Proceedings of the 5th International Conference on Nonlinear Dynamics
ND-KhPI2016
September 27-30, 2016, Kharkov, Ukraine

Liquid Nonlinear Oscillations in the U-Tube System

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Abstract
Dynamics of oscillation processes in a siphon U-tube is studied for the system of connected vessels filled with homogeneous liquid. The equations and phase paths describing the motion of viscous and non-viscous liquids are given, oscillation frequencies are considered. Oscillations are nonlinear in general case, but they turn into linear by setting specific parameter values of the system. Phase portraits are obtained and their dependences on parameters of the system are analyzed for both linear and non-linear cases.

Keywords
communicating vessels, oscillations of liquid, linear and non-linear motion equations, phase portraits

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Introduction

Studying the properties of motion of a liquid in a siphon U-tube has a long history and is still of an essential interest nowadays. System like communicating vessels has fundamental and practical importance. Here we can reference work [1] which examined transmission of liquid helium through superleak connecting two vessels in process of heating one of the containers. This process is interesting because a thermomechanical effect [2] in phonon regime takes place, which is also discussed in terms of an increase of quantum degenerancy in colder compartment. However, the way of changing the quantum degenerancy is not uniform; another method to cause it is adiabatic displacement of the wall dividing two compartments of homogeneous quantum fluid. On the thermomechanical effect a well-known process called helium fountain is based: if superfluid helium is heated, the flow of liquid can achieve velocity high enough for a short-time liquid excess the level of vertical vessel [3]. Practical importance of connected vessels is, for example, that their usage as a U-tube damper systems can lead to reduction of the vibration amplitude of high buildings; on ships these systems are used to reduce the rolling motion caused by waves [4]. Based on the principle of communicating vessels water locks of the rivers and channels operate, as well as the level-measuring tubes for water tanks. Siphon U-tubes are also used for determination of the volume of a non-magnetic body of a random shape [5]. There is also an interesting example of the system of communicating vessels which can be used in experimental studies of 4He equilibrium in its liquid/solid state, in such system one vessel is half-immersed into another [6]. Other important phenomena in this field are siphon properties of liquid, i. e. the ability of liquid to overcome a certain barrier without external mechanical action [7]. There exist analogies between liquid oscillations in a U-tube system and other physical phenomena like electric current [8], distribution of elastic waves in the condensed matters, and physical, mathematical pendulums [9].

Our article is devoted to the research of free vibrations of liquid in the system of communicating vessels including oscillations, which can lead to realization of the siphon mechanism. At the same time we do not take into account pouring and fountain effects. Actually, we examined such motion as oscillations, so for describing them differential equations and phase portraits were used. The equations are nonlinear in general case, but some specific values of the parameters of the system, for example the equality of the cross sections of vertical vessels, leads to motion described by linear equations. It should be noted that such a system is also analyzed in [8], where the relation between heights of the liquid in two vessels and time is obtained, and the period of oscillation is derived through a transformation formula of the elliptical integral of the second order. The main idea of the article [8] is using the unsteady Bernoulli equation for describing the liquid motion in the U-tank. We obtain the characteristics describing the motion of a liquid by means of Euler-Lagrange equations and the law of energy conversation.

1. Overview of the system

Here we study the behavior of incompressible liquid in U-tube in Earth's gravity field. Geometry of this system is shown in Fig.1. Index g corresponds to the wide container (tube), index s corresponds to the narrow container (tube), L corresponds to the communicating tube; S_g and S_s are cross sections of the vertical tubes, H_g and H_s are heights of liquid in these tubes. Zero coordinate of Z - axis is assigned to the equilibrium height H_0 of liquid under the upper line of the connecting tube, S_L is the cross section of this tube. The levels of the liquid never reach the height low enough to let the surrounding air enter the connecting tube. Length of the tube L is measured between the axes of vertical vessels, as shown in Fig.1.

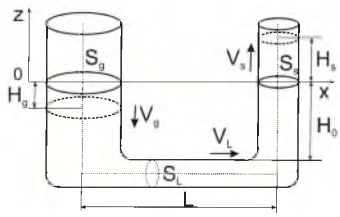

Figure 1. The U-tube system

In case of the free oscillations in the system disturbed from equilibrium state, motion equations for incompressible liquid could be written as a balance condition between volumes of liquid in vertical vessels or by means of parameters such as cross sections and height of the liquid:

$$S_g H_g = -S_s H_s \tag{1}$$

and after differentiation we obtain an equation for the speeds of the liquid in the different tubes of the system:

$$S_g v_g = -S_s v_s = -S_L v_L \tag{2}$$

Here we consider that constant of integration is zero. Directions of the speeds v_L и v_s are shown in Fig.1. Flow through the connecting tube S_L does not change the volume of liquid in it. Let us perform calculation of the main physical properties of this system using Lagrangian formalism.

