

SHORT-TIME STABILITY OF SCALAR VISCOUS SHOCKS IN THE
INVISCID LIMIT BY THE RELATIVE ENTROPY METHOD*KYUDONG CHOI[†] AND ALEXIS F. VASSEUR[‡]

Abstract. We consider inviscid limits to shocks for viscous scalar conservation laws in one space dimension, with strict convex fluxes. We show that we can obtain sharp estimates in L^2 for a class of large perturbations and for any bounded time interval. Those perturbations can be chosen big enough to destroy the viscous layer. This shows that the fast convergence to the shock does not depend on the fine structure of the viscous layers. This is the first application of the relative entropy method developed by N. Leger [*Arch. Ration. Mech. Anal.*, 199 (2011), pp. 761–778] and N. Leger and A. Vasseur [*Arch. Ration. Mech. Anal.*, 201 (2011), pp. 271–302] to the study of an inviscid limit to a shock.

Key words. viscous scalar conservation laws, inviscid limits, stability, relative entropy method, shocks

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1. Introduction and the main result. For any strictly convex flux function $A \in C^2(\mathbb{R})$, we consider the family of viscous scalar conservation laws in one space dimension:

$$(1.1) \quad \begin{cases} \partial_t U + \partial_x A(U) = \varepsilon \partial_{xx}^2 U & \text{for } t > 0, x \in \mathbb{R}, \\ U(0, x) = U_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

for any $\varepsilon > 0$ and $U_0 \in L^\infty$. Global unique solutions to (1.1) have been constructed by Hopf [17] and Oleřnik [29]. The inviscid case, $\varepsilon = 0$, is covered by the theory of Kruřkov [20]. Kuznetsov showed in [21] that, for fixed initial data U_0 , the solutions U^ε of (1.1) converge in L^1 , when ε goes to zero, to the solution U^0 of the inviscid scalar conservation law ((1.1) with $\varepsilon = 0$) with the rate $\sqrt{\varepsilon}$:

$$\|U^\varepsilon(t) - U^0(t)\|_{L^1} \leq C\sqrt{\varepsilon t} \text{TV}(U_0)$$

(for the proof, e.g., see either Serre [33] or Perthame [30]).

In this paper we consider the inviscid limit for general initial values and for any bounded time interval. We are particularly interested in the cases where the initial values carry too much entropy for the structure of the layer to be preserved in its vanishing viscosity limit. The shocks solutions of the inviscid case ($\varepsilon = 0$) can be described as follows. Consider two constants $C_L > C_R$, and the associated function defined by

$$(1.2) \quad S_0(x) = \begin{cases} C_L & \text{if } x < 0, \\ C_R & \text{if } x \geq 0. \end{cases}$$

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Then, the Rankine–Hugoniot conditions ensures that the function

$$(1.3) \quad S_0(x - \sigma t) \quad \text{with} \quad \sigma := \frac{A(C_L) - A(C_R)}{C_L - C_R}$$

is a solution to the inviscid equation (1.1) with $\varepsilon = 0$. The condition $C_L > C_R$ implies that they verify the entropy conditions, that is,

$$\partial_t \eta(U) + \partial_x G(U) \leq 0, \quad t > 0, \quad x \in \mathbb{R},$$

for any convex functions η , and $G' = \eta' A'$.

Our main result is the following.

THEOREM 1.1. *Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \quad \text{and} \quad \left(\frac{d}{dx} U_0 \right)_+ \in L^2(\mathbb{R}).$$

Then, there exists $\varepsilon_0 > 0$ such that for any $T > 0$, we have a constant $C^ > 0$ with the following:*

- I. *For any U solution to (1.1) with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0, T)$ such that $X(0) = 0$ and for any $0 < t < T$,*

$$(1.4) \quad \|U(t) - S(t)\|_{L^2(\mathbb{R})}^2 \leq \|U_0 - S_0\|_{L^2(\mathbb{R})}^2 + C^* \varepsilon \log(1/\varepsilon),$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (1.2).

- II. *Moreover, this curve satisfies*

$$(1.5) \quad |\dot{X}(t)| \leq C^* \quad \text{and}$$

$$(1.6) \quad |X(t) - \sigma t|^2 \leq C^* t^{2/3} \left(\|U_0 - S_0\|_{L^2(\mathbb{R})}^2 + \varepsilon \log(1/\varepsilon) \right).$$

- III. *The constant ε_0 depends only on $\|(\frac{d}{dx} U_0)_+\|_{L^2}$, C_L , C_R , $\|U_0\|_{L^\infty}$ and the flux function A , while C^* depends only on the same set as well as T .*

Remark 1.2. For any continuous function g , we define the function g_+ by $g_+(x) := g(x) \cdot \chi_{\{g>0\}}(x)$, where $\chi_{\{g>0\}}$ is the characteristic function on the positive part of the function g . In our theorem, the assumption $U_0 \in BV_{loc}$ ensures that $\frac{d}{dx} U_0$ is a Radon measure. Hence, $(\frac{d}{dx} U_0)_+$ is also a Radon measure, and the condition $(\frac{d}{dx} U_0)_+ \in L^2$ makes sense. Note that our estimates do not depend on any local BV norms of U_0 .

