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Scale-dependent behavior of scale equations

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We propose a new mathematical framework to formulate scale structures of general systems. Stack equations characterize a system in terms of accumulative scales. Their behavior at each scale level is determined independently without referring to other levels. Most standard geometries in mathematics can be reformulated in such stack equations. By involving interaction between scales, we generalize stack equations into scale equations. Scale equations are capable to accommodate various behaviors at different scale levels into one integrated solution. On contrary to standard geometries, such solutions often reveal eccentric scale-dependent figures, providing a clue to understand multiscale nature of the real world. Especially, it is suggested that the Gaussian noise stems from nonlinear scale interactions. © 2009 American Institute of Physics. [DOI: [10.1063/1.3207822](https://doi.org/10.1063/1.3207822)]

Most real world phenomena that interest us have multiscales in nature. In a system such as biological organs and composite materials, the system's characteristic behaviors vary with spatial/temporal scale levels and are governed by seemingly independent laws constrained to the corresponding level. However, in spite of such ubiquity of multiscales, there has been not much work toward developing a general theory on scale structures. In this paper, we introduce stack equations and scale equations as a mathematical framework to deal with interactions between scales and phenomena arising from them. Most standard mathematical functions can be reformulated in terms of accumulative scales (stacks). Their behavior at each scale level is determined in an isolated manner without referring to other levels. In this respect, conventional mathematical tools are not suitable for multiscale modeling. On the contrary, scale equations describe how adjacent scales are linked. They enable us to integrate various scale-dependent behaviors into one solution. In many cases, a single scale equation creates an eccentric figure that changes its behavior depending on scale levels. Further, scale equations are flexible enough to be combined with conventional modeling tools, for example, differential equations.

I. INTRODUCTION

It is a challenging task in scientific modeling to successfully capture the multiscale aspect of systems. Bridging distinct scales often brings great attentions, especially when those scales were previously regarded as unrelated and only accessible by different approaches.⁵ However, due to the difficulty entailed with studying interactions between scales, researchers often resign to focus on relatively narrow ranges of scale levels that characterize a specific process. This limit partially reflects the absence of appropriate mathematical language to deal with multiscale phenomena.²⁻⁴ Although

multiscale problems have long been studied in mathematics, we still lack proper tools formulating distinct scientific knowledge at different levels into their effect on the full scale.

Let us take differential equations as an example. Differential equations are one of the fundamental tools in mathematics and arise in many areas of science and other disciplines. To describe a phenomenon, one usually focuses on a specific scale level where it is observed. By doing so, one actually assumes that the subscale structures and their effects are negligible. On the contrary, if one wants to induce a large-scale behavior simply from a subscale formulation, one has to handle enormously many equations or otherwise adopt statistical assumptions to reduce complexity.

In this paper, we are proposing a new mathematical framework, scale equations, to describe behaviors varying with scale levels. Figure 1 shows an example of scale-dependent behavior of a solution of a scale equation. Although the first graph just looks like a normal sine curve, as it is gradually zoomed in, one can see quite different landscapes emerging at each level of magnification. This naturally reminds us of the multiscale nature of the real world which is hard to capture by conventional modeling tools.

In Secs. II and III, based on the Haar systems, we are going to show that most common mathematical objects such as polynomial and exponential functions can be reformulated in terms of accumulative scales (stacks). Since the equation determines the behavior of the curve at each scale level independently, that is, involving no interaction with other levels, we say that the curve has “laminarily stacked.” On the contrary, geometries that cannot be represented by stack equations are said to have “cross scales.” Cross-scaled curves generally show a much complex behavior and often create a fascinating figure that changes its behavior levelwise as in Fig. 1. We will establish scale equations as a tool to deal with such cross-scaled structures.

To put scale equations into a practical use, further study on physical interpretation of scale interaction should be followed. However, several examples illustrated in this paper

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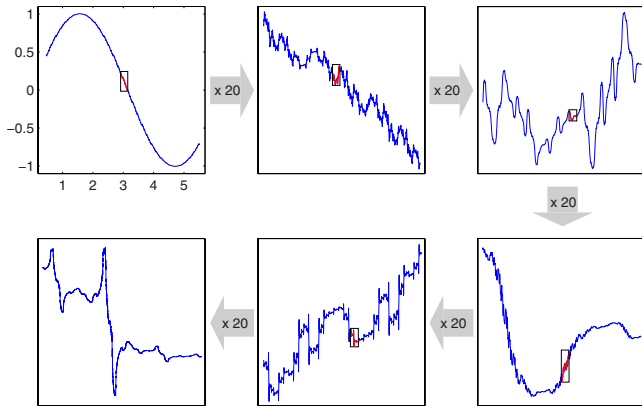


FIG. 1. (Color online) Gradual zoom-in on a solution curve of a scale equation. Along the arrows, each figure depicts 20 times of magnification in x of the boxed area in its predecessor. The graph is generated from the scale equation (5.2). The base stack condition is given as a sine curve $y = \sin x$ at a scale level 2.

provide clues to understand it. It is suggested in Sec. VI that nonlinear scale interactions are responsible for the Gaussian noise.

