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Spectral and structural stability properties of charged particle dynamics in coupled lattices[a]

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It has been realized in recent years that coupled focusing lattices in accelerators and storage rings have significant advantages over conventional uncoupled focusing lattices, especially for high-intensity charged particle beams. A theoretical framework and associated tools for analyzing the spectral and structural stability properties of coupled lattices are formulated in this paper, based on the recently developed generalized Courant-Snyder theory for coupled lattices. It is shown that for periodic coupled lattices that are spectrally and structurally stable, the matrix envelope equation must admit matched solutions. Using the technique of normal form and pre-Iwasawa decomposition, a new method is developed to replace the (inefficient) shooting method for finding matched solutions for the matrix envelope equation. Stability properties of a continuously rotating quadrupole lattice are investigated. The Krein collision process for destabilization of the lattice is demonstrated. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4920961]

I. INTRODUCTION

The transverse focusing lattice is one of the few crucial subsystems in modern accelerators and storage rings. Most contemporary accelerators and storage rings are designed based on an uncoupled linear transverse lattice,[1] where the two degrees of freedom in the transverse directions are decoupled. Well-known analyses of the effects of weak coupling on stability properties have left the incorrect impression that the coupling between the x-dynamics and y-dynamics always results in instabilities or other deleterious effects. It has been realized recently that coupled lattices are not necessarily more unstable than uncoupled lattices. On the contrary, it is believed that coupled lattice can be more advantageous in comparison with conventional uncoupled lattices, especially for high-intensity charged particle beams.[2–11] This is because the parameter space for coupled lattices is much larger than that for uncoupled lattices, and one can explore the larger parameter space for a coupled lattice to optimize the lattice design.

Of course, the most important consideration in lattice design is its stability properties. A thorough study of lattice stability requires one to distinguish two types of stability properties, spectral stability and structural stability.[12–17] The spectral stability of a linear periodic lattice is determined by the eigenvalues of the one-period map M of the lattice. If there exists a vector v such that $M^l v \to \infty$ as $l \to \infty$, then the map is spectrally unstable. Otherwise, it is spectrally stable. This is the most familiar stability property that is often analyzed. The structural stability of the lattice refers to the robustness of the spectral stability properties of the lattice with respect to a structural perturbation, such as imperfections in the magnets, or misalignment of the beam-line. A

lattice is structurally unstable if there exists a spectrally unstable lattice infinitesimally close-by. Otherwise, it is structurally stable.

Unfortunately, our understanding of the stability properties of coupled lattices is far from comprehensive due to the lack of an effective mathematical tool to describe the coupled dynamics. For 1D uncoupled dynamics, the *de facto* standard for parameterizing the focusing lattice is the Courant-Snyder (CS) theory,[1] which is mathematically elegant and directly linked to the physics of the beam dynamics. For coupled lattices, several parameterization schemes have been developed.[18–26] But none of these schemes is as effective for coupled lattices as the CS theory is for uncoupled lattices. Recently, we have developed a generalized Courant-Snyder theory for coupled lattices,[17,27–34] which generalizes every important aspect of the original CS theory to higher dimensions. Especially, the key components of the original CS theory, i.e., the envelope function (or the β function) and the associated envelope equation are generalized into a matrix envelope function and the associated matrix envelope equation.

In the present study, we apply the generalized CS theory to investigate the stability properties of coupled lattices. We prove an important proposition that a necessary condition for a periodic coupled lattice to be spectrally and structurally stable is that the generalized matrix envelope equation admits a matched solution. We also show how to apply the techniques of pre-Iwasawa decomposition[35,36] and normal form to construct a matched solution of the matrix envelope equation by simply solving the initial value problem once. This new method is of great value even for uncoupled lattices. Previously, one used the conventional shooting method to solve the initial value problem many times to search for a matched solution. Using the example of a continuously rotating quadrupole lattice, we illustrate in this paper how

the lattice becomes spectrally unstable through an interesting process called the Krein collision.

The paper is organized as follows. In Sec. II, we describe the spectral and structural stability properties of a generic linear periodic Hamiltonian system and the associated Krein collision. The generalized Courant-Snyder theory for coupled lattices is introduced in Sec. III, and the connection between stability properties and matched lattice functions is discussed in Sec. IV. The formalism developed here is applied to study the stability properties of a continuously rotating quadrupole lattice in Sec. V.

II. SPECTRAL AND STRUCTURAL STABILITY PROPERTIES OF LINEAR HAMILTONIAN SYSTEMS

The dynamics of a charged particle in a coupled or uncoupled periodic focusing system is completely specified by the one-period map. Because of the Hamiltonian nature of the dynamics, the one-period map is symplectic. For the linear focusing lattices considered in the present study, the one-period map is specified by a symplectic matrix M. In this section, we discuss the stability properties of M as a general symplectic matrix. The calculation and stability analysis of M for a specific choice of focusing lattice will be discussed in Secs. III and IV using the generalized Courant-Snyder theory.

