**Proceedings of the 5th International Conference on Nonlinear Dynamics**
**ND-KhPI2016**
**September27-30, 2016, Kharkov, Ukraine**

# Numerical Analysis of Stress-Strain State of Orthotropic Plates in the Form of Arbitrary Convex Quadrangle

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**Abstract**
*A numerical and analytical approach to solving problems of the stress-strain state of quadrangular orthotropic plates of complex shape has been proposed. Two-dimensional boundary value problem was solved using spline collocation and discrete orthogonalization methods after applying the appropriate domain transform.*
*The influence of geometric shape of plate in different cases of boundary conditions on the displacement and stress fields is considered according to the refined theory. The results were compared with available data from other authors.*

**Keywords**
Stress-strain state, quadrangular plate, boundary-value problem, spline collocation method, discrete orthogonalization method, orthotropy

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## Introduction

Stress-strain state analysis of plates with different fixing and load distribution options is a complex computational task. The problem is even more complicated for the composite anisotropic materials, which are becoming more widely used. Considering the diversity of possible shapes, one of the few universal approaches for the study of plates is a finite element method, which, however, has high requirements to computing resources and leaves many questions about the adequacy of the models and the choice of their parameters opened.

Many problems for plates with a relatively simple shape (circle, square, and rectangle) got their analytical or semi-analytical solutions, including the use of spline functions and series expansions [1–5]. In other cases, different numerical approaches have been proposed, often using parameterization that takes into account the specific shape of a particular plate [6, 7]. In [8] the analysis of the stress state of plates by R-functions method is considered. Some issues of application of the coordinate transformation for the analysis of complex-shaped plates have also been discussed in [9–12].

In this paper, we propose an approach to solving the problems of analysis of plates based on well-proven methods of spline collocation [13, 14] and discrete orthogonalization [14, 15]. The main advantages of using splines are stability against local perturbations in contrast to, for example, polynomial approximation, better convergence than that of polynomial approximation and simple and convenient computer implementation.

To be able to describe the complex area of almost any quadrilateral plate, it is proposed to use the appropriate coordinate transformation.

## 1. Basic Relations and Constitutive Equations

Let us consider the problem of stress-strain state of a rectangular plate in the Cartesian coordinates $x_1, x_2 (0 \le x_1 \le a,\ 0 \le x_2 \le b)$ with a thickness of $h$. According to the Timoshenko-Mindlin type refined plate theory, the equilibrium equations for a plate under transverse load are [2]:

$$Q_{1,1} + Q_{2,2} + q = 0$$
$$M_{1,1} + M_{12,2} - Q_1 = 0 \qquad (1)$$
$$M_{2,2} + M_{12,1} - Q_2 = 0$$

where $q$ is a transverse load, $Q_1$, $Q_2$ — shear forces, and $M_1$, $M_2$, $M_{12}$ — bending and twisting moments. For the moments and shear forces, the relations of elasticity are valid. In the case of orthotropic plate whose orthotropy axes coincide with the coordinate axes, these relations can be written as

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2, \qquad M_2 = D_{22}\kappa_2 + D_{12}\kappa_1$$
$$M_{12} = 2D_{66}\kappa_{12}, \; Q_1 = K_1\gamma_1, \; Q_2 = K_2\gamma_2 \qquad (2)$$

Here $\kappa_1, \kappa_2, \kappa_{12}$ — flexural strains of the coordinate surface, which can be determined by angles of rotation of the normal regardless of transverse shear $\theta_1, \theta_2$, angles of rotation of the normal due to transverse shear $\gamma_1, \gamma_2$and the complete angles of rotation of the rectilinear element $\psi_1, \psi_2$ as

$$\kappa_1 = \psi_{1,1}, \qquad \kappa_2 = \psi_{2,2}, \qquad 2\kappa_{12} = \psi_{1,2} + \psi_{2,1},$$
$$\gamma_1 = \psi_1 - \theta_1, \; \gamma_2 = \psi_2 - \theta_2, \; -\theta_1 = w_{,1}, \; -\theta_2 = w_{,2} \qquad (3)$$

Meaning by the $E_i, G_{ij}, \nu_i$ elastic and shear moduli and Poisson's ratios, the stiffness coefficients $K_i, D_{ij}$ can be determined as follows

$$D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, \qquad D_{12} = \nu_2 D_{11}, \qquad D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)},$$
$$D_{66} = \frac{G_{12}h^3}{12}, \qquad K_1 = \frac{5}{6}hG_{13}, \qquad K_2 = \frac{5}{6}hG_{23} \qquad (4)$$

