Kim and Jang *Advances in Difference Equations* 2013, 2013:371
http://www.advancesindifferenceequations.com/content/2013/1/371

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A semi-analytic method with an effect of memory for solving fractional differential equations

Kyunghoon Kim and Bongsoo Jang*

*Correspondence:
bsjang@unist.ac.kr
Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Ulsan, 689-798, Republic of Korea

Abstract

In this paper, we propose a new modification of the multistage generalized differential transform method (MsGDTM) for solving fractional differential equations. In MsGDTM, it is the key how to impose an initial condition in each sub-domain to obtain an accurate approximate solution. In several literature works (Odibat *et al.* in Comput. Math. Appl. 59:1462-1472, 2010; Alomari in Comput. Math. Appl. 61:2528-2534, 2011; Gökdoğan *et al.* in Math. Comput. Model. 54:2132-2138, 2011), authors have updated an initial condition in each sub-domain by using the approximate solution in the previous sub-domain. However, we point out that this approach is hard to apply an effect of memory which is the basic property of fractional differential equations. Here we provide a new algorithm to impose the initial conditions by using the integral operator that enhances accuracy. Several illustrative examples are demonstrated, and it is shown that the proposed technique is robust and accurate for solving fractional differential equations.

1 Introduction

In recent years, fractional differential equations have been considered as an important mathematical modeling in various fields of applied sciences and engineering because of describing memory properties. Several numerical methods for solving fractional differential equations have been introduced. Authors in [1, 2] presented the predictor-corrector approach based on the Adams-Bashforth-Moulton type numerical method that has been successful to obtain the stable approximate solutions for many fractional differential equations. Some of semi-analytic methods for solving fractional problems such as the Adomian decomposition method (ADM) [3–5], homotopy analysis method (HAM) [6–8], homotopy perturbation method (HPM) [9, 10], variational iteration method (VIM) [11–15] and generalized differential transform method (GDTM) [16–18] have been introduced to provide analytic or numeric approximations. In this paper, we propose a new modification of multistage generalized differential transform method (MsGDTM) to obtain an accurate approximate solution for solving fractional differential equation. The new proposed method gives an algorithm to impose an accurate initial condition in each sub-domain which contains the effect of memory. The paper is organized as follows. Section 2 introduces some definitions and notations of fractional calculus that we shall use. In Section 3, we present the basic ideas and some properties of GDTM. Difference between the standard MsGDTM and the proposed MsGDTM are described in Section 4. Several numerical illustrations are demonstrated, and they show the effectiveness of the proposed method in Section 5. Finally, we give a conclusion.

2 Preliminary

In this section we give some basic definitions and properties of the fractional calculus presented in this work.

Definition A real-valued function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number p, $p < \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\mu$, $t > a \geq 0$, $\mu \geq -1$, is defined by

$$J_a^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases}$$

The operator J_a^α satisfies the following properties: For $f(t) \in C_\mu$, $t > a \geq 0$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$,

1. $J_a^\alpha J_a^\beta f(t) = J_a^\beta J_a^\alpha f(t) = J_a^{\alpha+\beta} f(t)$,
2. $J_a^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}$.

The Riemann-Liouville fractional derivative is defined by

$$D_a^\alpha f(t) = \frac{d^m}{dt^m}\left(J_a^{m-\alpha} f(t)\right),$$

where $m-1 < \alpha \leq m$ and $m \in \mathbb{N}$. The Riemann-Liouville fractional derivative has been studied by many mathematicians. However, it is not suitable to model real world physical phenomena because it has difficulties to define the fractional order physical conditions such as initial condition. Here, we shall introduce a modified fractional differential operator D_a^α proposed by Caputo [19].

Definition The fractional derivative in the Caputo sense of $f(t)$, $f(t) \in C_{-1}^m$, $t > a \geq 0$, $m \in \mathbb{N}$, $t > 0$ is defined by

$$D_a^\alpha f(t) = \begin{cases} J_a^{m-\alpha}\left(\frac{d^m}{dt^m} f(t)\right), & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

Lemma 2.1 *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f(t) \in C_\mu^m$, $t > a \geq 0$, $\mu \geq -1$, then*

1. $D_a^\alpha J_a^\alpha f(t) = f(t)$,
2. $J_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k}{dt^k} f(a) \frac{(t-a)^k}{k!}$.

3 Generalized differential transform method

The differential transform method (DTM) that is based on the Taylor series has been successful to achieve accurate approximate solutions for the linear and nonlinear problems [20–24]. It differs from the traditional Taylor series in calculating coefficients. In DTM, all coefficients in Taylor series can be determined by solving the recursive equation that is induced from the given differential equation. In order to apply the DTM to solve fractional problems, we adopt the generalized Taylor formula that was introduced in [17]. In what follows we describe the basic idea of the generalized differential transform method (GDTM) and its properties. Let us introduce the generalized Taylor series formula with the Caputo fractional derivative.

Theorem 3.1 (Generalized mean value theorem [17]) *Suppose that* $f(t) \in C[a,b]$ *and* $D_a^\alpha f(t) \in C(a,b]$, *for* $0 < \alpha \leq 1$, *we have*

$$f(t) = f(a) + \frac{1}{\Gamma(\alpha)} D_a^\alpha f(\eta)(t-a)^\alpha$$

with $a \leq \eta \leq t$ *for all* $t \in (a,b]$.

