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Non-Iterative Rauscher Method for 1-DOF System: a New Approach to Studying Non-Autonomous System via Equivalent Autonomous One

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Abstract
In the paper a new non-iterative variant of Rauscher method is considered In its current statement the method can be used in analysis of forced harmonic oscillations in 1-DOF nonlinear system.
It is shown that three different types of equivalent authonomous dynamical systems can be built for a given 1-DOF non-autonomous one. Two of them (1st and 2nd type) have wider set of solutions than that of the initial system. These solutions correspond to various values of amplitude and phase of external excitation. Solutions of the equivalent system of 3rd type are exclusively periodic ones.
Based on the equivalent system of 3rd type such a function W(x,x') can be constructed that its level curves correspond to periodic orbits of the initial non-autonomous system. This function can be built a priori via computation of the invariant manifold of the equivalent system of 1st type. Using the same approach the Rauscher expansions cos(Ωt)=C(x,x'), sin(Ωt)=S(x,x') can also be constructed
It is also shown that equivalent systems can be investigated by means of harmonic balance method which allows construction of W(x,x'), C(x,x') and S(x,x') in semi-analytical manner.

Keywords
Rauscher method, equivalent autonomous system, periodic solutions, invariant manifolds methodology, continuation tecniques, harmonic balance method.

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Introduction

The key principle of Rauscher method is that in certain modes non-autonomous dynamical system may behave like an autonomous one. So instead of studying a non-autonomous system one can study some equivalent autonomous system. Consider, for example, non-authonomous dynamical system (1) having N degrees of freedom (DOFs):

$$\left\{x_i'' + F_i\left(x_1, x_2, .., x_n, x_1', x_2', .., x_n', t\right) = 0, \quad i = 1...N \right. \tag{1}$$

If the dependency $x_k = x_k(t)$ is somehow inverted to a form $t = t(x_k)$ (x_k is some pre-selected generalized coordinate), then the system (1) can be transformed into autonomous form:

$$\left\{x_i' + F_i\left(x_1, x_2, .., x_n, x_1', x_2', .., x_n', t(x_k)\right) = 0, \quad i = 1...N \right. \tag{2}$$

This idea was presented by Rauscher in [1]. Originally the Rauscher method could be applied for 1-DOF systems only, namely, for finding steady forced oscillations. However, it was generalized for multi-DOF systems [2-8] by means of theory of nonlinear normal vibration modes (NNMs).

The dependency $t = t(x_k)$ is constructed in such way that the systems (1) and (2) have the same solutions corresponding to steady forced oscillations (in this sense (2) can be called an equivalent system to (1)). For conservative systems (or close to conservative ones) the dependency $t = t(x_k)$ can be represented in the form of quadratures (see for example [3,4]).

In the papers [6,7] another approach is proposed. Equations of motion are supposed to be:

$$\left\{x_i' + F_i\left(x_1, x_2, .., x_n\right) = f_i \cos(\Omega t), \ (i = 1...N)\right. . \tag{3}$$

Then time t can be eliminated, according to [6,7], by means of constructing a dependency

$$\cos(\Omega t) = a_0 + a_1 x_k + a_2 x_k^2 + a_3 x_k^3 + ... \tag{4}$$

or $\cos(\Omega t) = C(x_k)$. Such expansion can be constructed if some initial approximation for x_k is known in form $x_k = A_0 + A_1\cos(\Omega t) + A_2\cos(2\Omega t) + \ldots$ It is proposed in [6] to rewrite the latter using trigonometric transformations as $x_k = A_0 - A_2 + (A_1 - 3A_3)\cos(\Omega t) + 2A_2\cos^2(\Omega t) + 4A_3\cos^3(\Omega t)\ldots$ Once (4) is substituted into this expression, coefficients of the same orders of x_k are equated, which leads to a system of algebraic equations that can be solved for unknowns $a_0, a_1, a_2, \ldots$

Once current approximation for $a_0, a_1, a_2, \ldots$ is known, expression (4) is substituted into (3). The new system does not contain time *t* explicitly:

$$\left\{x_i' + F_i(x_1, x_2, \ldots, x_n) = f_i\left(a_0 + a_1 x_k + a_2 x_k^2 + a_3 x_k^3 + \ldots\right),\ (i = 1\ldots N)\right. \tag{5}$$

It is analyzed via invariant manifolds methodology. If state space variables of a nonlinear system undergo *coherent changes*, one can speak of what is called nonlinear normal mode (NNM). NNM in the system (5) can be expressed as a set of such dependencies:

$$\left\{x_i = x_i(x_k, x_k'),\ x_i' = x_i'(x_k, x_k'),\ (i = 1, \ldots k-1, k+1, \ldots N)\right. . \tag{6}$$

These dependencies are invariant manifolds of the autonomous system (5) (they are also called NNMs by Shaw and Pierre). Geometrically they represent a set of surfaces (or a single hyper-surface) in the state space. In order to find NNM the autonomous system should be transformed into a system of partial differential equations taking x_k and x_k' as couple of independent variables. This approach was developed by S.Shaw and C. Pierre in [9,10]. Comprehensive overview of different NNM theories can be found in [11,12].