Combining (1) to (4) we obtain

$$K_1\psi_{1,1} + K_1 w_{,11} + K_2\psi_{2,2} + K_2 w_{,22} = -q$$
$$D_{11}\psi_{1,11} + D_{12}\psi_{2,12} + D_{66}\psi_{1,22} + D_{66}\psi_{2,12} - K_1\psi_1 - K_1 w_{,1} = 0 \qquad (5)$$
$$D_{22}\psi_{2,22} + D_{12}\psi_{1,12} + D_{66}\psi_{2,11} + D_{66}\psi_{1,12} - K_2\psi_2 - K_2 w_{,2} = 0$$

The boundary conditions on the sides $x_1 = const$ with clamped edges have the form

$$w = 0, \; \psi_1 = 0, \; \psi_2 = 0 \qquad (6)$$

while in the case of simply supported edges

$$w = 0, \; \psi_{1,1} = 0, \; \psi_2 = 0 \qquad (7)$$

For the sides $x_2 = const$ the boundary conditions can be written similarly.

## 2. Definition of Plates in the Form of Arbitrary Quadrangle

Let us consider the region in the space of Cartesian coordinates $x_1 0\, x_2$, limited by sides of a convex quadrangle, and set an objective to translate it into the normalised region $[0 \le \xi_1 \le 1], [0 \le \xi_2 \le 1]$ in the new coordinate system $\xi_1, \xi_2$. This transition is possible when using a change of variables in the form of

$$\bar{x} = T \cdot \bar{\varepsilon} \tag{8}$$

where vector $\bar{x}$ has components $\{x_1, x_2\}$, vector $\bar{\varepsilon}$ has components $\{1, \xi_1, \xi_2, \xi_1\xi_2\}$, and the components $t_{ij}$ of transition matrix $T$ are determined by the geometry of the plate. In general, for quadrangle with vertices $(x_{11}, x_{21})$, $(x_{12}, x_{22})$, $(x_{13}, x_{23})$, $(x_{14}, x_{24})$ components of the matrix $T$ will be equal to

$$\begin{gathered} t_{11} = x_{12}, \quad t_{12} = x_{13} - x_{12}, \quad t_{13} = x_{11} - x_{12}, \quad t_{14} = x_{14} - x_{13} + x_{12} - x_{11}, \\ t_{21} = x_{22},\ t_{22} = x_{23} - x_{22},\ t_{23} = x_{21} - x_{22},\ t_{24} = x_{24} - x_{23} + x_{22} - x_{21} \end{gathered} \tag{9}$$

Next, we can obtain the governing equations (5) in the new coordinates using the transformation (8), containing information about the geometry of the quadrangle. To do this, we introduce a vector $\bar{f}$ with 18 components $\{\psi_1, \psi_{1,1}, \psi_{1,2}, \psi_{1,11}, \psi_{1,22}, \psi_{1,12}, \psi_2, \dots, w_{,12}\}$ and the corresponding coefficient matrix $S$ size of $3 \times 18$. The equations (5) take the form

$$S \cdot \bar{f} = \bar{q} \tag{10}$$

where $\bar{q} = \{-q, 0, 0\}$ and non-zero components of the matrix $S$ are

$$\begin{gathered} s_{12} = K_1, \quad s_{19} = K_2, \quad s_{1.16} = K_1, \quad s_{1.17} = K_2, \\ s_{21} = -K_1,\ s_{24} = D_{11},\ s_{25} = D_{66},\ s_{2.12} = D_{12} + D_{66},\ s_{2.14} = -K_1, \\ s_{36} = D_{12} + D_{66}, \quad s_{37} = -K_2, \quad s_{3.10} = D_{66}, \quad s_{3.11} = D_{22}, \quad s_{3.15} = -K_2. \end{gathered} \tag{11}$$

To determine the elements of matrix $\bar{S}$, which is similar to $S$ and represents coefficients of (5) in the new coordinate system, we need to find expressions for all components of the vector $\bar{f}$ taking into account transformation (8).We derive equations for the partial derivatives by the example of deflection function $w(x_1, x_2)$.