Let us define $(D_a^\alpha)^n$ by

$$\left(D_a^\alpha\right)^n = D_a^\alpha \cdot D_a^\alpha \cdots D_a^\alpha \ (n\text{-times}).$$

Theorem 3.2 (Generalized Taylor's formula [17]) *Suppose that* $(D_a^\alpha)^k f(t) \in C(a,b]$ *for* $k = 0,1,\dots,n+1$, *where* $0 < \alpha \leq 1$, *then we have*

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(\left(D_a^\alpha\right)^i f\right)(a) + \frac{((D_a^\alpha)^{n+1} f)(\eta)}{\Gamma((n+1)\alpha+1)} (t-a)^{(n+1)\alpha}$$

with $a \leq \eta \leq t$, *for all* $t \in (a,b]$.

Let us define the generalized differential transform (GDT) of the kth derivative of $f(t)$ at $t = t_0$ as follows:

$$F(k) = \frac{1}{\Gamma(\alpha k+1)} \left[\left(D_{t_0}^\alpha\right)^k f(t)\right]_{t=t_0},$$

where $0 < \alpha \leq 1$, $k = 0,1,2,\dots$, and the generalized differential inverse transform of $F(k)$ is defined as follows:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^{\alpha k}.$$

In case of $\alpha = 1$, the GDT reduces to the classical differential transform.

Theorem 3.3 *Several fundamental properties of the GDT are listed below* [18].

1. *If* $f(t) = g(t) \pm h(t)$, *then* $F(k) = G(k) \pm H(k)$.
2. *If* $f(t) = ag(t)$, *then* $F(k) = aG(k)$, *where* a *is a constant.*
3. *If* $f(t) = g(t)h(t)$, *then* $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$.
4. *If* $f(t) = D_{t_0}^\alpha g(t)$, *then* $F(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} G(k+1)$.
5. *If* $f(t) = (t-t_0)^{n\alpha}$, *then* $F(k) = \delta(k-n)$, *where* $\delta(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$

6. *If* $f(t) = D_{t_0}^{\beta} g(t)$, $m-1 < \beta \le m$, *then*

$$F(k) = \frac{\Gamma(\alpha k + \beta + 1)}{\Gamma(\alpha k + 1)} G\left(k + \frac{\beta}{\alpha}\right).$$

4 The multistage generalized differential transform method with an effect of memory

4.1 The standard generalized multistage differential transform method

The basic idea of multistage generalized differential transform method (MsGDTM) is to apply the standard GDTM to the given problem in each sub-domain. In order to describe the MsGDTM, let us consider the following fractional initial value problem:

$$D_{t_0}^{\alpha} y(t) = f\big(t, y(t)\big), \quad t > 0, y(t_0) = y_0. \tag{1}$$

In order to apply the GDTM, we assume that the solution $y(t)$ is expanded by the generalized Taylor series at $t = t_0$

$$y(t) = \sum_{k=0}^{\infty} Y(k)(t - t_0)^{\alpha k},$$

where

$$Y(k) = \frac{1}{\Gamma(\alpha k + 1)} \big[\big(D_{t_0}^{\alpha}\big)^k f(t)\big]_{t=t_0}.$$

Some fundamental properties in Theorem 3.3 give the following recursive relation for the differential transform:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k + 1)} Y(k+1) = F(k), \quad k = 0, 1, 2, \ldots, \tag{2}$$

where $F(k)$ is the generalized differential transform of $f(t, y(t))$ at $t = t_0$. Combined with $Y(0)$, that is, the initial condition $y(t_0)$, the recursive equation (2) can be easily solved. Recalling that GDTM is based on the generalized Taylor series, an approximate solution can be obtained in a radius of convergence R

$$R = |t - t_0|^{\alpha} \lim_{n \to \infty} \left| \frac{\Gamma(n\alpha + 1)}{\Gamma((n+1)\alpha + 1)} \cdot \frac{(D_{t_0}^{\alpha})^{n+1} f(t)}{(D_{t_0}^{\alpha})^{n} f(t)} \right|_{t=t_0}.$$

In other words, it is impractical to achieve an accurate approximation by using the GDTM outside the radius of convergence R. To overcome this difficulty, the standard GDTM is applied in each sub-domain, which is called the MsGDTM.

For the equally spaced partition P: $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, where the nodes $t_i = i \cdot h$, $h = T/N$. On the ith sub-domain $\Omega_i \equiv (t_i, t_{i+1})$, $i = 0, 1, \ldots, N-1$, we define $y(t)|_{\Omega_i} \equiv y_i(t)$ and $f(t)|_{\Omega_i} \equiv f_i(t)$. The generalized differential transform $Y_i(k)$ of $y_i(t)$ at $t = t_i$ is defined by

$$Y_i(k) = \frac{1}{\Gamma(\alpha k + 1)} \big[\big(D_{t_i}^{\alpha}\big)^k y_i(t)\big]_{t=t_i}.$$

The generalized differential inverse transform of $Y_i(k)$ is defined by

$$y_i(t) = \sum_{k=0}^{\infty} Y_i(k)(t - t_i)^{\alpha k}.$$

Then the generalized differential transform $Y_i(k)$ is obtained by solving the following recursive relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y_i(k+1) = F_i(k), \quad k = 0, 1, 2, \ldots, \tag{3}$$

where $F_i(k)$ is the generalized differential transform of $f(t, y(t))$ at $t = t_i$. Suppose that s_{i,n_i} is the n_i-partial sum of $y_i(t)$ in Ω_i, that is,

$$y_i(t) \approx \sum_{k=0}^{n_i} Y_i(k)(t - t_i)^{\alpha k} \equiv s_{i,n_i}(t).$$

Then the solution $y(t)$ in (1) can be approximated by

$$y(t) \approx \sum_{i=0}^{N-1} \chi_{\Omega_i}(t) s_{i,n_i}(t),$$

where

$$\chi_{\Omega_i}(t) = \begin{cases} 1 & \text{if } t \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

In order to solve (3), it is necessary to have the initial condition $y_i(t_i)$ in each sub-domain Ω_i, $i \geq 0$. For $i = 0$, the initial condition gives $Y_0(0) = y(0)$. However, there are no given initial conditions $y_i(t_i)$, $i \geq 1$. In the standard multistage differential transform method (MsDTM) for solving differential equations with integer order, the initial condition $y_i(t_i)$ can be approximated by computing $s_{i-1,n_{i-1}}(t_i)$. In other words, $Y_i(0) = s_{i-1,n_{i-1}}(t_i)$, $i \geq 1$. This approach has been successful to obtain accurate approximate solutions [25]. In the MsGDTM for solving fractional differential equations the same technique has been employed in [26, 27].