Remark 1.3. The condition $(\frac{d}{dx} U_0)_+ \in L^2(\mathbb{R})$ can be replaced with $(\frac{d}{dx} U_0)_+ \in L^p(\mathbb{R})$ for any $1 < p \leq \infty$. Indeed, as in Lemma 3.2, it can be shown that the quantity $\|(\partial_x U(t))_+\|_{L^p(\mathbb{R})}$ is nonincreasing in time (see Remark 3.3). The only place where the assumption $(\frac{d}{dx} U_0)_+ \in L^2(\mathbb{R})$ is used is in the estimate (3.4) in the proof of Proposition 3.4. In order to use $(\frac{d}{dx} U_0)_+ \in L^p(\mathbb{R})$ for any $1 < p \leq \infty$, one needs to have $(\varepsilon \delta)^{1-1/p}$ instead of $\sqrt{\varepsilon \delta}$ in (3.4).

Remark 1.4. The term σt in the estimate (1.6) is meaningful when the time variable $t \gg (\varepsilon \log(1/\varepsilon))^3$.

This result shows a rate of convergence slightly worse than ε (to the log), for the inviscid limit to a shock, measured via the L^2 norm (squared). In the case of the limit to a regular solution of the inviscid case, the rate of convergence is $\sqrt{\varepsilon}$ (see [35], for instance). We also refer to Goodman and Xin [16], Bressan, Liu, and Yang [6], Lewicka [24], Bressan and Yang [5], and Christoforou and Trivisa [10].

An easy layer study shows that ε is the optimal rate for shocks with special initial data. Indeed, one can construct an associated steady viscous layer (see, for example, Il'in and Oleinik [18]) S_1 solution to

$$(1.7) \quad \begin{cases} A(S_1) - A(C_L) - \sigma(S_1 - C_L) = S_1', & x \in \mathbb{R}, \\ \lim_{x \rightarrow -\infty} S_1 = C_L, & \lim_{x \rightarrow +\infty} S_1 = C_R. \end{cases}$$

It is easy to show that $S_1((x - \sigma t)/\varepsilon)$ is a solution to (1.1) with initial data $S_1(x/\varepsilon)$. In this case, the rate of convergence is of order ε since

$$\int_{\mathbb{R}} |S_1((x - \sigma t)/\varepsilon) - S_0(x - \sigma t)|^2 dx = \varepsilon \int_{\mathbb{R}} |S_1(x) - S_0(x)|^2 dx = C\varepsilon.$$

This layer study can be extended to the case of a small initial perturbation where

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^p dx \leq C\varepsilon$$

for a $1 \leq p < \infty$. In this case, for a solution U to (1.1), we can consider

$$V(t, x) = U(\varepsilon t, \varepsilon x)$$

and study the asymptotic for large time. The function V is a solution to the equation

$$\begin{cases} \partial_t V + \partial_x A(V) - \partial_{xx}^2 V = 0, \\ V(0, x) = U(0, \varepsilon x). \end{cases}$$

The convergence to S_1 , up to a (constant) drift, in this setting, has been extensively studied (see, for instance, [18], Freistühler and Serre [14], and Kenig and Merle [19]). In this situation of small perturbation of the initial shock, those results show that the convergence with rate ε for the system (1.1) is due to the asymptotic limit in large time of the layer function $U(\cdot/\varepsilon)$ to $S_1((\cdot - \sigma t)/\varepsilon)$.

This layer study, however, collapses when

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^2 dx \gg \varepsilon.$$

In this situation, there is too much entropy for the asymptotic limit of the layer structure to be true. The physical layer may be destroyed. Theorem 1.1 shows that, nevertheless, the sharp convergence (up to the log) still holds for any bounded time interval.

Taking a limit as ε goes to 0 in Theorem 1.1, we recover the L^2 stability of shocks (up to a drift) first shown by Leger in [22]. Note that the stability result has to be up to a drift which depends on the solution itself (and may be not unique). This feature is also true for our result. The drift cannot be taken constant, as in the case of the layer problem.

Our result is based on the relative entropy method first used by Dafermos and DiPerna to show L^2 stability and uniqueness of Lipschitzian solutions to conservation laws [11, 12, 13]. They showed, in particular, that if \bar{U} is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0, T]$, then for any bounded weak entropic solution U it holds that

$$(1.8) \quad \int_{\mathbb{R}} |U(t) - \bar{U}(t)|^2 dx \leq C \int_{\mathbb{R}} |U(0) - \bar{U}(0)|^2 dx$$

for a constant C depending on \bar{U} and T .

The relative entropy method is also an important tool in the study of asymptotic limits ($\varepsilon \rightarrow 0$). The main idea is that convergence holds thanks to the strong stability of the solutions of the limit equations. Roughly speaking, if we have good consistency of ε models, with respect to the limit one, then nonlinearities are driven by the strong stability of the solution of the limit equation. Applications of the relative entropy method in this context began with the work of Yau [36] and have been studied by many others. For incompressible limits, see Bardos, Golse, and Levermore [1, 2], Lions and Masmoudi [25, 26], and Saint Raymond and others [15, 32, 27, 31]. For compressible models, see Tzavaras [34] in the context of relaxation and [4, 3, 28] in the context of hydrodynamical limits. However, in all those cases, the method works as long as the limit solution is Lipschitz. This is due to the fact that strong stability as (1.8) is not true when \bar{U} has a discontinuity. It has been proved in [22, 23], however, that some shocks are strongly stable up to a shift (see also related works from Chen and Frid, [7, 8] and Chen, Frid, and Li [9]). This article is the first extension of those results of stability to the study of inviscid limits to a shock. This is a part of the program initiated in [35].