2. Dynamical equations of the system of communicating vertical vessels

To obtain Lagrange function $L=T-U$ (where T is a kinetic and U is a potential energy) we first find potential energy U when liquid is disturbed from equilibrium state, which can be done by various ways: for example, by heating the liquid or by means of mechanical impact on liquid surface. In the narrow container the height of the liquid is $H_s/2$, and the change of its mass is positive $m_s = \rho\Delta V = \rho S_s H_s > 0$. For a large container the height of mass center of the liquid is $H_g/2$, and the change of its mass is negative $m_g = \rho\Delta V = \rho S_g H_g < 0$ because the liquid flows from large to narrow container. So the corresponding potential energies are:

$$U_s = \frac{1}{2}\rho g S_s H_s^2$$
$$U_g = \frac{1}{2}\rho g S_g H_g^2 \tag{3}$$

Kinetic energies of the liquid in the respective parts (tubes) of the vessel are:

$$T_s = \frac{1}{2}m_s v_s^2 = \frac{1}{2}\rho S_s\left(H_0 + H_s\right)v_s^2$$
$$T_g = \frac{1}{2}m_g v_g^2 = \frac{1}{2}\rho S_g\left(H_0 + H_g\right)v_g^2 \tag{4}$$
$$T_L = \frac{1}{2}m_L v_L^2 = \frac{1}{2}\rho S_L L v_L^2$$

and Lagrange function can be written as:

$$\mathrm{L} = T - U = -\frac{1}{2}\rho g S_s H_0^2\{y^2(1+\sigma_{sg}) - \dot{y}^2[y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}]\} \tag{5}$$

where $\sigma_{sg} = S_s/S_g$, $\sigma_{Lg} = S_L/S_g$, $\sigma_{Ls} = S_L/S_s$, $l = L/H_0$, $y = H_s/H_0$.

Dynamics of the system is described by Lagrange equation:

$$2\ddot{y}[y(1-\sigma_{sg}) + 1 + \frac{l}{\sigma_{Ls}(1+\sigma_{sg})}] + \dot{y}^2(1-\sigma_{sg}) + 2y = 0 \tag{6}$$

Nonlinear parts of motion equation appear because of variation of the height of liquid when it moves in the vessels with different cross sections. It is clear that when $\sigma_{sg} = 1$ ($S_g = S_s$) the equation (6) is linear:

$$\ddot{y}(1 + \frac{l}{2\sigma_{Ls}}) + y = 0 \tag{7}$$

So only equality of cross sections of vertical vessels is important for the linearity of the equation (6); the cross section of horizontal connecting tube thus can actually be of an arbitrary value. Equation (7) is linear with respect to the variable y and describes ordinary harmonic oscillations. Period of those oscillations can be written using reduced length $l_r = (H_0 + l/2\sigma_{Ls})$:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{H_0 + LS_s/2S_L}{g}} = 2\pi\sqrt{\frac{l_r}{g}} \tag{8}$$

As we see there takes place analogy with physical pendulum. In case of equality to zero of the parameter l the reduced length coincides with the equilibrium height of liquid H_0, and the physical pendulum becomes mathematical with oscillation period $T = 2\pi(H_0/g)^{1/2}$. The reduced length equals to equilibrium height of liquid in case of $\sigma_{Ls} = \infty$ too as then the component $LS_s/(2S_L)$ is zero. If cross sections of connecting tube and vertical tubes are of the same value, the reduced length differs from equilibrium liquid height H_0 on a half of length of connecting tube L. Otherwise at equality of all cross sections reduced length equals to the half of length of a connecting tube. The equation also becomes linear in the case of infinitesimal amplitude of oscillations ($y << 1$); then oscillation period is defined by equation (7).