4.2 Effect of memory in the standard multistage generalized differential transform method

Suppose that $y_i(t)$ is a solution of the following problem:

$$D_{t_i}^{\alpha} y_i(t) = f_i\big(t, y_i(t)\big), \quad t \in \Omega_i = (t_i, t_{i+1}), i \geq 1. \tag{4}$$

The analytical solution of (4) with an initial condition $y_i(t_i)$ can be obtained by taking the Riemann-Liouville integral operator as follows:

$$y_i(t) = y_i(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha - 1} f_i\big(\tau, y_i(\tau)\big)\, d\tau. \tag{5}$$

Lemma 4.1 *Suppose* $y_i(t) = \sum_{k=0}^{\infty} Y_i(k)(t-t_i)^{\alpha k}$ *and* $f_i(t) = \sum_{k=0}^{\infty} F_i(k)(t-t_i)^{\alpha k}$. *Then Eq.* (5) *has the following recursive relation*:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y_i(k+1) = F_i(k), \quad k = 0,1,2,\ldots.$$

Proof Since $J_{t_i}^{\alpha}(t-t_i)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-t_i)^{\gamma+\alpha}$, substituting the generalized Talyor series of $y_i(t)$ and $f_i(t)$ into (5) gives

$$\sum_{k=0}^{\infty} Y_i(k)(t-t_i)^{\alpha k} = y_i(t_i) + \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} F_i(k)(t-t_i)^{\alpha k+\alpha}.$$

Since $Y_i(0) = y_i(t_i)$, it completes the proof. □

This gives that Eq. (5) has the same recursive relation (3) with $Y_i(0) = y_i(t_i)$. Thus, the standard MsGDTM finds an approximate solution of (5) with an initial condition $y_i(t_i) \approx s_{i-1,n_{i-1}}(t_i)$. Solving the recursive relation, the solution $y_i(t)$ can be approximated by $y_i(t) \approx s_{i,n_i}(t)$. Therefore it is easy to see that the standard MsGDTM approximates the value of $y_i(t_{i+1})$ as follows:

$$\begin{aligned} y_i(t_{i+1}) &\approx y_i(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-\tau)^{\alpha-1} \tilde{f}_i\bigl(\tau, y_i(\tau)\bigr)\, d\tau \\ &\approx s_{i-1,n_{i-1}}(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-\tau)^{\alpha-1} \tilde{f}_i\bigl(\tau, y_i(\tau)\bigr)\, d\tau \\ &= s_{i,n_i}(t_{i+1}), \end{aligned} \tag{6}$$

where $\tilde{f}_i(t) = \sum_{k=0}^{n_i} F_i(k)(t-t_i)^{\alpha k}$.

In what follows we describe that the value of $y_i(t_{i+1})$ in the analytical approach does have an addition term compared to Eq. (6). This term results from the effect of memory for the fractional derivative operator. Taking the Riemann-Liouville integral operator to (1), the solution $y(t)$ can be computed at $t = t_{i+1}, t_i$ by

$$y(t_{i+1}) = y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{i+1}} (t_{i+1}-\tau)^{\alpha-1} f\bigl(\tau, y(\tau)\bigr)\, d\tau,$$

and

$$y(t_i) = y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_i} (t_i-\tau)^{\alpha-1} f\bigl(\tau, y(\tau)\bigr)\, d\tau.$$

Thus, for $i \geq 1$, the value of $y(t_{i+1})$ can be rewritten by

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1}-\tau)^{\alpha-1} f\bigl(\tau, y(\tau)\bigr)\, d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_i} \bigl\{(t_{i+1}-\tau)^{\alpha-1} - (t_i-\tau)^{\alpha-1}\bigr\} f\bigl(\tau, y(\tau)\bigr)\, d\tau. \end{aligned} \tag{7}$$

Hence the value of $y(t_{i+1})$ can be approximated by

$$y(t_{i+1}) \approx y(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^{\alpha-1} \tilde{f}_i(\tau, y(\tau))\, d\tau + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} \{(t_{i+1} - \tau)^{\alpha-1} - (t_i - \tau)^{\alpha-1}\} \tilde{f}_k(\tau, y(\tau))\, d\tau. \tag{8}$$

As seen in (6) and (8), it is clear that the standard MsGDTM is missing the memory term:

$$\text{memory} \equiv \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} \{(t_{i+1} - \tau)^{\alpha-1} - (t_i - \tau)^{\alpha-1}\} \tilde{f}_k(\tau, y(\tau))\, d\tau.$$

Thus it is difficult to obtain an accurate approximation by using the standard MsGDTM for solving the fractional differential equations.