When (6) is constructed from (5), the system (3) can be reduced to 1-DOF with respect to x_k. This allows one to obtain more precise trigonometrical approximation for x_k and therefore *iterative process is constructed*. This approach was extended by the author for more general case:

$$\left\{x_i'' + F_i(x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x', t) = f_i\cos(\Omega t) + g_i\sin(\Omega t),\quad i = 1\ldots N\right. \tag{7}$$

In order to make the system (7) non-autonomous one needs here to build such dependencies:

$$\cos(\Omega t) = C(x_k, x_k'),\quad \sin(\Omega t) = S(x_k, x_k') \tag{8}$$

In the papers [13-16] the latter are represented in the form of power series.

It also should be noted that different forms and various applications of the Rauscher method are discussed in [11,17]. Among recent works where the Rauscher method is used one can find the following ones: [18,19].

The method described above has a major drawback – iterative nature. So *the goal of the present paper is to find a non-iterative approach which corresponds to the primary idea of the Rauscher method: eliminating time from non-autonomous equations of motion in order to construct an equivalent autonomous system of equations which has the same periodic solutions as the initial one.*

In the present paper only 1-DOF mechanical system is considered:

$$x'' + hx' + \omega^2 x + \varphi(x) = f\cos(\Omega t) \tag{9}$$

The system (9) is supposed to have single equlibrium position at $x = 0$, the function $\varphi(x)$ is supposed to be an analytical function in the neighborhood of the equlibrium position and contain only nonlinear terms. Damping is considered to be small ($h \ll \omega^2$).

In Section 1 of the paper three different types of equivalent autonomous systems are considered and their properties are studied analytically. Geometrical interpretation of the obtained results is given as well. In Section 2 it is shown that some important results such as invariant surfaces of equivalent systems can be obtained semi-analytically via harmonic balance method. Section 3 contains illustrative examples.

1. Equivalent Autonomous Systems and Their Properties

If the system (9) is linear ($\varphi(x) = 0$) finding Rauscher approximations is quite an easy task. In such case one has: $x'' + hx' + \omega^2 x = f\cos(\Omega t)$ and $x''' + hx'' + \omega^2 x' = -f\Omega\sin(\Omega t)$. If only periodic solutions are considered, then $x'' = -\Omega^2 x$, $x''' = -\Omega^2 x'$ since $x = A\cos(\Omega t) + B\sin(\Omega t)$ for such case. Thus x'' and x''' can be eliminated which yields Rauscher approximations in the form (8). Periodical solution $x(t)$ can be found by means of the Rauscher expansions when they are solved for x and x'.

1.1. Equivalent dynamical system of 1st type

If the system (9) is nonlinear ($\varphi(x) \neq 0$), the relations $x'' = -\Omega^2 x$, $x''' = -\Omega^2 x'$ require some correcting function $y(t)$ to be introduced: $x'' = -\Omega^2 x + y$, $x''' = -\Omega^2 x' + y'$.

Therefore an additional equation for $y(t)$ is required. In order to find it let us differentiate (9) twice with respect to time *t* and then add termwise equation (9) multiplied by Ω^2 to the result:

$$x^{IV} + \Omega^2 x'' + h\left(x''' + \Omega^2 x'\right) + \omega^2\left(x'' + \Omega^2 x\right) + \Omega^2\varphi + \varphi''_{xx}\cdot\left(x'\right)^2 + \varphi'_x x'' = 0 \tag{10}$$

After change of variables this equation can be written in another way:

$$\begin{cases} x'' + \Omega^2 x = y; \\ y'' + hy' + \omega^2 y + \Phi\left(x, x', y\right) = 0 \end{cases} \tag{11}$$

where $\Phi\left(x, x', y\right) = \Omega^2\varphi + \varphi''_{xx}\cdot\left(x'\right)^2 + \varphi'_x\cdot\left(y - \Omega^2 x\right)$.

Both (10) and (11) represent a dynamical system which is equivalent to the initial system (9).

Since the new system has higher order than the initial one, it has more solutions than the initial one. Let us analyze these solutions. Initial equation (9) can be represented in the following form:

$$F\left(x'', x', x\right) = f\cos\left(\Omega t\right) \tag{12}$$

Therefore (10) can be written as $F'' + \Omega^2 F = 0$. This equation can be easily solved for F: $F\left(x'', x', x\right) = a\cos\left(\Omega t + \psi\right)$ where a and ψ are arbitrary constants. It can be concluded from this solution that *all possible solutions of equivalent systems (10) and (11) correspond to various possible combinations of amplitude and phase of external excitation*. Among others the solutions of autonomous system $F\left(x'', x', x\right) = 0$ are also included in that set (case $a = 0$).