II. HAAR SYSTEMS

We briefly review some necessary background in the Haar systems. Let us define a *dyadic interval* as⁹

$$I_j^i = (2^{-i}j, 2^{-i}(j + 1)).$$

Here, i and j are level and shift index, respectively. The length of the dyadic interval is denoted by $h^i = |I_j^i| = 2^{-i}$. A *dyadic step function* is a function $\phi_j^i(x)$ that has $2^{i/2}$ on I_j^i and 0 elsewhere. That is,

$$\phi_j^i(x) := 2^{i/2} \phi(2^i x - j)$$

$$\text{where } \phi(x) := \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We also define a *Haar function* ψ_j^i as

$$\psi_j^i(x) := 2^{i/2} \psi(2^i x - j)$$

$$\text{where } \psi(x) := \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily shown that for a fixed k , $\{\phi_j^k, \psi_j^i\}_{i \geq k, j}$ is an orthonormal set. Let V^i and W^i be spaces spanned by $\{\phi_j^i\}_j$ and $\{\psi_j^i\}_j$, respectively. Then we have a multiscale structure as

$$V^{i+1} = V^i \oplus W^i$$

and

$$\dots \subset V^{-2} \subset V^{-1} \subset V^0 \subset V^1 \subset V^2 \subset \dots \subset L^2(\mathbf{R}).$$

Moreover,

$$L^2(\mathbf{R}) = \bigoplus_{i \in \mathbf{Z}} W^i = V^k \cup \left(\bigoplus_{i \geq k} W^i \right) \text{ for any } k.$$

Therefore, for a fixed integer k , we can decompose a function $f \in L^2(\mathbf{R})$ as

$$f(x) = \sum_{j \in \mathbf{Z}} c_j^k \phi_j^k(x) + \sum_{i \geq k, j \in \mathbf{Z}} d_j^i \psi_j^i(x). \tag{2.1}$$

The coefficients c_j^k and d_j^i are uniquely determined by $c_j^k = \langle f, \phi_j^k \rangle$ and $d_j^i = \langle f, \psi_j^i \rangle$. In this paper, instead of these coefficients, we adopt normalized coefficients

$$a_j^k = 2^{i/2} \langle f, \phi_j^k \rangle \text{ and } b_j^i = 2^{i/2} \langle f, \psi_j^i \rangle \tag{2.2}$$

for simplified computation. Let us call the coefficient a_j^k as *accumulative-scale variable* or *stack variable* of $f(x)$ at the level i . The coefficient b_j^i is called *scale variables*. The stack variable a_j^k represents an average of $f(x)$ on the interval I_j^k , or in other words, a value of $f(x)$ at the k th scale level. In the limit, we have

$$\lim_{k \rightarrow \infty, x \in I_j^k} |a_j^k - f(x)| = 0 \tag{2.3}$$

for a fixed j .

Note that stack variables also satisfy

$$a_j^i = \frac{a_{2j}^{i+1} + a_{2j+1}^{i+1}}{2} \tag{2.4}$$

between two adjacent scale levels. Moreover, from the relation between the dyadic step function and the Haar function, stack and scale variables inherit the conditions

$$a_{2j}^{i+1} = a_j^i + b_j^i \text{ and } a_{2j+1}^{i+1} = a_j^i - b_j^i. \tag{2.5}$$

III. CURVES AND SURFACES WITH LAMINAR STACKS

In this section, we are going to show that fundamental curves that commonly occur in mathematics, such as polynomial, exponential, and sine curves, share a common property in terms of scales. In these curves, the behavior at each scale level is essentially isolated with no scale interaction involved in these curves. In this respect, we say that the curves are “laminar.”