In order to avoid this difficulty, it is necessary to introduce a new algorithm to impose the memory term at each sub-domain Ω_i. Here, as described in [1], we apply the piecewise linear interpolation of $f(\tau, y(\tau))$ in Ω_k at $k = 0, 1, \ldots, i+1$ to obtain the value of $y(t_{i+1})$ as follows:

$$\begin{aligned} y(t_{i+1}) &= y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{i+1}} (t_{i+1} - \tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau \\ &\approx y(t_0) + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{i} \int_{t_k}^{t_{k+1}} (t_{i+1} - \tau)^{\alpha-1} \hat{f}_k(\tau, y(\tau))\, d\tau \\ &= y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{i+1} a_{k,i+1} f(t_k, y(t_k)), \end{aligned}$$

where

$$\hat{f}_k(t) = \frac{t - t_{k+1}}{t_{k+1} - t_k} f_k(t_k) + \frac{t - t_k}{t_{k+1} - t_k} f_k(t_{k+1}),$$

$$a_{k,i+1} = \begin{cases} i^{\alpha+1} - (i - \alpha)(i+1)^\alpha, & \text{if } k = 0, \\ (i - k + 2)^{\alpha+1} + (i - k)^{\alpha+1} - 2(i - k + 1)^{\alpha+1}, & \text{if } 1 \le k \le i, \\ 1, & \text{if } k = i + 1. \end{cases}$$

Since the approximation of $y_i(t)$ in each sub-interval Ω_i can be obtained by using the GDTM, $y_i(t) \approx s_{i,n_i}(t)$, the initial condition $y(t_{i+1})$ can be evaluated by

$$y(t_{i+1}) \approx y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{i} a_{k,i+1} f(t_k, s_{k,n_k}(t_k)). \tag{9}$$

5 Numerical illustrations

In this section we demonstrate numerical results of several examples by using the standard MsGDTM and the proposed MsGDTM (P-MsGDTM) in (9) for various fractional order α. To confirm the numerical accuracy, we also present the numerical results obtained by the fractional Adams-Bashforth-Moulton method (FABM) [1]. To apply the GDTMs, we assume that the solution $y(t)$ belongs to $(D^\alpha)^k y(t) \in C(0, T)$, where $k > 5$ and $T > 1$. On

Table 1 Comparison of numerical results by the standard MsGDTM, the P-MsGDTM and the FABM for $\alpha = 0.9$ in Example 1

t	MsGDTM ($h = 0.001$)	P-MsGDTM ($h = 0.01$)	FABM ($h = 0.001$)
0.1	0.25272	0.15163	0.15071
0.2	0.59515	0.31730	0.31486
0.3	0.99844	0.50264	0.49866
0.4	1.40196	0.70268	0.69754
0.5	1.74493	0.90929	0.90367
0.6	1.99830	1.11312	1.10786
0.7	2.16673	1.30556	1.30143
0.8	2.27100	1.48018	1.47770
0.9	2.33272	1.63336	1.63561
1.0	2.36830	1.76411	1.76527

both the standard MsGDTM and the P-MsGDTM, the approximate solutions are obtained by using five generalized differential transforms. That is, $y_i(t) \approx \sum_{k=0}^{5} Y_i(k)(t-t_0)^{\alpha k}$. All solutions are computed up to time $t = 1.0$ when the fractional order α varies from 0.5 to 0.9.

Example 1 Consider the fractional Riccati equation [7]

$$D^\alpha y(t) = 2y(t) - y^2(t) + 1, \quad st > 0,$$

where $0 < \alpha \leq 1$, subject to the initial condition $y(0) = 0$.

Table 1 shows a comparison of the numerical results obtained by the standard MsGDTM, the P-MsGDTM and the FABM for the fractional order $\alpha = 0.9$. The time step is chosen as $h = 0.001$ in the standard MsGDTM and the FABM, and $h = 0.01$ in the P-MsGDTM. The numerical results in the P-MsGDTM are in a good agreement with the ones in FABM at each time $t = 0.1, \dots, 0.9$. However, the standard MsGDTM gives inaccurate approximate solutions. For $\alpha = 0.5, 0.6, 0.7$ and 0.8, the comparisons of the numerical results are shown in Figure 1. It is easy to see that the P-MsGDTM gives accurate numerical solutions for all α, but the standard MsGDTM does not. As the fractional order α is getting smaller, the standard MsGDTM gives an inaccurate approximate solution in a shorter range of time.

Example 2 Consider the fractional differential equation [6]

$$D^\alpha y(t) = -y(t),$$

where $0 < \alpha \leq 1$, subject to the initial conditions $y(0) = 1$.

The exact solution can be written analytically, $y(t) = E_\alpha(-t^\alpha)$, where $E_\alpha(z)$ is the one-parameter Mittag-Leffler function as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + 1)}.$$

The numerical results obtained by the standard MsGDTM, the P-MsGDTM are compared with the exact solution for the fractional order $\alpha = 0.9$ in Table 2. It is shown that the numerical approximations by the P-MsGDTM agree with the exact solution in three

