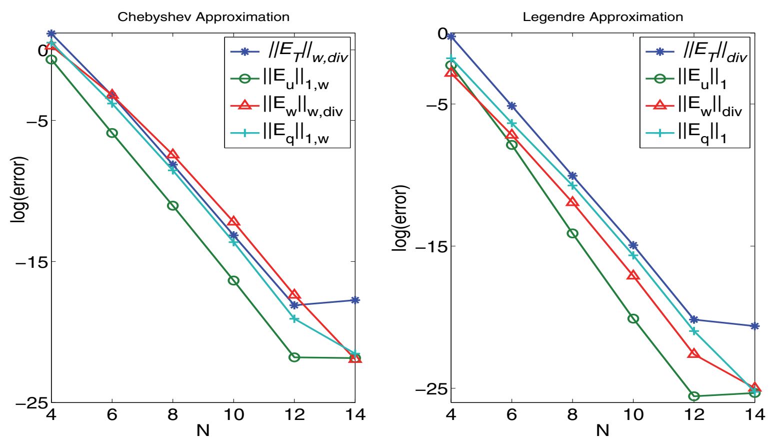

FIG. 3. *Error discretization for Example* 1.

Example 2. We consider the following velocity and pressure:

$$\begin{cases} \mathbf{u}_1 = \exp(x+y) + y, \\ \mathbf{u}_2 = -\exp(x+y) - x, \\ p = \cos(\pi x)\cos(\pi y) + x - \frac{1}{2}, \\ q = \exp(x+y) - \cos(\pi x) + yx \end{cases} \tag{4.2}$$

as exact solutions of the Stokes–Darcy equations. Error discretization of Example 2 for Legendre and Chebyshev approximation are given in Figures 4 and 5, which show spectral convergence of the errors.

Example 3. In this example, let exact solutions of the Stokes–Darcy equations be

$$\begin{cases} \mathbf{u}_1 = -\cos(\pi x)\sin(\pi y), \\ \mathbf{u}_2 = \sin(\pi x)\cos(\pi y), \\ p = \sin(\pi x) - \frac{2}{\pi}, \\ q = y\sin(\pi x). \end{cases} \tag{4.3}$$

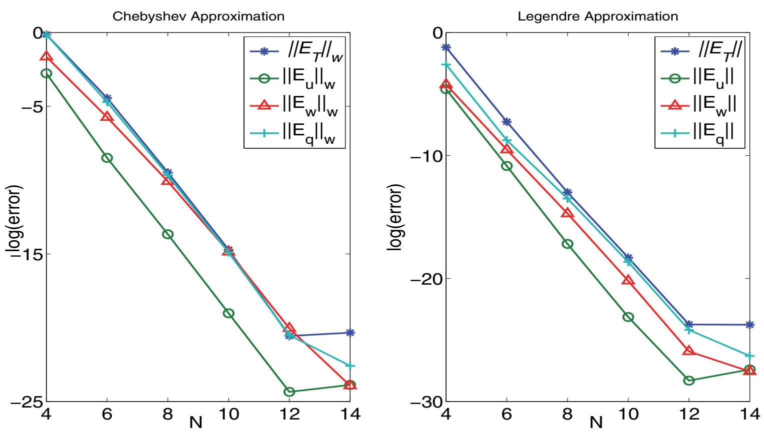

FIG. 4. *L^2_w-error discretization for Example* 2.

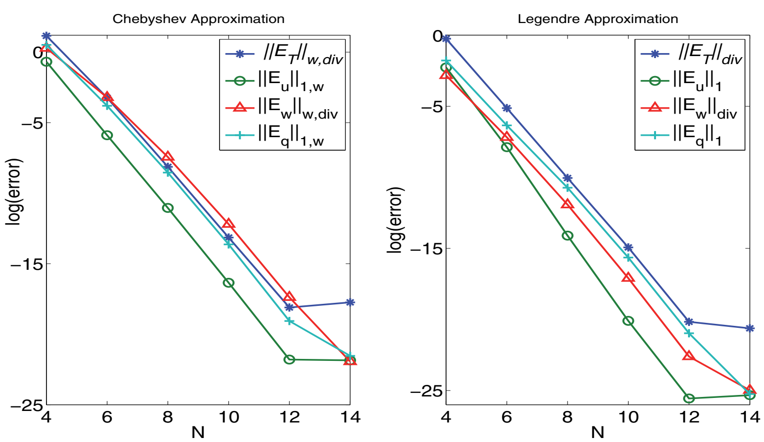

FIG. 5. *Error discretization for Example* 2.

