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Perturbed Rotations of a Rigid Body Close to the Lagrange Case under the Action of Unsteady Perturbation Torques

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Abstract
Perturbed rotations of a rigid body close to the Lagrange case under the action of perturbation torques slowly varying in time are investigated. Conditions are presented for the possibility of averaging the equations of motion with respect to the nutation angle and the averaged system of equations of motion is obtained. In the case of the rotational motion of the body in the linear-dissipative medium the numerical integration of the averaged system of equations is conducted

Keywords
Perturbed motion, averaging, torque.

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Introduction

The authors investigated new problem of the motion of a rigid body about a fixed point under the action of perturbation torques of forces of different physical nature. The motion with the torque of external forces in Lagrange's case is considered as a nonperturbed motion. The influence of the perturbations is determined by the averaging method for the Lagrange-Poisson motion [1, 2]. Papers [3-8] were devoted to the investigation of perturbed motions close to Lagrange motion. Paper [3] is a brief survey of some theoretical results in area of dynamics of the rigid body with one fixed point from view point of the applications to the mechanics of space flight. The authors investigated perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case when the restoring torque is constant [2, 4] and when the restoring torque depends on the nutation angle [5]. Perturbed rotations of a rigid body close to the regular precession in the Lagrange case under the action of restoring torque depending on slow time and nutation angle, as well as perturbation torque slowly varying in time, was studied in [6]. The motion of a symmetric heavy rigid body about a fixed point when the body is subjected to frictional forces due to a surrounding dissipative medium was considered in [7]. The motion of a slightly asymmetric heavy rigid body in the viscous medium was studied in [8].

In our paper a new approach is developed for the investigation of perturbed motions of Lagrange top for perturbation torques slowly varying in time. We develop an averaging procedure for system of the equations of motion of a rigid body under arbitrary initial conditions for perturbations admitting of averaging with respect to the nutation angle θ. An actual mechanical model of the perturbations, corresponding to the body's motion in a medium with linear dissipation, is considered.

1. Statement of the problem and the unperturbed motion

Consider the motion of a dynamically symmetric heavy rigid body about a fixed point O under the action of perturbation torques of arbitrary nature. The equations of motion have the form:

$$
\begin{aligned}
&A\dot{p} + (C - A)\, qr = \mu \sin\theta \cos\varphi + \varepsilon M_1 \\
&A\dot{q} + (A - C)\, pr = -\mu \sin\theta \sin\varphi + \varepsilon M_2 \\
&C\dot{r} = \varepsilon M_3, \;\; M_i = M_i(p, q, r, \psi, \theta, \varphi, \tau), \;\; i = 1,2,3 \\
&\dot{\psi} = (p\sin\varphi + q\cos\varphi)\operatorname{cosec}\theta, \;\; \tau = \varepsilon t \\
&\dot{\theta} = p\cos\varphi - q\sin\varphi, \;\; \dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\theta
\end{aligned}
\tag{1}
$$

Here p, q and r are the projections of the vector of angular velocity of the body onto the principal axes of inertia passing through the point O. The values εM_i, $i = 1,2,3$ are the projections of the vector of the perturbation torques onto the same axes. They depend on the slow time $\tau = \varepsilon t$, where t is time and ε is a small parameter ($\varepsilon \ll 1$). The torques εM_i are also 2π-periodic functions of the Eulerian angles ψ, φ and θ. Here, A is the equatorial and C is the axial moments of inertia of the body about the point O, $A \neq C$. It is assumed that the body is acted upon by a restoring torque whose maximum value is equal to μ and that is generated by a force of constant magnitude and direction, applied at the some point of the axis of dynamic symmetry. In the case of a heavy top we have $\mu = mgl$. Here m is the mass of the body, g is the acceleration due to gravity, and l is the distance from fixed point O to center of gravity of the body.

The problem is post of investigating the behavior of the solution of system (1) for nonzero values of the small parameter ε on a sufficiently long time interval $t \sim \varepsilon^{-1}$. The averaging method [9] is used for solving the problem.