The result can be extended to any entropy in the following way. Fix any strictly convex function $\eta \in C^2$ as an entropy. We define the associated relative entropy functional $\eta(\cdot|\cdot)$ as

$$\eta(x|y) := \eta(x) - \eta(y) - \eta'(y)(x - y).$$

We then have the following extension.

THEOREM 1.5. *Consider a strictly convex entropy functional $\eta \in C^2(\mathbb{R})$. Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \quad \text{and} \quad \left(\frac{d}{dx} U_0 \right)_+ \in L^2(\mathbb{R}).$$

Then, there exists $\varepsilon_0 > 0$ such that for any $T > 0$, we have a constant $C^ > 0$ with the following:*

- I. *For any U solution to (1.1) with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0, T)$ such that $X(0) = 0$, and for any $0 < t < T$, and for any α verifying $\varepsilon \leq \alpha \leq \varepsilon_0$, we have*

$$(1.9) \quad \int_{\{|x-X(t)| \geq C^* \alpha\}} \eta(U(t, x)|S(t, x)) dx \leq \int_{\mathbb{R}} \eta(U_0(x)|S_0(x)) dx + C^* e^{-\alpha/\varepsilon},$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (1.2).

- II. *Moreover, this curve satisfies*

$$(1.10) \quad |\dot{X}(t)| \leq C^* \quad \text{and}$$

$$(1.11) \quad |X(t) - \sigma t|^2 \leq C^* t^{2/3} \left(\int_{\mathbb{R}} \eta(U_0(x)|S_0(x)) dx + \varepsilon \log(1/\varepsilon) \right).$$

- III. *The constant ε_0 depends only on $\|(\frac{d}{dx} U_0)_+\|_{L^2}$, C_L , C_R , $\|U_0\|_{L^\infty}$, the flux function A , and the entropy functional η , while C^* depends only on the same set as well as T .*

Remark 1.6. As in Remark 1.3, the condition $(\frac{d}{dx}U_0)_+ \in L^2(\mathbb{R})$ can be replaced with $(\frac{d}{dx}U_0)_+ \in L^p(\mathbb{R})$ for any $1 < p \leq \infty$. Theorem 1.1 is a direct application of Theorem 1.5 with $\eta(x) := x^2$, and $\alpha = \varepsilon \log(1/\varepsilon)$. Indeed, in this case we have $\eta(x|y) = (x - y)^2$, and

$$\int_{\{|x-X(t)| \leq C^*\alpha\}} \eta(U(t,x)|S(t,x)) dx \leq C|\{|x - X(t)| \leq C^*\alpha\}| \leq CC^*\alpha.$$

For the rest of the paper, we will assume that the initial value U_0 lies in not only BV_{loc} but also C^1 . It allows us to work with smooth solutions $U \in C^1([0, T] \times \mathbb{R})$. The general BV_{loc} case can be obtained by a density argument.

The idea of the proof is to study the evolution of the relative entropy of the solution with respect to the shock, outside of a small region centered at $X(t)$ (this small region corresponds to the layer localization):

$$(1.12) \quad \int_{-\infty}^{X(t)-\delta\varepsilon} \eta(U(t,x)|C_L) dx + \int_{X(t)+\delta\varepsilon}^{\infty} \eta(U(t,x)|C_R) dx.$$

The change in time involves two effects. One is due to the hyperbolic part of the equation, and the second involves the parabolic part (or order ε). In [22], it was shown that for the hyperbolic case $\varepsilon = 0$, with zero layer width $\delta = 0$, the quantity (1.12) is nonincreasing when we choose wisely the drift $X(t)$. When considering the viscous term, the layer with width $(\delta\varepsilon)$ is introduced to avoid the effect of the viscous term on the layer (see Lemma 2.1). The idea is then that the stability induced by the hyperbolic part is enough to counterbalance the effect of the parabolic term, provided that we consider a layer fat enough (see Proposition 3.4 and the proof of Proposition 4.1). For technical considerations, we will fix $\delta = \log(1/\varepsilon)$. The drift $X(t)$ is still chosen with respect to the hyperbolic part of the equation in a similar way as in [22]. Stability is preserved, despite the nonzero layer width, thanks to a monotonicity property induced in the layer by the additional assumption $(\frac{d}{dx}U_0)_+ \in L^2(\mathbb{R})$.

2. Evolution of the relative entropy. For $\delta > 0$, we consider a Lipschitz nondecreasing function ϕ to localize the layer, verifying

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq \delta. \end{cases}$$

To get the optimal result, we will later fix a special function (see (4.1)).