Let the dimension of M be $2n \times 2n$, and let λ_l $(l = 1, \ldots, 2n)$ be the eigenvalues of M. It is straightforward to prove that if λ is an eigenvalue of a symplectic matrix, then its inverse $1/\lambda$ and its complex conjugate $\bar{\lambda}$ are also eigenvalues. Then, the eigenvalue distribution can be divided into four categories:

(1) All eigenvalues are distinct and on the unit circle of the complex plane, i.e., $|\lambda_l| = 1$ and $\lambda_l \neq \lambda_m$ for $l \neq m$.
(2) All eigenvalues are on the unit circle. There are repeated eigenvalues. But the geometric multiplicity for all eigenvalues are the same as the algebraic multiplicity, i.e., $Mul_g(\lambda_l) = Mul_a(\lambda_l)$ for all l.
(3) All eigenvalues are on the unit circle. There are repeated eigenvalues with algebraic multiplicity greater than geometric multiplicity, i.e., $Mul_g(\lambda_l) < Mul_a(\lambda_l)$ for some l.
(4) There exits at least one eigenvalue not on the unit circle, i.e., $|\lambda_l| \neq 1$ for some l.

Here, an eigenvalue λ_l of M is a root of the characteristic polynomial $Det(I - \lambda M)$, and the algebraic multiplicity of an eigenvalue $Mul_a(\lambda_l)$ is the order of the root. The geometric multiplicity of an eigenvalue $Mul_g(\lambda_l)$ is the number of independent eigenvectors corresponding to the eigenvalue. In general, $Mul_g(\lambda_l) \leq Mul_a(\lambda_l)$. According to the basic theory of linear algebra, cases (1) and (2) are spectrally stable, and cases (3) and (4) are spectrally unstable. For cases (1) and (2), we would like to know whether their spectral stability will be sustained under a small structural perturbation. Case (1) can also be shown to be structurally stable by considering the symplectic nature of M. As the structure of the system is perturbed, the eigenvalues will move accordingly. However, they cannot move off the unit circle due to a small structural perturbation on M for case (1). This is because for every

eigenvalue λ off the circle, there exits another eigenvalue $1/\bar{\lambda}$, which is on the opposite side of the unit circle. If one of the eigenvalues of case (1) were allowed to move off the unit circle, then there would be more than $2n$ eigenvalues. This forbidden situation is illustrated in Fig. 1 for $n = 2$. When there are repeated eigenvalues for case (2), the constraints on the locations of the eigenvalues do not prohibit the eigenvalues moving off the unit circle due to structural perturbations, which is the so-called Krein collision,[12–17] as illustrated in Fig. 2 for $n = 2$. Krein collisions preserve the symmetry of the eigenvalue distribution with respect to the real axis and the unit circle and are the only possible pathways in parameter space for a spectrally stable system to become spectrally unstable. When this happens, the system is structurally unstable. What is more interesting is that not all possibilities in case (2) are structurally unstable. Case (2) needs to be further divided into two sub-categories:

(2.1) For all repeated eigenvalues, the corresponding eigenvectors have the same signatures.
(2.2) There is at least one repeated eigenvalue whose eigenvectors have different signatures.

Here, the signature of an eigenvector ψ of M is defined to be the sign of its self-product $\langle \psi, \psi \rangle = \psi^* iJ\psi$. Here, J is the $2n \times 2n$ unit symplectic matrix, i.e.,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1}$$

The product between two vectors ψ and ϕ in general is defined to be

$$\langle \psi, \phi \rangle \equiv \psi^* iJ\phi, \tag{2}$$

where ψ^* is the complex conjugate of ψ^T. Krein,[12] Gel'fand and Lidskii,[13] and Moser[14] proved that case (2.1) is structurally stable, and that case (2.2) is structurally unstable. This is the celebrated Krein-Gel'fand-Lidskii-Moser theorem.

Let us use the example of a 1D uncoupled lattice $(n = 1)$ to demonstrate the process of a Krein collision. For a 1D uncoupled periodic lattice, the one-period transfer map M is[17]
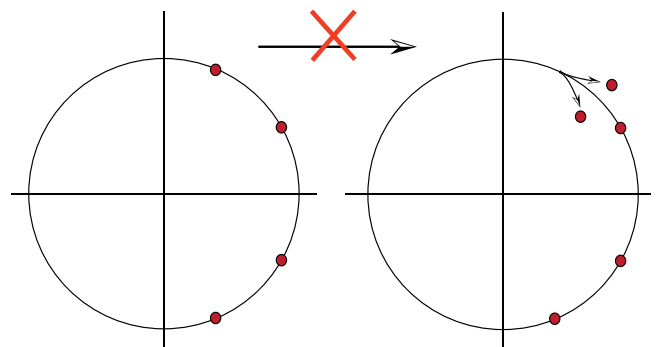
$$M = S_0^{-1} P S_0, \tag{3}$$