It can be noted that changes in ψ do not affect shape of periodical orbits of the solutions of the equation $F\left(x'', x', x\right) = a\cos\left(\Omega t + \psi\right)$. Thus even more important conclusion arises:

▲ Statement 1. All possible periodic orbits on the phase plane (x, x') and in the state space (x, x', y, y') found by means of equivalent systems (10) or (11) correspond to various values of a single parameter - amplitude *f* of external excitation.

For convenience equivalent systems (10) or (11) will be called *equivalent systems of 1st type* in the subsequent considerations.

1.2. Equivalent dynamical system of 2nd type

Differentiating (12) with respect to time *t* yields: $\left[F\left(x'', x', x\right)\right]' = -\Omega f\sin\left(\Omega t\right)$. Taking both this equation and (12) one can obtain : $\left(F'\right)^2 + \Omega^2 F^2 = \Omega^2 f^2$ or in extended form:

$$\left(x''' + hx'' + \omega^2 x' + \varphi'_x\cdot x'\right)^2 + \Omega^2\left(x'' + hx' + \omega^2 x + \varphi\left(x\right)\right)^2 = \Omega^2 f^2 \tag{13}$$

Equation $\left(F'\right)^2 + \Omega^2 F^2 = \Omega^2 f^2$ can be solved for F as $F\left(x'', x', x\right) = f\cos\left(\Omega t + \psi\right)$ where ψ is an arbitrary constant. It can be concluded from such solution that *all possible solutions of the equivalent system (13) correspond to all possible variations of the phase of external excitation*.

▲ Statement 2. All possible periodic orbits on the phase plane (x, x') found by means of equivalent system (13) are exactly the same as in the initial system (9).

For convenience the equivalent system (13) will be called the *equivalent system of 2nd type* in the subsequent considerations.

1.3. Equivalent system investigation via invariant manifolds methodology

When investigation of system (9) is focused on periodical solutions only, it can be noted that the solution itself and all its derivatives change in time in a coherent manner. Therefore one may expect to find the following dependencies in (11):

$$\begin{cases} y = y\left(x, x'\right); \\ y' = z = z\left(x, x'\right) \end{cases} \tag{14}$$

Following [10] these dependencies can be constructed using invariant manifolds methodology. It can be applied to the equivalent system of 1st type because it consists of two ODEs of 2nd order.

Let us denote $x = u, x' = v$. The system (11) is now written in standard form:

$$\left\{u' = v;\quad v' = -\Omega^2 u + y;\quad y' = z;\quad z' = -\left(hy' + \omega^2 y + \Phi(u,v,y)\right)\right. \tag{15}$$

Differentiation with respect to time t now becomes a partial differential operator: $\frac{d}{dt} = v\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}$, which leads to the following PDEs:

$$\left\{u\frac{\partial y}{\partial u} + \left(y - \Omega^2 u\right)\frac{\partial y}{\partial v} = z;\qquad u\frac{\partial z}{\partial u} + \left(y - \Omega^2 u\right)\frac{\partial z}{\partial v} = -\left(hz + \omega^2 y + \Phi(u,v,y)\right)\right. \tag{16}$$

Such PDEs can be solved in different ways (the solution can be written in form of power series [10] or found via Galerkin method [20]). Here the solution is found in power series form:

$$\left\{y = a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2 + \ldots;\quad z = b_1 u + b_2 v + b_3 u^2 + b_4 uv + b_5 v^2 + \ldots\right. \tag{17}$$

Solution (17) is substituted into (16). (At this stage the functions $\Phi(u,v,y)$ and $\varphi(x)$ are considered to be polynomials or they should be expanded in power series otherwise). When terms of the same power of *u* and *v* are equated in the obtained equalities this leads to a recurrent system of algebraic equations with respect to unknown coefficients a_i, b_i. Among others there exists a closed subsystem of nonlinear equations with respect to a_1, a_2, b_1, b_2. Once a_1, a_2, b_1, b_2 are found, all other coeffisients are evaluated in *unique way*. Equations with respect to a_1, a_2, b_1, b_2 are:

$$\left\{a_1 a_2 - \Omega^2 a_2 = b_1;\quad b_2 a_1 - \Omega^2 b_2 = -hb_1 - \omega^2 a_1;\quad a_1 + a_2^2 = b_2;\quad b_1 + b_2 a_2 = -hb_2 - \omega^2 a_2\right. \tag{18}$$

Consecutive elimination of unknowns produces an equation of the 6-th degree with respect to a_1. When solved,this equation allows one to find two real roots (in the small damping case $h \ll \omega^2$):