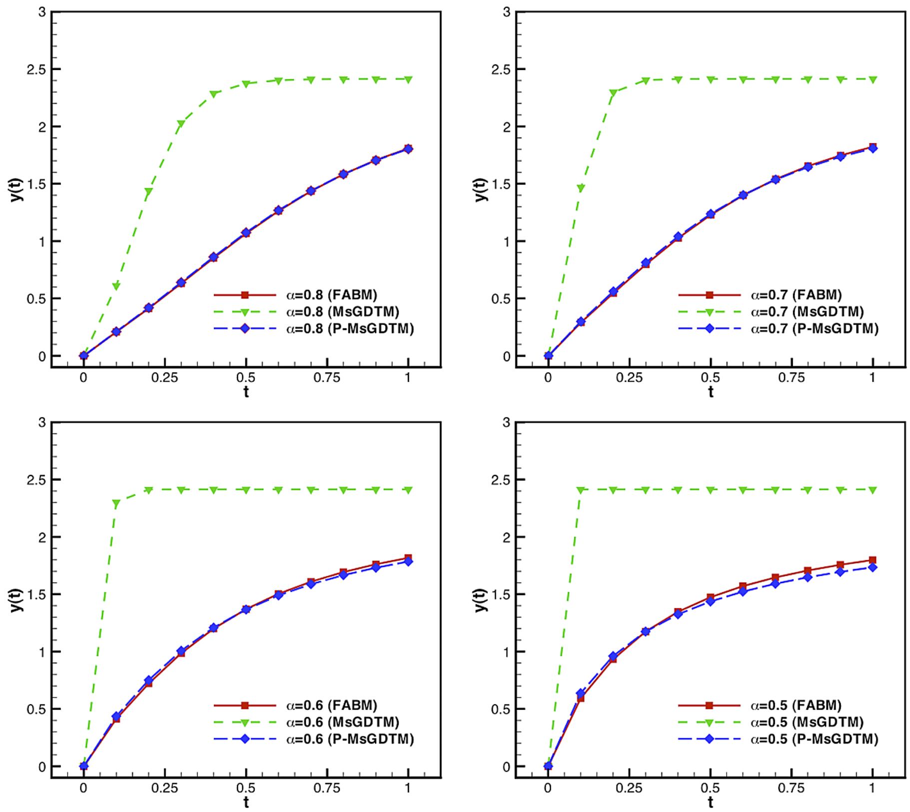

Figure 1 Numerical comparisons for the fractional order $\alpha = 0.5, 0.6, 0.7$ and 0.8: FABM ($h = 0.001$), MsGDTM ($h = 0.001$) and P-MsGDTM ($h = 0.01$) in Example 1.

Table 2 Comparison of numerical results by the standard MsGDTM, the P-MsGDTM and the exact solution for $\alpha = 0.9$ in Example 2

t	MsGDTM ($h = 0.005$)	P-MsGDTM ($h = 0.005$)	Exact solution
0.1	0.83816	0.87835	0.87809
0.2	0.70252	0.78628	0.78576
0.3	0.58883	0.70881	0.70807
0.4	0.49354	0.64201	0.64109
0.5	0.41367	0.58367	0.58261
0.6	0.34672	0.53227	0.53111
0.7	0.29061	0.48673	0.48549
0.8	0.24358	0.44618	0.44488
0.9	0.20416	0.40994	0.40859
1.0	0.17112	0.37744	0.37606

decimal places for all t. However, the standard MsGDTM gives only reliable approximations when the time is close to zero and the error is getting larger as the time increases. Figure 2 presents the comparisons of the numerical results by the standard MsGDTM and P-MsGDTM and the exact solutions for the fractional order $\alpha = 0.5, 0.6, 0.7$ and 0.9.

Example 3 Consider the following fractional differential equation:

$$D^{\alpha} y(t) = \exp[y(t)] - 2y(t),$$

where $0 < \alpha \leq 1$, subject to the initial conditions $y(0) = 0$.

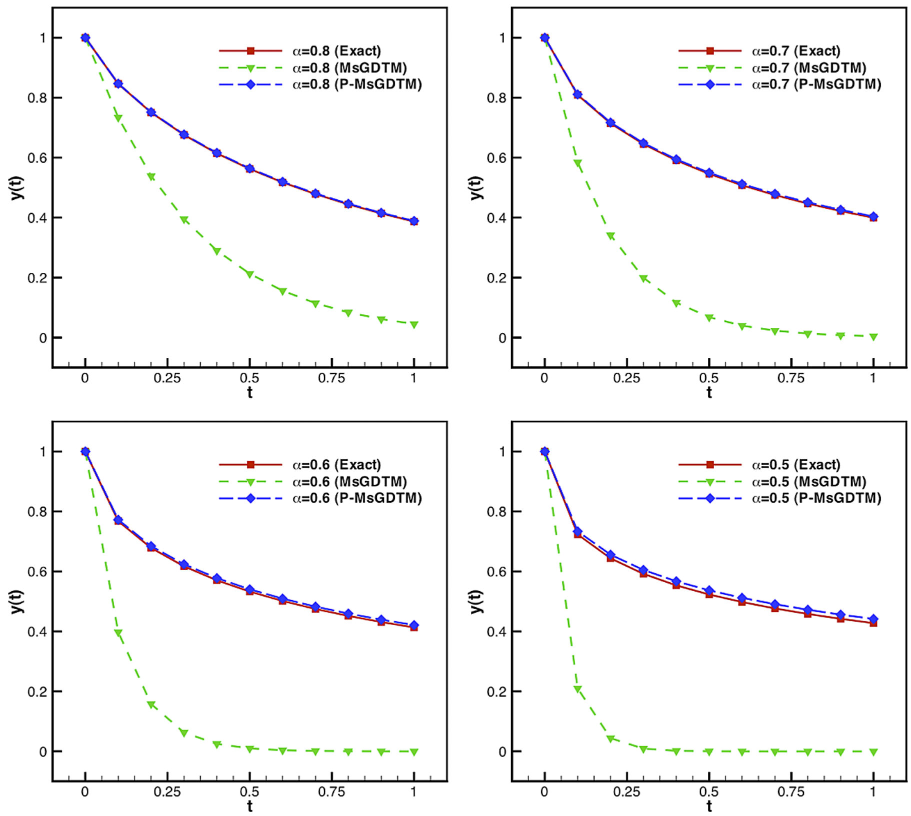

Figure 2 Numerical comparisons for the fractional order $\alpha = 0.5, 0.6, 0.7$ and 0.8: MsGDTM ($h = 0.005$) and P-MsGDTM ($h = 0.005$) in Example 2.