3. Phase portraits. Analysis of nonlinear oscillations of the system

Let us consider behavior of the system in general and integrate the equation (6). Without friction the integral of motion is a total energy of the system:

$$W = T + U = \frac{1}{2}\rho g S_s H_0^2[y^2(1+\sigma_{sg}) + \dot{y}^2[y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] = const \tag{9}$$

Constant of integration is defined by maximum value of liquid height in narrow vertical vessel $y_0 = H_{max}/H_0$ (turning point when $y = const$), i.e.:

$$y^2(1+\sigma_{sg}) + \dot{y}^2[y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] = y_0^2(1+\sigma_{sg}) \tag{10}$$

For studying of phase portraits it is convenient to rewrite the equation (10) into:

$$\dot{y}^2 = \frac{y_0^2 - y^2}{y(1-\sigma_{sg}) + 1 + \dfrac{l}{\sigma_{Ls}(1+\sigma_{sg})}} \tag{11}$$

This rearrangement allows to find period of nonlinear oscillations as the integral of this equation [9]. The equation (11) defines a phase portrait of the system generally.

3.1 Phase portraits of the linear oscillations of the system

At equality of cross sections of vertical tubes oscillations of the liquid are described by linear equation (7), and, thus, in the denominator of expression (11) there is no dependence on coordinate y. In this case phase portrait of the system is ellipse with semiaxes $a^2 = y_0^2$, $b^2 = y_0^2/(1 + l/\sigma_{Ls})$, and eccentricity $\varepsilon^2 = 1/(l + \sigma_{Ls})$:

$$\frac{y^2}{a^2} + \frac{\dot{y}^2}{b^2} = 1 \tag{12}$$

Obviously, if we have harmonic oscillations when parameters of the system are fixed, all ellipses are similar for different oscillation amplitudes. Direction of rotation is topological invariant (see Fig. 2).

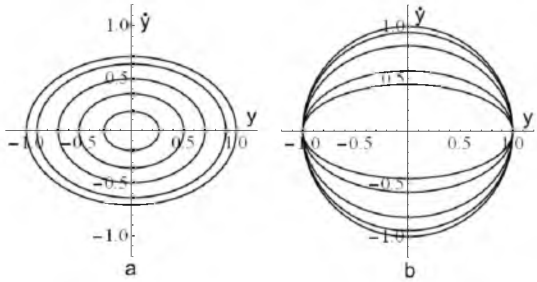

Figure 2. The harmonic oscillations of a liquid in the system: a) $l/\sigma_{Ls} = 2$, $y_0=0.2; 0.5; 0.7; 0.9$; 1; b) $y_0 = 1$, $l/\sigma_{Ls} = 0; 0.125; 0.5; 2; 4$.

Interesting dependence on parameter $l/\sigma_{Ls} = LS_s/(S_LH_0)$ follows from the equation (12): in extremely minimum case $l/\sigma_{Ls} = 0$ phase portrait on plane $y - y_0$ is a circle. This parameter may become zero (or closely near to zero) in two ways: when equilibrium liquid height in vertical vessels much exceeds the length on connecting tube ($l = 0$), or when two vessels are connected by sealed channel with large cross section ($\sigma_{Ls} = \infty$). The increase in parameter l/σ_{Ls} leads to the flattened phase portrait in Fig. 2 (b), and the decreasing of the frequency of oscillations.

3.2 Phase portraits of nonlinear oscillations

In case of the maximum amplitude $y_0 = H_{smax}/H_0 = 1$ at big difference in sections of vertical tubes ($\sigma_{sg}=0$), and under the condition $l/\sigma_{Ls} = 0$ the equation (11) is reduced to:

$$\dot{y}^2 = 1 - y \tag{13}$$

On the phase plane it is equation of the parabola (Fig. 3 (a)) closed by a vertical segment. Analyzing the equation (11) it is possible to see that if the length of a connecting tube is increased, the parabola is flattened and turns into an ellipse. The graphic analysis of fluctuations for the case $l/\sigma_{Ls}= 0$ is also given in [4] where two phase paths in the limiting cases are presented: when the cross sections of vertical tubes are identical and when they are much different. When amplitude changes within $0 < y_0 < 1$ (displacement from $-y_0$ to y_0) the following evolution of phase paths takes place in the system: semiaxes of ellipses at $y_0 \to 0$ increase with a growth of amplitudes, ellipses being strongly deformed when a fluid level reaches the bottom points; corners of paths on phase plane during approximation to a parabolic form are sharped (Fig. 3 (b)).