More precisely, a curve is said to be *laminarly stacked* if its stack variables $\{a_j^i\}_j$ at each level i satisfy a certain relation like

$$F(h^i; a_{j+r_1}^i, \dots, a_{j+r_m}^i) = 0, \quad i, j \in \mathbf{Z}, \tag{3.1}$$

where $r_1, \dots, r_m, m \geq 1$ are some distinct integers. Note that the equation consists of only stack variables at the same level i , excluding scale variables. We demand that the function F is continuous with respect to $h^i > 0$ and has a maximal rank m for the arguments $a_{j+r_1}^i, \dots, a_{j+r_m}^i$. An equation (3.1) that defines a relation between stack variables at the same scale level is called a *stack equation*.

For example, consider a quadratic function

$$y = C_1 x^2 + C_2 x + C_3, \tag{3.2}$$

where C_1, C_2, C_3 are arbitrary constants. This is actually a laminar curve. By integrating the function according to the

definition of stack variables in Eq. (2.2), one can obtain the system of equations for the four consecutive stack variables

$$\begin{aligned}
 a_j^i &= C_1\left(\frac{1}{3} + j + j^2\right)h^2 + C_2\left(\frac{1}{2} + j\right)h^i + C_3, \\
 a_{j+1}^i &= C_1\left(\frac{7}{3} + 3j + j^2\right)h^2 + C_2\left(\frac{3}{2} + j\right)h^i + C_3, \\
 a_{j+2}^i &= C_1\left(\frac{19}{3} + 5j + j^2\right)h^2 + C_2\left(\frac{5}{2} + j\right)h^i + C_3, \\
 a_{j+3}^i &= C_1\left(\frac{37}{3} + 7j + j^2\right)h^2 + C_2\left(\frac{7}{2} + j\right)h^i + C_3.
 \end{aligned}
 \tag{3.3}$$

Elimination of C_1, \dots, C_4 reduces Eq. (3.3) into a stack equation

$$a_j^i - 3a_{j+1}^i + 3a_{j+2}^i - a_{j+3}^i = 0, \quad i, j \in \mathbb{Z}.
 \tag{3.4}$$

In fact, the stack equation (3.4) can be regarded as an equivalent expression for Eq. (3.2). Once any three consecutive stack variables are fixed at a scale level i , the equation (3.4) generates all other stack variables one by one and the feature at the corresponding scale level is completely determined. Note that one does not have to assign three values at each scale level. Due to the requirement that the equation (3.4) should be across entire levels, $i \in \mathbb{Z}$, and also due to the condition (2.4), after the first three stack variables at a scale level i are fixed, stack variables at other levels are also automatically determined. This agrees well with the fact that the general quadratic curve in Eq. (3.2) also has three degrees of freedom. It is easy to extend the above result to arbitrary higher order polynomials.

It turned out that most fundamental functions commonly used in mathematics are laminarly stacked. Let us take an example of an exponential function

$$y = C \exp(x)
 \tag{3.5}$$

with a constant C . The corresponding stack equation is

$$a_{j+1}^i = \exp(h^i)a_j^i.
 \tag{3.6}$$

Another example is a cosine and sine curve,

$$y = C_1 \cos x + C_2 \sin x.
 \tag{3.7}$$

Regardless of the constants C_1 and C_2 , at the scale level i , the curve satisfies the stack equation

$$a_j^i = \frac{1}{2} \sec h^i (a_{j-1}^i + a_{j+1}^i).
 \tag{3.8}$$

Note that stack equations (3.4), (3.6), and (3.8) can be regarded as invariants under the transformation between adjacent scale levels in Eq. (2.4).

Above examples also suggest that solutions of many differential equations are laminar. Note that three curves (3.2), (3.5), and (3.7) are actually general solutions of differential equations

$$y''' = 0, \quad y' = y, \quad \text{and} \quad y'' = y,
 \tag{3.9}$$

respectively. Although it is generally not true that solutions of differential equations are laminarly stacked, they are bound to be *asymptotically laminar*. This means that there exists a certain function F satisfying

$$F(h^i; a_{j+k_1}^i, \dots, a_{j+k_m}^i) = O(h^i), \quad \text{as } i \rightarrow \infty.
 \tag{3.10}$$

Here, the only difference from the previous definition of stack equation is the symbol O adopted to describe asymptotic upper bound of the magnitude of F .

It is simple to confirm that solutions of differential equations, if any, are asymptotically laminar. For instance, a second order ordinary differential equation

$$y'' + y' + y = 0$$

allows

$$\frac{a_{j+2}^i - 2a_{j+1}^i + a_j^i}{h^2} + \frac{a_{j+2}^i - a_j^i}{2h} + a_{j+1}^i = O(h)$$

as an asymptotic stack equation. Such asymptotic stack equations can be obtained by the conventional discretization used in numerical analysis. It is also straightforward to show that solutions of partial differential equations are asymptotically laminar once we extend the Haar systems to a high dimensional space.