FIG. 1. Eigenvalues are forbidden to move off the unit circle for case 1. Illustrated here is the case of $n = 2$.
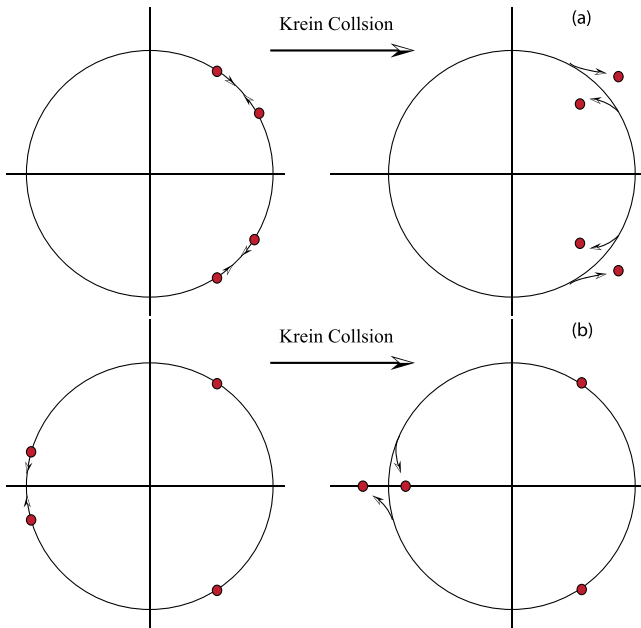
FIG. 2. Eigenvalues are allowed to move off the unit circle for case 2 when there are repeated eigenvalues. This is the so-called Krein collision. Illustrated in (a) and (b) are two possibilities for the case of $n = 2$. For each possibility, the eigenvalue distribution on the left is before the collision, and the one on the right is after the collision.

$$P = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \qquad (4)$$

$$S_0 = \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}, \qquad (5)$$

$$\phi = \int_0^T \frac{dt}{\beta(t)}, \qquad (6)$$

$$\beta(t) = w^2(t), \qquad (7)$$

where $w(t)$ is a matched envelope function, $w_0 = w(0)$ and $\dot{w}_0 = \dot{w}(0)$ are the initial conditions for $w(t)$, and ϕ is the one-period phase advance. The envelope function $w(t)$ is determined by the envelope equation

$$\ddot{w} + \kappa_q(t)w = w^{-3}. \qquad (8)$$

Since M is similar to P, and the eigenvalues and signatures are preserved by a similarity transformation [see Eq. (32)], the spectral and structural stability properties of the lattice are completely determined by the phase advance matrix P. The eigenvalues, eigenvectors, and signatures are

$$\lambda_+ = \cos\phi + i\sin\phi, \quad \psi_+ = (1, i)^T, \quad \sigma_+ = -1, \qquad (9)$$

$$\lambda_- = \cos\phi - i\sin\phi, \quad \psi_- = (1, -i)^T, \quad \sigma_- = 1. \qquad (10)$$

Evidently, the system is spectrally and structurally stable when $\phi \neq n\pi$, which corresponds to case (1). As the system parameters vary, the phase advance ϕ changes. A Krein collision occurs at $\phi = n\pi$, where $\lambda_+ = \lambda_-$ and $\sigma_+ = -\sigma_-$. This is case (2.2), which is structurally unstable. Starting from a stable lattice with a small phase advance, we can increase the focusing strength and thus the phase advance

gradually. The system is stable until the phase advance approaches π.

III. GENERALIZED COURANT-SNYDER THEORY

To prepare for the investigation of the stability properties for general coupled lattice, we briefly summarize here the generalized Courant-Snyder theory, a thorough description of which can be found in Ref. 17.

The linear dynamics of a charged particle relative to the fiducial orbit are governed by a general time-dependent Hamiltonian[37] of the form

$$H = \frac{1}{2} z^T \Lambda z, \quad \Lambda = \begin{pmatrix} \kappa(t) & R(t) \\ R(t)^T & m^{-1}(t) \end{pmatrix}. \qquad (11)$$

Here, $z = (x, y, p_x, p_y)^T$ are the transverse phase space coordinates, and $\kappa(t)$, $R(t)$, and $m^{-1}(t)$ are time-dependent 2×2 matrices. The matrices $\kappa(t)$, $m^{-1}(t)$, and Λ are also symmetric. In this general Hamiltonian, the quadrupole component is in the diagonal terms of the $\kappa(t)$ matrix. The off-diagonal terms of $\kappa(t)$ contain the skew-quadrupole and dipole components. The solenoidal component and the torsion of the fiducial orbit[38] are included in the $R(t)$ matrix. The symplectic matrix specifying the map between z_0 and $z = M(t)z_0$ is

$$M(t) = S^{-1}P^T S_0, \qquad (12)$$

$$S = \begin{pmatrix} w^{-T} & 0 \\ (wR - \dot{w})m & w \end{pmatrix}, \qquad (13)$$

where subscript "0" denotes initial conditions at $t = 0$, and w is a 2×2 envelope matrix function satisfying the matrix envelope equation

$$\frac{d}{dt}\left(\frac{dw}{dt}m - wRm\right) + \frac{dw}{dt}mR^T + w(\kappa - RmR^T) - (w^T wmw^T)^{-1} = 0. \qquad (14)$$