- $a_1 = \Omega^2 - \omega^2$ *This is a parasitic solution.* After all transformations it produces the expansion $y = -hx' - \omega^2 x + \Omega^2 x - \varphi(x)$. Taking into account that $y = x'' + \Omega^2 x$ this leads to equation $x'' = -hx' - \omega^2 x - \varphi(x)$. So *the current solution corresponds to free oscillations, not forced ones.*
- $a_1 = 0$ - when substituted into other equations this root leads to the trivial solution of the system (18): $\{a_1 = a_2 = b_1 = b_2 = 0\}$ – when substituted further, this solution allows one to find *nonzero* $a_3, a_4, a_5, b_3, b_4, b_5$ and so on.

The above analysis leads to an important conclusion:

▲ Statement 3. In the state space (x, x', y, y') there exist unique hypersurface (14) which corresponds to forced oscillations in the initial system (9) and passes through the equilibrium position of that system in the space of variables (x, x', y, y').

Consider now periodic solutions if the initial system (9). If amplitude of external excitation *f* is varied, periodic orbits form a hypersurface in the space of variables (x, x', y, y'). This surface passes through the equilibrium position. The periodic solutions of (9) are at the same time the solutions of the equivalent system (11). This leads to the conclusion:

▲ Statement 4. For each point on the surface (14) which satisfy equations (16) there exist a closed trajectory in the space of variables (x, x', y, y') which passes through that point and corresponds to periodic solution of (9) under some value of external excitation *f*.

1.4. Rauscher expansions construction. Equivalent dynamical system of 3rd type.

It can be derived from (9) that $\cos(\Omega t) = \left(x'' + hx' + \omega^2 x + \varphi\right)/f$ and $\sin(\Omega t) = -\left(x''' + hx'' + \omega^2 x' + \varphi'_x \cdot x'\right)/\Omega f$. Taking into account dependencies (14) obtained via the *equivalent system of 1st type* one has the following Rauscher approximations:

$$\left\{\begin{array}{l}\cos(\Omega t) = \left(y(x,x') + hx' + \left(\omega^2 - \Omega^2\right)x + \varphi\right)/f \\ \sin(\Omega t) = -\left(z(x,x') + h\left(y(x,x') - \Omega^2 x\right) + \left(\omega^2 - \Omega^2\right)x' + \varphi'_x \cdot x'\right)/\Omega f\end{array}\right. \tag{19}$$

which exactly corresponds to the form (8) used by other Rauscher-like methods.

In fact, the system (19) can be considered as a couple of algebraic equations with respect to x and x'.The solution of this couple of equations - x and x' - depends on t as it is a parameter which is introduced in (19) through periodical functions. This means that x and x' can only be periodical functions if the dependencies (19) take place simultaneously.

Taking into account the identity $\cos^2(\Omega t)+\sin^2(\Omega t)=1$ one can obtain from (19):

$$\frac{1}{\Omega^2}\left(z(x,x')+h\left(y(x,x')-\Omega^2 x\right)+\left(\omega^2-\Omega^2\right)x'+\varphi'_x\cdot x'\right)^2+\left(y(x,x')+hx'+\left(\omega^2-\Omega^2\right)x+\varphi\right)^2=f^2 \quad (20)$$

This expression is not changed if the phase of external load is varied by an arbitrary constant. This equivalent system of a new type can be also considered as the result of substitution (14) into (13) and division of the latter by Ω^2. Due to the considerations above the system (20) has remarkable properties: *all possible solutions of the equivalent system (20) are exclusively periodic solutions of the initial system (9) wich correspond to various values of the phase of external load.*

▲ **Statement 5. The only trajectories on the phase plane (x,x') which can be found by means of the equivalent system (20) are periodic orbits of the initial system (9).**

The equivalent system (20) will be called the *equivalent system of 3rd type*.

1.5. Geometrical interpretation of equivalent system of 3rd type.

Let us denote left hand side of (20) as W_0 and left-hand side minus right-hand side of (20) as W:

$$W_0=\frac{1}{\Omega^2}\left(z(x,x')+h\left(y(x,x')-\Omega^2 x\right)+\left(\omega^2-\Omega^2\right)x'+\varphi'_x\cdot x'\right)^2+\left(y(x,x')+hx'+\left(\omega^2-\Omega^2\right)x+\varphi\right)^2 \quad (21)$$

$$W=W_0-f^2$$

In such case it is clear that equation (20) is true only when $W=0$. Therefore periodic orbits of the system (9) are the lines of intersection of function W and plane (Fig. 1,a).