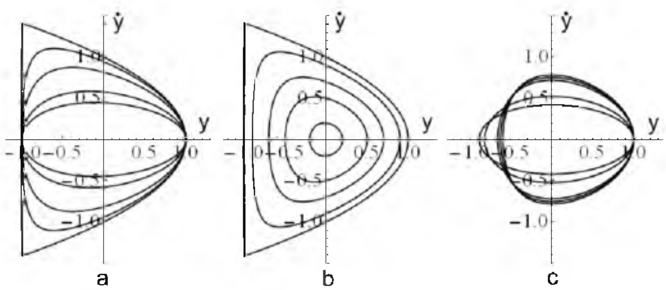

Figure 3. Phase portraits of nonlinear oscillations of a liquid in the system: a) $y_0^2=1$, $\sigma_{Ls} = 0$, $l/\sigma_{Ls} = 0; 0.125; 0.5; 2; 4$; b) $\sigma_{sg} = 0$, $l/\sigma_{Ls}=0$, $y_0 = 0.2; 0.5; 0.7; 0.9; 1$; c) amplitude is small $y^2 \to 0$, $\sigma_{sg} = 0.5$, $l/\sigma_{Ls} = 0; 0.125; 0.5; 2; 4$.

Taking into account a numerical factor the behavior of the system in cases $\sigma_{sg} = 1$ and $y_0 << 1$ coincide. When there is small difference between cross sections, the phase paths are ellipses, and in the process of decreasing of the parameter they are imposed at each other, almost merging into one curve. Further, if the oscillation amplitude is small ($y << 1$), we have the linear description of liquid oscillations in vessels with a period (8). Assuming the amplitude to be a small, but such that we neglect only an item contain square of derivative from y, we obtain a nonlinear equation. In Fig. 3 (c) change of phase paths in process of increase in parameter l/σ_{Ls} is the following: elliptic paths are flattened to an ordinate axis having one generic point.

4. Oscillation frequencies in the system

For studying nonlinear oscillations of the system and analyzing oscillation period we find the integral of equation (11):

$$t - t_0 = \sqrt{(1-\sigma)} \int dy \sqrt{\frac{y+q}{y_0^2 - y^2}} \tag{14}$$

where t_0 is the constant of integration and limits of integration are $-y_0$ and y_0:

$$t = \sqrt{1-\sigma_{sg}} \times 2\left(E(\varphi,k)\sqrt{y_0+q} - \sqrt{\frac{(y_0-y)(y+y_0)}{y+q}} \right) \tag{15}$$

where $E(\varphi, k)$ is an elliptical integral of the second kind. Here notations are:

$$\sin\phi = \sqrt{\frac{(y_0+q)(y+y_0)}{2y_0(y+q)}},\ q = \frac{1}{1-\sigma_{sg}} + \frac{l}{\sigma_{Ls}\left(1-\sigma_{sg}^2\right)},\ k = \sqrt{\frac{2y_0}{y_0+q}}$$

The half-period of oscillations is a time span of the oscillations amplitude changing from its minimum to maximum point (from $-y_0$ to y_0). We see that if we substitute the limits into the expression (15), the second term becomes zero both at minimum and maximum value. Furthermore we see that in the case of $y = -y_0$, $\sin\varphi = 0$ and thus, $\varphi = 0$, and if $y = y_0$ then $\sin\varphi = 1$, and thus $\varphi = \pi/2$. Because $E(0,k) = 0$, then oscillation frequency is:

$$\omega = \pi \Big/ \left(2\sqrt{(1-\sigma_{sg})(y_0+q)}\, E\left(\frac{\pi}{2}, k\right) \right) \tag{16}$$

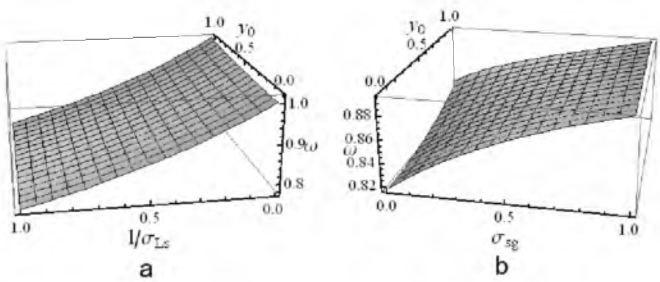

Figure 4. The dependence of oscillation frequency on a) initial amplitude y_0 and parameter l/σ_{Ls} at $\sigma_{sg} = 0.5$; b) initial amplitude y_0 and parameter σ_{sg} at $l/\sigma_{Ls} = 0.5$.