As a final remark, we point out that laminar curves are not limited to differentiable functions. Fractal curves generated by an affine transform naturally possess laminar stacks due to their invariance along scale levels, i.e., self-similarity. Roughly speaking, if integration of the function in the Haar system determines a relation of consecutive stack variables at each scale level independently, the function is laminar. Most L_2 functions we commonly use in mathematics are laminar. Let us consider the Weierstrass function

$$y = \sum_{j \in \mathbb{Z}} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is an odd integer, and $ab > 1 + \frac{3}{2}\pi$. This is a fractal function famous for being continuous everywhere, but differentiable only on a set of points of measure zero. By integrating it on a dyadic interval at a certain scale level, one can show that the stack variables at the corresponding scale level are completely determined, without consideration of stack variables at other levels. Therefore the Weierstrass function is laminarly stacked.

IV. SCALE EQUATIONS

In Sec. III, we showed that most standard geometries in mathematics are laminarly stacked and are characterized by stack equations. Since the laminar structures are generally not capable to handle scale-dependent variation, we may attribute difficulties in multiscale modeling to lack of mathematical tools to deal with scale interactions.

The goal of this section is to find a way to construct curves that are not laminar and behave variously in a scale-dependent manner. We say that a curve has *cross scales* if the curve is not (asymptotically) laminar. The curve in Fig. 1 has cross scales because there exists no stack equation for it.

Consider a function F of h^i , stack variables $a_{j+r_1}^i, \dots, a_{j+r_m}^i$, and scale variables $b_{j+s_1}^i, \dots, b_{j+s_n}^i$. Here r_1, \dots, r_m and s_1, \dots, s_n are sequences of distinct integers.

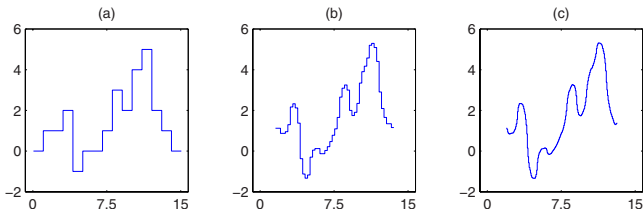


FIG. 2. (Color online) Construction of a solution of a scale equation from a base stack condition. From arbitrarily chosen base stack condition at a scale level 0 in (a), the graphs (b) and (c) at finer scale levels are derived by keeping applying the scale equation (4.3). Note that there are gradual loss of the curve at both ends as the stack equation is applied to a base condition given on a finite interval.

Assume that F satisfies the same condition (continuity and maximal rank) as in the definition of the stack equation (3.1). Now the equation

$$F(h^i; a_{j+r_1}^i, \dots, a_{j+r_m}^i; b_{j+s_1}^i, \dots, b_{j+s_n}^i) = 0, \quad i, j \in \mathbb{Z} \quad (4.1)$$

is called a *scale equation*.

It is notable that scale equations can be regarded as generalization stack equations, in that there is a corresponding scale equation for each stack equation. Stack equations form an exceptionally small subset of scale equations. It is hard to figure out under what condition a scale equation is reducible to a stack equation. This is understandable in that classification of laminarly stacked curves and cross-scaled curves is fundamental and might not be easily identified by other mathematical concepts.

Let us revisit the example of the quadratic polynomial (3.2). Integration according to the definition of a scale equation (2.2) gives

$$b_j^i = -\frac{1}{4}C_1(1 + 2j)h^{i^2} - \frac{1}{4}C_2h^i. \quad (4.2)$$

Combining this with the equations (3.3), we obtain a scale equation

$$b_j^i = \frac{a_{j-1}^i - a_{j+1}^i}{8}. \quad (4.3)$$

Note that all curves generated from the stack equation (3.4) are also solutions of the scale equation (4.3).

Another example of scale equations is

$$b_j^i = \frac{1}{8} \sec^2\left(\frac{h^i}{4}\right) \sec\left(\frac{h^i}{2}\right) (a_{j-1}^i - a_{j+1}^i), \quad (4.4)$$

which matches with the stack equation (3.8).