We derive some necessary relations for the unperturbed motion [2, 10], when $\varepsilon = 0$.

The first integrals of the equations of motion for the unperturbed system (1) are

$$
\begin{aligned}
&G_z = A\sin\theta\left(p\sin\varphi + q\cos\varphi\right) + Cr\cos\theta = c_1 \\
&H = \frac{1}{2}\left[A\left(p^2 + q^2\right) + Cr^2\right] + \mu\cos\theta = c_2, \;\; r = c_3
\end{aligned}
\tag{2}
$$

Here G_z is the projection of the moment of momentum vector onto the vertical Oz, H is the body's total energy, r is the projection of the angular velocity vector onto the axis of dynamic symmetry, $c_i, i = 1,2,3$ are arbitrary constants $(c_2 \geq -\mu)$.

The expression for the nutation angle θ in the unperturbed motion as a function of time t, of the motion integrals (2) and of arbitrary phase constant β is known [2, 10]

$$
\begin{aligned}
&\cos\theta = u_1 + (u_2 - u_1)\operatorname{sn}^2(\alpha t + \beta), \quad -1 \leq u_1 \leq u_2 \leq 1 \leq u_3 < +\infty \\
&\alpha = \left[\mu\left(u_3 - u_1\right)/(2A)\right]^{1/2}, \quad \operatorname{sn}(\alpha t + \beta) = \sin am(\alpha t + \beta, k) \\
&k^2 = (u_2 - u_1)(u_3 - u_1)^{-1}, \quad 0 \leq k^2 \leq 1
\end{aligned}
\tag{3}
$$

Here sn is the elliptic sine [11], k is the modulus of the elliptic functions, and u_1, u_2, u_3 are real roots of the cubic polynomial

$$
Q(u) = A^{-2}[(2H - Cr^2 - 2\mu u)(1 - u^2)A - (G_z - Cru)^2]
\tag{4}
$$

Relations between the roots of the polynomial $Q(u)$ and first integrals (2) can be written in the following manner:

$$
\begin{aligned}
&u_1+u_2+u_3=\frac{H}{\mu}-\frac{Cr^2}{2\mu}+\frac{C^2r^2}{2A\mu}\\
&u_1u_2+u_1u_3+u_2u_3=\frac{G_zCr}{A\mu}-1\\
&u_1u_2u_3=-\frac{H}{\mu}+\frac{Cr^2}{2\mu}+\frac{G_z^2}{2A\mu}
\end{aligned}
\tag{5}
$$

2. The averaging procedure

Let us reduce the equations of perturbed motion (1) to a form admitting of the application of the averaging method [9]. We pick out the slow and the fast variables. The first integrals (2) are the slow variables for perturbed motion (1). The fast variables are the angles of proper rotation φ, of nutation θ, and of precession ψ.

We reduce the first three equations in (1) after several transformations to the form

$$
\begin{aligned}
&\dot{G}_z=\varepsilon\left[\left(M_1\sin\varphi+M_2\cos\varphi\right)\sin\theta+M_3\cos\theta\right]\\
&\dot{H}=\varepsilon\left(M_1p+M_2q+M_3r\right)\\
&\dot{r}=\varepsilon C^{-1}M_3,\ M_i=M_i\left(p,q,r,\psi,\theta,\varphi,\tau\right),\ i=1,2,3
\end{aligned}
\tag{6}
$$

Here and in the last three equations in (1) it is implicit that the variables $p,\ q,\ r$ have been expressed as functions of $G_z, H,\ r,\ \psi, \theta, \varphi$ and have been substituted into (1) and (6). The initial values of the slow variables G_z, H, r can be computed from (2).