On the other hand, it is clear from (21) that equation (20) is true when $\sqrt{W_0}=|f|$. Therefore *level curves of the surface $\sqrt{W_0}$ are periodic orbits of the system (9) corresponding to different levels of external excitation (Fig. 1,b). Similar statement is correct for W too.*

The conditions $W=0$ and $\sqrt{W_0}=|f|$ can be treated as a certain type of periodicity conditions for the solutions of the initial system. It also should be noted here that surfaces W and W_0 posess certain *invariance properties* with respect to forced periodic motions in (9). In particular, while all parameters of the system are fixed, different periodic motions of the system correspond to the same set of level curves of these surfaces (of the same level).

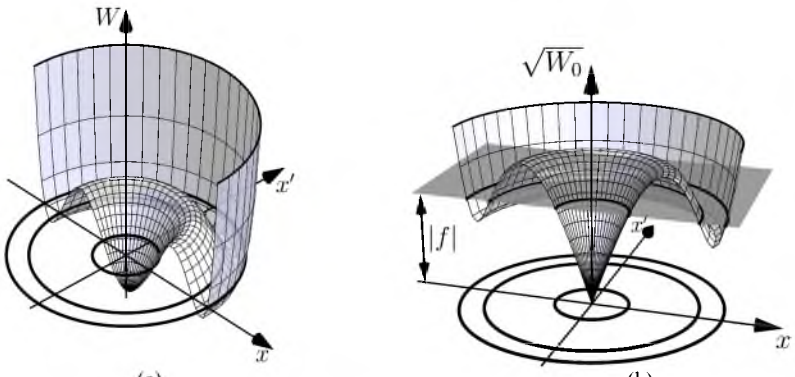

Figure 1. Geometrical interpretation of surfaces W – (a) and W_0 – (b).

2. Investigation via harmonic balance method. Construction of W and W_0 via level curves.

A priori construction of the surfaces W and W_0 may provide investigator with some valuable information about the dynamical system being investigated. If the surfaces (14) are obtained analytically one can use expressions (21) to define W and W_0. On the other hand, if one desires to

study large-amplitude motions this makes evaluation of (14) via power series inappropriate while other methods of solving PDEs (16) may require some sophisticated computations.

However, if some periodic solution $x(t)$ in (11) is found then one can build the corresponding *section* (level curve) of W and W_0 since $W(x,x') = const, W_0(x,x') = const$ for every such solution.

So W and W_0 can be obtained section by section via their level curves (see for example Fig. 3,b or Fig. 6). One can then interpolate through the obtained sections. This is an alternative approach to solving the problem of W and W_0 construction. Since the level curves of W and W_0 are closed curves corresponding to various periodic orbits of the *equivalent dynamical system of 1st type* (11), harmonic balance method can be engaged to calculation of such orbits in many cases.

2.1 Application of harmonic balance method to equivalent dynamical system of 1st type

The solution of the *equivalent dynamical system of 1st type* (11) can be written as truncated Fourier series taking into account harmonics up to n-th one:

$$\begin{aligned} x &\approx A_0 + A_1\cos(\Omega t) + \sum_{k=2}^{n} A_k\cos(k\Omega t) + B_k\sin(k\Omega t) \\ y &\approx \Omega^2\left(A_0 + \sum_{k=2}^{n}(1-k^2)\left(A_k\cos(k\Omega t) + B_k\sin(k\Omega t)\right)\right) \end{aligned} \tag{22}$$

Note that due to arbitrariness in choosing time reference point for (11) the expression (22) is chosen is such way that it does not contain term $B_1\sin(\Omega t)$.

The first equation of (11) is satisfied by (22) automatically. If relations (22) are substituted into the second equation of (17), one can obtain *2n-1* algebraic equations of harmonic balance method with respect to unknowns A_k, B_k. The number of equations is *2n-1*, not *2n* as it may be expected.

An additional equation is required here in order to select some particular periodic trajectory in the state space (x,x',y,y') among others. It is proposed here to construct this additional equations in the following way: $\bar{h}(x,x') = \lambda$, where $\bar{h}$ - averaged total energy of the system per period which can be written in the following form:

$$\bar{h}(t) = \frac{\Omega}{2\pi}\int_{t}^{t+\frac{2\pi}{\Omega}}\left[T(x'(\tau)) + \Pi(x(\tau))\right]d\tau \tag{23}$$

where T and Π - are kinetic and potential energy respectively. If periodic motion $x(t)$ with period $\frac{2\pi}{\Omega}$ is considered then $\bar{h}(t) = const\ \forall t$. So the particular value λ of averaged total energy per period $\bar{h}$ can be taken as a continuation parameter for evaluating periodic trajectories.

Complete system of algebraic equations of harmonic balance method can be written as follows:

$$\begin{cases} \Phi_i(A_0...A_n, B_0...B_n) = 0 \ \}, \ (i = 1...\ 2n-1) \\ \bar{h}(x,x') = \lambda \end{cases} \tag{24}$$

Equations (24) can be writted in general form as $\{\Phi_i([A],\lambda) = 0, \ (i = 1...\ 2n)$ where $[A] = \{A_0...A_n, B_0...B_n\}^T = \{a_1,...,a_{2n+1}\}^T$ - vector of unknowns.