For any fixed $\delta > 0$ and $X \in C^1([0, T])$, we are interesting in the evolution of

$$(2.1) \quad H(t) := \int_{-\infty}^{\infty} \phi^2(|x - X(t)|/\varepsilon) \eta(U(t,x)|S(t,x)) dx,$$

where $S(t,x) := S_0(x - X(t))$ and where S_0 is defined in (1.2). A special value of δ (depending on ε) and of the function X will be chosen later. Note that $H(t)$ controls the quantity (1.12). In fact, we have (1.12) $\leq H(t)$.

Let us denote $F(\cdot, \cdot)$ the flux of the relative entropy $\eta(\cdot|\cdot)$ defined by

$$(2.2) \quad F(x, y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)).$$

Note that

$$\begin{aligned} \partial_U \eta(U|C) &= \eta'(U) - \eta'(C), \\ \partial_U F(U, C) &= G'(U) - \eta'(C)A'(U) = (\eta'(U) - \eta'(C))A'(U). \end{aligned}$$

So, for any solution U of (1.1) and any constant C , multiplying (1.1) by $\eta'(U) - \eta'(C)$, we get

$$(2.3) \quad \partial_t \eta(U|C) + \partial_x F(U, C) = \varepsilon(\eta'(U) - \eta'(C))\partial_{xx}^2 U.$$

We have the following lemma.

LEMMA 2.1. *The function H , defined in (2.1), satisfies the following on $(0, T)$:*

$$\begin{aligned} \frac{dH}{dt}(t) &= \int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon} \phi\left(\frac{-x+X(t)}{\varepsilon}\right) \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U(t,x)|C_L) \right. \\ &\quad \left. - F(U(t,x), C_L) \right] dx \\ &\quad + \varepsilon \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \partial_{xx}^2 U(t,x)(\eta'(U(t,x)) - \eta'(C_L)) dx \\ &\quad - \int_{X(t)}^{X(t)+\delta\varepsilon} \frac{2}{\varepsilon} \phi\left(\frac{x-X(t)}{\varepsilon}\right) \phi'\left(\frac{x-X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U(t,x)|C_R) \right. \\ &\quad \left. - F(U(t,x), C_R) \right] dx \\ &\quad + \varepsilon \int_{X(t)}^{\infty} \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right) \right]^2 \partial_{xx}^2 U(t,x)(\eta'(U(t,x)) - \eta'(C_R)) dx \\ &:= (L)Hyp + (L)Dif + (R)Hyp + (R)Dif. \end{aligned}$$

Proof. First we split the term $H(t)$ into the two parts:

$$\begin{aligned} H(t) &= \int_{-\infty}^{\infty} \left[\phi\left(\frac{|x-X(t)|}{\varepsilon}\right) \right]^2 \eta(U|S) dx \\ &= \int_{-\infty}^{\infty} \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 + \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right) \right]^2 \right) \eta(U|S) dx \\ &= \int_{-\infty}^{\infty} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \eta(U|C_L) dx + \int_{-\infty}^{\infty} \left[\phi\left(\frac{x-X(t)}{\varepsilon}\right) \right]^2 \eta(U|C_R) dx \\ &:= H^L + H^R. \end{aligned}$$

To compute $\frac{d}{dt}(H^L)$, we put $C = C_L$ in (2.3), multiply by $\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2$, and integrate in x . Then we have

$$\begin{aligned} \frac{d}{dt}(H^L) &= \int_{-\infty}^{\infty} \partial_t \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \right) \eta(U|C_L) dx \\ &\quad + \int_{-\infty}^{\infty} \partial_x \left(\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \right) F(U, C_L) dx \\ &\quad + \varepsilon \int_{-\infty}^{\infty} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \partial_{xx}^2 U(\eta'(U) - \eta'(C_L)) dx \\ &= \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon} \right) \cdot \phi\left(\frac{-x+X(t)}{\varepsilon}\right) \phi'\left(\frac{-x+X(t)}{\varepsilon}\right) \left[\dot{X}(t)\eta(U|C_L) - F(U, C_L) \right] dx \\ &\quad + \varepsilon \cdot \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \right]^2 \partial_{xx}^2 U(\eta'(U) - \eta'(C_L)) dx = (L)Hyp + (L)Dif \end{aligned}$$

We get the result for $\frac{d}{dt}(H^R) = (R)Hyp + (R)Dif$ in the same way. \square

3. Control of the hyperbolic terms. In this section, we show that by choosing a special drift function $X(\cdot)$, the hyperbolic effects become nonpositive. This will be used in section 4 to control the parabolic effects.

Following [22], we define the normalized relative entropy flux $f(\cdot, \cdot)$ by

$$f(x, y) := \frac{F(x, y)}{\eta(x|y)}.$$

We have the following properties.

LEMMA 3.1. *For any $L > 0$, there exists a constant $\Lambda > 0$, such that for any x, y with $|x|, |y| \leq L$, we have*

$$(3.1) \quad \begin{aligned} 1/\Lambda &\leq \eta''(x) \leq \Lambda, \\ \frac{1}{2\Lambda}(x - y)^2 &\leq \eta(x|y) \leq \frac{1}{2}\Lambda(x - y)^2, \\ |F(x, y)| &\leq \Lambda(x - y)^2, \\ 0 &\leq (\partial_x f)(x, y) \leq \Lambda, \\ 1/\Lambda &\leq (\partial_y f)(x, y). \end{aligned}$$

The proof of this lemma can be found in [22].