The right hand sides of (6) contain the three fast variables, which presents a difficulty for the application of the averaging method. To eliminate this difficulty we require that the right hand sides of (6) depend on only one fast variable, the nutation angle θ, and be periodic functions of θ of period 2π, and have following structural properties of perturbed torque of forces

$$
\begin{aligned}
&M_1\sin\varphi+M_2\cos\varphi=M_1^*(G_z,H,r,\tau,\theta)\\
&M_1p+M_2q=M_2^*(G_z,H,r,\tau,\theta)\\
&M_3=M_3^*(G_z,H,r,\tau,\theta)
\end{aligned}
\tag{7}
$$

$$
M_1=pf,\quad M_1=qf,\quad M_3=M_3^*,\quad f=f(G_z,H,r,\theta,\tau)
\tag{8}
$$

We assume the fulfilment of the necessary and sufficient conditions (7) or, in particular, of the sufficient conditions (8), which encures the validity of relations (7). Then system (6) can be presented in the form

$$
\begin{aligned}
&\dot{G}_z=\varepsilon F_1\left(G_z,H,r,\tau,\theta\right),\quad F_1=M_1^*\sin\theta+M_3^*\cos\theta\\
&\dot{H}=\varepsilon F_2\left(G_z,H,r,\tau,\theta\right),\quad F_2=M_2^*+M_3^*r\\
&\dot{r}=\varepsilon F_3\left(G_z,H,r,\tau,\theta\right),\quad F_3=C^{-1}M_3^*
\end{aligned}
\tag{9}
$$

Here F_1, F_2, F_3 are 2π –periodic functions of θ.

We propose to carry out the investigation of the perturbed motion in the slow variables $u_i,\ i=1,2,3$. The slow variables G_z, H and r can be expressed in terms of u_i from (5) as follows [1, 2]

$$
\begin{aligned}
G_z &= \delta_2 (A\mu)^{1/2} (u_1 + u_2 + u_3 + u_1 u_2 u_3 + \delta_1 R)^{1/2} \operatorname{sign}(1 + u_1 u_2 + u_1 u_3 + u_2 u_3) \\
H &= \frac{1}{2}\mu\left[(u_1 + u_2 + u_3)(1 + AC^{-1}) + (\delta_1 R - u_1 u_2 u_3)(1 - AC^{-1})\right] \\
r &= \delta_2 C^{-1} (A\mu)^{1/2} (u_1 + u_2 + u_3 + u_1 u_2 u_3 - \delta_1 R)^{1/2} \\
R &= \left[(1 - u_1^2)(1 - u_2^2)(u_3^2 - 1)\right]^{1/2}, \quad \delta_1 = \operatorname{sign}(G_z^2 - C^2 r^2), \quad \delta_2 = \operatorname{sign} r
\end{aligned}
\tag{10}
$$

At the initial instant the quantities δ_1 and δ_2 are determined from the initial conditions for G_z and r. If during the motion one or both of the quantities $G_z^2 - C^2 r^2$ and r pass through zero, a change of sign is possible for δ_1 and δ_2, to determine which we can make use of the original system (9).

The desired system of equations for the slow variables takes the form after some transformations

$$
\begin{aligned}
&\frac{du_i}{dt} = \varepsilon V_i(u_1, u_2, u_3, \tau, \theta), \quad u_i(0) = u_i^0, \quad i = 1,2,3 \\
&V_i = V_{i1} F_1^* + V_{i2} F_2^* + V_{i3} F_3^*, \quad V_{ij} = V_{ij}(u_1, u_2, u_3), \quad j = 1,2,3
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
V_{11} &= \frac{G_z - Cru_1}{A\mu(u_1 - u_2)(u_1 - u_3)} \\
V_{12} &= \frac{u_1^2 - 1}{\mu(u_1 - u_2)(u_1 - u_3)} \\
V_{13} &= \frac{C}{\mu(u_1 - u_2)(u_1 - u_3)}\left[(CA^{-1} - 1) r u_1^2 - G_z A^{-1} u_1 + r\right]
\end{aligned}
\tag{12}
$$

Here, the functions V_{2j}, V_{3j}, $j = 1,2,3$ are obtained from the corresponding expressions (12) for the same values of j by cyclic permutation of the indices on the quantities u_i. The functions F_i^* are obtained by substituting into the F_i from (9) the expressions (10). The initial values u_i^0 for variables u_i are computed from the initial data G_z^0, H^0, r^0 with the aid of relations (5).