Table 3 Differential transforms $G_i(k)$ for $g(t) = \exp[y(t)]$, $k = 0, 1, 2, 3, 4$

k	$G_i(k)$
0	$\exp[Y_i(0)]$
1	$Y_i(1)\exp[Y_i(0)]$
2	$((Y_i(1))^2/2 + Y_i(2))\exp[Y_i(0)]$
3	$((Y_i(1))^3/3! + Y_i(1)Y_i(2) + Y_i(3))\exp[Y_i(0)]$
4	$((Y_i(1))^4/4! + (Y_i(1))^2 Y_i(2)/2! + (Y_i(2))^2/2 + Y_i(1)Y_i(3) + Y_i(4))\exp[Y_i(0)]$

Table 4 Comparison of numerical results by the standard MsGDTM, the P-MsGDTM and the FABM for $\alpha = 0.9$ in Example 3

t	MsGDTM ($h = 0.01$)	P-MsGDTM ($h = 0.1$)	FABM solution ($h = 0.001$)
0.1	0.15248	0.12301	0.12229
0.2	0.28534	0.21661	0.21653
0.3	0.40432	0.29700	0.29824
0.4	0.51386	0.36926	0.37165
0.5	0.61766	0.43587	0.43916
0.6	0.71902	0.49846	0.50237
0.7	0.82132	0.55820	0.56252
0.8	0.92843	0.61608	0.62056
0.9	1.04548	0.67294	0.67735
1.0	1.18035	0.72956	0.73367

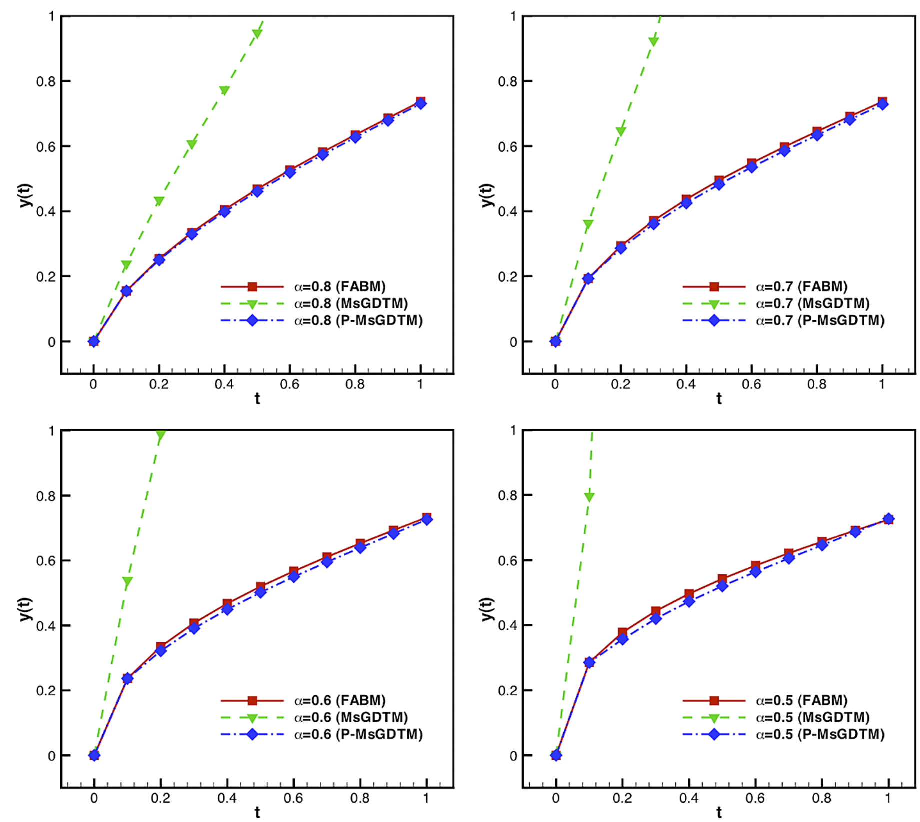

Figure 3 Numerical comparisons for the fractional order $\alpha = 0.5, 0.6, 0.7$ and 0.8: MsGDTM ($h = 0.01$) and P-MsGDTM ($h = 0.1$) in Example 3.

Table 5 Differential transforms $G_i(k)$ for $g(t) = y(t)\ln[y(t)]$, $k = 0, 1, 2, 3, 4$

k	$G_i(k)$
0	$Y_i(0)\ln[Y_i(0)]$
1	$Y_i(1)\ln[Y_i(0)] + Y_i(1)$
2	$Y_i(2)\ln[Y_i(0)] + Y_i(2) + \frac{(Y_i(1))^2}{2Y_i(0)}$
3	$Y_i(3)\ln[Y_i(0)] + Y_i(3) + \frac{Y_i(1)Y_i(2)}{Y_i(0)} - \frac{(Y_i(1))^3}{6(Y_i(0))^2}$
4	$Y_i(4)\ln[Y_i(0)] + Y_i(4) + \frac{2Y_i(1)Y_i(3)+(Y_i(2))^2}{2Y_i(0)} - \frac{(Y_i(0))^2 Y_i(2)}{2(Y_i(0))^2} + \frac{(Y_i(1))^4}{12(Y_i(0))^3}$

In order to obtain a recursive relation for the nonlinear term $\exp[y(t)]$, we adopted the method in [28] using the Adomian polynomials. Let us assume $g(t) = \exp[y(t)] = \sum_{k=0}^{\infty} G_i(k)(t-t_i)^k$, then the Adomian polynomial gives the differential transforms $G_i(k)$ in Table 3. Numerical results by the standard MsGDTM, the P-MsGDTM and the FABM for $\alpha = 0.9$ are shown in Table 4. It is shown that the numerical approximations by the P-MsGDTM agree well with the results by the FABM for all t. Here, the time step $h = 0.1$ is used to obtain the approximation in the P-MsGDTM, whereas the FABM employs the time step $h = 0.001$. For $\alpha = 0.5, 0.6, 0.7$ and 0.8, the numerical comparisons are shown in Figure 3. In the standard MsGDTM the numerical results are dramatically increasing as the fractional order is getting smaller.