As mentioned before, the stack equation (3.4) is essentially the same as the quadratic polynomial (3.2) and therefore only allows a three-parametrized family of curves. However, it is less restrictive for the scale equation (4.3) and we can assign any five values at a certain level. This implies that quadratic polynomial is just a special case among possible solutions. Similarly, sine and cosine curves obtained from the stack equation (3.8) only make a small subset of solution of Eq. (4.4).

We furthermore have liberty to choose range of scale levels to apply scale equations. While stack equations are necessarily applied to all scale levels, $i \in \mathbb{Z}$ to keep consis-

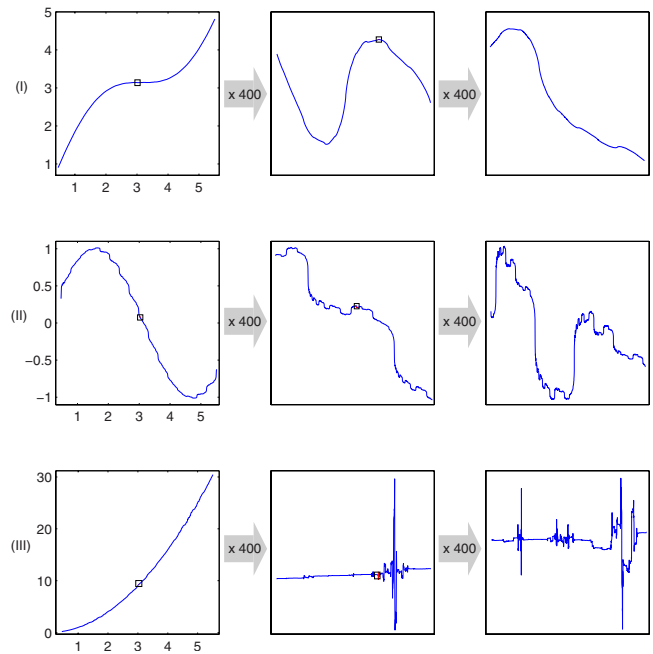


FIG. 3. (Color online) Three typical solutions of scale equations (4.4)–(4.6), respectively, and their gradual magnifications in x -axis. In each figure, the scale in y -axis is automatically adjusted to fit in the square frame. The base stack conditions are given at a scale level 2 as $y=x+\sin x$, $y=\sin x$, and $y=\exp x$ for Eqs. (4.4)–(4.6), respectively.

tency, one can apply scale equations to a certain scale level and finer, that is, $i \geq k$, instead of applying it to the whole scale levels. This practically gives infinite degree of freedom in constructing a solution curve from a given scale equation.

Once we determine stack variables $\dots a_{-1}^k, a_0^k, a_1^k, \dots$ at a certain base level k , then the stack variables at finer levels $k+1, k+2, \dots$ will be determined by the scale equation (4.1) and the condition (2.5). Obviously, the stack variables at coarser scale levels $k-1, k-2, \dots$ are also fixed by the condition (2.4). Such assignment of stack variables at a certain scale k is said to be a *base stack condition*. Base stack conditions in scale equations indicate predetermined behavior at a large scale level. They are a given condition and play a similar role of boundary conditions in partial differential equations. Figure 2 depicts the construction procedure of one possible solution of the equation (4.3) from a base stack condition.

Due to these flexibilities, solutions of scale equations often take a complex form across scale levels. One of typical solutions is illustrated in Fig. 3(I). Although the curve is seemingly not very different from a simple sine curve in the large scale level, 400 times of magnification shows that there are some fluctuations in a certain segment of the curve. These fluctuations occur irregularly depending on site and scale level and cause cross-scaled structure.

Another two typical examples of cross-scaled curves generated by scale equations and base conditions are shown in Fig. 3. The graphs in Fig. 3(II) are a solution of the scale equation

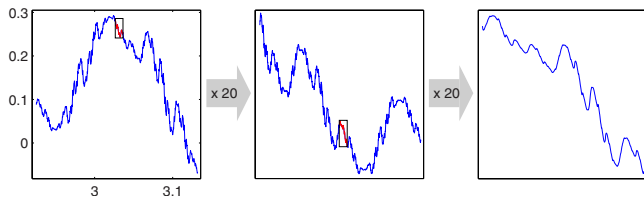


FIG. 4. (Color online) Fading away of fractality in gradual magnification of the scale equation (5.1). The base stack condition $y=\sin x$ is given at a scale level 2.

$$b_j^i = \frac{a_{j-1}^i - a_j^i}{16}. \tag{4.5}$$

The corresponding base condition is generated from a sine curve at a certain scale level.