Into the right side of system (11) we substitute the fast variable θ from expression (3) for the unperturbed motion.

The right hand sides of system (11) will be the periodic functions of t with period $2K(k)/\alpha$, where k and α are defined by relations (3). Averaging the right hand sides of the resultant system with respect to phase of the nutation angle θ, we obtain, in the slow time $\tau = \varepsilon t$, the averaged system of first approximation

$$
\begin{aligned}
&\frac{du_i}{d\tau} = U_i(u_1, u_2, u_3, \tau), \quad u_i(0) = u_i^0, \quad i = 1,2,3 \\
&U_i(u_1, u_2, u_3, \tau) = \frac{\alpha}{2K(k)} \int_0^{2K/\alpha} V_i(u_1, u_2, u_3, \tau, \theta(t))\, dt
\end{aligned}
\tag{13}
$$

After investigating and solving system (13) for u_i, the original slow variables G_z, H, r are recovered from formulas (10). The slow variables u_i and G_z, H, r are determined with an error of order ε.

3. Perturbed motion of a rigid body under linear dissipative torques

We investigate the perturbed Lagrange motion with torques applied to the body from the surrounding medium. This is the case, for example, for a medium the viscous properties of which change due to changes in the density, temperature, and composition of the medium. We assume that the perturbed torques are linearly dissipative and have the form

$$M_1 = -a(\tau)p, \quad M_2 = -a(\tau)q, \quad M_3 = -b(\tau)r, \quad a(\tau),\ b(\tau) > 0,\ \tau = \varepsilon t \tag{14}$$

Here $a(\tau)$ and $b(\tau)$ are positive integrable functions depending on the medium's properties and the body's shape.

Torques (14) satisfy the sufficient conditions (7) for the possibility of averaging with respect only the nutation angle θ. System (6) can be written as follows

$$\begin{aligned} &\dot{G}_z = -\varepsilon\left[\left(a(\tau)p\sin\varphi + a(\tau)q\cos\varphi\right)\sin\theta + b(\tau)r\cos\theta\right] \\ &\dot{H} = -\varepsilon\left[a(\tau)\left(p^2+q^2\right) + b(\tau)r^2\right] \\ &\dot{r} = -\varepsilon C^{-1}b(\tau)r \end{aligned} \tag{15}$$

Having integrated the third equation in (15), we obtain (r^0 is the arbitrary initial value of the axial rotation velocity)

$$r = r^0 \exp(-\varepsilon C^{-1}\int_0^t b(\varepsilon t)dt) \tag{16}$$

Consider a case where $a(\tau),\ b(\tau)$ have the form

$$a(\tau) = a_0 + a_1\tau, \qquad b(\tau) = b_0 + b_1\tau, \quad a_0,\ a_1,\ b_0,\ b_1 - const \tag{17}$$

An averaged system (13) after several transformations, with reference to (14), have the form

$$\begin{aligned} \frac{du_1}{d\tau} &= \frac{-1}{A\mu(u_1-u_2)(u_1-u_3)}\Big\{a(\tau)\left[A^{-1}(G_z - Cru_1)(G_z - Crv) + (u_1^2-1)(2H - Cr^2 - 2\mu v)\right] + \\ &\qquad + b(\tau)r(G_z - Cru_1)(v-u_1)\Big\} \\ \frac{du_2}{d\tau} &= \frac{-1}{A\mu(u_2-u_3)(u_2-u_1)}\Big\{a(\tau)\left[A^{-1}(G_z - Cru_2)(G_z - Crv) + (u_2^2-1)(2H - Cr^2 - 2\mu v)\right] + \\ &\qquad + b(\tau)r(G_z - Cru_2)(v-u_2)\Big\} \\ \frac{du_3}{d\tau} &= \frac{-1}{A\mu(u_3-u_2)(u_3-u_1)}\Big\{a(\tau)\left[A^{-1}(G_z - Cru_3)(G_z - Crv) + (u_3^2-1)(2H - Cr^2 - 2\mu v)\right] + \\ &\qquad + b(\tau)r(G_z - Cru_3)(v-u_3)\Big\} \end{aligned} \tag{18}$$