Table 6 Comparison of numerical results by the standard MsGDTM, the E-MsGDTM and the FABM for $\alpha = 0.9$ in Example 4

t	MsGDTM ($h = 0.01$)	P-MsGDTM ($h = 0.01$)	FABM solution ($h = 0.001$)
0.1	0.88925	0.81180	0.81250
0.2	1.19751	1.04318	1.04599
0.3	1.39697	1.21115	1.21592
0.4	1.51304	1.32935	1.33529
0.5	1.57693	1.41164	1.41800
0.6	1.61111	1.46902	1.47529
0.7	1.62911	1.50939	1.51529
0.8	1.63853	1.53817	1.54357
0.9	1.64343	1.55901	1.56389
1.0	1.64597	1.57436	1.57873

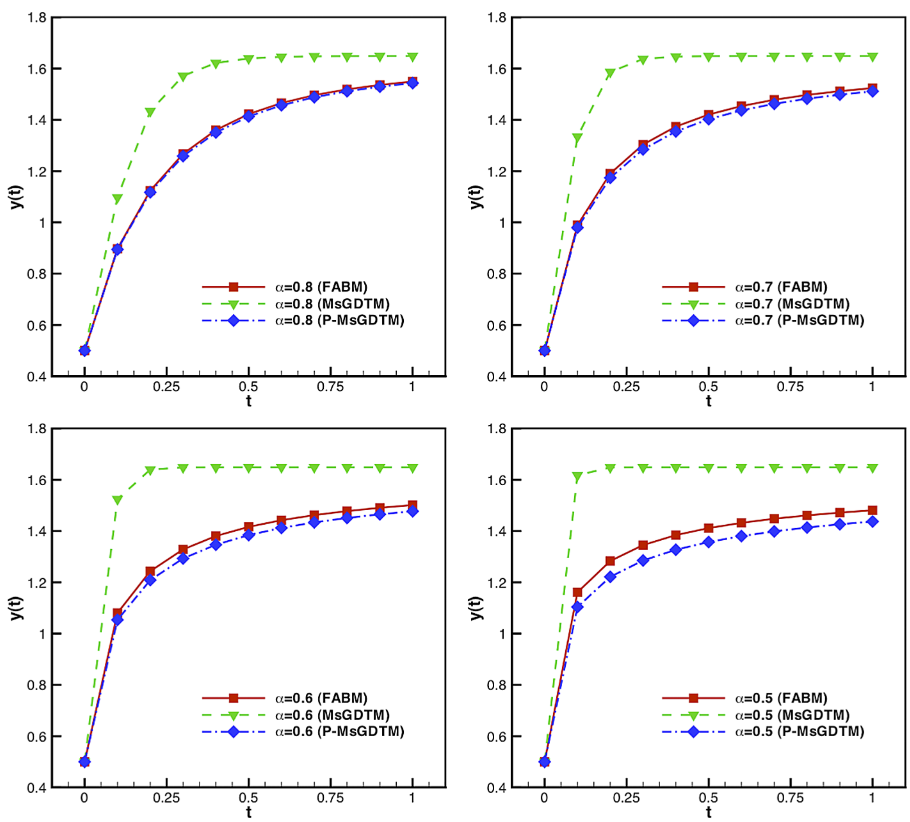

Figure 4 Numerical comparisons for the fractional order $\alpha = 0.5, 0.6, 0.7$ and 0.8: MsGDTM ($h = 0.01$) and P-MsGDTM ($h = 0.01$) in Example 4.

Example 4 Consider the following fractional differential equation:

$$D^{\alpha} y(t) = 2y(t) - 4y(t)\ln\bigl[y(t)\bigr],$$

where $0 < \alpha \leq 1$, subject to the initial conditions $y(0) = 0.5$.

For the nonlinear term $\ln[y(t)]$, we obtained the differential transforms by using the Adomian polynomials in [28]. Assuming $g(t) = y(t)\ln[y(t)] = \sum_{k=0}^{\infty} G_i(k)(t - t_i)^k$, the differential transforms $G_i(k)$ are listed in Table 5. For the fractional order for $\alpha = 0.9$, the numerical results by the standard MsGDTM ($h = 0.01$), the P-MsGDTM ($h = 0.01$) and the FABM

($h = 0.001$) are shown in Table 6. For $\alpha = 0.5, 0.6, 0.7$ and 0.8, the approximate solutions are depicted in Figure 4. Even if the numerical results by the P-MsGDTM with small fractional order have some difference with the results by the FABM, it will be overcome with a small time step. In fact, if we consider the numerical error for $\alpha = 0.5$ at $t = 1$, the error is about 0.066 for $h = 0.01$ but 0.047 for $h = 0.005$.