We now define the shift function X . It is the solution of the following ODE:

$$(3.2) \quad \begin{cases} \dot{X}(t) = f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right), \\ X(0) = 0. \end{cases}$$

Note that for any $\varepsilon > 0$, $U \in C^1([0, T] \times \mathbb{R})$ (since $U_0 \in C^1(\mathbb{R})$). The existence and uniqueness of X comes from the Cauchy–Lipschitz theorem.

First, X is Lipschitz, since we have from Lemma 3.1

$$(3.3) \quad |\dot{X}(t)| \leq \frac{\left|F\left(U(t, X(t)), \frac{C_L + C_R}{2}\right)\right|}{\eta\left(U(t, X(t))\left|\frac{C_L + C_R}{2}\right.\right)} \leq 2\Lambda^2,$$

where we used the fact $\|U(t)\|_{L^\infty} \leq \|U_0\|_{L^\infty}$ for $t > 0$. It proves (1.10).

Note that from the definition of X , if the velocity U were constant in the layer (that is, $U(t, x) \sim U(t, X(t))$ for $-\delta\varepsilon \leq x - X(t) \leq \delta\varepsilon$), then, from the last property of Lemma 3.1, we would have directly that

$$(L)H_{yp} + (R)H_{yp} \leq -\frac{C_L - C_R}{\Lambda}(\eta(U(t, X(t))|C_L) + \eta(U(t, X(t))|C_R)).$$

However, this is too much to hope, since the layer characterizes the region where the function $U(t, \cdot)$ is expected to drop from about C_L to about C_R . We still can show that the hyperbolic terms are negative, provided that the behavior of U in the layer is not too oscillatory (the values can drop, but do not bounce back much). This last property of U is proved in the following lemma, which can be seen as a weak version of the Oleřnik’s principle.

LEMMA 3.2. $\|(\partial_x U(t))_+\|_{L^2(\mathbb{R})} \leq \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}$ for any $t > 0$.

Proof. We differentiate (1.1) w.r.t. x , multiply $(\partial_x U)_+$, and integrate in x to get

$$\begin{aligned} 0 &= \int (\partial_x U)_+ \left[\partial_t \partial_x U + A''(U) |\partial_x U|^2 + A'(U) \partial_{xx}^2 U - \varepsilon \partial_{xxx}^3 U \right] dx \\ &= \int \left[\frac{1}{2} \partial_t ([(\partial_x U)_+]^2) + A''(U) (\partial_x U)_+^3 \right. \\ &\quad \left. + A'(U) \partial_x \left(\frac{[(\partial_x U)_+]^2}{2} \right) + \varepsilon |\partial_x ((\partial_x U)_+)|^2 \right] dx. \end{aligned}$$

Then, we use integration by parts to get

$$\begin{aligned} &= \int \left[\frac{1}{2} \partial_t ([(\partial_x U)_+]^2) + \frac{1}{2} A''(U) (\partial_x U)_+^3 + \varepsilon |\partial_x ((\partial_x U)_+)|^2 \right] dx \\ &\geq \frac{1}{2} \frac{d}{dt} \int [(\partial_x U)_+]^2 dx. \quad \square \end{aligned}$$

Remark 3.3. The result of the above lemma can be extended up to the case L^p for any $1 \leq p \leq \infty$. Indeed, for any finite p , we just multiply $((\partial_x U)_+)^{p-1}$ instead of $(\partial_x U)_+$ in the proof. Then the limit case $p = \infty$ follows directly.

We now prove the main proposition of this section.

PROPOSITION 3.4. *Let $(L)_{Hyp}$ and $(R)_{Hyp}$ be such as in Lemma 2.1. There exists a constant $\theta > 0$ such that, for any ε, δ satisfying*

$$\varepsilon \delta \leq \theta,$$

we have

$$\begin{aligned} &(L)_{Hyp} + (R)_{Hyp} \\ &\leq -\frac{\theta}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \phi \left(\frac{|x - X(t)|}{\varepsilon} \right) \phi' \left(\frac{|x - X(t)|}{\varepsilon} \right) (U(t, x) - S(t, x))^2 dx. \end{aligned}$$

Proof. We use the definition of $X(t)$ to get

$$(L)_{Hyp} = \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon} \right) \cdot \phi \left(\frac{-x + X(t)}{\varepsilon} \right) \cdot \phi' \left(\frac{-x + X(t)}{\varepsilon} \right) \cdot \eta(U|C_L) \cdot h(t, x) dx,$$

where $h(t, x) := [f(U(t, X(t)), \frac{C_L + C_R}{2}) - f(U(t, x), C_L)]$.

In order to make the function $h(t, x)$ strictly negative over the domain of the above integral, we use the condition $(\frac{d}{dx} U_0)_+ \in L^2(\mathbb{R})$. Indeed, we observe that, for any $x \in [X(t) - \delta\varepsilon, X(t)]$,

$$\begin{aligned} (3.4) \quad U(t, X(t)) - U(t, x) &= \int_x^{X(t)} (\partial_x U)(t, y) dy \leq \int_x^{X(t)} (\partial_x U)_+(t, y) dy \\ &\leq \|(\partial_x U(t))_+\|_{L^2(\mathbb{R})} \sqrt{|X(t) - x|} \leq \left\| \left(\frac{d}{dx} U_0 \right)_+ \right\|_{L^2(\mathbb{R})} \sqrt{\delta\varepsilon}, \end{aligned}$$

where we used that $\|(\partial_x U(t))_+\|_{L^2}$ is not increasing (see Lemma 3.2).