Here $v = u_3 - (u_3 - u_1)E(k)/K(k)$, $K(k), E(k)$ are the complete elliptic integrals of the first and second kinds. The expressions (3), (10) are substituted in the place of $G_z,\ H,\ r,\ k$.

The averaged system (18) was integrated numerically for $t \gg 0$ under various initial conditions and problem parameters. Let us present the calculation results for three cases corresponding to the following initial data:

$$\text{a)}\ u_1^0 = 0.913,\ u_2^0 = 0.996,\ u_3^0 = 1.087,\ \theta^0 = 5^0 \tag{19}$$

$$\text{b)}\ u_1^0 = 0,\ u_2^0 = 0.5,\ u_3^0 = 2,\ \theta^0 = 60^0 \tag{20}$$

$$\text{c)}\ u_1^0 = -0.932,\ u_2^0 = -0.866,\ u_3^0 = 2.932,\ \theta^0 = 150^0 \tag{21}$$

The data presented correspond to a spinning top receiving at the initial instant an angular rotation velocity equal to $r^0 = \sqrt{3}$ around the dynamic symmetry axis and deviated from the vertical by the angle θ^0. In addition, we take $A = 1.5,\ C = 1,\ \mu = 0.5,\ a_0 = 0.125,\ b_0 = 0.1,\ a_1 = b_1 = 1$. Using the values of u_i found as a result of the numerical integration, we determine the variables from formulas (10). The graphs of functions, $G_z, H, r, u_i,\ i = 1,2,3$ are shown in Figs. 1-3 for the three cases mentioned.

The total energy H decreases monotonically and asymptotically approaches the value $H = -\mu = -0.5$. The projection of the moment of momentum vector onto the vertical G_z in cases a and b decreases monotonically, while in case c it increases monotonically, tending to zero in all three cases. The quantities u_1 and u_2 decrease monotonically and tend to -1, while u_3 asymptotically approaches +1. In this connection as follows from (3), we have $\cos\theta \to -1\ (\theta \to \pi)$. Thus, under the action of external dissipation the rigid body, for the initial condition, tends to the unique stable (lower) equilibrium position.

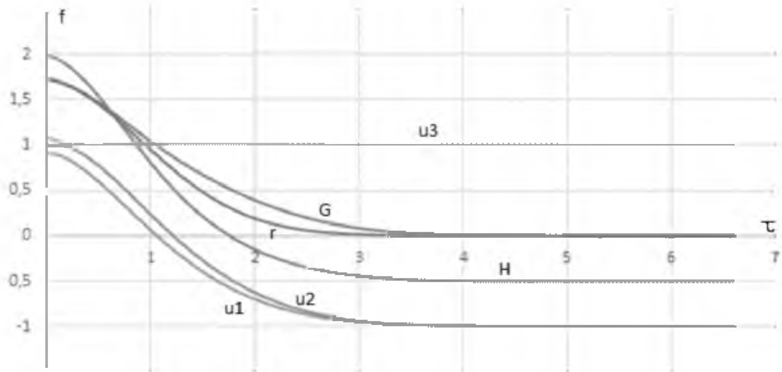

Figure 1. The graphs of functions $G_z, H, r, u_i,\ i = 1,2,3$ for the case a).