6 Conclusion

In this paper, we proposed a new modified multistage generalized differential transform method for solving fractional differential equations. We have pointed out that the standard MsGDTM has difficulty to handle the effect of memory in solving fractional differential equations. However, the proposed MsGDTM (P-MsGDTM) can deal with the memory effectively. Several illustrative examples showed that the P-MsGDTM obtained accurate numerical approximations, but the standard MsGDTM failed to get robust approximations for all examples. It is concluded that the proposed MsGDTM is very simple and effective for solving fractional problems. Here, all numerical results were performed by using Mathematica 8.0.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BJ established the scheme, performed all the numerical examples in Section 5 and drafted the manuscript. KK helped to inspect the manuscript and designed the figures. All authors read and approved the final manuscript.

Acknowledgements
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0004869).

Received: 13 May 2013 Accepted: 20 November 2013 Published: 18 Dec 2013

References

1. Diethelm, K, Ford, NJ, Freed, D: A predictor-corrector approach for the numerical solution of fractional differential equation. Nonlinear Dyn. **29**, 2-22 (2009)
2. Baleanu, D, Diethelm, K, Scalas, E, Trujillo, J: Fractional Calculus Models and Numerical Methods. World Scientific, Boston (2012)
3. Saha Ray, S, Bera, RK: An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method. Appl. Math. Comput. **167**, 561-571 (2005)
4. Hu, Y, Luo, Y, Lu, Z: Analytical solution of the linear fractional differential equation by Adomian decomposition method. J. Comput. Appl. Math. **215**, 220-229 (2008)
5. Duan, J-S, Rach, R, Baleanu, D, Wazwaz, A-M: A review of the Adomian decomposition method and its applications to fractional differential equations. Commun. Fract. Calc. **3**, 73-99 (2012)
6. Hashim, I, Abdulaziz, O, Momani, S: Homotopy analysis method for fractional IVPs. Commun. Nonlinear Sci. Numer. Simul. **14**, 674-684 (2009)
7. Odibat, Z, Momani, S, Xu, H: A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations. Appl. Math. Model. **34**, 593-600 (2010)
8. Elsaid, A: Homotopy analysis method for solving a class of fractional partial differential equations. Commun. Nonlinear Sci. Numer. Simul. **16**, 3655-3664 (2011)
9. Abdulaziz, O, Hashim, I, Momani, S: Application of homotopy-perturbation method to fractional IVPs. J. Comput. Appl. Math. **216**, 574-584 (2008)
10. Abdulaziz, O, Hashim, I, Momani, S: Solving systems of fractional differential equations by homotopy-perturbation method. Phys. Lett. A **372**, 451-459 (2008)
11. Das, S: Analytical solution of a fractional diffusion equation by variational iteration method. Comput. Math. Appl. **57**, 483-487 (2009)
12. Guo, S, Mei, L: The fractional variational iteration method using He's polynomials. Phys. Lett. A **375**, 309-313 (2011)
13. Wu, G-c: A fractional variational iteration method for solving fractional nonlinear differential equations. Comput. Math. Appl. **61**, 2186-2190 (2011)
14. Wu, GC, Baleanu, D: Variational iteration method for the Burgers flow with fractional derivatives-New Lagrange multipliers. Appl. Math. Model. **37**, 6183-6190 (2012)
15. Wu, GC, Baleanu, D: Variational iteration method for fractional calculus - a universal approach by Laplace transform. Adv. Differ. Equ. **2013**, Article ID 18 (2013)
16. Arikoglum, A, Ozkol, I: Solution of fractional differential equations by using differential transform method. Chaos Solitons Fractals **34**, 1473-1481 (2007)

17. Odibat, Z, Shawagfeh, N: Generalized Talyor's formula. Appl. Math. Comput. **186**, 286-293 (2007)
18. Odibat, Z, Momani, S, Erturk, VS: Generalized differential transform method: application to differential equations of fractional order. Appl. Math. Comput. **197**, 467-477 (2008)
19. Caputo, M: Linear models of dissipation whose *Q* is almost frequency independent. Part II. Geophys. J. R. Astron. Soc. **13**, 529-539 (1967)
20. Jang, MJ, Chen, CL, Liu, YC: On solving the initial-value problems using the differential transformation method. Appl. Math. Comput. **115**, 145-160 (2000)
21. Jang, MJ, Chen, CL, Liu, YC: Two-dimensional differential transform for partial differential equations. Appl. Math. Comput. **121**, 261-270 (2001)
22. Ravi Kanth, ASV, Aruna, K: Differential transform method for solving the linear and nonlinear Klein-Gordon equation. Comput. Phys. Commun. **180**, 708-711 (2009)
23. Jang, B: Comments on 'Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method'. J. Comput. Appl. Math. **233**, 224-230 (2009)
24. Jang, B: Solving linear and nonlinear initial value problems by the projected differential transform method. Comput. Phys. Commun. **181**, 848-854 (2010)
25. Odibat, Z, Bertelle, C, Aziz-Alaoui, MA, Duchamp, GHE: A multi-step differential transform method and application to non-chaotic or chaotic systems. Comput. Math. Appl. **59**, 1462-1472 (2010)
26. Alomari, AK: A new analytic solution for fractional chaotic dynamical systems using the differential transform method. Comput. Math. Appl. **61**, 2528-2534 (2011)
27. Gökdoğan, A, Yildirim, A, Merdan, M: Solving a fractional order model of HIV infection of CD4+ T cells. Math. Comput. Model. **54**, 2132-2138 (2011)
28. Elsaid, A: Fractional differential transform method combined with the Adomian polynomials. Appl. Math. Comput. **218**, 6899-6911 (2012)

10.1186/1687-1847-2013-371
Cite this article as: Kim and Jang: A semi-analytic method with an effect of memory for solving fractional differential equations. *Advances in Difference Equations* 2013, 2013:371