Error discretization of Example 3 for Legendre and Chebyshev approximation are presented in Figures 6 and 7, which show that the spectral errors decay exponentially with respect to N.

5. Conclusion. We have presented and analyzed the first order system least squares method for coupled Stokes–Darcy equations using Legendre and Chebyshev pseudospectral methods, which are known to be very accurate. The least squares functionals are defined as the sum of squared L^2_w-norm of residuals in the first order system of coupled Stokes–Darcy equations. While the boundary conditions are imposed on the solution spaces, the Beaver–Joseph–Saffmann interface conditions are treated differently. The squared L^2_w-norm of residuals of the interface conditions are also added to the least squares functional. Continuous and discrete homogeneous least squares functionals for Legendre and Chebyshev pseudospectral methods are shown to be equivalent to $\|\mathbf{T}\|^2_{w,\text{div},\Omega_S}+\|\mathbf{T}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|\mathbf{u}\|^2_{1,w,\Omega_S}+\|\mathbf{w}\|^2_{w,\text{div},\Omega_D}+\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|q\|^2_{1,w,\Omega_D}$. The spectral convergence for both Legendre and Chebyshev pseudospectral methods is

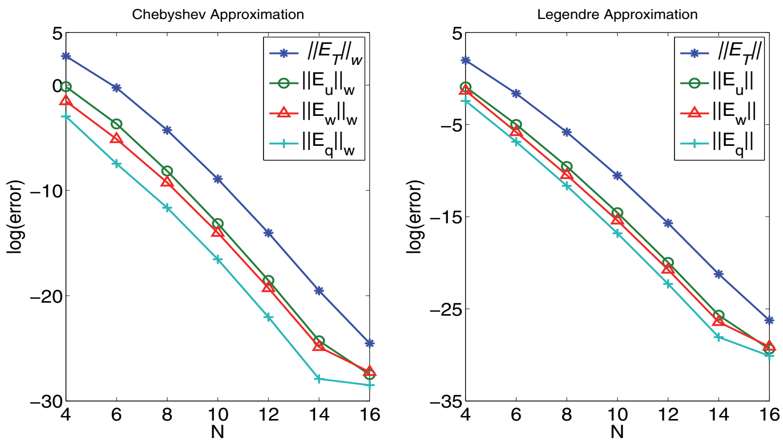

FIG. 6. *L^2_w-error discretization for Example* 3.

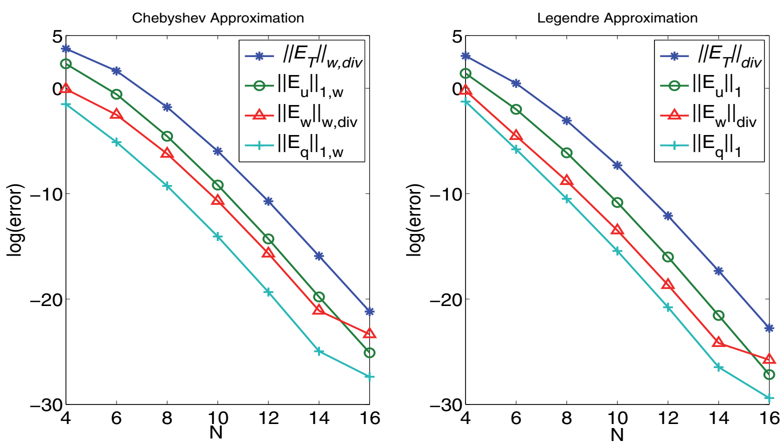

FIG. 7. *Error discretization for Example* 3.

derived. Some numerical experiments are given to demonstrate the analysis. The proposed method can be employed in the Stokes–Darcy problem in the three-dimensional case as well as the Navier–Stokes–Darcy equation in two and three dimensions. However, it involves more technicalities.