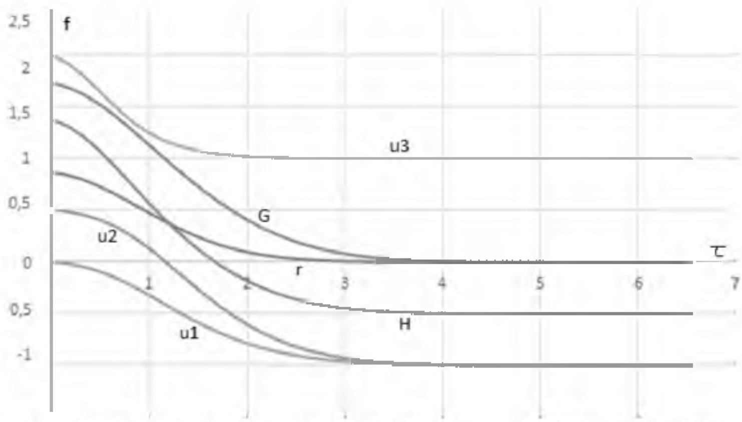

Figure 2. The graphs of functions $G_z, H, r, u_i,\ i = 1,2,3$ for the case b).

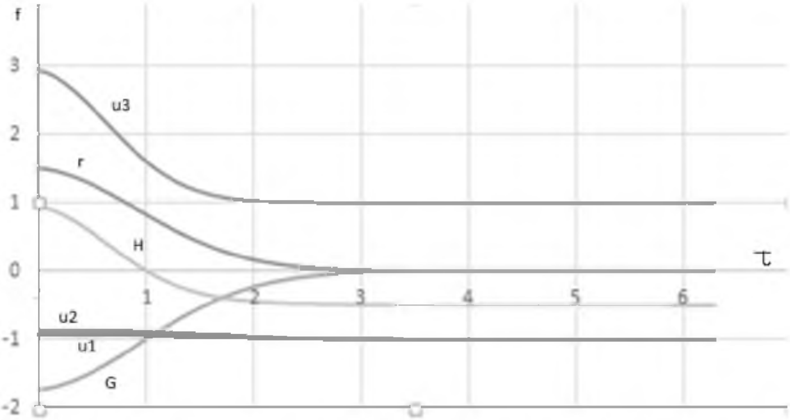

Figure 3. The graphs of functions G_z, H, r, u_i, $i=1,2,3$ for the case c).

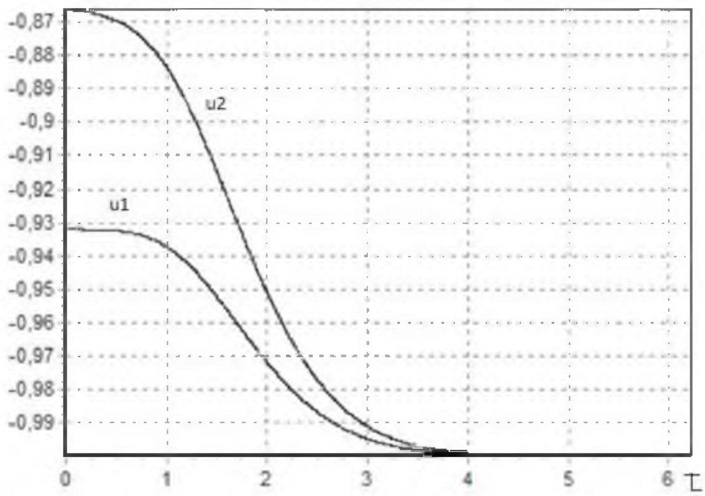

Figure 4. The graphs of functions u_1, u_2 with smaller scale.

The graphs of functions u_1 and u_2 in Fig. 3 are coincide. The graphs of functions u_1, u_2 in Fig. 4 with smaller scale at the ordinate axes show that the quantities u_1 and u_2 decrease. The correctness of the calculation was monitored by the fact that the values r as obtained from the numerical data and from formulas (10) practically coincided with the exact solution (17).

Conclusions

Comparison of obtained results with the results of [1, 2] shows that the perturbation torque slowly varying in time smooth out the variation of u_i, $i=1,2,3$, G_z, H in the calculation results. The rigid body under the action of perturbation torque (14) tends to the stable equilibrium position more quickly than it was obtained in [1, 2].

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