The first derivatives can be obtained from the system of equations based on well-known expressions for the partial derivative of a composite function (hereinafter the derivatives with respect to $\xi_i$denote the index after the semicolon):

$$w_{;1} = w_{,1}x_{1;1} + w_{,2}x_{2;1},\ w_{;2} = w_{,1}x_{1;2} + w_{,2}x_{2;2} \tag{12}$$

Solution of this expression system is

$$w_{,1} = Aw_{;1} + Bw_{;2},\ w_{,2} = Cw_{;1} + Dw_{;2} \tag{13}$$

where $A$, $B$, $C$, $D$ are expressions in $\xi_1, \xi_2$:

$$\begin{gathered} A = x_{2;2}/\left(x_{1;1}x_{2;2} - x_{1;2}x_{2;1}\right), \quad B = -x_{2;1}/\left(x_{1;1}x_{2;2} - x_{1;2}x_{2;1}\right), \\ C = -x_{1;2}/\left(x_{1;1}x_{2;2} - x_{1;2}x_{2;1}\right),\ C = x_{1;1}/\left(x_{1;1}x_{2;2} - x_{1;2}x_{2;1}\right) \end{gathered} \tag{14}$$

or explicitly

$$\begin{gathered} A = (t_{24}\xi_1 + t_{23})/\chi, \quad B = -(t_{24}\xi_2 + t_{22})/\chi, \\ C = -(t_{14}\xi_1 + t_{13})/\chi, \quad D = (t_{14}\xi_2 + t_{12})/\chi \end{gathered} \tag{15}$$

with the designation $\chi = (t_{12}t_{24} - t_{22}t_{14})\xi_1 + (t_{14}t_{23} - t_{24}t_{13})\xi_2 + (t_{12}t_{23} - t_{22}t_{13})$.

With the expression for the first order (13), second order partial derivatives can be obtained by replacing the function $w$ by $w_{,1}$ or $w_{,2}$ in the right side of the equations (13):

$$\begin{aligned} w_{,11} &= (AA_{;1} + BA_{;2})w_{;1} + (AB_{;1} + BB_{;2})w_{;2} + A^2 w_{;11} + B^2 w_{;22} + 2ABw_{;12} \\ w_{,22} &= (CC_{;1} + DC_{;2})w_{;1} + (CD_{;1} + DD_{;2})w_{;2} + C^2 w_{;11} + D^2 w_{;22} + 2CDw_{;12} \\ w_{,12} &= (AC_{;1} + BC_{;2})w_{;1} + (AD_{;1} + BD_{;2})w_{;2} + ACw_{;11} + BDw_{;22} + (AD + BC)w_{;12} \end{aligned} \tag{16}$$

Consider a vector $\bar{m}$ with components $\{w_{,1}, w_{,2}, w_{,11}, w_{,22}, w_{,12}\}$, corresponding vector $\bar{m}^*$ with components $\{w_{;1}, w_{;2}, w_{;11}, w_{;22}, w_{;12}\}$, and the transition matrix $L$, which satisfy the following relationship

$$\bar{m} = L \cdot \bar{m}^* \tag{17}$$

With (13) and (16), we obtain the non-zero elements of the transition matrix $L$ in form of

$$\begin{gathered} l_{11} = A, \quad l_{12} = B, \quad l_{21} = C, \quad l_{22} = D, \\ l_{31} = AA_{;1} + BA_{;2}, \quad l_{32} = AB_{;1} + BB_{;2}, \quad l_{33} = A^2, \quad l_{34} = B^2, \quad l_{35} = 2AB, \\ l_{41} = CC_{;1} + DC_{;2}, \ l_{42} = CD_{;1} + DD_{;2}, \ l_{43} = C^2, \ l_{44} = D^2, \ l_{45} = 2CD, \\ l_{51} = AC_{;1} + BC_{;2}, \quad l_{52} = AD_{;1} + BD_{;2}, \\ l_{53} = AC, \quad l_{54} = BD, \quad l_{55} = AD + BC \end{gathered} \tag{18}$$

Expressions for $A$, $B$, $C$, $D$ are given in (15) and their derivatives $A_{;1}, A_{;2}, \dots, D_{;2}$ have the form

$$\begin{gathered} A_{;1} = \big(t_{24}\chi - (t_{24}\xi_1 + t_{23})(t_{12}t_{24} - t_{22}t_{14})\big)/\chi^2 \\ A_{;2} = -(t_{24}\xi_1 + t_{23})(t_{14}t_{23} - t_{24}t_{13})/\chi^2 \\ B_{;1} = (t_{24}\xi_2 + t_{22})(t_{12}t_{24} - t_{22}t_{14})/\chi^2 \\ B_{;2} = -\big(t_{24}\chi - (t_{24}\xi_2 + t_{22})(t_{14}t_{23} - t_{24}t_{13})\big)/\chi^2 \\ C_{;1} = -\big(t_{14}\chi - (t_{14}\xi_1 + t_{13})(t_{12}t_{24} - t_{22}t_{14})\big)/\chi^2 \\ C_{;2} = (t_{14}\xi_1 + t_{13})(t_{14}t_{23} - t_{24}t_{13})/\chi^2 \\ D_{;1} = -(t_{14}\xi_2 + t_{12})(t_{12}t_{24} - t_{22}t_{14})/\chi^2 \\ D_{;2} = \big(t_{14}\chi - (t_{14}\xi_2 + t_{12})(t_{14}t_{23} - t_{24}t_{13})\big)/\chi^2 \end{gathered} \tag{19}$$