It is straightforward to extend scale equations to multi-variable systems. Suppose that $(\{a_j^i\}, \{b_j^i\})$ and $(\{\tilde{a}_j^i\}, \{\tilde{b}_j^i\})$ are pairs of stack and scale variables for $y=f(x)$ and $y=\tilde{f}(x)$, respectively, and also suppose that they satisfy coupled scale equations

$$b_j^i = \frac{a_{j-1}^i - a_{j+1}^i}{4}, \tag{4.6}$$

$$\tilde{b}_j^i = \frac{\tilde{a}_{j-1}^i - \tilde{a}_{j+1}^i}{8} \frac{4h^i}{32h^i - a_{j-1}^i + a_{j+1}^i}.$$

The figures in Fig. 3(III) depict the graph of $y=\tilde{f}(x)$. Being zoomed in, the curve reveals more complex details consisting of peaks and dislocations.

We extend the definition (4.1) to the second order as

$$F(h^i; a_{j+r_1}^i, \dots, a_{j+r_m}^i; b_{j+s_1}^i, \dots, b_{j+s_n}^i; b_{j+t_1}^{i+1}, \dots, b_{j+t_l}^{i+1}) = 0,$$

$$i, j \in \mathbb{Z}$$

by involving scale variables $b_{j+t_1}^{i+1}, \dots, b_{j+t_l}^{i+1}$ at the next finer scale level $(i+1)$, where t_1, \dots, t_l are some distinct integers. Scale equations of second order or higher are especially called *scale interaction equations*. Examples of scale interaction equations are presented in Sec. VI.

V. CURVES WITH CROSS SCALES

In his famous book, Mandelbrot⁶ suggested that fractal is a universal principle underlying behind the irregular features of nature. However, there has been controversy on physical existence of fractals. Avnir *et al.*¹ argue that most experimental data reported as fractals actually carry the power law in only very limited range of scale levels.⁷ They claim that the real question on fractality of nature should be ‘‘Why are these limited-range fractal so common?’’ We can approach this issue in the broader context of scale-dependent behaviors of multiscale systems using scale equations.

Scale equations of the first order typically lead to a fractal-like shape as in Fig. 3(II). By having a scale equation involved with h^i , one can create various fractal features, including limited range of power law. Consider the scale equation

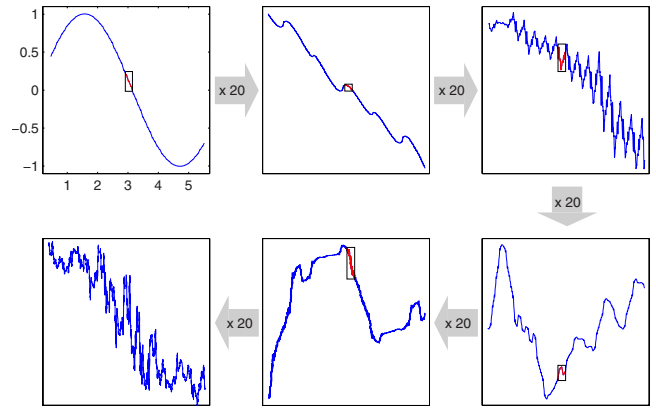


FIG. 5. (Color online) Scale-dependent behavior of a solution of Eq. (5.2). The base stack condition $y=\sin x$ is given at the scale level 2.

$$b_j^i = \frac{1}{1 + e^{-\lambda h^i}} \frac{a_{j-1}^i - a_{j+1}^i}{4}, \tag{5.1}$$

where λ is a positive constant. This equation converges to Eq. (4.3) as $h^i \rightarrow 0$. Figure 4 shows a solution of the equation and its magnified details when $\lambda = 10^5$. Gradually zooming in the curve, one can see that similar figures appear repeatedly along first several orders of magnifications. But this self-similarity soon fades away and the curve eventually turns into differential ones under a certain scale level.

As mentioned before, the concept of cross scales brings us beyond fractals or limited range of fractals. Figure 1 in Sec. I is an example of such cross-scaled curves that behave in a complex manner depending on scale levels. When using scale equations, one can create such complex cross-scaled curves, especially by dealing with h^i as a parameter in the equations. Let us put another example of such scale equations,

$$b_j^i = \left(1 + 2 \sin\left(\frac{\lambda}{h^i}\right) \right) \frac{a_{j-1}^i - a_{j+1}^i}{8}. \tag{5.2}$$

Since $\sin(\lambda/h^i)$ oscillates fast as h^i approaches zero, the first factor plays as an irregular perturbation on Eq. (4.3). Figure 5 depicts various levelwise behaviors of a solution of Eq. (5.2). The parameter λ is chosen properly for best visualization.