We can rewrite the function h as

$$h(t, x) = f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right) - f\left(U(t, x), \frac{C_L + C_R}{2}\right) + f\left(U(t, x), \frac{C_L + C_R}{2}\right) - f(U(t, x), C_L).$$

Since f is increasing with respect to the first variable, we have

$$h(t, x) \leq f\left(U(t, x) + \left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}\sqrt{\delta\varepsilon}, \frac{C_L + C_R}{2}\right) - f\left(U(t, x), \frac{C_L + C_R}{2}\right) + f\left(U(t, x), \frac{C_L + C_R}{2}\right) - f(U(t, x), C_L).$$

Then, thanks to Lemma 3.1, we get

$$h(t, x) \leq \Lambda\left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}\sqrt{\delta\varepsilon} - \frac{C_L - C_R}{2\Lambda} \leq -\theta < 0$$

for $\sqrt{\delta\varepsilon}$ and θ small enough.

Since $\phi(\cdot), \phi'(\cdot)$ and $\eta(\cdot) \geq 0$, we get

$$(L)_{Hyp} \leq -\theta \int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon} \phi\left(\frac{-x + X(t)}{\varepsilon}\right) \phi'\left(\frac{-x + X(t)}{\varepsilon}\right) \eta(U|C_L) dx.$$

Then, from Lemma 3.1, we have (changing the constant θ if necessary)

$$(L)_{Hyp} \leq -\theta \int_{X(t)-\delta\varepsilon}^{X(t)} \left(\frac{2}{\varepsilon}\right) \phi\left(\frac{-x + X(t)}{\varepsilon}\right) \phi'\left(\frac{-x + X(t)}{\varepsilon}\right) (U - C_L)^2 dx.$$

In a similar way, we obtain the following estimate on $(II)_{Hyp}$:

$$(R)_{Hyp} \leq -\theta \int_{X(t)}^{X(t)+\delta\varepsilon} \left(\frac{2}{\varepsilon}\right) \phi\left(\frac{x - X(t)}{\varepsilon}\right) \phi'\left(\frac{x - X(t)}{\varepsilon}\right) (U - C_R)^2 dx.$$

Combining the two last inequalities gives the desired result. \square

4. Control of the parabolic terms. For any $\delta \geq 1/\theta$, we now fix the function ϕ in the following explicit way:

$$(4.1) \quad \phi(x) = \begin{cases} \theta e^{1-\theta\delta} x & \text{for } x \in [0, 1/\theta], \\ e^{\theta(x-\delta)} & \text{for } x \in [1/\theta, \delta]. \end{cases}$$

We will use the straightforward computation

$$(4.2) \quad \int_0^\delta (\phi'(x))^2 \chi_{\{\phi' > \theta\phi\}} dx = C_\theta \cdot e^{-2\theta\delta}.$$

This section is dedicated to the proof of the following proposition.

PROPOSITION 4.1. *There exists constants $\theta, C > 0$ such that for any ε, δ verifying*

$$\frac{1}{\theta} \leq \delta \quad \text{and} \quad \varepsilon\delta \leq \theta,$$

we have

$$\frac{dH(t)}{dt} \leq Ce^{-\theta\delta}.$$

Proof. First, we estimate the term $(L)_{Dif}$. Integrating by parts, we obtain

$$\begin{aligned} (L)_{Dif} &= \int_{-\infty}^{X(t)} 2\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\phi'\left(\frac{-x + X(t)}{\varepsilon}\right)\partial_x U(\eta'(U) - \eta'(C_L))dx \\ &\quad - 2\varepsilon \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 \eta''(U)|\partial_x U|^2 dx. \end{aligned}$$

Then, using Hölder’s inequality and Lemma 3.1, we get

$$\begin{aligned} (L)_{Dif} &\leq \frac{2\varepsilon}{\Lambda} \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 |\partial_x U|^2 dx \\ &\quad + \frac{\Lambda}{8\varepsilon} \int_{-\infty}^{X(t)} \left[2\phi'\left(\frac{-x + X(t)}{\varepsilon}\right)(\eta'(U) - \eta'(C_L))\right]^2 dx \\ &\quad - \frac{2\varepsilon}{\Lambda} \int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 |\partial_x U|^2 dx \\ &\leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)} \left[\phi'\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 |U - C_L|^2 dx. \end{aligned}$$

In the same way, we obtain the following estimate for $(R)_{Dif}$:

$$(R)_{Dif} \leq \frac{C}{\varepsilon} \int_{X(t)}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{x - X(t)}{\varepsilon}\right)\right]^2 |U - C_R|^2 dx.$$

Combining the two last inequalities, we find

$$(4.3) \quad (L)_{Dif} + (R)_{Dif} \leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{|x - X(t)|}{\varepsilon}\right)\right]^2 |U(t, x) - S(t, x)|^2 dx.$$