The influence caused by initial amplitude y_0 and by parameters of the system l/σ_{Ls} and σ_{sg} is shown in Fig. 4 (a) and in Fig. 4 (b). Fig. 4 (a) presents the dependence of oscillation frequency on initial amplitude and parameter l/σ_{Ls} when the relation between cross sections is constant value. It is obvious that oscillation frequency increases with initial amplitude growth and the exponential law describes decreases with growth of parameter l/σ_{Ls}, and dependence of frequency on parameter l/σ_{Ls}. In Fig. 4 (b) there is a dependence of the oscillation frequency on initial amplitude y_0 and a parameter l/σ_{Ls}. As we can see, oscillation frequency changes with an increase of an initial amplitude y_0 as an exponential function and with an increase of relation between cross sections σ_{sg} as a logarithmic function. To the linear cases of the oscillations there correspond lines with a constant frequency. For example, if $k = 0$ (i. e. $y_0 = 0$) $E(\pi/2,k) = \pi/2$ and then frequency is:

$$\omega = 1\Big/\sqrt{\frac{H_0}{g}}\sqrt{1+\frac{S_s S_g}{H_0 S_L (S_g + S_s)}} \tag{17}$$

In addition, k becomes zero at $q \rightarrow \infty$ (i. e. at $\sigma_{sg} = 1$), and then we receive the expression for the frequency. If the cross sections of vertical tubes considerably differ, the oscillations period does not depend on a cross section of a vertical tube with a large diameter.

$$\frac{T}{2} = \sqrt{\frac{H_0}{g}}\,2\sqrt{y_0 + 1 + \frac{LS_s}{S_L H_0}}\,E\left(\frac{\pi}{2}, \sqrt{\frac{2y_0}{y_0 + 1 + \frac{LS_s}{H_0 S_L}}}\right) \tag{18}$$

4. Dynamics of viscous liquid

In the above section the system of communicating vessels which is filled with a perfect liquid was considered. Actually ideal non-viscous liquid almost never exists in nature, so considering practical application, it is necessary to take into account the dissipation of energy in the system, which occurs in viscous (non-ideal) liquid. The review of viscous liquid is also interesting because of the above-mentioned thermomechanical effect, which can be seen for superfluid helium, since it has the normal (viscous) component and superfluid component in which there is no dissipation of energy.

Let us consider viscosity in the simplest case, one whereby a member that takes into account the attenuation of oscillations in the system of proportional speed and equal to $-\eta(v_s + v_g + v_L)$, where η is a parameter that is expressed through the coefficient of dynamic viscosity. Considering the ratio between speeds in three knees, we got that the equations of motion are:

$$\frac{\rho g S_s H_0^2}{2}\left\{2\ddot{y}[y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] + \dot{y}^2(1-\sigma_{sg}^2) + 2y\left(1+\sigma_{sg}\right)\right\} = -\eta\sqrt{gH_0}\,\dot{y}\left(1-\sigma_{sg} + \frac{1}{\sigma_{Ls}}\right) \tag{19}$$

After some transformations, equation has the form

$$2\ddot{y}\left(y + \frac{1}{1-\sigma_{sg}} + \frac{l}{\sigma_{Ls}\left(1-\sigma_{sg}^2\right)}\right) + \dot{y}^2 + k\left(\frac{1}{\sigma_{Ls}\left(1-\sigma_{sg}^2\right)} + \frac{1}{1+\sigma_{sg}}\right)\dot{y} + \frac{2y}{1-\sigma_{sg}} = 0 \tag{20}$$

where $k = 2\eta(gH_0)^{1/2}/(\rho g S_s H_0^2)$.

If we consider a system in which the cross sections of vertical tubes are equal, then we have an equation that is similar to the classical equation of the damped oscillation of the pendulum:

$$\ddot{y} + \frac{k}{2\sigma_{Ls}\left(2 + l/\sigma_{Ls}\right)}\dot{y} + \frac{2}{2 + l/\sigma_{Ls}}y = 0 \tag{21}$$

in this case, depending on relations between the parameters of the system we have: a regular periodic oscillations (when $k < 4\sigma_{Ls}(2(2 + l/\sigma_{Ls}))^{1/2}$) with frequency

$$\omega = \frac{2\sigma_{Ls}}{(2 + l/\sigma_{Ls})}\sqrt{2(2 + l/\sigma_{Ls}) - k^2} \tag{22}$$

We also can have an aperiodic oscillations when $k > 4\sigma_{Ls}(2(2 + l/\sigma_{Ls}))^{1/2}$ and the so-called "critical damping" when $k = 4\sigma_{Ls}(2(2 + l/\sigma_{Ls}))^{1/2}$.