Now some continuation tecnique can be applied to (24) in order to find the solution $[A]$. Or equations (24) may be transformed into differential form with respect to independent variable λ and solved via numerical integration.

Once $A_0...A_n, B_0...B_n$ are obtained for some fixated value of λ, one can evaluate $x(t)$ and then build corresponding section of the W and W_0 surfaces. Since $x(t)$ obtained via (22) is an *approximate* periodic solution of (11) the conditions $W(x,x') = const, W_0(x,x') = const$ are not fully satisfied. Instead one has $W(x,x') = W^* + \Delta(t), W_0(x,x') = W_0^* + \Delta(t)$ where W^* and W_0^* are unknown exact values of W and W_0, and $\Delta(t)$ is the residual. If large enough number of harmonics n is taken, then

$\Delta(t)$ is a small oscillating function. So the averaged values of $W(x,x')$ over oscillations' period can be taken as approximation for W^*. Let us denote averaged values of W and W_0 as $\overline{w}$ and $\overline{w}_0$ respectively.

Consider quantity $\overline{w}$ - averaged value of W for some solution $x(t)$ of the system (11). It follows from the above that $\overline{w}$ is a function of continuation parameter: $\overline{w}=\overline{w}(\lambda)$. This dependency has the same major property as function W: if the periodic orbit corresponding to $x(t)$ is at the same time a periodic orbit of the ininial system (9), then $\overline{w}(\lambda)=0$. Therefore if some zero of the function $\overline{w}=\overline{w}(\lambda)$ is evaluated (denote it as $\lambda=\lambda_0$), then corresponding set of coefficients $A_0...A_n,B_0...B_n$ can be found. Thus approximate solution $x_{appr}(t)$ can be found and corresponding periodic orbit of the initial system can be built (see Example 2 in Section 3).

3. Examples

As an example Duffing equation is considered: $mx''+\beta x'+cx+\tilde{\gamma}_1x^2+\tilde{\gamma}_2x^3=\tilde{f}\cos(\Omega t)$. It can be written in the form (9):

$$x''+hx'+\omega^2x+\gamma_1x^2+\gamma_2x^3=f\cos(\Omega t) \tag{25}$$

where $h=\beta/m$, $\omega^2=c/m$, $\gamma_1=\tilde{\gamma}_1/m$, $\gamma_2=\tilde{\gamma}_2/m$, $f=\tilde{f}/m$

3.1 Example 1

Parameters of the system are taken as follows: $m=1$, $c=1$, $\beta=0.05$, $\tilde{\gamma}_1=0$, $\tilde{\gamma}_2=0.3$, $\tilde{f}=0.1$, $\Omega=1.2$. *Equivalent system of the 1st type* for such system is the following:

$$\begin{cases} x''+\Omega^2x=y \\ y''+hy'+\omega^2y+6\gamma x(x')^2+3\gamma x^2\left(y-\Omega^2x\right)+\gamma\Omega^2x^3=0 \end{cases} \tag{26}$$

Invariant manifold corresponding to forced oscillations can be built for this equivalent system by means of the following PDEs:

$$\left\{u\frac{\partial y}{\partial u}+\left(y-\Omega^2u\right)\frac{\partial y}{\partial v}=z;\ u\frac{\partial z}{\partial u}+\left(y-\Omega^2u\right)\frac{\partial z}{\partial v}=-\left(hz+\omega^2y+6\gamma uv^2+3\gamma u^2\left(y-\Omega^2u\right)+\gamma\Omega^2u^3\right)\right. \tag{27}$$

where $u=x, v=x'$.

The invariant manifold corresponding to forced oscillations is calculated using power series (terms up to 5th degree were kept):

$$\begin{aligned} y=&-0.072224u^3-0.002717u^2v+0.150468uv^2+0.000629v^3-0.002751u^5-7.86463210^{-7}u^4v- \\ &-0.007306u^3v^2-0.000378u^2v^3+0.006466uv^4+0.000053v^5 \\ z=&\ 0.003913u^3-0.650020u^2v-0.008152uv^2+0.150468v^3+0.000197u^5-0.014442u^4v+ \\ &+0.000265u^3v^2-0.013890u^2v^3-0.000660uv^4+0.006467v^5 \end{aligned} \tag{28}$$

The surfaces $y(u,v)$ and $z(u,v)$ obtained analytically are plotted on Figure 2. Having (28) calculated one can construct functions W and W_0 according to the formulae (21). Graphical representation of W is given on Figure 3. Condition $W=0$ results in periodic orbits of the system (25) on the phase plane. They are shown on Figure 4,a.