Using Lemma 2.1, Proposition 3.4, and (4.3), we find

$$(4.4) \quad \frac{dH(t)}{dt} \leq \frac{1}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi'(C\phi' - \theta\phi)\right] \left(\frac{|x - X(t)|}{\varepsilon}\right) |U(t, x) - S(t, x)|^2 dx.$$

Using that $U - S$ is a bounded function, and doing the change of variables $z = (x - X(t))/\varepsilon$, we find

$$\begin{aligned} \frac{dH(t)}{dt} &\leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[(\phi')^2 \chi_{\{C\phi' - \theta\phi > 0\}}\right] \left(\frac{|x - X(t)|}{\varepsilon}\right) |U(t, x) - S(t, x)|^2 dx \\ &\leq \frac{C\|U(t) - S(t)\|_{L^\infty}^2}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[(\phi')^2 \chi_{\{C\phi' - \theta\phi > 0\}}\right] \left(\frac{|x - X(t)|}{\varepsilon}\right) dx \\ &\leq C \int_0^\delta (\phi')^2(z) \chi_{\{C\phi' - \theta\phi > 0\}}(z) dz. \end{aligned}$$

Changing the constant θ if needed, and using (4.2), gives the desired result. □

5. Proof of Theorem 1.5. This section is devoted to the proof of Theorem 1.5. Integrating the estimate of Proposition 4.1 between 0 and $t \in (0, T)$ gives the result of (I). Indeed, for any ε, δ with $\frac{1}{\theta} \leq \delta$ and $\varepsilon\delta \leq \theta$, where θ is the constant from Proposition 4.1, we have

$$\begin{aligned} \int_{\{|x-X(t)| \geq \delta\varepsilon\}} \eta(U(t, x)|S(t, x)) \, dx &\leq H(t) \leq H(0) + \int_0^t \frac{d}{dt} H(s) \, ds \\ &\leq \int_{\mathbb{R}} \eta(U_0|S_0) \, dx + CT e^{-\theta\delta}. \end{aligned}$$

By taking $\varepsilon_0 := \theta^2$, we have for any $\varepsilon \leq \alpha \leq \varepsilon_0$,

$$\int_{\{|x-X(t)| \geq \alpha/\theta\}} \eta(U(t, x)|S(t, x)) \, dx \leq \int_{\mathbb{R}} \eta(U_0|S_0) \, dx + CT e^{-\alpha/\varepsilon}.$$

It proves our main estimate (1.9) by taking C^* large enough.

Observe that

$$\int_{\mathbb{R}} \eta(U|S) \, dx = \int_{\{|x-X(t)| \geq C^*\alpha\}} \eta(U|S) \, dx + \int_{\{|x-X(t)| < C^*\alpha\}} \eta(U|S) \, dx$$

and the second term is bounded by $CC^*\alpha$. Thus, by taking $\alpha = \varepsilon \log(1/\varepsilon)$, we obtain for any $t \in (0, T)$

$$(5.1) \quad \int_{\mathbb{R}} \eta(U|S) \, dx \leq \int_{\mathbb{R}} \eta(U_0|S_0) \, dx + C^* \varepsilon \log(1/\varepsilon)$$

for any $\varepsilon \leq \varepsilon_0$ (changing ε_0 and C^* if needed).

It only remains to prove (1.11). We define first

$$\psi(x) := \begin{cases} 0 & \text{if } |x| > 2, \\ 1 & \text{if } |x| \leq 1, \\ 2 - |x| & \text{if } 1 < |x| \leq 2. \end{cases}$$

Let $s \in (0, t)$ and $R > 0$. We multiply $\Psi_R(s, x) := \psi(\frac{x-X(s)}{R})$ to (1.1) and integrate in x to get

$$\begin{aligned} 0 &= -\frac{d}{ds} \int \Psi_R \cdot U \, dx + \int \partial_x(\Psi_R) A(U) \, dx + \int \partial_t(\Psi_R) U \, dx + \varepsilon \int \Psi_R \cdot \partial_{xx}^2 U \, dx \\ &= -\underbrace{\frac{d}{ds} \int \psi\left(\frac{x-X(s)}{R}\right) \cdot U(s, x) \, dx}_{(I)} \\ &\quad + \underbrace{\frac{1}{R} \int \psi'\left(\frac{x-X(s)}{R}\right) \cdot (A(U(s, x)) - \dot{X}(s)U(s, x)) \, dx}_{(II)} \\ &\quad - \varepsilon \underbrace{\frac{1}{R} \int \psi'\left(\frac{x-X(s)}{R}\right) \cdot \partial_x U(s, x) \, dx}_{(III)}. \end{aligned}$$

By using the above observation, we have

$$\begin{aligned} (\sigma - \dot{X}(s)) &= \frac{1}{C_L - C_R} \left(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) \right) \\ &= \frac{1}{C_L - C_R} \left(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (\text{II}) + (\text{I}) + (\text{III}) \right). \end{aligned}$$

Then we integrate the above equation in time on $[0, t]$ to get

$$\begin{aligned} (5.2) \quad |\sigma t - X(t)| &\leq C \left(t \cdot \max_{s \in (0, t)} \underbrace{\left| A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (\text{II}) \right|}_{(\text{II}')} \right. \\ &\quad \left. + \left| \int_0^t (I) ds \right| + t \cdot \max_{s \in (0, t)} \left| (\text{III}) \right| \right). \end{aligned}$$