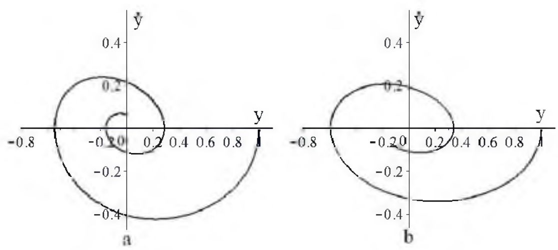

Figure 5. Phase portraits of the vibrations of viscous liquid with dimensionless viscosity $k = 0.5$: a) $\sigma_{sg} = 0.2, l = 1, \sigma_{Ls} = 0.4$; б) $\sigma_{sg} = 0.2, l = 2, \sigma_{Ls} = 0.4$.

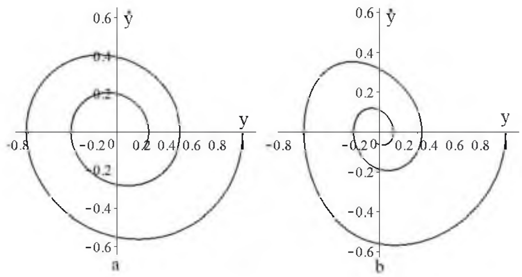

Figure 6. Phase portraits of the vibrations of viscous liquid with dimensionless viscosity $k = 0.5$: a) $\sigma_{sg} = 0.95, l = 1, \sigma_{Ls} = 0.4$; б) $\sigma_{sg} = 0.2, l = 1, \sigma_{Ls} = 1$.

In the case of vessels of different cross sections graphical analysis of the equation is possible. Clearly, as for conventional oscillator with dissipation of energy, phase portrait of liquid oscillations is transformed from a closed curve into an open curve, with increasing viscosity spiral spins faster and the distance between the coils is increased. The decreasing of parameter σ_{sg} also leads to more rapid damping vibrations, as shown in Fig. 5 (a) and Fig. 6 (a), where are presented phase portraits of oscillations for different parameters of the system for the same initial amplitude: Fig. 5 (a) phase trajectory corresponds to a significant asymmetry in the system, while in Fig. 6 (a) the difference between the vertical sections of tube is low. When increasing the length of the connecting tube (decrease equilibrium height) the distance between the turns of the spiral decreases, i.e. damping rate decreases, as shown in Fig. 5 (b). In Fig. 6 (b) it is shown that the curve twists slowly with increasing of the parameter σ_{Ls} too. In other words, increase in the asymmetry of the system is similar to the increase of the viscosity of a fluid and has damping effect.

Conclusions

Oscillations of liquid in the system of communicating vessels are studied. Frequencies and periods of free oscillation are described for the general case through elliptical integral.

Phase portraits are plotted for the general case of oscillations when different parameters of the system are changed, i. e. dependence of liquid motion on one parameter of the system when other parameters are fixed. When oscillations are linear, phase paths are elliptical, while in the case of nonlinear oscillations we have parabolas closed by a vertical segment. The same kind of phase portraits also correspond to the vertical motion of a ball elastically bouncing off a horizontal surface, which suggests analysis of the strong analogies between the various mechanical motions and the oscillations of the liquid in the communicating vessels.

In the sections 2 and 3 of this work the internal friction (viscosity) is neglected, which have an influence on real liquids' motion. If we consider influence of viscosity, the general motion equations and, therefore, oscillation periods will differ from those which were obtained for an ideal liquid. It is of interest to further analyze the general expression for oscillations frequency because in the current work analysis is presented for the case of the different cross sections of two vessels, which is described by the nonlinear equations. The case of the identical cross sections of two vessels was considered earlier in the work [9]. It is also prospective to consider the siphon mechanism within this system and study the fountain effect caused by special initial conditions. Using discussed system of communicating vessels such extraordinary phenomena as superfluid flow (for example, liquid helium or diluted quantum gas) and supersolid (^{4}He) could be experimentally studied.

In the section 4 of this work we take into account the influence of the viscosity, to consider the situation more approximate to the real liquids. The several phase portraits of liquid oscillations are given. Accounting an influence of viscous leads to nonlinear motion equations that are not solved by quadrature, but can be reduced into Abel's differential equations.

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