When expressions (28) are known, Rauscher expansions can be built via formulae (19). Their correctness may be checked in the following way. If values of t, x and x' are taken from the results of numerical integration and substituted into the Rauscher expansions $\cos(\Omega t)=C(x,x')$, $\sin(\Omega t)=S(x,x')$, these expressions must be satisfied. Such check is illustrated on Figure 4,b. Solid lines correspond to the left-hand side of the Rauscher expansions and dots correspond to right-hand side. Calculations are done for the maximum amplitudes mode on Fig.4,a.

If the system (26) is investigated semi-analytically by means of harmonic balance approach

(Section 2), the results of that study confirm analytical ones. For example, on Figure 2 the surfaces $y(u,v)$ and $z(u,v)$ obtained semi-analytically are shown as set of concentric lines. When plotted together with analytically obtained results, these graphs fit well (Fig. 2). Graphical presentation of W obtained semi-analytically section-wise is given on Figure 3,b.

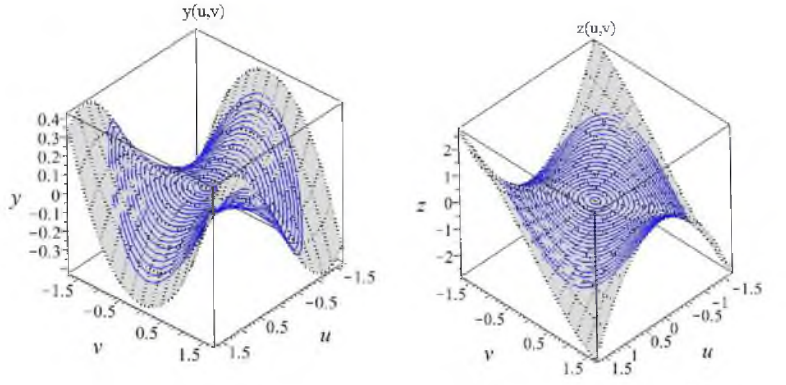

Figure 2. Comparison of the invariant manifold surfaces of the equivalent system of the 1st type (26) obtained analytically (grey surfaces) and via harmonic balance (concentric lines): $y(u,v)$ - (a), $z(u,v)$ - (b).

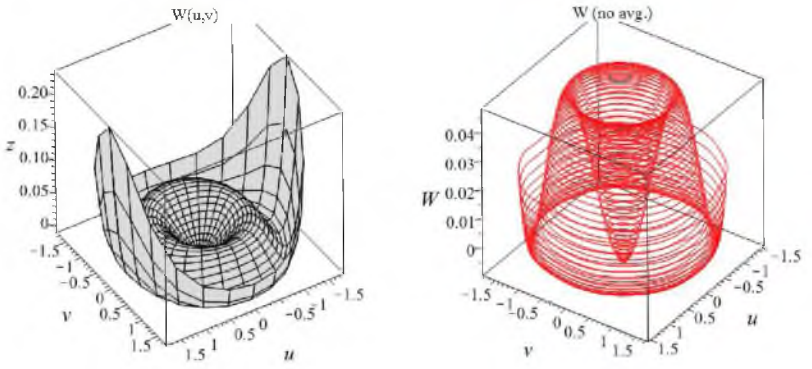

Figure 3. Graphical representation of W obtained analytically (a) and semi-analytically (b)

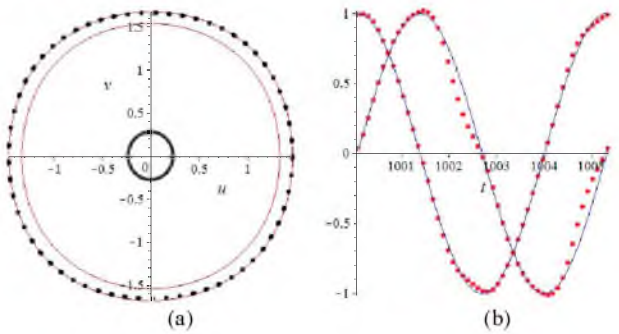

Figure 4. (a) - Periodical orbits obtained analytically (thin lines) are compared with the results of numerical integration (dots); (b) - Rauscher expansions quality check.

3.2 Example 2

Parameters of the system are taken as follows: $m=1, c=1, \beta=0.02, \tilde{\gamma}_1=0.35, \tilde{\gamma}_2=0.04, \tilde{f}=0.1, \Omega=0.8$. This system is studied via harmonic balance approach. On the Figure 5,a the dependency $\overline{w}(\lambda)$ is

shown when $\lambda = \bar{h}$ - averaged total energy of the system according to (23). The dependency $\bar{w}(\lambda)$ is built for different number of harmonics used. Harmonic balance equations were converted to differential ones and integrated numerically.