We introduce the vector $\bar{f}^*$ with 18 components $\{\psi_1, \psi_{1;1}, \psi_{1;2}, \psi_{1;11}, \psi_{1;22}, \psi_{1;12}, \psi_2, \dots, w_{;12}\}$ similar to $\bar{f}$ in (10), and the transition matrix $P$ such that satisfy

$$P \cdot \bar{f}^* = \bar{f} \tag{20}$$

The structure of the components of the vectors $\bar{f}$ and $\bar{f}^*$ includes derivatives of the three functions, for each of which (17) is applicable. Then the matrix $P$ is of the form

$$P = \begin{pmatrix} 1 & o_r & 0 & o_r & 0 & o_r \\ o_c & L & o_c & O & o_c & O \\ 0 & o_r & 1 & o_r & 0 & o_r \\ o_c & O & o_c & L & o_c & O \\ 0 & o_r & 0 & o_r & 1 & o_r \\ o_c & O & o_c & O & o_c & L \end{pmatrix} \tag{21}$$

where $O$ — zero matrix of size 5×5, $o_c$ — zero column vector of five components, $o_r$ — zero row vector of the five components.

In view of (20) equation (10) is written in a new coordinate system as

$$S \cdot (P \cdot \bar{f}^*) = \bar{q}^* \tag{22}$$

or

$$\bar{S} \cdot \bar{f}^* = \bar{q}^* \tag{23}$$

where $\bar{f}^*$ and $\bar{q}^*$ are analogs of the vectors $\bar{f}$ and $\bar{q}$ in the new coordinate system and $\bar{S} = S \cdot P$.

Equations (23) are the system of governing equations (5) in coordinates $\xi_1, \xi_2$ and contain information about geometry of the quadrangular plate. Since the original domain in the form of an arbitrary quadrangle in the new coordinates takes the form of a square, to solve the boundary value problem it is possible to use the corresponding numerical methods.

It should be noted that the use of boundary conditions containing derivatives of deflection $w$ and angles $\psi_1, \psi_2$ (simply supported, free edge), is required to solve the problem taking into account changes due to transition to the new system of coordinates according to (17). In particular, the boundary conditions in the case of simply supported edges on $\xi_1 = const$ sides of a the plate take the form of

$$w = 0,\ A\psi_{1;1} + B\psi_{1;2} = 0,\ \psi_2 = 0 \tag{24}$$

with similar amendments to $\psi_2$ on the sides $\xi_2 = const$. Expressions $A$ and $B$ are meant in form (15).

## 3. Numerical Results and Discussion

System of equations (23) in conjunction with different boundary conditions form a two-dimensional boundary value problem with respect to the deflection $w$ and angles $\psi_1, \psi_2$. The latter can be solved by the methods of spline collocation and discrete orthogonalization.

Spline approximation based on B-splines of the third degree with 30 pairs of collocation points on the $\xi_2$-axis was used. The resolving system of higher order ordinary differential equations was solved by discrete orthogonalization method with 1500 integration points. These parameters were the same for all calculations.

### 3.1 Trapezium shaped plates

Generally, variable coefficients in the equations (23) are dependent on the coordinates $\xi_1, \xi_2$ and the proposed approach makes it possible to use it for the analysis of a stress-strain state of orthotropic plates. Despite this, the proposed calculation scheme was tested in the case of isotropic material. The results were compared with those of work [6].

**Table 1.** The coordinates of the vertices of the trapezoids under review

| object No. | $x_{11}$ | $x_{21}$ | $x_{12}$ | $x_{22}$ | $x_{13}$ | $x_{23}$ | $x_{14}$ | $x_{24}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 47.15 | 4.13 | 47.15 | –4.13 | 57.15 | –5.00 | 57.15 | 5.00 |
| 2 | 8.66 | 2.32 | 8.66 | –2.32 | 18.66 | –5.00 | 18.66 | 5.00 |
| 3 | 52.15 | 4.56 | 52.15 | –4.56 | 57.15 | –5.00 | 57.15 | 5.00 |
| 4 | 8.74 | 3.18 | 8.74 | –3.18 | 13.74 | –5.00 | 13.74 | 5.00 |
| 5 | 0.96 | 0.80 | 0.96 | –0.80 | 5.96 | –5.00 | 5.96 | 5.00 |