One can see that unique landscapes keep appearing and disappearing along scale levels. It is not hard to develop a scale equation so that intended effect occurs at a certain scale level.

VI. GAUSSIAN NOISE GENERATED BY SCALE INTERACTION

In this section, we claim that there is a notable connection between nonlinear scale interaction and noise, especially the Gaussian noise. The Gaussian noise is statistical noise that follows the normal distribution. It is a common hypothesis in physics and telecommunication to assume that noise of a measured signal takes a normal distribution.⁸ If the noise is generated from a collection of a great number of indepen-

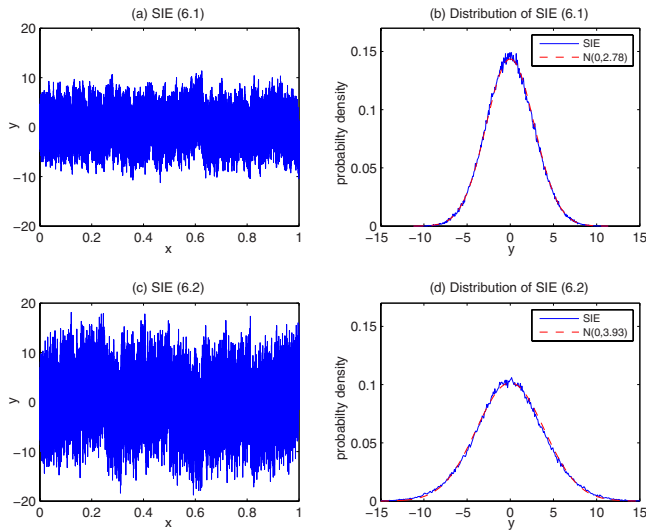


FIG. 6. (Color online) The graphs of two scale interaction equations (6.1) and (6.2) and the distributions of the values. The base stack condition $a_0^0=0$ at the scale level 0 is adopted in both cases.

dent random phenomena with properly bounded second moment, this hypothesis is supported by the central limit theorem.

Let us consider the two scale equations of the second order,

$$b_{j+1}^i = \sin(b_{[j/2]}^{i-1} + b_j^i + \overline{b_{2j}^{i+1}}) \tag{6.1}$$

and

$$b_{j+1}^i = \arctan(b_{[j/2]}^{i-1} + b_j^i + \overline{b_{2j}^{i+1}}), \tag{6.2}$$

where $\overline{b_{2j}^{i+1}} = b_{2j}^{i+1} + b_{2j+1}^{i+1}$. Note that there appear three adjacent scale levels ($i-1$), i , and ($i+1$) in Eqs. (6.1) and (6.2). The equations can be viewed as evolutionary equations, in that scale variable b_{j+1}^i at the scale level i is determined by its predecessors in three scale levels. In this respect, second order scale equations are often called scale interaction equations. Figures 6(a) and 6(c) depict the graphs of the scale equations (6.1) and (6.2), respectively, at the 18th scale level. Their distribution based on $2^{18}=26144$ stack variables a_j^{18} is shown in Figs. 6(b) and 6(d). The base stack condition is $a_0^0=0$ at the scale level 0. The tendency of graphs toward the Gaussian noise is well shown in the figures and becomes even clearer as one raises the scale levels. It turns out that this tendency is persistent under change in initial conditions.

Occurrence of the noise is not limited to the above equations. If a nonlinear scale equation involves scale variables at the different scale levels and its graph is properly bounded, the Gaussian noise is likely to appear. Note that the above scale equations are completely deterministic and are not associated with any conventional (pseudo) random number generation. One might think of these scale equations as a kind of random number generators based on accumulation of nonlinear computations. However, besides the fact that the equations do not involve modulus operations as most conventional random number generators, there is a major difference between them. While those generators are expected to produce “white” noises which have 0 autocorrelation, the