In Eq. (12), $P \in Sp(4) \cap SO(4) = U(2)$ is a symplectic rotation, which is the generalized phase advance, determined by

$$\dot{P} = -P\begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}, \qquad (15)$$

$$\mu \equiv (wmw^T)^{-1}. \qquad (16)$$

Alternatively and preferably, the transfer matrix $M(t)$ can be expressed in terms of a symmetric envelope matrix $u(t)$, which is defined to be the symmetric component of $w(t)$ in its polar decomposition

$$u(t) \equiv \sqrt{\beta(t)}, \qquad (17)$$

$$\beta(t) \equiv w^T(t)w(t). \qquad (18)$$

In terms of u, the transfer matrix is

$$M(t) = S_u^{-1}P_u^{-1}S_{u0}, \qquad (19)$$

$$S_u \equiv \begin{pmatrix} u^{-1} & 0 \\ (uR - Du - \dot{u})m & u \end{pmatrix}, \qquad (20)$$

$$D \equiv L_{umu}^{-1}[(um\dot{u} - \dot{u}mu) + u(Rm - mR^T)u], \quad (21)$$

where $P_u \in Sp(4) \cap SO(4) = U(2)$ is a symplectic rotation determined by the differential equation

$$\dot{P}_u = -P_u \left[\begin{pmatrix} 0 & \mu_u \\ -\mu_u & 0 \end{pmatrix} - \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \right], \quad (22)$$

$$\mu_u \equiv (umu^T)^{-1}. \quad (23)$$

In Eq. (21), L_V^{-1} is the inverse of the Lyapunov operator defined as

$$L_V(X) = VX + XV, \quad (24)$$

for a symmetric, positive-definite matrix V. A detailed discussion of the Lyapunov operator can be found in Ref. 17. The envelope matrix $u(t) \equiv \sqrt{\beta(t)}$ is determined from the differential equation for β

$$\ddot{\beta} = 2e - fg - g^T f^T - \beta h - h^T \beta + 2m^{-1}\beta^{-1}m^{-1}, \quad (25)$$

$$e \equiv \dot{u}Du + \dot{u}^2 - uD^2u - uD\dot{u}, \quad (26)$$

$$f \equiv uDu + u\dot{u}, \quad (27)$$

$$g \equiv (\dot{m} - Rm + mR^T)m^{-1}, \quad (28)$$

$$h \equiv (\kappa - RmR^T - \dot{R}m - Rm)m^{-1}, \quad (29)$$

$$\dot{u} = L_{\sqrt{\beta}}^{-1}(\dot{\beta}). \quad (30)$$

Equation (25) is a second-order ordinary differential equation for β, since every term on the right-hand-side is a function of β and $\dot{\beta}$.

For every t, $M(t)$ is specified by two $n \times n$ symmetric matrices β and $\dot{\beta}$, and a $U(n)$ matrix P_u. The dimension of $M(t)$ is thus $(n^2 + n)/2 + (n^2 + n)/2 + n^2 = n(2n + 1)$, as expected for symplectic matrices. Another important advantage of using the symmetric envelope matrix u over the unsymmetric envelope matrix w is that Eq. (19) enables the application of advanced techniques of pre-Iwasawa decomposition and normal form to find a matched solution for the β matrix without using the (inefficient) shooting method.

IV. STABILITY ANALYSIS AND MATCHED LATTICE FUNCTIONS

For practical applications of coupled lattices, it is desirable to design a coupled lattice that belongs to cases (1) and (2.1), which are both spectrally and structurally stable. As mentioned previously, the parameter space satisfying this condition for a coupled lattice is larger than that for an uncoupled lattice. The generalized Courant-Snyder theory described in Sec. III provides an effective tool to study the stability properties of coupled lattices. One of the important results is that if a matched solution for β exists, then the stability property of a general coupled lattice is completely determined by the phase advance matrix P_u. Using a matched solution for β, the one-period map is

$$M(T) = S_{u0}^{-1} P_u(T)^T S_{u0}, \quad (31)$$

which indicates that $M(T)$ is similar to the inverse of the phase advance $P_u(T)^T$ and thus has the same eigenvalues and multiplicity as $P_u(T)^T$. Because $P_u(T)^T$ is a rotation, its eigenvalues are on the unit circle. Now we show that the phase advance $P_u(T)$ also determines the structural stability of $M(T)$. For an eigenvector ψ of $M(T)$, $S_{u0}\psi$ is an eigenvector of $P_u(T)^T$, and the product between the two eigenvectors defined in Eq. (2) is preserved by the similarity transformation, i.e.,

$$\langle S_{u0}\psi, S_{u0}\phi \rangle = \psi^* S_{u0}^T iJS_{u0}\phi = \psi^* iJ\phi = \langle \psi, \phi \rangle. \quad (32)$$

Therefore, $P_u(T)^T$ and $M(T)$ have the same eigenvalues, signatures, and thus structural stability properties.

As in the case of a 1D uncoupled lattice, matched solutions for β are much more preferable than unmatched solutions, because the lattice functions can be completely determined by a matched β solution in one lattice period, i.e., the lattice functions are periodic in terms of the lattice period. On the other hand, if an unmatched β solution is used, the β function and other lattice functions have to be solved in the entire time domain of $0 < t < \infty$. Does a matched β solution always exist? The answer is negative. What are the conditions for the existence of a matched β solution? We prove now that for cases (1) and (2.1), Eq. (25) admits a matched solution for β. The proof utilizes the technique of normal form and pre-Iwasawa decomposition[35,36] for symplectic matrices.