**Table 2.** Results of calculation of the deflection $\hat{w}$ for the trapezoids

| object No. | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\hat{w}$ | 108889 | 59309 | 17128 | 15628 | 7911 |
| $\hat{w}$, [6] | 112400 | 60960 | 17030 | 15630 | 7703 |
| δ, % | 3.12 | 2.71 | 0.58 | 0.01 | 2.70 |

As objects for analysis, five plates in the form of trapezoids were selected. Constant load $q = q_0$ evenly distributed on the surface while conditions on the edges assume rigid fixation (clamped edge). The coordinates of the vertices of the trapezoids are shown in Table 1. The thickness of the plates $h$ is 0.1, and the Poisson's ratio $\nu_1 = \nu_2 = 0.3$.

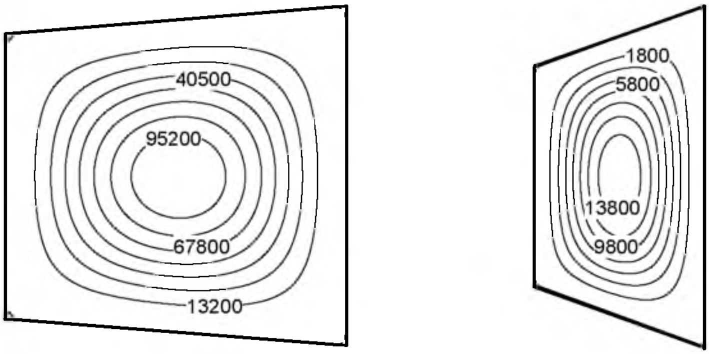


**Figure 1.** General view of the deflection surface of the plate with a different geometry

The results of calculations are shown in Table 2 (the first row) as a value of the maximum deflection $\hat{w} = wE/q_0$. The data of the article [6] presented below, as well as the values of the relative differences between the results δ. The calculated results are in good agreement for all the considered options. General view of deflection surfaces $\hat{w}(x_1, x_2)$ for some of them is shown in Fig. 1. The deflection surface corresponds to the symmetry of the plate and the absolute value of deflection decreases with decrease of the surface area.

### 3.2 Plate having the shape of an irregular quadrangle

As a more complex object for analysis a plate of an irregular convex quadrangle with vertices (1.0, 4.0), (4.0, 1.0), (7.0, 5.0), (2.0, 4.5) was selected. For comparison, we analysed three identical plates, deployed at 120 degrees relative to each other. All calculation parameters and physical constants of the objects except for the plate size correspond to those defined above for the trapezoids. Bounding conditions imply clamped edges.

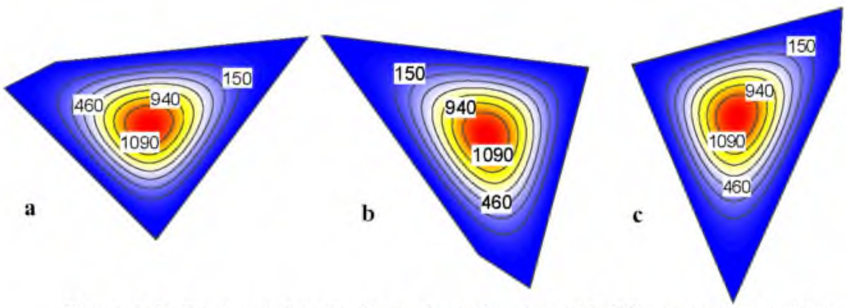


**Figure 2.** Deflection surface for the quadrangular plate with different angular position

Fig. 2 shows a view of the deflection surfaces $\hat{w} = wE/q_0$ for all three cases: original plate (Fig. 2a), rotated by 120 degrees CCW (Fig. 2b), and rotated by 120 degrees CW (Fig. 2c). As expected, the results are almost identical: the maximum deflection in the central part of the plate is 1245.06. It is worth noting that the use of proposed approach does not cause difficulties in the calculations in the neighbourhood of the corner points, despite the fact that the shape of the quadrangular plate is close to the triangle.

## Conclusions

A numerical and analytical approach to solving problems about the stress-strain state of orthotropic rectangular plates of complex shape is proposed. The system of governing equations is written in the new coordinates, based on transformation that take into account the complex shape of the quadrangular plate. Resulting two-dimensional boundary value problem is solved by the methods of spline collocation and discrete orthogonalization. Usage of these methods allows us to study the isotropic and orthotropic plates for both types of analysis, static and dynamic.

Calculated results agree well with the data of the other authors. The texture of deflection plate surfaces shows the possibility to perform calculations in the close proximity to the corner points.

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