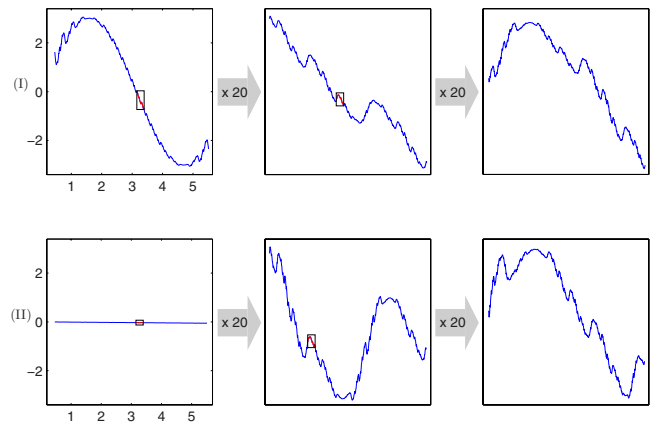


FIG. 7. (Color online) Two solutions of Eq. (7.1) generated from different base stack conditions. The base stack conditions $y=\sin x$ and $y=-0.01x$ are given at the scale level 2 for (I) and (II), respectively. Note that magnification in y-axis is adjusted to fit in the square frames.

noise generated by scale equations has large autocorrelation and therefore is not white noise. In this respect, we suspect that the origin of the nonwhite Gaussian noise is connected to nonlinear scale interactions. Further study on noise occurrence from scale interactions will follow in the future.

VII. COMBINING SCALE EQUATIONS AND DIFFERENTIAL EQUATIONS

Scale equations are very flexible in that one can easily combine them with traditional modeling methods. A possible combination of scale equations and differential equations is briefly sketched in this section.

Let us remind that the development of a cross-scaled curve generated by a first order scale equation requires a base stack condition. However, for a given scale equation, a qualitative feature of the curve at each scale level is relatively free from its base condition. This mutual independence of scale equations and base stack conditions are illustrated in Fig. 7. Both curves (I) and (II) are the solutions of the scale equation

$$b_j^i = \frac{a_{j-1}^i - a_{j+1}^i}{5}. \tag{7.1}$$

A base stack condition for the curve (I) is generated from a sine curve, while that for (II) from a slightly slanted line. Although two curves are totally different in a large scale level, they carry qualitatively similar features in finer levels.

This suggests that scale equations are naturally capable to combine with other modeling methods, especially differential equations. Although the notion of differential equations is based on the infinitesimal analysis, in practical cases, we often focus on their solution at a certain scale level where the model is expected to fit best. If we can turn the differential equations into a finite difference one at that scale level, this actually provides a base stack condition for scale equations. In other words, we apply differential equations at a certain scale level and up and scale equations at the level down. Therefore the qualitative aspect in finer scale levels is dominated by the scale equation, while the behavior at large scale levels is governed by the differential equations.

In this respect, curves (I) and (II) in Fig. 7 can be regarded as a combination of the scale equation (7.1) and the differential equations

$$y'' = y \quad \text{and} \quad y'' = 0,$$

respectively.

We believe that the concepts of laminarly stacked/cross-scaled structures expand the reservoir of mathematical functions. The future study on scale equations will focus on their connections to physical phenomena and development of a framework to handle various multiscale models, broadening our understanding of complex phenomena in nature.

¹Avnir, D., Biham, O., Lidar, D., and Malcai, O., "Is the geometry of nature fractal?" *Science* **279**, 39 (1998).

²Brandt, A., in *Multiscale and Multiresolution Methods: Theory and*

Applications, edited by Barth, T. J., Chan, T. F., and Haimes, R. (Springer-Verlag, Berlin, 2001), pp. 1–96.

³Cross, M., Croft, T. N., Slone, A. K., Williams, A. J., Christakis, N., Patel, M. K., Bailey, C., and Pericleous, K., "Computational modeling of multiphysics and multiscale processes in parallel," *Int. J. Comput. Methods Eng. Sci. Mech.* **8**, 63 (2007).

⁴Dolbow, J., Khaleel, M. A., and Mitchell, J., "Multiscale mathematics initiative: A roadmap," DOE Report No. PNNL-14966, 2004.

⁵E. W. and Engquist, B., "Multiscale modeling and computation," *Not. Am. Math. Soc.* **50**, 1062 (2003).

⁶Mandelbrot, B. B., *The fractal Geometry of Nature* (Freeman, San Francisco, 1982).

⁷Mandelbrot, B. B., Pfeifer, P., Biham, O., Malcai, O., Lidar, D. A., and Avnir, D., "Is nature fractal?," *Science* **279**, 783 (1998).

⁸Refregier, P., *Noise Theory and Application to Physics: From Fluctuations to Information* (Springer, New York, 2004).

⁹Wojtaszczyk, P., *A Mathematical Introduction to Wavelets* (Cambridge University Press, Cambridge, England, 1996).