First, let us invoke the established result that for case (1), M can always be transformed into the following normal form with a symplectic matrix A

$$M = ANA^{-1}, \quad (33)$$

$$N = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \dots & \\ & & & R_n \end{pmatrix}, \quad (34)$$

$$R_l = \begin{pmatrix} \cos\phi_l & \sin\phi_l \\ -\sin\phi_l & \cos\phi_l \end{pmatrix}. \quad (35)$$

Obviously, $N \in Sp(2n) \cap SO(2n) = U(n)$ is a symplectic rotation. This fact is proved[16] from the existence of a complete set of $2n$ orthonormal eigenvectors (ψ_l, ψ_{-l}), $(l = 1, 2, \dots, n)$, satisfying

$$\langle \psi_l, \psi_m \rangle = \delta_{lm}, \quad (36)$$

$$\langle \psi_{-l}, \psi_{-m} \rangle = -\delta_{lm}, \quad (37)$$

$$\langle \psi_l, \psi_{-m} \rangle = 0. \quad (38)$$

Here, ψ_l and $\psi_{-l} = \bar{\psi}_l$ are a pair of eigenvectors corresponding to the eigenvalues λ_l and $\lambda_{-l} = \bar{\lambda}_l$, respectively. Equations (36) and (37) state that ψ_l and ψ_{-l} have different signatures. The normal form is actually explicitly constructed. The transfer matrix A is given as[16]

$$A = \sqrt{2}(\xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_n, \eta_n), \quad (39)$$

where ξ_l and η_l are real and imaginary components of the eigenvector ψ_l, i.e., $\psi_l = \xi_l + i\eta_l$.

We now show that for case (2.1), such a set of orthonormal bases exists as well. For a repeated eigenvector λ with $Mul_g(\lambda) = Mul_a(\lambda)$, the corresponding eigenvectors span a subspace M_λ of R^{2n}. Because the signature never vanishes in M_λ, we can always select a set of orthonormal bases for M_λ through a Gram-Schmidt process. The subspace $M_{-\lambda}$, a complex conjugate image of M_λ, has the same structure except that the signature has the opposite sign. Therefore, for both case (1) and case (2.1), the normal form given by Eq. (33) exists.

Second, we apply the pre-Iwasawa decomposition to the symplectic matrix A. According to the theory of Iwasawa decomposition,[35,36] a symplectic matrix G can always be uniquely factored as

$$G = P \begin{pmatrix} Y & 0 \\ QY & Y^{-1} \end{pmatrix}, \tag{40}$$

where $P \in Sp(2n) \cap SO(2n) = U(n)$ is a symplectic rotation, and Y and Q are symmetric. The statement is true as well if the decomposition is defined alternatively to be

$$G = \begin{pmatrix} Y & 0 \\ QY & Y^{-1} \end{pmatrix} P. \tag{41}$$

Let the unique pre-Iwasawa decomposition of A be

$$A = P_A S_A, \tag{42}$$

$$S_A = \begin{pmatrix} Y & 0 \\ QY & Y^{-1} \end{pmatrix}_A. \tag{43}$$

Then, the transfer matrix is $M = S_A^{-1} P_A^{-1} N P_A S_A$. We choose the initial conditions for β and $\dot{\beta}$ such that

$$S_{u0} = S_A, \tag{44}$$

and the solution of β will give the same transfer map

$$M = S_u^{-1} P_u^{-1} S_A = S_A^{-1} P_A^{-1} N P_A S_A. \tag{45}$$

Thus

$$S_u^{-1} P_u^{-1} = S_A^{-1} P_A^{-1} N P_A. \tag{46}$$

The uniqueness of pre-Iwasawa decomposition requires that

$$S_u^{-1} = S_A^{-1}, \tag{47}$$

$$P_u^{-1} = P_A^{-1} N P_A. \tag{48}$$

Equations (44) and (47) prove that $S_u^{-1} = S_{u0}^{-1}$, i.e., the β solution is matched. Thus we have proven the proposition that for cases (1) and (2.1), Eq. (25) admits matched solutions. This proof is a constructive one, which can be used as an effective method to find a matched solution for β. The conventional method for finding matched solutions is the shooting method, where one takes an initial condition for β and solves the ordinary differential equation once in one period. In general, the solution is not matched, i.e., $\beta(T) \neq \beta(0)$ or $\dot{\beta}(T) \neq \dot{\beta}(0)$. The shooting method requires one to estimate a better initial condition based on the size of the mismatch,

and solve the differential equation again. This iteration is carried out many times until a matched solution is found. The new method suggested by the above constructive proof of the existence of matched solution only requires solving Eq. (25) once with an arbitrary initial condition. We can construct the one-period map $M(T)$ using any matched or unmatched solution of Eq. (25), then the eigenvectors of $M(T)$ can be calculated. When the set of bases satisfying Eqs. (36)–(38) exists, the initial condition for a matched solution is uniquely given by Eq. (44). This new method applies to both 1D uncoupled lattices and coupled lattices in higher dimensions. Of course, this procedure fails when the set of bases satisfying Eqs. (36)–(38) does not exist. However, for the desirable cases (1) and (2.1), such a set of bases exists. Another practical implication of the existence of a matched β solution is that when a matched solution for β cannot be found, the lattice must be unstable.