We observe

$$\begin{aligned} (\text{II}') &\leq \underbrace{\left| A(C_L) - A(C_R) - \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot A(U) dx \right|}_{(\text{II}'_1)} \\ &\quad + \underbrace{\left| - (C_L - C_R)\dot{X}(s) + \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot (\dot{X}(s)U(s, x)) dx \right|}_{(\text{II}'_2)}. \end{aligned}$$

For (II'_1) , we compute

$$\begin{aligned} (\text{II}'_1) &= \left| A(C_L) - \frac{1}{R} \int_{-2R+X(s)}^{-R+X(s)} A(U) dx - A(C_R) + \frac{1}{R} \int_{R+X(s)}^{2R+X(s)} A(U) dx \right| \\ &\leq \frac{1}{R} \left[\int_{-2R+X(s)}^{-R+X(s)} |A(C_L) - A(U)| dx + \int_{R+X(s)}^{2R+X(s)} |A(U) - A(C_R)| dx \right]. \end{aligned}$$

We use $|A(y) - A(z)| \leq C|y - z|$ for $|y|, |z| \leq M_1$ to get

$$\leq \frac{C}{R} \int_{-2R+X(s)}^{2R+X(s)} |U - S| dx.$$

We use Hölder's inequality and Lemma 3.1 to get

$$(\text{II}'_1)^2 \leq \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s)|S(s)) dx.$$

Likewise, for the second term (II'_2) , we have

$$\begin{aligned} (\text{II}'_2) &= \left| \dot{X}(s) \cdot \left| - (C_L - C_R) + \frac{1}{R} \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot U(s, x) dx \right| \right| \\ &\leq \frac{C}{R} \int_{-2R+X(s)}^{2R+X(s)} |U - S| dx \leq \frac{C}{\sqrt{R}} \cdot \|U(s) - S(s)\|_{L^2(\mathbb{R})}, \end{aligned}$$

where we used $|\dot{X}(s)| \leq C$. Thus we get

$$(5.3) \quad (\text{II}')^2 \leq \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s)|S(s)) dx.$$

On the other hand, we compute

$$\begin{aligned} \left| \int_0^t (\text{I}) ds \right| &= \left| \int \psi \left(\frac{x - X(t)}{R} \right) \cdot U(t, x) dx - \int \psi \left(\frac{x}{R} \right) \cdot U_0(x) dx \right| \\ &= \left| \int \psi \left(\frac{x - X(t)}{R} \right) \cdot (U(t, x) - S(t, x)) dx + \int \psi \left(\frac{x - X(t)}{R} \right) \cdot S(t, x) dx \right. \\ &\quad \left. - \int \psi \left(\frac{x}{R} \right) \cdot S_0(x) dx - \int \psi \left(\frac{x}{R} \right) \cdot (U_0(x) - S_0(x)) dx \right|. \end{aligned}$$

Note that $\int \psi \left(\frac{x - X(t)}{R} \right) \cdot S(t, x) dx = \int \psi \left(\frac{x}{R} \right) \cdot S_0(x) dx$. Thus, we have

$$\leq \left| \int \psi \left(\frac{x - X(t)}{R} \right) \cdot (U(t, x) - S(t, x)) dx \right| + \left| \int \psi \left(\frac{x}{R} \right) \cdot (U_0(x) - S_0(x)) dx \right|.$$

We use Hölder's inequality and Lemma 3.1 to get

$$(5.4) \quad \left| \int_0^t (\text{I}) ds \right|^2 \leq CR \left(\int_{\mathbb{R}} \eta(U(t)|S(t)) dx + \int_{\mathbb{R}} \eta(U_0|S_0) dx \right).$$

Also, we have

$$(5.5) \quad \begin{aligned} |(\text{III})| &= \frac{\varepsilon}{R} \left| \int \psi' \left(\frac{x - X(s)}{R} \right) \cdot \partial_x U(s, x) dx \right| \\ &= \frac{\varepsilon}{R} \left| \int_{-2R+X(s)}^{-R+X(s)} \partial_x U(s, x) dx - \int_{R+X(s)}^{2R+X(s)} \partial_x U(s, x) dx \right| \\ &\leq \frac{\varepsilon}{R} \cdot 4 \cdot \|U(s)\|_{L^\infty} \leq \frac{C \cdot \varepsilon}{R}. \end{aligned}$$

Finally, by using (5.1), we combine (5.3), (5.4), and (5.5) with (5.2) to get, for any $t \in (0, T)$,

$$|\sigma t - X(t)|^2 \leq C \left(\frac{t^2}{R} + R \right) \cdot \left(\int_{\mathbb{R}} \eta(U_0|S_0) dx + \varepsilon \log \left(\frac{1}{\varepsilon} \right) \right) + \frac{C \cdot \varepsilon^2 \cdot t^2}{R^2}.$$

Since the above estimate holds for any $0 < R < \infty$, the estimate (1.11) follows once we take $R := t^{2/3}$ (changing C^* if needed).

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