Appendix A. Gordon–Hall transformation. In this section, we briefly review the Gordon–Hall transformation. Let $\mathbf{F}$ be a vector-valued function of two independent variables $\hat{x}$ and $\hat{y}$ over a domain $\mathbf{S}$ in the $\hat{x}\hat{y}$-plane whose range is Ω in $\mathbf{R}^2$. We assume that $\mathbf{F}$ is a continuous transformation which maps $\mathbf{S}$ one-to-one onto a simply connected bounded region Ω in $\mathbf{R}^2$ such that $\mathbf{F} : \partial\mathbf{S} \to \partial\Omega$. For convenience, suppose that $\mathbf{S} = [0, h] \times [0, h]$.

We would like to construct a one-to-one function $\mathbf{T} : \mathbf{S} \to \Omega$ which matches $\mathbf{F}$ on the boundaries of $\mathbf{S}$, the so-called boundary interpolant of F, such that

$$\text{(A.1)} \qquad \begin{cases} \mathbf{T}(0, \hat{y}) = \mathbf{F}(0, \hat{y}), & \mathbf{T}(h, \hat{y}) = \mathbf{F}(h, \hat{y}), \qquad 0 \le \hat{y} \le h, \\ \mathbf{T}(\hat{x}, 0) = \mathbf{F}(\hat{x}, 0), & \mathbf{T}(\hat{x}, h) = \mathbf{F}(\hat{x}, h), \qquad 0 \le \hat{x} \le h. \end{cases}$$

By using the similar arguments given in [10, 11], we can choose the following simple transfinite bilinear Lagrange interpolant of $\mathbf{F}$:

$$
\begin{aligned}
\mathbf{T}(\hat{x},\hat{y}) &= \begin{bmatrix} x(\hat{x},\hat{y}) \\ y(\hat{x},\hat{y}) \end{bmatrix} \\
&:= (1-\hat{x}/h)\mathbf{F}(0,\hat{y}) + (\hat{x}/h)\mathbf{F}(h,\hat{y}) \\
&\quad +(1-\hat{y}/h)\mathbf{F}(\hat{x},0) + (\hat{y}/h)\mathbf{F}(\hat{x},h) \\
&\quad -(1-\hat{x}/h)(1-\hat{y}/h)\mathbf{F}(0,0) - (1-\hat{x}/h)(\hat{y}/h)\mathbf{F}(0,h) \\
&\quad -(1-\hat{y}/h)(\hat{x}/h)\mathbf{F}(h,0) - (\hat{y}/h)(\hat{x}/h)\mathbf{F}(h,h).
\end{aligned} \tag{A.2}
$$

It is to be noted that in practice we do not need the function $\mathbf{F}$ in the transfinite interpolation $\mathbf{T}$. The only thing we need is the geometric description of Ω in terms of its boundary which is subdivided into four parametric curve segments. It is necessary that the transfinite interpolation $\mathbf{T}$ has to be one-to-one in the interior of Ω. If $\mathbf{T}$ is one-to-one, it is invertible [27]. By the implicit function theorem [27], if the Jacobian of the transformation $\mathbf{T}$

$$
\left|\frac{\partial(x,y)}{\partial(\hat{x},\hat{y})}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial \hat{x}} & \dfrac{\partial x}{\partial \hat{y}} \\ \dfrac{\partial y}{\partial \hat{x}} & \dfrac{\partial y}{\partial \hat{y}} \end{vmatrix} \tag{A.3}
$$

is nonzero and $\mathbf{T}$ is continuously differentiable in the interior of $\mathbf{S}$, then $\mathbf{T}$ has a local inverse at each point of Ω. Then we have the transformed function $\hat{u}(\hat{x},\hat{y}) := u(\mathbf{T}(\hat{x},\hat{y}))$ defined on the rectangular domain $\mathbf{S}$, and the elliptic equation defined on curved domain Ω is also transformed to an elliptic equation defined on the rectangular domain $\mathbf{S}$. For more details and examples, see [12, 15].

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