V. CONTINUOUSLY ROTATING QUADRUPOLE LATTICES

As an illustrative application of the theoretical formalism developed in Secs. III and IV, we investigate here the stability properties of a continuously rotating quadrupole lattice.[2–8] The Hamiltonian of a charged particle in such a lattice is[31,34]

$$H = \frac{1}{2} z^T \Lambda z, \quad \Lambda = \begin{pmatrix} \kappa(t) & 0 \\ 0 & I \end{pmatrix}, \tag{49}$$

$$\kappa(t) = \kappa_{q0} \begin{pmatrix} \cos(2\pi t/T) & \sin(2\pi t/T) \\ -\sin(2\pi t/T) & \cos(2\pi t/T) \end{pmatrix}. \tag{50}$$
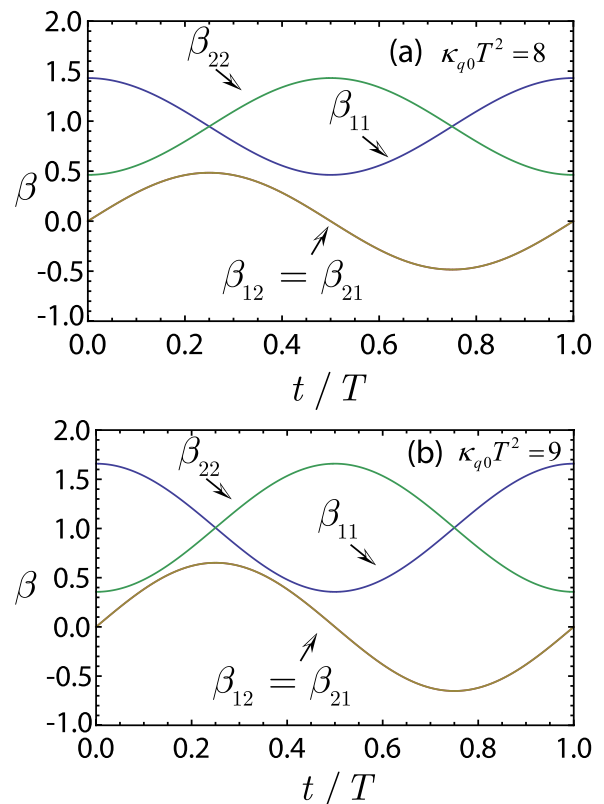
FIG. 3. Matched solutions for the cases of (a) $\kappa_{q0}T^2 = 8$ and (b) $\kappa_{q0}T^2 = 9$.

We will smoothly vary $\kappa_{q0}T^2$ and observe the movement of the eigenvalues of the one-period map. For a given value of $\kappa_{q0}T^2$, we find a matched β solution using the procedure described above. The calculation is carried out using a code developed in Ref. 34. The matched β solutions for the cases of $\kappa_{q0}T^2 = 8$ and $\kappa_{q0}T^2 = 9$ are plotted in Fig. 3. The eigenvalue distributions for different values of $\kappa_{q0}T^2$ are plotted in Fig. 4. For $\kappa_{q0}T^2 = 8$ and $\kappa_{q0}T^2 = 9$, it belongs to case (1). The two eigenvalues on the left half circle have different signatures, and move towards one another as $\kappa_{q0}T^2$ increases. At $\kappa_{q0}T^2 = \pi^2$, these two eigenvalues collide at $\lambda = -1$. Since their signatures are different, this is an unstable Krein collision and thus structurally unstable. Right after the collision at $\kappa_{q0}T^2 = 10$, these two eigenvalues moves off the unit circle, and lead to an unstable lattice. This is the scenario depicted in Fig. 2(b). These analyses can be straightforwardly carried out for any coupled lattice, such as the N-rolling lattice[31] and the Möbius accelerator.[9] We note that to calculate the eigenvalues displayed in Fig. 4, it is not necessary to find the matched β solutions. Any solution of the β matrix over one lattice period can be used. Matched solutions are preferable when the β matrix is used as a lattice function for the machine.