The calculation predicts 5 periodical solutions for the given parameter set. This is confirmed when one constructs amplitude-frequency response of the system (Fig. 5,b). Corresponding periodic orbits are depicted on Figure 5,c (circles) and compared with the results of numerical simulation (lines).

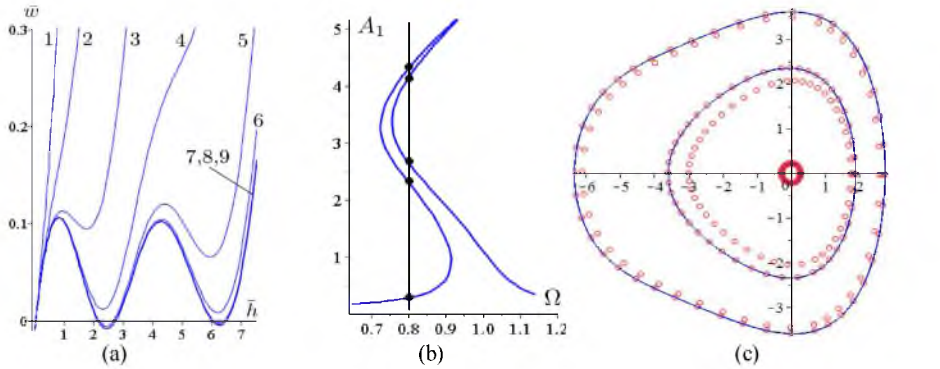

Figure 5. (a) - The dependencies $\bar{w}(\lambda)$ obtained for the case λ is the averaged total energy. Numbers show the number of harmonics used; (b) - amplitude-frequency response of the system (1st harmonic); (c) - closed trajectories corresponding to $\Omega = 0.8$

Surface W obtained section-wise is shown on Figure 6. The results on Fig. 6,a are obtained without averaging values for each section. The results on Fig. 6,b are obtained after averaging. The sections with to much 'wavyness' on Fig. 6,a indicate that the number of harmonics taken into account may become insufficient if one desires to investigate regimes with even higher amplitudes.

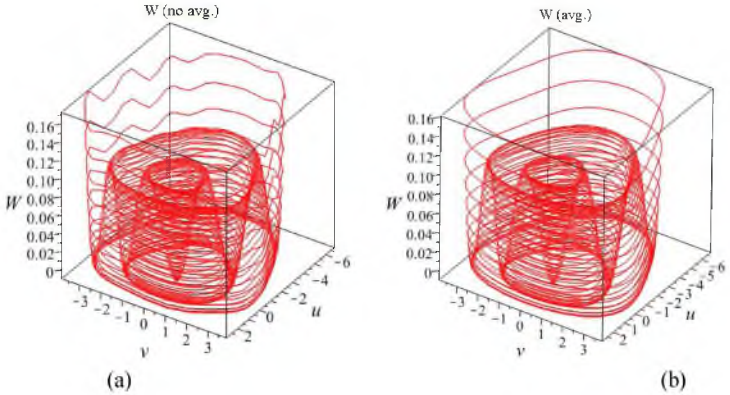

Figure 6. Sections of the W surface obtained without (a) and with (b) averaging the results.

Conclusions

In the present paper a new non-iterative variant of Rauscher method is considered. It does not have convergence issues as opposed to iterative approaches. This method can be used in analysis of forced oscillations in 1-DOF nonlinear systems described by analytical functions.

It is shown that three different types of equivalent authonomous dynamical systems can be built for a given 1-DOF non-autonomous one. Based on the equivalent system of 3rd type such functions $W(x,x')$ and $W_0(x,x')$ can be constructed so that their level curves correspond to periodic orbits of the initial non-autonomous system. When represented graphically, the condition $\sqrt{W_0(x,x')} = |f|$ shows exact correspondence between the amount of the external excitation f and the number and shape of existing periodical orbits of the system.

Once W and W_0 are carefully computed one can: (i) - obtain different periodical orbits which correspond to forced oscillations in the initial system; (ii) - estimate amplitudes of vibrations for

these regimes; (iii) - track bifurcations of periodical solutions of (9) with respect to change of amplitude of external excitation f Functions $W(x,x')$ and $W_0(x,x')$ can be built *a priori* via finding the invariant manifold of the *equivalent system of 1st type*. The same is correct for the Rauscher expansions that can be constructed in the form $\cos(\Omega t) = C(x,x')$, $\sin(\Omega t) = S(x,x')$.

It is also shown that the *equivalent system of 1st type* can be investigated by means of harmonic balance method. Then both the functions $W(x,x')$ and $W_0(x,x')$ can be built section-wise.

Despite the domain of applicability of the proposed method is the matter of further investigations, the results of the current investigations are promising, especially if this approach is generalized for multi-DOF systems.

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