VI. CONCLUSIONS AND FUTURE WORK

We have studied in this paper the spectral and structural stability of charged particle dynamics in a coupled focusing lattice as a Hamiltonian system. The recently developed generalized Courant-Synder theory for coupled lattices has been applied. It is has been demonstrated that for coupled lattices that are spectrally and structurally stable, the matrix envelope equation must admit matched solutions. A new method is presented to determine a matched solution for the matrix envelope using the technique of normal form and pre-Iwasawa decomposition. If a matched solution exists, this method is able to determine the matched solution simply by solving the envelope equation once without using the (inefficient) shooting procedure. As an example, stability properties of a continuously rotating quadrupole lattice are investigated. The Krein collision process for destabilization of the lattice has also been demonstrated.
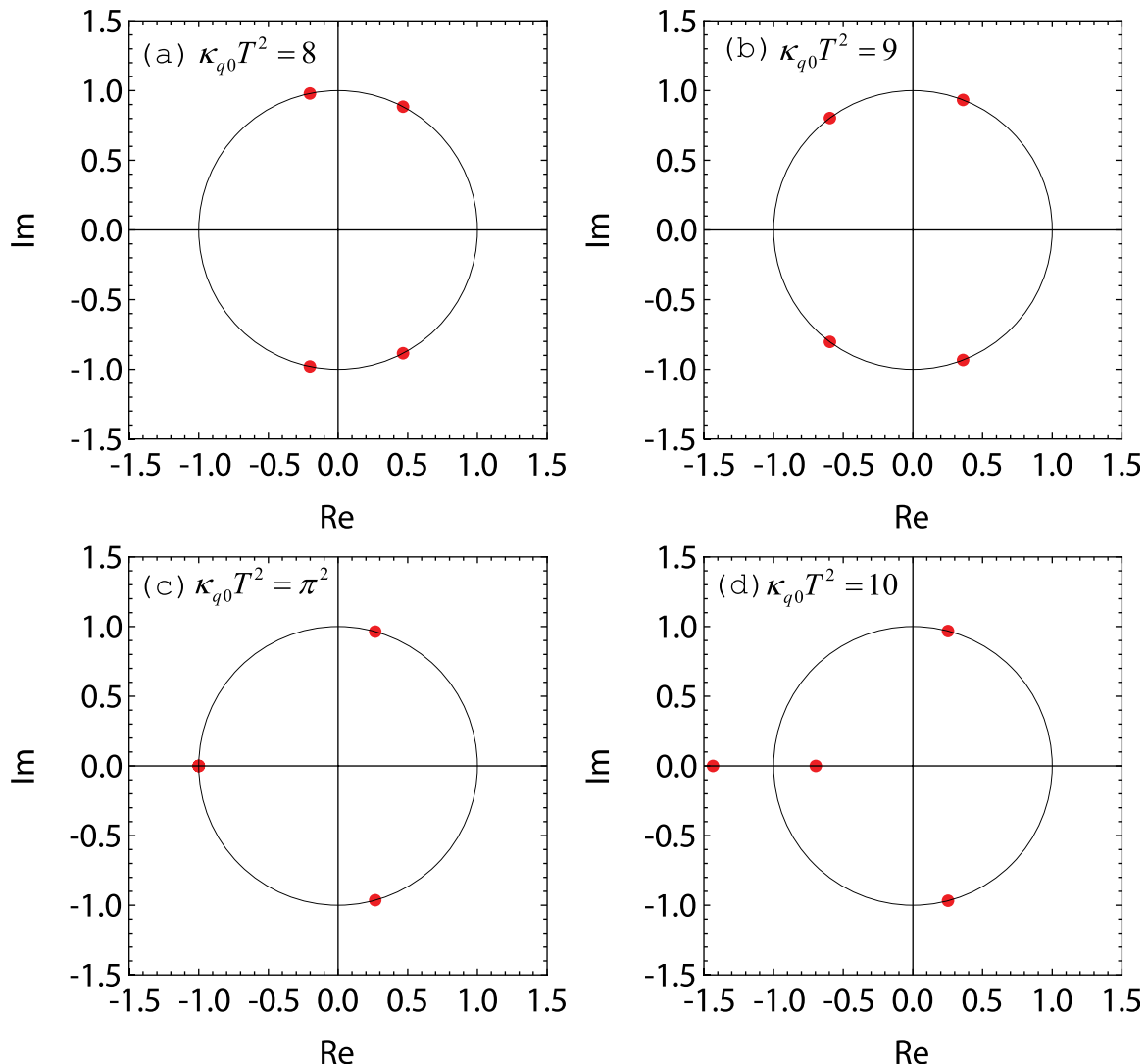

FIG. 4. Eigenvalues of continuously rotating quadrupole lattices. The two eigenvalues on the left half circle have different signatures in (a), and move towards one other as $\kappa_{q0}T^2$ increases in (b). The Krein collision occurs at $\kappa_{q0}T^2 = \pi^2$ in (c), after which the two eigenvalues on the left move off the unit circle and lead to an unstable lattice in (d).

The application of coupled lattices to high-intensity charged particle beams[39–43] will be investigated in future studies. The theoretical framework and analytical tools developed in the present study are also expected to be effective for the current theoretical and experimental investigations of emittance exchange technologies.[44–50]

ACKNOWLEDGMENTS

[1]E. Courant and H. Snyder, Ann. Phys. **3**, 1 (1958).

[2]R. L. Gluckstern, in *Proceedings of the 1979 Linear Accelerator Conference* (Brookhaven National Laboratory, 1979), pp. 245–248.

[3]C. Roberson, A. Mondelli, and D. Chernin, Part. Accel. **17**, 79 (1985).

[4]D. Chernin, Part. Accel. **24**, 29 (1988).

[5]J. Petillo, D. Chernin, and A. Mondelli, in *Proceedings of the 1989 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 1989), pp. 1055–1057.

[6]J. Krall, S. Slinker, M. Lampe, and G. Joyce, J. Appl. Phys. **77**, 463 (1995).

[7]J. Smith, V. Bailey, H. Lackner, and S. Putnam, in *Proceedings of the 1997 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 1998), Vol. 1, pp. 1251–1253.

[8]J. Petillo, C. Kostas, D. Chernin, and A. Mondelli, in *Proceedings of the 1991 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 2002), pp. 613–615.

[9]R. Talman, Phys. Rev. Lett. **74**, 1590 (1995).

[10]J. J. Barnard, in *Proceedings of the 1995 Particle Accelerators Conference* (IEEE, Piscataway, NJ, 1996), p. 3241.

[11]R. A. Kishek, J. J. Barnard, and D. P. Grote, in *Proceedings of the 1999 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 1999), p. 1761.

[12]M. Krein, Dokl. Akad. Nauk SSSR N.S. **73**, 445 (1950).

[13]I. M. Gel'fand and V. B. Lidskii, Usp. Mater. Nauk **10**, 3 (1955).

[14]J. Moser, Commun. Pure Appl. Math. **XI**, 81 (1958).

[15]V. Yakubovich and V. Starzhinskii, *Linear Differential Equations with Periodic Coefficients* (Wiley, 1975), vol. I.

[16]A. J. Dragt, "Lie methods for nonlinear dynamics with applications to accelerator physics," (unpublished).

[17]H. Qin, R. C. Davidson, J. W. Burby, and M. Chung, Phys. Rev. ST Accel. Beams **17**, 044001 (2014).

[18]L. C. Teng, NAL Report FN-229, 1971.

[19]D. A. Edwards and L. C. Teng, IEEE Trans. Nucl. Sci. **20**, 885 (1973).

[20]L. C. Teng, in *Proceedings of the 2003 Particle Accelerator Conference*, Piscataway, NJ (2003), p. 2895.

[21]G. Ripken, Tech. Rep. R1-70/04, DESY, 1970.

[22]H. Wiedemann, *Particle Accelerator Physics* (Springer-Verlag, 2007), pp. 614–620.

[23]V. A. Lebedev and S. A. Bogacz, J. Instrum. **5**, P10010 (2010).

[24]G. Dattoli, C. Mari, M. Richetta, and A. Torre, Nuovo Cimento B **107**, 269 (1992).

[25]G. Dattoli, G. Gallerano, C. Mari, A. Torre, and M. Richetta, Nuovo Cimento B **107**, 1151 (1992).

[26]G. Dattoli, C. Mari, L. Mezi, and A. Torre, Nucl. Instrum. Methods Phys. Res. A **321**, 447 (1992).

[27]H. Qin and R. C. Davidson, Phys. Rev. Spec. Top.– Accel. Beams **12**, 064001 (2009).

[28]H. Qin and R. C. Davidson, Phys. Plasmas **16**, 050705 (2009).

[29]H. Qin, M. Chung, and R. C. Davidson, Phys. Rev. Lett. **103**, 224802 (2009).

[30]M. Chung, H. Qin, and R. C. Davidson, Phys. Plasmas **17**, 084502 (2010).

[31]H. Qin and R. C. Davidson, Phys. Plasmas **18**, 056708 (2011).

[32]H. Qin, R. C. Davidson, M. Chung, and J. W. Burby, Phy. Rev. Lett. **111**, 104801 (2013).

[33]H. Qin and R. C. Davidson, Phys. Rev. Lett. **110**, 064803 (2013).

[34]M. Chung, H. Qin, E. P. Gilson, and R. C. Davidson, Phys. Plasmas **20**, 083121 (2013).

[35]K. Iwasawa, Ann. Math. **50**, 507 (1949).

[36]M. de Gosson, *Symplectic Geometry and Quantum Mechanics* (Birkhauser Verlag, 2006), pp. 42–44.

[37]L. Michelotti, *Intermediate Classical Dynamics With Applications to Beam Physics* (John Wiley & Sons Inc., 1995), p. 166.

[38]G. H. Hoffstaetter, in *Proceedings of 1995 Particle Accelerator Conference* (1995), pp. 2707–2710.

[39]T. F. Wang and L. Smith, Part. Accel. **12**, 247 (1982).

[40]A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley, New York, 1993).

[41]R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001), p. 284.

[42]S. Lund and B. Bukh, Phys. Rev. Spec. Top.– Accel. Beams **7**, 024801 (2004).

[43]F. Zimmermann, Phys. Rev. Spec. Top. Accel. Beams **7**, 124801 (2004).

[44]M. Cornacchia and P. Emma, Phys. Rev. Spec. Top. Accel. Beams **5**, 084001 (2002).

[45]P. Emma, Z. Huang, K.-J. Kim, and P. Piot, Phys. Rev. Spec. Top. Accel. Beams **9**, 100702 (2006).

[46]H. Qin, R. C. Davidson, M. Chung, J. J. Barnard, and T. F. Wang, in *Proceedings of 2011 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 2011), p. 758.

[47]L. Groening, Phys. Rev. ST Accel. Beams **14**, 064201 (2011).

[48]C. Xiao, O. Kester, L. Groening, H. Leibrock, M. Maier, and P. Rottländer, Phys. Rev. ST Accel. Beams **16**, 044201 (2013).

[49]L. Groening, M. Maier, C. Xiao, L. Dahl, P. Gerhard, O. K. Kester, S. Mickat, H. Vormann, M. Vossberg, and M. Chung, Phys. Rev. Lett. **113**, 264802 (2014).

[50]M. Chung, H. Qin, L. Groening, R. C. Davidson, and C. Xiao, Phys. Plasmas **22**, 013109 (2015).